On the numerical solution of the Boussinesq equations

by

Geir Pedersen
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Abstract

A compact theory for weakly nonlinear, dispersive surface waves is presented. Different formulations of the final equations are discussed in view of numerical methods, and a difference technique is applied to one particular formulation. We then proceed to describe the implementation of different boundary conditions and finally develop a corrected difference technique.

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## 1 Introduction

The literature of hydrodynamics offers a large number of equations for weakly dispersive waves of finite amplitude. Generally, these equations are essentially similar and may be recognized as modifications of Boussinesq or KdV equations. The development of the theories follow many different paths and reach various destinations depending on the authors original motivation and personal taste. In the present paper we are concerned with numerical solution of the Boussinesq equations and approach the topic thereafter. For completeness and convenience we have included a derivation of such equations even though the final results are obtainable from other papers like Wu (1981) [13] and Peregrine (1972) [10].

Usually Boussinesq or KdV equations are obtained by expansions in two small parameters. In the present report the parameters are named $\alpha$ and $\epsilon$ and are measures of nonlinearity and dispersion, respectively. The expansions are generally carried out one step beyond leading order in both parameters, but some authors (Pedersen & Gjevik 1983 [6], Ertekin et al. 1984 [1], 1986 [2]) have made a point of retaining higher order terms in $\alpha$. Therefore, even though the significance of the extra terms are questionable, we will work them out at the first few basic steps of the derivation. However, our final equations always consist of terms of order 1, $\alpha$ and $\epsilon$. It should also be noted that all error terms generally contain $\epsilon$ as a factor which means that the fully nonlinear hydrostatic equations are contained within the description.

It is announced above that our goal is to achieve formulations which are convenient for numerical solution. We will mainly be guided by the following points:

- Simplicity in the sense that the number and complexity of the terms of the equations are kept at a satisfactory level.

- We have to solve implicit sets of equations at each time-step. The size and number of such sets should of course be reduced as much as possible. Even more important is to avoid dealing with ill-conditioned sets. Generally the implicit sets are similar to the Helmholtz' equation:

$$ f - \gamma \nabla^2 f = g $$

where $g$ is known. Formulations giving negative $\gamma$ should generally be avoided.

- The order of the equations should be kept low. This is not crucial for finite differences, but may be essential for effective use of finite elements.
Often depth values must be obtained from interpolation between discrete measured values etc. Higher order derivatives of the depth may thus be hard to define. Derivation beyond second order is not to be recommended.

2 Basic equations.

Weakly dispersive and nonlinear shallow water equations have been derived in numerous articles and text books. We will generalize the procedure used by Pedersen (1988) [9] to allow for large spatial depth variations. The equations are formulated in a coordinate system with horizontal axes, \( ox^* \) and \( oy^* \), in the undisturbed water level and the vertical axis, \( oz^* \), pointing upwards. The asterisks indicate dimensional quantities. The fluid is confined to \(-h^* < z^* < \eta^*\) and the velocity potential is denoted by \( \Phi^* \). We introduce a characteristic depth \( h_0 \), wavelength \( \ell \), amplitude: \( a h_0 \), and dimension-less variables according to

\[
\begin{align*}
    z^* &= h_0 z \\
    t^* &= \ell (g h_0)^{-\frac{1}{2}} t \\
    h^* &= h_0 h(x,y,t) \\
    \eta^* &= a h_0 \eta \\
    \Phi^* &= \alpha \ell (g h_0)^{\frac{1}{2}} \Phi \\
    p^* &= \alpha \rho g h_0 p
\end{align*}
\]

where \( p^* \) is an external pressure applied to the surface, \( \rho \) is the density of the fluid and \( g \) is the constant of gravity. We note that the variations of the depth with time are prescribed to be of the same order as the variations of the surface. This requirement can be met in two ways:

1. The amplitudes of the depth variations are small in the sense that we may write \( h = h^{(i)}(x,y) + \alpha \tau(x,y,t) \). In this case we have \( \partial^n h / \partial t^n = O(\alpha) \) for all \( n \geq 1 \).

2. The time scale of the depth variations are much larger than the typical wave period which means: \( \partial^n h / \partial t^n = O(\alpha^n) \)

We will assume the first, which is the weakest of these restrictions.

2.1 General formulations

In dimension-less form the governing equations for irrotational, incompressible flow become

\[
\varepsilon \nabla^2 \Phi = -\Phi_{zz} \quad -h < z < \alpha \eta
\]

\[
\Phi_t + \frac{1}{2} \alpha (\nabla \Phi)^2 + \frac{1}{2} \alpha \varepsilon^{-1} (\Phi_z)^2 + \eta + p = 0 \\
\varepsilon (\eta_t + \alpha \nabla \Phi \cdot \nabla \eta) = \Phi_z \\
\varepsilon (h_t + \nabla \Phi \cdot \nabla h) = -\Phi_z
\]

where \( \varepsilon \) is a small parameter representing the viscous effects.
where indices denote partial differentiation, $\nabla$ is the horizontal component of the gradient operator and $\epsilon = \frac{h_0^2}{\ell^2}$ which, according to the long wave assumption, is small. Provided the lateral boundary conditions and the initial conditions are independent of $z$ to zeroth order in $\epsilon$, the above equations imply:

$$\Phi(x, y, z, t) = \Phi(x, y, 0, t) + O(\epsilon) \quad (7)$$

A depth averaged velocity potential is defined by:

$$\phi = (h + \alpha \eta)^{-1} \int_{-h}^{\alpha \eta} \Phi \, dz \quad (8)$$

We also define $\psi$ as the value of $\Phi$ at the free surface:

$$\psi \equiv \Phi|_{z=\alpha \eta} \quad (9)$$

From the requirement of volume conservation in a vertical fluid column we find:

$$\eta_t + r_t = -\nabla \cdot \mathbf{Q} \quad (10)$$

where $\mathbf{Q}$ is the integrated, horizontal volume flux density. Integration of equation (3) combined with (5) and (7) gives:

$$\Phi = \psi + \epsilon R \quad (11)$$

where $R$ is defined by:

$$R \equiv -(\alpha \eta - z)\mathcal{D} - \frac{1}{2}(\alpha \eta - z)^2 \nabla^2 \psi + O(\epsilon^2) \quad (12)$$

and the differential operator $D$ is:

$$D \equiv \frac{\partial}{\partial t} + \alpha \nabla \Phi(x, y, \alpha \eta) \cdot \nabla \quad (13)$$

The relationship between $\psi$ and $\phi$ becomes:

$$\phi = \psi + \epsilon \mathcal{R}$$
$$= \psi - \epsilon (\frac{1}{2} H \mathcal{D} \eta + \frac{1}{6} H^2 \nabla^2 \phi) + O(\epsilon^2) \quad (14)$$
$$= \psi - \epsilon (\frac{1}{2} h \eta_t + \frac{1}{6} h^2 \nabla^2 \phi) + O(\epsilon^2, \alpha \epsilon)$$

where $H = h + \alpha \eta$ denotes the total depth. For the flux density $\mathbf{Q}$ we obtain:

$$\mathbf{Q} = H \nabla \Phi = H \nabla (\psi + \epsilon \mathcal{R}) + \alpha \epsilon \mathcal{R} \nabla \eta$$
$$\quad + \epsilon (\mathcal{R} - R(x, y, -h)) \nabla h \quad (15)$$
The quantity between the first two braces is recognized as $\phi$. Substituting for $R$ from equation (12) we find:

$$\bar{Q} = H \nabla \psi - \epsilon H \left( \frac{1}{2} H \nabla D \eta + \frac{1}{6} H^2 \nabla (\nabla^2 \psi) \right)$$

$$- \epsilon \alpha H \left( D \eta - \frac{1}{2} H \nabla^2 \psi \right) \nabla \eta + O(\epsilon^2)$$

$$= H \nabla \psi - \epsilon h \nabla \eta + \frac{1}{6} h \nabla^2 (\nabla^2 \psi) + O(\epsilon^2, \epsilon \alpha)$$

(16)

In terms of the depth averaged potential, $\phi$, the flux becomes:

$$\bar{Q} = H \nabla \phi + \epsilon H \left( \frac{1}{2} D \eta \nabla h + H \nabla^2 \phi \nabla \left( \frac{1}{3} H + \frac{1}{2} \alpha \eta \right) \right) + O(\epsilon^2)$$

(17)

$$= H \nabla \phi + \epsilon h \left( \frac{1}{2} \eta_t + \frac{1}{3} h \nabla^2 \phi \nabla h + O(\epsilon^2, \epsilon \alpha) \right)$$

