Application of Compressed Sensing to Reconstruction of 3-D Charge Collection Efficiency in Silicon Sensors

Munir Yassin

Department of Physics
University of Oslo, Norway

This dissertation is submitted for the degree of
Magister Scientiae, M.Sc in Physics.

High Energy Physics Section,
ATLAS group
May 2016
"Have those who disbelieved not considered that the heavens and the earth were a joined entity, and we ripped them apart and made from water every living thing? Then will they not believe?" Quran, 21:30.

I would like to dedicate this thesis to my loving parents . . .
Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements. This dissertation contains fewer than 65,000 words including appendices, bibliography, footnotes, tables and equations and has fewer than 150 figures.

Munir Yassin
May 2016
Acknowledgements

First and most of all I would like to thank my supervisor Alexander Lincoln Read and subsidiary supervisor Ole Myren Røhne, for their support, patience, flexibility, encouragement, enthusiasm, suggestions and ideas that were essential for each step of the long road in the making of this thesis. They believe in my potential did keep me going in the hard times of my researcher’s work. No words of gratitude are enough to express my thanks.

I also would like to thank each one on the ATLAS section at Oslo University for their friendliness and kindness, especially Knut Oddvar Høie Vadla and Ida Marie Bentsen. We had some very pleasant memories together as Master students. I did appreciate your company significantly. I regard myself lucky to have been considered part of this wonderful group with the caring leadership of Farid Ould-Saada.

There are so many others that had a hand in the shaping of this thesis through their help and support and deserve all thanks and gratitude. I would like to start by thanking Thibaud Humain, for sharing with me the essential software for 3DCCE simulation and his help, I’m looking forward to meeting you in person, and thanks go to our common friend the brilliant HEP theoretician Adil Jueid that did introduce us.

A very warm thanks go to Øyvind Ryan for helping me with the understanding of Compressed Sensing. I have to thank Andreas Solbrå for his fantastic Master thesis warmly, it was the first breakthrough in understanding and appreciating CS theory, and it was only through his thesis that I got to the RSAF method that did change the course of my thesis toward what have been unthinkable. This work had great benefit from the previous work done in this matter by the 3DCCE group at CERN. I would like to thank Nasser M. Abbasi for his fantastic CT Matlab program that has been a great help in the earliest stage of my Masters.

Finally, I would like to thank all the professors at UiO that I find exceedingly engaged and enthusiastic in their effort to convey their knowledge and love of science, specially Morten Hjorth-Jensen and Arnt Inge Vistnes.

’Nanos gigantum humeris insidentes’
‘If I have seen further it is by standing on the sholders [sic] of Giants.’ Isaac Newton’s remark in a letter, 1676.
Abstract

CERN is leading the quest of pushing the limits of experimental High Energy Particle Physics with major upgrades and creating innovative new particle sensors and analysis tools. In this spirit, we have taking up the work done by 3DCCE CERN group where they investigated extending the hit Charge Collection Efficiency characterization of sensors from 2 dimensions to a full spatial 3-D efficiency map. 3DCCE could provide an excellent tool to study local inefficiencies, spacial trapping effects and design impacts of the 3-D and diamond sensors. The goal of this thesis was to improve the reconstruction algorithm, to reduce the measurements time needed for volumetric reconstruction based on traditional Computed Tomography principles. With the new reconstruction method, known as Compressed Sensing, the measurement time could be reduced from approximately a month to a week or less, fitting better the normal length of a test-beam experiment period. In this thesis, we show the results obtained by using Compressed Sensing to reconstruct phantoms representing Charge Collection Efficiency of double sided 3-D sensors. Furthermore, we optimized one particular implementation of Compressed Sensing, known as reconstruction via Recursive Spatial Adaptive Filtering for those sensors, achieving theoretical reconstruction that points toward a possible measurement time reduction to a day or less.
# Table of contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>List of figures</td>
<td></td>
<td>xiii</td>
</tr>
<tr>
<td>List of tables</td>
<td></td>
<td>xvii</td>
</tr>
<tr>
<td>Nomenclature</td>
<td></td>
<td>xix</td>
</tr>
<tr>
<td>Introduction</td>
<td></td>
<td>xxi</td>
</tr>
<tr>
<td><strong>1</strong> Standard Model and ATLAS</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1.1</td>
<td>Elementary Particles of SM</td>
<td>1</td>
</tr>
<tr>
<td>1.2</td>
<td>ATLAS and LHC</td>
<td>3</td>
</tr>
<tr>
<td>1.2.1</td>
<td>Magnet system</td>
<td>4</td>
</tr>
<tr>
<td>1.2.2</td>
<td>Muon Spectrometer</td>
<td>5</td>
</tr>
<tr>
<td>1.2.3</td>
<td>Calorimeter</td>
<td>7</td>
</tr>
<tr>
<td>1.2.4</td>
<td>Inner Detector</td>
<td>8</td>
</tr>
<tr>
<td><strong>2</strong> 3D Charge Collection Efficiency and Computed Tomography</td>
<td>11</td>
<td>11</td>
</tr>
<tr>
<td>2.1</td>
<td>Test-beam experiment</td>
<td>11</td>
</tr>
<tr>
<td>2.2</td>
<td>3D pixel sensor</td>
<td>12</td>
</tr>
<tr>
<td>2.3</td>
<td>2-D Charge Collection Efficiency</td>
<td>13</td>
</tr>
<tr>
<td>2.4</td>
<td>Why 3-D Charge Collection Efficiency</td>
<td>15</td>
</tr>
<tr>
<td>2.5</td>
<td>Computed Tomography</td>
<td>17</td>
</tr>
<tr>
<td><strong>3</strong> Compressed Sensing</td>
<td>23</td>
<td>24</td>
</tr>
<tr>
<td>3.1</td>
<td>Linear Inverse Problem</td>
<td>24</td>
</tr>
<tr>
<td>3.2</td>
<td>Reconstruction from incomplete spectral</td>
<td>24</td>
</tr>
<tr>
<td>3.3</td>
<td>Compressed Sensing</td>
<td>26</td>
</tr>
<tr>
<td>3.4</td>
<td>Normed spaces</td>
<td>30</td>
</tr>
<tr>
<td>3.5</td>
<td>Approximation to l0 minimization</td>
<td>38</td>
</tr>
</tbody>
</table>
# Table of contents