We note that

$$\bar{Q} = H \nabla \phi + O(\epsilon^2, \epsilon \alpha)$$

(18)

provided $\nabla h = O(\alpha)$

From the definition of $\psi$ and equation (5) we have:

$$\nabla \psi (x, y, t) = \nabla \Phi (x, y, \alpha \eta, t) + O(\alpha \epsilon)$$

$$\psi_t (x, y, t) = \Phi_t (x, y, \alpha \eta, t) + O(\alpha \epsilon)$$

(19)

The leading error terms are easily calculated from (5) but are of no interest at present. From the Bernoulli equation (4) we now find:

$$\psi_t + \frac{1}{2} \alpha (\nabla \psi)^2 + \eta + p = + O(\alpha \epsilon)$$

(20)

which, together with equation (10), constitutes a basic set of Boussinesq equations. To the lowest order we thus have:

$$\eta_t + \tau_t = -\nabla \cdot (h \nabla \psi) + O(\alpha, \epsilon)$$

(21)

$$\psi_t + \eta + p = O(\alpha)$$

(22)

These two equations, together with the expressions for $\bar{Q}$ and the relation between $\psi$ and $\phi$, can be used to derive a variety of Boussinesq equations. One should also notice that $\psi$ in the above relation may be replaced by $\phi$ without reducing the accuracy.

### 2.2 Boundary conditions at a rigid wall

At a vertical and impermeable wall, with unit normal $\vec{n}$, we have the boundary condition:

$$0 = \Phi_n \equiv \nabla \Phi \cdot \vec{n}$$

(23)
which can be combined with (4) to give:

\[ \eta_n = -p_n \]  
\[ (24) \]

For \( p_n = 0 \) this will turn into a symmetry condition for \( \eta \). From (23) it immediately follows:

\[ \psi_n = 0 \]  
\[ (25) \]

Expressed in terms of \( \phi \) the condition becomes slightly more complicated. The simplest way to obtain an equation for \( \phi \) is to combine the integrated form of the no-flux condition:

\[ \tilde{Q} \cdot \tilde{n} = 0 \]  
\[ (26) \]

and (17) to find:

\[
\phi_n = -\varepsilon(\frac{1}{2}D\eta + \frac{1}{3}H\nabla^2\phi)h_n - \varepsilon\alpha\frac{1}{2}H^2\nabla^2\phi p_n + O(\varepsilon^2) \\
= \varepsilon(\frac{1}{6}h\nabla^2\phi + \frac{1}{2}\nabla h \cdot \nabla \phi + \frac{1}{2}r_t)h_n + O(\varepsilon^2, \alpha \varepsilon) 
\]  
\[ (27) \]

## 2.3 Formulation for \( \eta \) and \( \psi \)

The Bernoulli equation can be used in its very simple form (20), and at rigid boundaries we have simple symmetry the conditions (23) and (24). Thus, most of the problems will in this case be associated with the continuity equation. The major disadvantage using the expression for \( \tilde{Q} \) given in (16), is the presence of the third derivatives of \( \psi \) which will give rise to fourth derivatives when substituted into (10). From (21) we may obtain:

\[
\nabla(\nabla^2\psi) = -\nabla(\frac{\eta_t + r_t}{h} + \nabla \psi \cdot \nabla h) + O(\varepsilon, \alpha) 
\]  
\[ (28) \]

which gives:

\[
\tilde{Q} = H\nabla \psi - \varepsilon h\{\frac{1}{3}h\nabla^2\eta_t - \frac{1}{6}(h\nabla r_t + (\eta_t + r_t)\nabla h + \nabla(\nabla h \cdot \nabla \psi))\} + O(\varepsilon^2, \alpha \varepsilon) 
\]  
\[ (29) \]

This equation will simplify substantially for constant depth, but there is generally no gain in reducing the order of the highest \( \psi \) derivative from 4 to 3, especially not when the price is introduction of third derivatives of \( h \).

Another possibility is to remove the higher order \( \eta \) term from the expression for \( \tilde{Q} \). Using (21) we find:

\[
\tilde{Q} = H\nabla \psi + \frac{1}{2}ch^2(\nabla \nabla \cdot (h\nabla \psi) + \nabla r_t - \frac{1}{3}h\nabla(\nabla^2 \psi)) + O(\varepsilon^2, \alpha \varepsilon) 
\]  
\[ (30) \]

This form of \( \tilde{Q} \) gives a set of Boussinesq equations which may be solved numerically by an almost explicit technique, though at the cost of severe restrictions on the time increments.
2.4 Formulation for $\eta$ and $\phi$

In this case the appropriate form of the Bernoulli equation is obtained by combining (14) and (20):

$$ \phi_t + \frac{1}{2} \alpha (\nabla \phi)^2 + \eta + p - \epsilon (\overline{R})_t = O(\epsilon^2, \alpha \epsilon) \quad (31) $$

By once again using (21) we find:

$$ \phi_t + \frac{1}{2} \alpha (\nabla \phi)^2 + \eta + p - \epsilon \left( \frac{1}{2} h r_t + \frac{1}{2} h \nabla \cdot (h \nabla \phi) - \frac{1}{6} h^2 \nabla^2 \phi \right)_t = O(\epsilon^2, \alpha \epsilon) \quad (32) $$

Since the depth is independent of time to order 1, this equation simplifies to:

$$ \phi_t + \frac{1}{2} \alpha (\nabla \phi)^2 + \eta + p - \epsilon \left( \frac{1}{2} h r_t + \frac{1}{2} h \nabla \cdot (h \nabla \phi) - \frac{1}{6} h^2 \nabla^2 \phi \right)_t = O(\epsilon^2, \alpha \epsilon) \quad (33) $$

The representation (17) of $\bar{Q}$ will again give rise to derivatives of the potential which is of higher than second order. However, this time the expression is easily rewritten by use of (21)

$$ \bar{Q} = H \nabla \phi + \epsilon h \left( \frac{1}{2} \eta_t - \frac{1}{3} r_t - \frac{1}{3} \nabla h \cdot \nabla \phi \right) \nabla h + O(\epsilon^2, \alpha \epsilon) \quad (34) $$

We note that no $h$ derivatives of higher than second order will appear in the continuity equation. The implementation of a no-flux boundary condition is, on the other hand, somewhat awkward. The Laplacian may be removed from (27) by use of (33) and (21):

$$ h (2 - \frac{1}{2} \epsilon h^2) \phi_{nt} = h_n \{ \phi_t + \frac{1}{2} \alpha (\nabla \phi)^2 + \eta + p + \frac{1}{2} \epsilon h (r_t + h \phi_{st}) \} + O(\epsilon^2, \alpha \epsilon) \quad (35) $$

where the index $s$ denotes differentiation in the direction that is tangential to the boundary. For constant depth, $h = 1$, we obtain the set:

$$ \eta_t = - \nabla \cdot \{(1 + \alpha \eta) \nabla \phi\} + O(\alpha \epsilon) \quad (36) $$

$$ \phi_t + \frac{1}{2} \alpha (\nabla \phi)^2 + \eta + p - \frac{1}{3} \epsilon \nabla^2 \phi_t = O(\epsilon^2, \alpha \epsilon) \quad (37) $$

which is the set that is solved numerically in [9] and by Wang, Wu & Yates (1988) [12].

2.5 Formulation in terms of velocities

An averaged horizontal velocity $\bar{u}$ can be defined by:

$$ \bar{Q} = H \bar{u} \quad (38) $$

From (10) it immediately follows:

$$ \eta_t + r_t = - \nabla \cdot (H \bar{u}) \quad (39) $$
Noting that $\nabla \psi = \bar{u} + O(\epsilon)$ equation (16) implies:

$$\nabla \psi = u - \frac{1}{2} \epsilon h (\nabla \cdot (h \bar{u})) - \frac{1}{3} h \nabla \cdot (u + \nabla \eta) + O(\epsilon^2, \alpha \epsilon)$$  \hspace{1cm} (40)

Invoking this relation after applying $\nabla$ to (20) yields:

$$\bar{u}_t + \frac{\alpha}{2} \nabla (\bar{u})^2 + \nabla \eta + \nabla p - \frac{1}{2} \epsilon h (\nabla \cdot (h \bar{u}_t)) - \frac{1}{3} h \nabla \cdot (\bar{u}_t + \nabla \eta_t) = O(\epsilon^2, \alpha \epsilon)$$  \hspace{1cm} (41)

which is consistent with the equations that are given by [10] and solved numerically by Pedersen & Rygg (1987) [8] and Rygg (1988).

Relative to the formulation of the previous section, the advantage of a simple continuity equation is minor compared to the introduction of an extra unknown, which has to found by solving a coupled set of equations at each time. The loss of efficiency will be most pronounced for finite element methods, where the idea of staggered grids is non-trivial to implement and complication in the representation of the boundary conditions may occur. However, Boussinesq equations similar to (41) can be obtained also when the flow is rotational. Relevant examples are waves influenced by a rotational mean current and waves on a stratified fluid. Therefore, the study of numerical solutions of (39) and (41) is of general interest.

3 Properties of the equations on constant depth.

As demonstrated in the previous section, the higher order terms of the Boussinesq equations may be rewritten into various different forms. Although being consistent to the appropriate order, the different forms give different solutions. Sometimes authors claim one particular version to be superior to the rest. For linearized waves on constant depth it is easily accepted that different KdV and Boussinesq equations can be ranked in accordance with their dispersion relations. On the other hand, the matter is much more complicated for general cases involving finite amplitude waves and nonuniform bottom topography. In 1984 Ertekin et al. presented evidence that one particular formulation of the Boussinesq equations predicted soliton propagation speeds that were in excellent agreement with experiments, even for quite high waves. The set in question was the so-called “Green-Nadghi” equations in which many $O(\alpha \epsilon)$ terms are maintained. However, agreement for one single wave property as soliton propagation speed is not a sound foundation for any reliable conclusion. The ability to reproduce shape and genesis of solitons, behaviour of cnoidal waves etc. and finally the effects of nonuniform topography must also be considered. Thus, until further documentation on the superiority of the “Green-Nadghi” is available, it does not seem worthwhile bothering with all the extra $O(\alpha \epsilon)$ terms. Anyway, in the remainder of the present section we present a brief discussion of dispersion relations and solitary wave solutions.
3.1 Dispersion relations

In this subsection we assume constant depth, zero surface pressure and linearized equations. Choosing $\psi$ and $\eta$ as variables equation (20), (10) and the various expressions for $Q$ give:

$$\eta_t = -\nabla^2 \psi + \varepsilon \{ \gamma \nabla^2 \eta + (\gamma - \frac{1}{3}) \nabla^4 \psi \} + O(\alpha, \varepsilon^2)$$ (42)

$$\psi = -\eta + O(\alpha, \varepsilon^2)$$ (43)

or by elimination of $\psi$:

$$(1 - \varepsilon \gamma \nabla^2) \eta_{tt} = (1 + \varepsilon \{ \frac{1}{3} - \gamma \} \nabla^2) \nabla^2 \eta$$ (44)

The representations (16), (29) and (30) correspond to $\gamma = \frac{1}{2}$, $\gamma = \frac{1}{3}$ and $\gamma = 0$ respectively. If we linearize (36), (37) and eliminate $\phi$ we find:

$$(1 - \frac{1}{3} \varepsilon \nabla^2) \eta_{tt} = \nabla^2 \eta$$ (45)

We note that (44) and (45) are identical when $\gamma = \frac{1}{3}$. Denoting the frequency by $\omega$ and the total wave number by $a$ we find from (44):

$$\omega^2 = \frac{a^2 + \varepsilon (\gamma - \frac{1}{3}) a^4}{1 + \varepsilon a^2} = a^2 - \frac{1}{3} \varepsilon a^4 + \frac{1}{3} \varepsilon^2 a^6 + O(a^8)$$ (46)

In the present scaling the exact inviscid dispersion relation reads:

$$\omega^2 = a \varepsilon^{- \frac{1}{2}} \tan(\varepsilon^{\frac{1}{2}} a) = a^2 - \frac{1}{3} \varepsilon a^4 + \frac{2}{15} \varepsilon^2 a^6 + O(a^8)$$ (47)

Comparison of the $a^6$ terms of the two dispersion relations shows that $\gamma = \frac{1}{3}$, which also corresponds to (45), gives a slightly better result than $\gamma = \frac{1}{2}$ and that $\gamma = 0$ is the poorest option.

3.2 Solitary waves

For a solitary wave we may write:

$$\psi, \phi = \int U(\xi) d\xi \quad \eta = Y(\xi)$$ (48)

where $\xi = x - ct$ and $U$ and $Y$ satisfies the boundary conditions:

$$\lim_{\xi \to \pm \infty} \frac{d^n Y}{d\xi^n} = \lim_{\xi \to \pm \infty} \frac{d^n U}{d\xi^n} = 0 \quad ; n = 0, 1, 2...$$ (49)
The solitary wave solution for the set (36), (37) are derived in [8] and [9]. When (48) is invoked in the nonlinear counterpart of the set (42), (43) we obtain:

\[ cY' = ((1 + \alpha Y)U' + \varepsilon \gamma Y'' - \varepsilon (\gamma - \frac{1}{3})U'' \]  
\[ cU = \frac{1}{2} \alpha U^2 + Y \]  

Integration of the first of these equations and use of the conditions at infinity gives:

\[ cY = (1 + \alpha Y)U + \varepsilon \gamma Y'' - \varepsilon (\gamma - \frac{1}{3})U'' \]  

which by elimination of \( Y \) yields:

\[ (c^2 - 1)U - \frac{3}{2} \alpha cU^2 + \frac{1}{2} \alpha^2 U^3 = \varepsilon \{(c^2 - 1)\gamma + \frac{1}{3}U''\} - \frac{1}{2} \varepsilon \alpha \gamma (U^2)'' \]  

It would of course be consistent with earlier approximations to neglect all terms of order \( \varepsilon \alpha \) and \( \alpha^2 \) in this equation, but we will retain all terms because we want to examine the differences between various soliton solutions. For testing of numerical techniques it is also advantageous to have exact solutions of the analytical equations. The equation (53) is of the form:

\[ R(U) = \nu U'' + \kappa (U^2)'' \]  

where \( \nu \) and \( \kappa \) are constants. We multiply both sides of (54) by \( I = dH(U)/d\xi = H'U' \) intending to determine an \( H \) which makes \( I \) an integrating factor. After some manipulations we obtain:

\[ R(U)\{H(U)\}' = \left[ \frac{1}{2} \nu + \kappa U \right] \frac{dH}{dU} - 2\kappa \{(U')^2\}' + 2\kappa \{(U')^2 H\}' \]  

Hence, the desired \( H \) is one satisfying:

\[ \frac{dH}{dU} = \frac{2\kappa}{\frac{1}{2} \nu + \kappa U} H \]  

We may thus use:

\[ H = (\nu + 2\kappa U)^2 \]  

From (55) it then follows:

\[ (U')^2 = \frac{1}{2\kappa H} \int R(U) \frac{dH}{dU} dU \]  

Integration and use of the boundary conditions at infinity gives:

\[ \varepsilon (U')^2 = \{(c^2 - 1)\gamma + \frac{1}{3} - \alpha c\gamma U\}^{-2} \left\{ \frac{1}{2} (c^2 - 1)(c^2 - 1)\gamma + \frac{1}{3} \right\} U^2 \]

\[ - \frac{\alpha}{6} c(5(c^2 - 1)\gamma + 1)U^3 + \frac{\alpha^2}{8} (4c^2 - 1)\gamma + \frac{1}{3} U^4 - \frac{\alpha^3}{10} c U^5 \} \]
Table 1: The first “insignificant” term in the expansions of the speed of solitons.

<table>
<thead>
<tr>
<th>Solution</th>
<th>q</th>
</tr>
</thead>
<tbody>
<tr>
<td>KdV equation</td>
<td>0</td>
</tr>
<tr>
<td>ψ-formulation, γ = 0</td>
<td>0</td>
</tr>
<tr>
<td>ψ-formulation, γ = 1/4</td>
<td>−1/24</td>
</tr>
<tr>
<td>ψ-formulation, γ = 1/3</td>
<td>−1/15</td>
</tr>
<tr>
<td>ψ-formulation (36, 37)</td>
<td>−1/24</td>
</tr>
<tr>
<td>Green-Nadghi (c = √1 + A)</td>
<td>−1/8</td>
</tr>
</tbody>
</table>

If α is chosen as the ratio $U_{max}^2/(gh_0)^{1/4}$, the maximum value of $U$ becomes equal to 1. Requiring that $U' = 0$, $U = 1$ fits the above relation we then find a relation between $\alpha$ and $c$. Expansion in a Taylor series in $\alpha$ gives:

$$c = 1 + \frac{1}{2} \alpha - \frac{\gamma}{8} \alpha^2 + O(\alpha^3)$$  \hspace{1cm} (60)

It is generally more convenient to express $c$ in terms of $A = \eta_{max}/h_0$. From (51) we easily find that $A = \alpha + O(\alpha^3)$, which means that $\alpha$ can be replaced by $A$ in (60).