## 4 Reconstruction via Recursive Spatially Adaptive Filtering

4.1 Block Matching 3D filter

4.1.1 BM3D procedure steps

4.1.2 Measures of reconstruction quality

4.1.3 BM3D performance

4.2 Recursive Spatially Adaptive Filtering

4.2.1 Iterative algorithm with stochastic approximation

## 5 Results

5.1 RSAF vs reconstitution via $l_0$ and $l_1$-minimization

5.2 Our optimization of RSAF

5.3 3-DCCE Phantoms

5.4 3-DCCE reconstruction with classical methods

5.5 3-DCCE reconstruction with Compressed Sensing

5.6 3-DCCE of challenging phantoms

## 6 Conclusion

References
List of figures

1.1 Elementary Particles table .................................................. 2
1.2 LHC ................................................................. 3
1.3 ATLAS Drawing .......................................................... 4
1.4 Magnets .............................................................. 5
1.5 Spectrometer ............................................................ 6
1.6 Calorimeters ............................................................... 7
1.7 InnerDetector ............................................................ 9
2.1 Layout of the Timepix Telescope mechanics, ...................... 12
2.2 The 3-D etched columns from the pixel sensor design of FBK and CNM fabrication .................................................. 13
2.3 Sketch of the top and vertical view of a Pixel sensor ............. 14
2.4 Example of 2DCCE of a 3-D sensor. .................................... 15
2.5 Sinogram illustration. ...................................................... 17
2.6 Demonstration of a simple back projection. ......................... 19
2.7 Illustration of Fourier slice theorem ................................. 20
2.8 The intersection between Polar coordinates and Cartesian coordinates. .. 21
3.1 Illustration of high compressing using wavlet transform. ......... 27
3.2 Compressing with wavelet transform. ............................... 28
3.3 Sinusoidal signal made of 3 sine tones with frequencies 50 Hz, 120Hz, 250 Hz and corresponding amplitudes of 0.7, 1 and 0.5. ........... 31
3.4 Reconstruction with the Basis Pursuit, $l_1$-minimization from 30 measurements. ........................................... 32
3.5 Reconstruction with $l_2$-minimization from 30 measurements. ...... 32
3.6 Reconstruction with the Basis Pursuit, $l_1$-minimization from 23 measurements. ........................................... 33
3.7 Reconstruction with $l_2$-minimization from 300 measurements. ...... 34
3.8 Reconstruction with $l_2$-minimization from 430 measurements. ...... 34
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.9</td>
<td>Reconstruction with $l_2$-minimization from 490 measurements.</td>
</tr>
<tr>
<td>3.10</td>
<td>Reconstruction with $l_2$-minimization from 500 measurements.</td>
</tr>
<tr>
<td>3.11</td>
<td>Sketch illustrating the closest point of the line to the origin accord. $l_1$ norm sphere and The $l_2$ norm.</td>
</tr>
<tr>
<td>3.12</td>
<td>Comparing the reconstruction of Shepp-Logan phantom using $l_2$-minimization, $l_1$-minimization, and $l_0$ minimization approximation.</td>
</tr>
<tr>
<td>4.1</td>
<td>Flowchart of the general BM3D procedure.</td>
</tr>
<tr>
<td>4.2</td>
<td>Illustrating selection of blocks in BM3D.</td>
</tr>
<tr>
<td>4.3</td>
<td>Filtering comparison of BM3D, MLP, GSM, KSVD.</td>
</tr>
<tr>
<td>4.4</td>
<td>Flowchart of the system based on the reconstruction via Recursive Spatially Adaptive Filtering algorithm.</td>
</tr>
<tr>
<td>5.1</td>
<td>Shepp-Logan phantom of size $(256 \times 256)$.</td>