As stated at the beginning of this section, different weakly nonlinear and dispersive theories give different relations $c(A)$. The first two terms of an expansion in powers of $A$ have to be identical for all consistent formulations and we may assume in general:

$$c = 1 + \frac{1}{2} A + q A^2 + O(A^3)$$  \hspace{1cm} (61)

In table 1 we have listed values of $q$ for some different solitary wave solutions. The KdV equation referred to in the table is given by:

$$\eta_t + \left(1 + \frac{3}{2} \alpha \eta\right) \eta_x + \frac{\epsilon}{6} \eta_{xxx} = 0$$  \hspace{1cm} (62)

4 Numerical methods

The numerical method presented in this section is an extension of the one reported in [9]. A technique for solving a set of velocity based Boussinesq equations derivable from (39) and (41) is described in great detail in [8]. Rygg (1988) compared this method to parabolized equations and experiments for a topographical lens problem. Large parts of the present methods are similar to corresponding parts of these earlier works and will be described very briefly. The relationship between the techniques in [9] and [8] is discussed in section 4.3.

4.1 The basic difference technique

The approximation to a quantity $f$ at a grid-point with coordinates $(\beta \Delta x, \gamma \Delta y, \kappa \Delta t)$ where $\Delta x$, $\Delta y$ and $\Delta t$ are the grid increments, is denoted by $f^p_{\beta,\gamma}$. To improve the
readability of the difference equations we introduce the symmetric difference operator, $\delta_z$:

$$\delta_z f^{(\alpha)} = \frac{1}{\Delta z} (f^{(\alpha)}_{i+\frac{1}{2},\gamma} - f^{(\alpha)}_{i-\frac{1}{2},\gamma})$$

(63)

and the midpoint average operator $\bar{z}$ by:

$$\bar{z} f^{(\alpha)}_{i,\gamma} = \frac{1}{2} (f^{(\alpha)}_{i-\frac{1}{2},\gamma} + f^{(\alpha)}_{i+\frac{1}{2},\gamma})$$

(64)

Difference and average operators with respect to the other coordinates $y$ and $t$ are defined correspondingly. We note that all combinations of these operators are commutative. To abbreviate the expressions further we also group terms of identical indices inside square brackets, leaving the super- and subscripts outside the bracket. These notations are adopted from [8].

The components of the flux density vector $\vec{Q}$ are introduced according to:

$$\vec{Q} = U\hat{i} + V\hat{j}$$

(65)

The present method, as well as the one reported by [8], is based on the spatial configuration shown in figure 1 and the grid are staggered in time. In the latter method averaged velocities were specified at the flux-nodes. The discrete quantities are denoted by:

$$w_{i,j}^{(n+\frac{1}{2})}, \quad \dot{w}_{i,j}^{(n)}, \quad h_{i,j}^{(n)}, \quad V_{i,j}^{(n+\frac{1}{2})}, \quad U_{i,j}^{(n+\frac{1}{2})}$$

(66)

where $w$ denotes either $\phi$ or $\psi$. The quantity $\dot{w}$ corresponds to the time derivative of $w$ and is defined by:

$$\dot{w} = \delta_t w_{i,j}^{(n)}$$

(67)

If the depth is time dependent, we assume that values of the depth function itself and its derivatives can be obtained at any time necessary. The continuity equation (10) is discretized according to:

$$[\delta_t n + \delta_t h = -\delta_z U - \delta_y V_{i,j}^{(n+\frac{1}{2})}]$$

(68)

which expresses volume conservation for the cells depicted in figure 1. Naturally $U$ and $V$ have to be related to the potentials through discretization of relations like (16) etc. The Bernoulli equation ((20) or (33)) is approximated by:

$$[\dot{u} + T + \eta + p - \epsilon S = 0]_{i,j}^{(n)}$$

(69)

where

$$[T_{i,j}^{(n)} = \frac{\alpha}{2} \{ (\delta_x w^x)^{(n+\frac{1}{2})} (\delta_x w^x)^{(n+\frac{1}{2})} + (\delta_y w^y)^{(n+\frac{1}{2})} (\delta_y w^y)^{(n+\frac{1}{2})} \}]_{i,j}$$

(70)

The two options $S = 0$ and $S = (\mathcal{R})_{i,j}$ corresponds to $u = \psi$ and $u = \phi$ respectively.

Rigid boundaries are preferably located to flux-nodes to simplify the implementation of the boundary conditions. If the fluid is confined to $y \leq (j + \frac{1}{2})\Delta y$, say, we use:

$$V_{i,j}^{(n+\frac{1}{2})} = 0$$

(71)
Figure 1: The spatial distribution of grid points. o denotes the locations of $\eta$ and $\phi(\psi)$ nodes, - shows the positions of U-points and | where the V values are sought. The dashed line corresponds to the boundaries of the volume conservation cell.
in (68) and implement symmetry conditions by introduction of fictitious quantities according to:

\[ \delta \psi = 0 \big|_{i,j}^{(n+\frac{1}{2})} \quad \delta \nu = 0 \big|_{i,j}^{(n+\frac{1}{2})} \]  \tag{72}

whereas the treatment of the condition for \( \phi \) is less straightforward.

### 4.2 Difference equations involving \( \phi \)

The difference equations described in this subsection are already implemented on a computer and are generalizations of those used in [9]. Since we are using a finite difference model it is not necessary to rewrite (17). For convenience and completeness we will give the equations that are actually discretized:

\[ \eta_t + r_t = -\nabla \cdot \{(h + \alpha \eta) \nabla \phi + e \left( \frac{1}{2} \eta_t + \frac{1}{3} h \nabla^2 \phi \right) \nabla h \} + O(\epsilon^2, \epsilon \alpha) \quad \tag{73} \]

\[ \phi_t + \frac{1}{2} \alpha (\nabla \phi)^2 + \eta + p - \epsilon \left( \frac{1}{2} h r_{tt} + \frac{1}{2} h \nabla \cdot (h \nabla \phi_t) - \frac{1}{6} h^2 \nabla^2 \phi_t \right) = O(\epsilon^2, \epsilon \alpha) \quad \tag{74} \]

The prior equation is the continuity equation, and the latter is the Bernoulli equation. Both equations are, at least implicitly, given before. We note that the embraced expression on the right hand side of the continuity equation, equals the flux density \( \bar{Q} \).

Values for the flux components are obtained according to:

\[ [U = (\bar{h}^e + \alpha \bar{\eta}^e \delta_e) \delta_e \phi + \epsilon \left( \frac{1}{2} \bar{h}^e \delta_i \bar{\eta}^e + \frac{1}{3} (\bar{h}^e)^2 (\delta_e^2 + \delta_y^2) \bar{\phi} \right) \delta_e h - \beta C(U)^{(n+\frac{1}{2})}]_{i,j}^{(n+\frac{1}{2})} \quad \tag{75} \]

where \( -\beta C(U) \) is a correction term which is discussed in a subsequent section. Throughout the present section \( \beta \) is used to mark similar correction terms, putting \( \beta = 0 \) switches the correction off. The other flux-component, \( V \), is treated correspondingly. Equation (68) and (75) defines a linear set of equations for \( \eta_{i,j}^{(n+1)} \) which is solved by line by line iteration. The iterative procedure is similar to that which is used for the Bernoulli equation (see below). A difference version of the Bernoulli equation (33) reads:

\[ [\phi_t + T + \eta + p]
- \frac{1}{2} e \{ r_{tt} + \delta_e (\bar{h}^e \delta_e \phi) \delta_y (\bar{h}^e \delta_y \phi) - \frac{1}{3} (\delta_e^2 + \delta_y^2) \bar{\phi} \} - \beta B = 0 \big|_{i,j}^{(n)} \quad \tag{76} \]

where we have assumed that some appropriate value for \( r_{tt} \) is available. The discrete Bernoulli equation, (76), gives an Helmholtz type equation for \( \phi \), which is solved by the same ADI iteration as is used in [8] and [9]. The details are thus omitted. We start the iteration procedure using values that are obtained from the previous time step. Unless a shift is used in the ADI procedure, very short waves (wavelength comparable to the grid scale) display slow convergence and instabilities may arise (see [9]). Since this shift in any other respect is unfavorable we will instead apply smoothing to the initial values of the iteration. No smoothing of any kind is applied to the solutions for \( \phi \) or \( \eta \). At
internal points smoothed values for \( f \), denoted by \( \tilde{f} \), are obtained by the two steps:

\[
\begin{align*}
[g] & = \tilde{f} \text{smooth}_{i,j} \\
[\tilde{f}] & = \text{smooth}_{i,j}
\end{align*}
\] (77)

If a harmonic component \( f = Ae^{i(k_x \Delta x + k_y \Delta y)} \) is subjected to this procedure the amplitude of \( \tilde{f} \) becomes \( \cos^2(\frac{k_x \Delta x}{2}) \cos^2(\frac{k_y \Delta y}{2})A \). We note that long waves are only slightly affected whereas waves of the “Nyquist-type” \( (k_x \Delta x = \pi \text{ or } k_y \Delta y = \pi) \) are nihilated. At a boundary point, \( j = 0 \) say, the second step is modified according to:

\[
\tilde{f}_{i,0} = \frac{1}{4}(3g_{i,0} + 2g_{i,1} - g_{i,2})
\] (78)

Again the “Nyquist” waves vanish. When the prescribed initiation is used, two iterations at each time step generally suffice both for the continuity and the Bernoulli equation, as long as \( \Delta x \approx \epsilon \). For smaller grid increments there may be a gap in the spectrum between the long-wave components, that converge under the unshifted ADI iteration, and the short waves, for which the growth is inhibited by the prescribed smoothing. This problem is most easily circumvented by starting the iteration from \( \phi = 0 \), and apply four iterations to the Bernoulli equation.

In the previous section we discussed the implementation of a no-flux condition at straight rigid wall. If the fluid is located to \( y < (M + \frac{1}{2}) \Delta y \), the flux \( V^{(n+\frac{1}{2})} \) is set to zero and the fictitious quantity \( \eta_{i,M+\frac{1}{2}} \) do not appear in any equation. However, according to (75) the values of \( \phi^{(n+\frac{1}{2})}_{i,M+\frac{1}{2}} \) are present in the expressions for \( V^{(n+\frac{1}{2})}_{i,M-\frac{1}{2}} \) and \( U^{(n+\frac{1}{2})}_{i+\frac{1}{2},M} \), as well as in (76) for \( j = M \). Because \( \phi^{(n+\frac{1}{2})}_{i,M+\frac{1}{2}} \) only appears in terms of order \( \alpha \) or \( \epsilon \), the idea of applying a symmetry condition also for \( \phi \) is tempting. A local analysis shows that relative errors of magnitude \( O(\frac{\Delta y}{\Delta x}) \) must be expected. Therefore, the condition (35) is discretized by means of midpoint differences and averaging to give:

\[
\begin{align*}
\left\{2h(1 - \frac{\epsilon}{4}(\delta_y h)^2) - \frac{\alpha}{\Delta t} \delta_x \phi^{(n-\frac{1}{2})} \delta_y h \phi\right\} & - \frac{\delta \phi^{(n-\frac{1}{2})} \delta_y h}{\Delta t} + \frac{\delta \phi^{(n-\frac{1}{2})}\delta y h}{\Delta t} \\
& - \frac{\delta \phi^{(n-\frac{1}{2})}\delta y h}{\Delta t} + \frac{\delta \phi^{(n-\frac{1}{2})}\delta y h}{\Delta t} \\
& = \frac{1}{2} \alpha((\delta_y \phi^{(n-\frac{1}{2})})^2 + \delta y h \phi^{(n-\frac{1}{2})}) - \frac{\delta \phi^{(n-\frac{1}{2})}\delta y h}{\Delta t}\right\}
\end{align*}
\] (79)

where we have omitted all superscripts that are equal to \( n \). The result is a family of equations that are applied at the boundary in the ADI procedure.

At a boundary we may define an input wave by specifying either the value of \( \phi \) or its normal derivative which are denoted by \( \hat{\phi} \) and \( \hat{u} \) respectively. Preferably the depth should be constant in the vicinity of an input boundary, which implies that \( \hat{u} \) has interpretation as a velocity. If \( \phi \) is specified it is convenient to have the boundary coinciding with \( \phi \)-nodes, at say \( x = 0 \). In this case the discrete condition becomes

\[
[\phi = \hat{\phi} + \beta C^{(\phi)}_{j0,i}]^{(n+\frac{1}{2})}
\] (80)

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On the other hand, if the normal derivative is given, the boundary should be located at flux-nodes. The difference counterpart to the condition $\partial \phi / \partial x = \dot{u}$ at $x = \frac{1}{2} \Delta x$ reads:

$$[\delta_x \phi = \ddot{u} + \beta C(u)]_{j}^{(n+\frac{1}{2})}$$

(Because of the nonlinear terms in the flux densities we have to specify a condition for $\eta$ in addition to (80) or (81). Proper care should be taken to assure that this extra condition is completely consistent with the values of $\phi$ or $\dot{u}$ according to the actual incoming wave form.}

A Sommerfeldts radiation condition can be written:

$$L(\phi) \equiv (\kappa_1 + \kappa_2) \phi_x + \kappa_1 \frac{c}{\sin \theta} \phi_x + \kappa_2 \frac{c}{\cos \theta} \phi_y = 0$$

(82)

where $\kappa_1$ and $\kappa_2$ are arbitrary, $c$ is the phase speed of the outgoing wave and $\theta$ is the angle between the boundary and the direction of wave advance. We assume that the boundary is parallel to the $y$-axis, with the fluid to the left. A condition of this simple form enables description of only a few of the many types of open boundaries that may arise. The following requirements should at least be satisfied:

1. All outgoing waves should be associated with the same single speed $c$ and direction $\theta$. If $\kappa_2$ is chosen to be zero, only the ratio $c / \sin \theta$ has to be common.

2. $c$ and $\theta$, as well as the shape of the waves, have to vary slowly in time. This requirement is most easily met if the waves are nearly fully developed solitons or slightly modulated cnoidal wave trains, and the depth is nearly constant in the vicinity of the boundary.

3. Reliable values for $\theta$ and $c$ have to be available. Sometimes asymptotic considerations may provide values for $\theta$ and $c$, but generally some sort of “numerical measurement” has to be applied (interpolation etc.).

As long as $\theta = O(1)$ a convenient choice for the coefficients are $\kappa_1 = 1, \kappa_2 = 0$. However, for small $\theta$ the factor $c / \sin \theta$ becomes very sensitive to small deviations in $\theta$. Thus, small errors in $\theta$ due to imperfection of “measurements” (point 3 above) may cause a large error in the radiation condition. If we assume

$$\dot{\phi} = \Phi(\sin \theta_1 x + \cos \theta_1 y - ct)$$

(83)

where $\theta_1 = \theta - \Delta \theta$. A simple analysis of (82) yields:

$$E \equiv L(\dot{\phi}) = -c((\kappa_1 \cot \theta - \kappa_2 \tan \theta) \Delta \theta + \frac{O(\Delta \theta^2)}{\cos \theta}) \dot{\Phi}$$

(84)

For $\kappa_2 = 0$ we find that $E = O(\Delta \theta)$ if $\theta = O(1)$, and that $E = O(1)$ if $\theta = O(\Delta \theta)$. From the expression for $E$ it immediately follows that an optimal choice for the coefficients is
given by \( \kappa_2 / \kappa_1 = \cot^2 \theta \), which implies \( E = O(\Delta \theta^2) \) provided \( \theta - \frac{1}{2} \pi = O(1) \). For the radiation condition we obtain:

\[
\phi_t + c \sin \theta \phi_x + c \cos \theta \phi_y = 0 \tag{85}
\]

which means that \( \phi \) is differentiated along the direction of wave propagation. It must be noted that \( \cot \theta \) may be inaccurate and that very small \( \kappa_1 / \kappa_2 \) may be unsound. Eq. (82) is discretized according to:

\[
[(\kappa_1 + \kappa_2) \delta_t \phi_x^e + \kappa_1 \frac{c}{\sin \theta} \delta_x \phi_x^e + \kappa_2 \frac{c}{\cos \theta} \delta_x \phi_y^e = \beta C^{(r)}(n)]_{i+\frac{1}{2},j} \tag{86}
\]

The same radiation condition that is used for \( \phi \) has to be used for \( \eta \), because \( \eta_{o,j}^{(n)} \) appears in the discrete continuity equation at \( x = \Delta x \).