</tr>
<tr>
<td>5.2</td>
<td>RSAF vs Conventional CS methods, reconstruction from 12 projections.</td>
</tr>
<tr>
<td>5.3</td>
<td>RSAF vs Conventional CS methods, reconstruction from 9 projections.</td>
</tr>
<tr>
<td>5.4</td>
<td>Original RSAF vs our optimization, reconstruction from 8 projections.</td>
</tr>
<tr>
<td>5.5</td>
<td>Original RSAF vs our optimization, reconstruction from 7 projections.</td>
</tr>
<tr>
<td>5.6</td>
<td>Original RSAF vs our optimization, reconstruction from 6 projections.</td>
</tr>
<tr>
<td>5.7</td>
<td>3-DCCE phantoms, by CERN 3-DCCE group.</td>
</tr>
<tr>
<td>5.8</td>
<td>3-DCCE phantom with spherical shaped CCE.</td>
</tr>
<tr>
<td>5.9</td>
<td>3-DCCE phantom with highly irregular CCE pattern.</td>
</tr>
<tr>
<td>5.10</td>
<td>Dobus phantom of 2 and 3 um resolution reconstructed with filtered Back-projection using 360 projections.</td>
</tr>
<tr>
<td>5.11</td>
<td>Dobus phantom of 2 and 3 um resolution reconstructed with filtered Back-projection using 180 projections.</td>
</tr>
<tr>
<td>5.12</td>
<td>Challenging phantoms reconstructed with filtered Back-projection using 360, 180 and 36 projections.</td>
</tr>
<tr>
<td>5.13</td>
<td>2-D Fourier transform of a Dobos phantom.</td>
</tr>
<tr>
<td>5.14</td>
<td>Dobos phantom 3um rasterized reconstruction from 38 projections with original, optimized and quasi-random diamo RSAF.</td>
</tr>
<tr>
<td>5.15</td>
<td>Dobos phantom 3um rasterized reconstruction from 12 projections with quasi-random diamo RSAF and symmetric diam RSAF.</td>
</tr>
<tr>
<td>5.16</td>
<td>Dobos phantom 3um rasterized reconstruction from 12 projections with quasi-random diamo RSAF and symmetric diam RSAF.</td>
</tr>
<tr>
<td>5.17</td>
<td>Reconstruction from limited angle projection.</td>
</tr>
<tr>
<td>5.18</td>
<td>Irregular phantom reconstruction.</td>
</tr>
</tbody>
</table>
5.19 Spherical phantom reconstruction. .......................... 79
List of tables

2.1 General specifications of the 3-D pixels. . . . . . . . . . . . . . . . . . . . . . . . . . . 14
Nomenclature

Acronyms / Abbreviations

2-DCCE  2-dimensional Charge Collection Efficiency
3-DCCE  3-dimensional Charge Collection Efficiency
ATLAS  A toroidal LHC ApparatuS
BM3D  Block Matching 3-dimensional
BP   Basis Pursuit
CCE    Charge Collection Efficiency
CS    Compressed Sensing
CSFT  Continuous Spatial Fourier Transform
CT    Computed Tomography
CTFT  Continuous Time Fourier Transform
DDTC  Double Side, Double Type Columns
DUT   Device Under Test
FFT   Fast Fourier Transfrom
GSM   Gaussian Scale-Mixture
HEP   High Energy Physics
IBL   Insertable B-Layer
Nomenclature

LAr  Liquid Argon
LHC  Large Hadron Collider
LIP  Linear Inverse Problem
MAD  Mean Absolute Deviation
MRT  Magnetic resonance imaging
MSE  Mean Squared Error
PSNR Peak Signal-to-Noise Ratio
RIP  Restricted Isometry Property
RMAD Relative Mean Absolute Difference
RSAF Recursive Spatial Adaptive Filtering
FBP  Filtered Back-projection
SPS  Super Proton Synchrotron
TOT  Time Over Threshold
TV   Total Variation
Introduction

The Standard Model (SM) is the theory of the fundamental particles and the interactions between them. The crown jewel of its success, the discovery of the Higgs boson, was announced 4 July 2012 by CERN. The discovery was confirmed independently by the two experiments ATLAS, with a mass $126.0 \pm 0.6 \text{ GeV}/c^2$, and CMS, with a mass $125.3 \pm 0.6 \text{ GeV}/c^2$. This was followed by an awarding of the Nobel Prize in Physics to Peter Higgs and François Englert, for their work on predicting the Higgs mechanism, in December 2013.