### 4.3 Correspondence between potential and velocity based formulations on flat bottom

In this subsection we are going to demonstrate the close relation between the numerical method in [8] and the one presented in section 4.2.

If the equations (4.1) and (4.2) in [8], part 2, are rewritten according to the scaling in the present report, and the depth is assumed constant we obtain:

\[
\bar{u}_t + \frac{\alpha}{2} (\bar{u}^2)_x = -\nabla \eta + \frac{1}{3} \epsilon \nabla^2 \bar{u}_t \tag{87}
\]

\[
\eta_t = -\nabla \cdot ((1 + \alpha \eta) \bar{u}) \tag{88}
\]

where \( \bar{u} \) is the averaged horizontal velocity. In terms of the numbering and notations introduced in the previous subsections, the difference method discussed in [8] gives:

\[
[\delta_t (1 - \frac{1}{3} \epsilon \Lambda) u = -\delta_x (\eta + T)]_{i+\frac{1}{2},j}^{(n)} \tag{89}
\]

\[
[\delta_t (1 - \frac{1}{3} \epsilon \Lambda) v = -\delta_y (\eta + T)]_{i,j+\frac{1}{2}}^{(n)} \tag{90}
\]

where \( \bar{u} = u + v \), \( \Lambda \) denotes \( \delta_x^2 + \delta_y^2 \) and

\[
[T^{(n)} = \frac{1}{2} \{(\bar{u}^x)^{(n-\frac{1}{2})}(\bar{u}^x)^{(n+\frac{1}{2})} + (\bar{v}^y)^{(n-\frac{1}{2})}(\bar{v}^y)^{(n+\frac{1}{2})}\}]_{i,j} \tag{91}
\]

We note that the quantities involved are:

\[
\eta_{i,j}^{(n)} , \ u_{i+\frac{1}{2},j}^{(n+\frac{1}{2})} , \ v_{i,j+\frac{1}{2}}^{(n+\frac{1}{2})} \tag{92}
\]

A discrete vorticity is defined by:

\[
[\zeta = \delta_x v - \delta_y u_{i+\frac{1}{2},j+\frac{1}{2}}^{(n+\frac{1}{2})} \tag{93}
\]
From (89) and (90) it then follows:

\[ (1 - \frac{1}{3} \epsilon \Lambda) \zeta^{(n+\frac{1}{2})} = C_{i+\frac{1}{2},j+\frac{1}{2}} \tag{94} \]

where the C's are "constants of summation". Even if \( \zeta = 0 \) initially, rotational boundary layers may occur due to unsound boundary conditions. This is a point always to be considered when discretizing boundary conditions for \( u \) and \( v \). Provided all boundary conditions are consistent with \( \zeta = 0 \), we may assume \( \zeta = 0 \) at every point of the grid and be encouraged to introduce a discrete potential \( \phi^{(n+\frac{1}{2})}_{i,j} \) according to:

\[
[\delta_x \phi^{(n+\frac{1}{2})}]_{i,j} \quad [\delta_y \phi^{(n+\frac{1}{2})}]_{i,j+\frac{1}{2}} \tag{95}
\]

Analogous to the non-discrete case, the potential is single valued if and only if the circulation along every closed path, like the one indicated in figure 2, is zero. The circulation along the set of four velocity nodes surrounding \( \zeta^{(n+\frac{1}{2})} \) is given by:

\[
\Gamma^{(n+\frac{1}{2})}_{i+\frac{1}{2},j+\frac{1}{2}} = \Delta x u^{(n+\frac{1}{2})}_{i+\frac{1}{2},j} + \Delta y v^{(n+\frac{1}{2})}_{i+\frac{1}{2},j+\frac{1}{2}} - \Delta x v^{(n+\frac{1}{2})}_{i+\frac{1}{2},j+1} - \Delta y v^{(n+\frac{1}{2})}_{i,j+\frac{1}{2}} \tag{96}
\]

From (93) it immediately follows that \( \Gamma^{(n+\frac{1}{2})}_{i+\frac{1}{2},j+\frac{1}{2}} \equiv 0 \). Provided the domain is simply connected etc., this implies that the circulation of any closed path is zero. Invoking \( \phi \) in (89) and (90) we obtain:

\[
[\delta_i (1 - \frac{1}{3} \epsilon \Lambda) \phi + \eta + T^{(n)}]_{i,j} = D \tag{97}
\]

where \( D \) is a constant which may be set to zero. Observing that

\[
[T^{(n)} = \frac{1}{2} \{(\delta_x \phi^x)^{(n-\frac{1}{2})}(\delta_x \phi^x)^{(n+\frac{1}{2})} + (\delta_y \phi^y)^{(n-\frac{1}{2})}(\delta_y \phi^y)^{(n+\frac{1}{2})}\}]_{i,j} \tag{98}
\]

we recognize (97) as identical to what is obtained from (68), (75) and (76) by substituting 1 for \( h \) and deleting the correction terms. It is easily seen that also the continuity equations are equivalent. Most of the analysis in [8] is thus relevant to the method described in section 4.2 as well.

4.4 The effect of a staircase boundary

When modeling lakes or coastal waters one usually have to deal with highly irregular boundaries. A rough approximation to such boundaries, that is commonly used for the hydrostatic equations, is a staircase boundary resembling the one in figure 3. This representation is only of first order accuracy in the grid increments, and if used in combination with the Boussinesq equations the \( O(\epsilon) \) terms of (27) have to be abandoned. However, any representation of a ragged coastal line has to be at least locally inaccurate, and
Figure 2: Definition sketch of the grid. $\circ$ : nodes for $\eta$ and $\phi$, $-$ : nodes for $u$, $\ast$ : nodes for $v$, $\bullet$ : nodes for the vorticity $\zeta$. The dotted line indicates a closed path for which we may compute a discrete circulation.
wave breaking at the shore may cause a significant energy dissipation that is difficult
to account for. Hence, it may still be worthwhile considering the crude, but simple,
"saw-tooth" boundaries. We will address the combination of staircase boundaries and
dispersive equations by performing a simple analysis valid for linearized equations and
constant depth.

Pedersen (1986) [7] investigated the effects of a "saw-tooth" boundary on solutions
of the discretized shallow water equations. The spatial arrangement of the grid used in
that paper equals that of the present report. In this section we will perform a similar
analysis for the linearized Boussinesq equations. The equations that are to be analyzed
read:
\[
\delta \eta = - \Lambda \phi |^{(n+\frac{1}{2})}_{i,j} \quad \left[ (1 - \frac{1}{3} \epsilon \Lambda) \delta \phi + \eta = 0 \right]_i^{(n)} \tag{99}
\]
where the notations are as in the previous subsection. From (99) we may easily eliminate \( \eta \) to obtain:
\[
\left\{ (1 - \frac{1}{3} \epsilon \Lambda) \delta t^2 - \Lambda \right\} \phi |^{(n+\frac{1}{2})}_{i,j} = 0 \tag{100}
\]
According to section 4.3, (99) is consistent with:
\[
\delta \eta = - \delta_x u - \delta_y v |^{(n+\frac{1}{2})}_{i,j} \\
\left[ (1 - \frac{1}{3} \epsilon \Lambda) \delta t u = - \delta_x \eta |^{(n)}_{i+\frac{1}{2},j} \right] \\
\left[ (1 - \frac{1}{3} \epsilon \Lambda) \delta t v = - \delta_y \eta |^{(n)}_{i,j+\frac{1}{2}} \right] \tag{101}
\]
where the relation between \( \phi \) and \( u, v \) is given by (95). At a straight boundary defined
by \( y = x - \frac{1}{2} \Delta x \) (see figure 3) the noflux condition may be accounted for by setting:
\[
u |^{(n+\frac{1}{2})}_{j+\frac{1}{2},j} = 0 \\
\nu |^{(n+\frac{1}{2})}_{j, j-\frac{1}{2}} = 0 \tag{102}
\]
or by introducing fictitious quantities \( \phi |^{(n)}_{j+1, j} \) defined according to:
\[
\delta \phi |^{(n+\frac{1}{2})}_{j+\frac{1}{2},j} = 0 \\
\delta \phi |^{(n+\frac{1}{2})}_{j, j-\frac{1}{2}} = 0 \tag{103}
\]
Implementation of (103) calls for some caution because each fictitious \( \phi \)-value can be
computed by both conditions, and must be regarded as double-valued. The safe proce­
dure is to use the conditions only in the form given in (103) and thereby avoid dealing
with the fictitious quantities explicitly. Since the boundary conditions (102) and (103)
are equivalent (see equation (95)), all results obtained for the set (99) will be valid also
for the formulation (101). Combining (103) and (100) we find:
\[
\left[ (1 - \frac{1}{3} \epsilon \Lambda) \delta t^2 - \Lambda \right] \phi |^{(n+\frac{1}{2})}_{i,j} = 0 \tag{104}
\]
where
\[
[\Lambda \phi = - \frac{1}{\Delta x} \delta \phi |_{j-\frac{1}{2},j} + \frac{1}{\Delta y} \delta \phi |_{j,j+\frac{1}{2}}]^{(n+\frac{1}{2})} \tag{105}
\]

\[20\]
Figure 3: The staircase boundary analyzed in section 4.4. The circles to the right of the boundary are fictitious nodes. The marks have the same interpretations as in figure 2.
Equation (104) is the boundary condition for (100) which is actually analyzed. For simplicity we assume $\Delta x = \Delta y$ and define incoming and reflected waves by:

$$\phi_{\text{inc}} = A e^{i(k(x-\frac{1}{2}\Delta x) + \ell y - \omega t)}$$
$$\phi_{\text{ref}} = B e^{i(\ell (x-\frac{1}{2}\Delta x) + ky - \omega t + \theta)}$$

where $i$ is the imaginary unit, $A$, $B$ are real positive numbers and $\omega$, $k$, and $\ell$ are related by the numerical dispersion relation obtainable from (100):

$$\tilde{\omega}^2 = \frac{\tilde{\alpha}^2}{1 + \frac{1}{3} \epsilon \tilde{\alpha}^2}$$

where

$$\tilde{\omega} = \frac{2}{\Delta t} \sin \left( \frac{\omega \Delta t}{2} \right)$$
$$\tilde{k} = \frac{2}{\Delta x} \sin \left( \frac{k \Delta x}{2} \right)$$
$$\tilde{\ell} = \frac{2}{\Delta y} \sin \left( \frac{\ell \Delta y}{2} \right)$$

The bars correspond to the use of capitalized letters in [8], part 1, section 2.1. Substitution into (105) gives:

$$RA + R^* B e^{\theta} = 0$$

where $R = \tilde{\alpha}^2 + \frac{1}{\Delta x^2} (e^{-i\Delta x} + e^{i\Delta x} - 2)$ and $R^*$ is the complex conjugate of $R$. Observing that the phase shift $\theta$ is denoted by $\delta$ in [7], we find that (108) is identical to equation (20) in [7] except for the replacement of $\tilde{\omega}^2$ by $\tilde{\alpha}^2$ in the definition of $R$. Provided $\tilde{\alpha}$ and $\theta$ is inserted for $\tilde{\omega}$ and $\delta$ respectively, the results from [7] concerning a single boundary will thus apply in the present case as well.

Unfortunately, there are a few errors in the results of [7] which were pointed out to the author by Kristian B. Dysthe. The errors concern the results for the spuriously trapped wave mode:

$$\phi = A e^{i(k(x-\frac{1}{2}\Delta x) + \ell y - \omega t)}$$

where $k = \sigma - i\gamma$ ($\gamma > 0$) and $\ell = \sigma + i\gamma$ fit the dispersion relation (107). The wave mode satisfies the boundary condition (105) provided $R$ (defined below (108)) is zero. This leads to:

$$0 = \tilde{\alpha}^2 + \frac{2}{\Delta x^2} (e^{-\gamma \Delta x} \cos(\sigma \Delta x) - 1)$$

which replaces the incorrect equation (25) in [7]. Taylor series expansion yields:

$$\gamma = \frac{1}{2} \sigma^2 \Delta x + O(\Delta x^2)$$

and the e-folding distance in the direction normal to the boundary becomes:

$$E = \frac{\lambda^2}{\sqrt{2\pi^2} \Delta x} + O(1)$$

where $\lambda = \sqrt{2\pi}/\sigma$ is the wavelength.
The simple results that are presented, indicate that saw-tooth boundaries can be applied to the Boussinesq equations, at least when linearized. In the nonlinear case the presence of such boundaries might cause a production of noise that can be fatal, unless a non-trivial smoothing procedure is applied. Experience from simulation of cnoidal wave diffraction at a convex corner suggests that such problems in fact do occur.

5 Correction terms

The basic idea for correcting the methods is to find additional terms that will give fourth order accuracy for $\eta$ when the nonlinear and dispersion terms are omitted. If the same terms are invoked in the full Boussinesq equations, we will have relative errors that are $O(\Delta x^4, \Delta y^4, \Delta t^4, \alpha \Delta x^2, \ldots, \varepsilon^2, \alpha \varepsilon)$ compared to the fully non-viscid equations. Provided $\Delta x, \Delta y, \Delta t = O(\varepsilon^{1/3})$ the error becomes $O(\varepsilon^2, \alpha \varepsilon)$ and the discretization involves no principal loss of accuracy. We note that $\Delta x = O(\varepsilon^{1/3})$ corresponds to $\Delta z^*$ being comparable to the depth.

We may derive many different sets of correction terms that give the desired accuracy. The correction terms should satisfy two requirements:

- The terms should preferably be similar to the dispersion terms that are already present in the equations (see [8], part 1, section 4). Higher order differences which demand extra fictitious values at the boundaries will be particularly disadvantageous.

- The convergence of the ADI iteration should not be violated.

The second points calls for a brief discussion. After discretization, the implicit equation sets are generally similar to:

$$f - (\gamma^{(x)} \delta_x^2 + \gamma^{(y)} \delta_y^2) f = g$$

which is a generalization of the finite difference counterpart of (1). According to the introduction (section 1) negative $\gamma$'s should not be permitted. As stated below, this is somewhat too restrictive. According to equation (3.5) through (3.9) in the first part of [8], the ADI procedure will converge for (113) provided:

$$- \gamma^{(x)} < \frac{\Delta x^2}{8} \quad - \gamma^{(y)} < \frac{\Delta y^2}{8}$$

Thus, Boussinesq equations that give a negative $\gamma$ are unsound, whereas correction terms giving negative $\gamma$'s can be applied provided (114) is satisfied. Correction terms to a second order method will of course be proportional to the squares of the grid increments. The case of variable coefficients is far more difficult to analyze. An indication of the limits for stability may however be obtained by replacing the coefficients with constants that give the "worst case" and then apply (114).
In the proceeding subsections we will make extensive use of the relations:

\[
\begin{align*}
F_{\theta} &= F + \frac{\Delta q^2}{8} F_{\theta \theta} + O(\Delta q^4) \\
\delta_q F &= F_q + \frac{\Delta q^2}{24} F_{\theta \theta \theta} + O(\Delta q^4) \\
\delta^2_q F &= F_{q q} + \frac{\Delta q^2}{12} F_{\theta \theta \theta \theta} + O(\Delta q^4) \\
\delta_q (G \delta_q F) &= (GF_q)_q + \frac{\Delta q^2}{24} \{(GF_{\theta \theta})_q + (GF_q)_{\theta \theta \theta}\} + O(\Delta q^4) \\
\delta_q (G^2 \delta_q F) &= (GF_q)_q + \Delta q^2 \{\frac{1}{6} (GF_q)_{\theta \theta \theta} - \frac{1}{12} (GF_{\theta \theta})_q - \frac{1}{4} (G_q F_{\theta \theta})_q\} + O(\Delta q^4)
\end{align*}
\]

where \(F\) and \(G\) are smooth functions of \(q\). The above relations are easily obtained by Taylor series expansions.

Procedures similar to the present are presented in [8] and [9]. For simplicity we will restrict the discussion to time independent depth.

5.1 Correction terms for the interior

The linearized and hydrostatic equations read:

\[
\begin{align*}
\eta_t &= -\nabla \cdot (h \nabla w) + O(\epsilon, \alpha) \\
w_t + \eta + p &= O(\epsilon, \alpha)
\end{align*}
\]

where \(w\) denotes any representation of the potential. Discrete versions are obtained from (69) and (68):

\[
\begin{align*}
[\delta_t \eta &= -\Box w + C]^{(n+\frac{1}{2})}_{i,j} \\
[\delta_t w + \eta + p &= B]^{(n)}_{i,j}
\end{align*}
\]

where \(B\) and \(C\) are correction terms proportional to \(\Delta x^2\), \(\Delta y^2\) and \(\Delta t^2\) and

\[
\Box \equiv \delta_x h^x \delta_x + \delta_y h^y \delta_y
\]

Preferably \(C\) should be expressible as:

\[
C = \delta_x C^{(n)} + \delta_y C^{(U)} + \delta_y C^{(V)}
\]

which means that the continuity equation remains in conservative form. Elimination of \(w\) from the difference equations (122) and (123) yields:

\[
[\delta_t^2 \eta = \Box (\eta + p) + \delta_t C - \Box B]^{(n)}_{i,j}
\]

\[24\]
which is a discretized version of the second order equation:

$$\eta_{tt} = \nabla \cdot (h\nabla (\eta + p))$$

(127)

From (119) we easily find an $\Box_E$ that fits:

$$\Box F = \nabla \cdot h\nabla F + \Box_E F + O(\Delta x^4, \Delta y^4)$$

(128)

for any function $F(x, y)$. Invoking the analytical solution $\hat{\eta}$, we find that (126) is satisfied to fourth order accuracy provided:

$$[-\delta_t C + \Box B = \Box_E (\hat{\eta} + p) - \frac{\Delta t^2}{12} \hat{\eta}_{ttt} + O(\Delta t^4, \ldots)]_{i,j}$$

(129)

Thus, if we may find expressions for $B$ and $C$ in terms of $\hat{\omega}, \hat{\eta}, p$ and $h$ that satisfy the above equation, then (122) and (123) give fourth order accuracy for $\eta$. The right hand side of (129) can be rewritten:

$$[-\delta_t C + \Box B = \Box_E w_t - \frac{\Delta t^2}{12} \Box (p + \hat{\eta})_{tt} + O(\Delta t^4, \ldots)]_{i,j}$$

(130)

One group of the many options for $C$ and $B$ that fits this equation is given by:

$$C^{(t)} = 0$$

$$C^{(V)} = \left(\frac{\Delta t^2}{12} - \kappa_1\right)h \eta_{zt} + \left(\frac{\Delta t^2}{12} - \kappa_2\right)h p_{zt} - \frac{1}{4} \Delta x^2 w_{zz} h_x + \frac{\Delta y^2}{12} h w_{yy}$$

$$-\frac{1}{6} \Delta x^2 \eta_{zt} - \frac{1}{6} \Delta y^2 \eta_{yt} + O(\Delta x^4, \ldots)$$

$$C^{(V)} = \left(\frac{\Delta t^2}{12} - \kappa_1\right)h \eta_{yt} + \left(\frac{\Delta t^2}{12} - \kappa_2\right)h p_{yt} - \frac{1}{4} \Delta y^2 w_{yy} h_y + \frac{\Delta x^2}{12} h w_{xx}$$

$$-\frac{1}{6} \Delta y^2 \eta_{yt} - \frac{1}{6} \Delta x^2 \eta_{xt} + O(\Delta x^4, \ldots)$$

$$B = \kappa_1 \nabla \cdot (h \nabla w_t) - \kappa_2 p_{tt} + \frac{\Delta x^2}{12} w_{zz} + \frac{\Delta y^2}{12} w_{yy} + O(\Delta x^4, \ldots)$$

where the coefficients $\kappa_1$ and $\kappa_2$ may be chosen as any combination of $\Delta x^2$, $\Delta y^2$ and $\Delta t^2$ that does not lead to a violation of (114). For small $h$ and course grids this may be difficult to achieve. The derivatives in the above equation may be replaced by any second order difference approximation.

5.2 Correction terms involving a refined mesh for the depth

An alternative to the correction procedure of the previous section, is obtained by assuming depth values to be available at the flux nodes. This means essentially that the number of depth values has to be doubled. For $\nabla \cdot h \nabla$ we now use the difference approximation:

$$\Box = \delta_h h \delta_x + \delta_y h \delta_y$$

(132)
Using (118) we obtain an expression for \( \Box_E = \Box - \nabla \cdot h \nabla \) which leads to:

\[
C^{(t)} = 0
\]

\[
C^{(u)} = \left( \frac{\Delta t^2}{12} - \kappa_1 \right) h_\eta + \frac{\Delta t^2}{12} - \kappa_2 \right) h_\rho_t - \frac{\Delta y^2}{24} h w_y \\
\frac{1}{24} \Delta x^2 \eta_{xt} + \frac{1}{24} \Delta y^2 \eta_{yt} + O(\Delta x^4, \ldots)
\]

(133)

\[
C^{(v)} = \left( \frac{\Delta t^2}{12} - \kappa_1 \right) h_\eta + \frac{\Delta t^2}{12} - \kappa_2 \right) h_\rho_t - \frac{\Delta x^2}{24} h w_x \\
\frac{1}{24} \Delta y^2 \eta_{yt} + \frac{1}{24} \Delta x^2 \eta_{xt} + O(\Delta x^4, \ldots)
\]

\[
B = \kappa_1 \nabla \cdot (h \nabla w_t) - \kappa_2 p_{tt} - \frac{\Delta x^2}{24} \rho_{xt} - \frac{\Delta y^2}{24} \rho_{yt} + O(\Delta x^4, \ldots)
\]

These correction terms will never cause a conflict with (114) as long as \( \kappa_1, \kappa_2 \leq \Delta t^2/12 \).

### 5.3 Corrected boundary conditions

We start this discussion by noting that a condition of symmetry does not need any correction, and that (79) implies symmetry in the linear and hydrostatic approximation. For simplicity we assume that the surface pressure \( p \) is zero in the vicinity of the boundary in question.

At first it might seem that we simply could find fourth order difference representations for the boundary conditions. Alas, this straightforward approach will not work because we do not have a complete fourth order method – it is only the solution for \( \eta \) that inherits this accuracy. The correct procedure is to combine the discrete boundary conditions and the discrete Bernoulli and continuity equation, to obtain a condition for \( \eta \) alone from which we claim high accuracy. The actual conditions are:

\[
[w = \hat{w} + C^{(u)}]_{0, j}^{(n+\frac{1}{2})}
\]

(134)

\[
[\delta_x w = \hat{u} + C^{(u)}]_{1, j}^{(n+\frac{1}{2})}
\]

(135)

\[
[\delta_t w - c \delta_x \hat{w}^t = C^{(v)}]_{j}^{(n)}
\]

(136)

When working in the linear and hydrostatic approximation there is no need to assign values to \( \eta \) at the boundaries. Still, it is convenient to do so by demanding (123) to be satisfied also at the fictitious (or boundary) nodes. Combining (134) and (123) we find

\[
[B - \eta = \delta_t \hat{w} + \delta_t C^{(u)}]_{1, j}^{(n)}
\]

(137)

which is desired to give fourth order accuracy when combined with (126). This is obtained if the boundary condition itself gives fourth order accuracy when applied to the analytical solution. The analytical solution satisfies \( \tilde{\eta} = -\tilde{w}_t \) which imply:

\[
[\delta_t C^{(u)} = B - \frac{\Delta t^2}{24} \hat{w}_{tt} + O(\Delta t^4, \ldots)]_{1, j}^{(n)}
\]

(138)

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A discrete "integral" of $B$ is defined by:

$$I^{(n+\frac{1}{2})}_{i,j} = \sum_{i=1}^{n} B^{(k)}_{i,j}$$  \hspace{1cm} (139)

According to (131) and (133), $I$ is easily found as a simple expression and the above equation gives:

$$[C^{(w)} = I - \frac{\Delta t^2}{24} \dot{\hat{w}}_{tt} + O(\Delta t^4,\ldots)]^{(n+\frac{1}{2})}_{i,j}$$  \hspace{1cm} (140)

The other conditions can be treated similarly provided that their validity can be extended throughout a region near the boundary. For the radiation condition (136) this simply means that the analytical counterpart is valid both for $\phi$ and $\eta$, with the same value for $c$, also at the neighbouring points. The Neumann condition is extended by assuming the existence of a $\tilde{w}(x,y,t)$ which equals the analytical $\tilde{\phi}_x$ in the vicinity of the boundary. We find:

$$[C^{(u)} = \delta_x I + \frac{1}{24} (\Delta x^2 \dot{u}_{xx} - \Delta t^2 \dot{u}_{tt}) + O(\Delta t^4,\ldots)]^{(n+\frac{1}{2})}_{i,j}$$  \hspace{1cm} (141)

$$[C^{(c)} = B + c \delta_x I + \frac{1}{24} (\Delta x^2 \eta_{xx} + \Delta t^2 \eta_{tt}) - c \frac{\Delta t^2}{8} \eta_{xt}]^{(n)}_{i,j}$$  \hspace{1cm} (142)

The latter equation may be simplified further by repeated use of the radiation condition to rewrite the right hand side.

References