However, with the finding of the Higgs boson, a chapter of experimental particle physics is about to be closed and the search for the bridge to the next chapter of the story is on. Theories that develop a deeper understanding of physics beyond the Standard Model may be discovered as a result of the upgrades to the Large Hadron Collider (LHC), where the Higgs boson was discovered, and its detectors. For silicon trackers, it means they have to provide tracking in a more challenging environment. Some of the new challenges are a higher number of collisions per bunch crossing, which requires increased granularity, and an increase in integrated luminosity, which requires the selection of more radiation hard silicon sensor material and a reduction of the material in the tracking volume for better performance.

CERN researchers are already engaged in the development of several different types of sensors to meet the new challenges, like 3-D pixel, planar silicon, and diamond sensors. We presume that 3-dimensional measurements of the Charge Collection Efficiency (3-DCCE) will benefit the development and evaluation the sensor designs by making it possible to see the detailed behavior of charge efficiency inside the sensor.

Obtaining 3-DCCE has been highly challenging due to the unreasonable time needed to achieve it, based on the classical principles of Computed Tomography (CT). In this thesis we show that applying the principles of Compressed Sensing reduce the measurement time needed to obtain the 3-DCCE drastically, making it affordable to run 3-DCCE during the regular test-beam experiment period. With CS, we need just 10% of the measurements (number of projections) typically required by classical CT methods. During this thesis, we have optimized a particular CT method and adapted it to the 3-D pixel sensors with penetrating cathodes, reducing the number of projections needed even further. The reconstructions are of
intensity images that represent the pattern of a 2-D slice of the 3-DCCE, which we expect a Double Sided pixel sensor to have.

The thesis starts with a short presentation of SM and ATLAS in Chapter 1, followed by a presentation of the CT principles and the classical methods used to obtain a reconstruction. We include in Chapter 2 an introduction to 3-D pixel sensors. In Chapter 3, we formulate the basis of Compressed Sensing, while in Chapter 4, we explain the CS algorithm used. In Chapter 5, we show the results of the reconstruction with the various improvements we have added to the CS algorithm.
Chapter 1

Standard Model and ATLAS

“...the Standard Model, no one pretends that it is the final word on the subject, but at least we are now playing with a full deck of cards. Since 1978, when the Standard Model achieved the status of ‘orthodoxy’, it has met every experimental test. Moreover, it has an attractive aesthetic feature: all of the fundamental interactions derive from one general principle, the requirement of local gauge invariance.”

—David Griffiths, Introduction to Elementary Particles.

1.1 Elementary Particles of SM

Elementary particles are the particles that make up all matter that we know of (excluding here Dark Matter that we only are aware of it existence). The matter itself consist of two types. The first type that makes up the universe, we call just matter. The second type we call anti-matter, and it is made up of antiparticles. That have the same mass as the particles of ordinary matter but opposite charges. When particles and antiparticles of the same type get together, they annihilate.

Luckily there is barely a trace of antimatter in the universe. Satellite experiments have found evidence of positrons (antiparticles corresponding to electrons) and a few antiprotons in primary cosmic rays, amounting to less than 1% of the particles in primary cosmic rays.¹

In High Energy Particle Physics (HEPP) experiments, antimatter and matter have the same level of occurrence and numerous experiments are using antiprotons to investigate the properties of antimatter. Antimatter is clearly an important and very real part of High Energy Particle Physics.

¹Source for the ratio of anti matter is Wikipedia, antimatter
The elementary particles belong to two classes. The first is fermions, particles obeying the Pauli exclusion principle having spin $s=1/2$ for the elementary ones. The other class consists of particles that follow Bose–Einstein statistics. Those bosons have integer spin $s=\{0,1,2,...\}$. The elementary bosons have $s=1$ and are mediators of the interaction forces, electromagnetic, weak and strong. They are photons, $Z$ and $W^{\pm}$, and gluons. The Higgs particle is a boson with $s=0$ and is a consequence of the mechanism responsible for the masses of the Elementary Particles. For all the elementary particles see Figure 1.1.

The fermions are divided into two groups, quarks, and leptons, and each group consists of three generations. Each generation contains elements of the same characteristic but different mass. The second and third generation of the particles have the same properties as the first generation but they are much heavier. The heavy generation particles decay with very short half-lives to light particles and are observed only in very high-energy environments.
1.2 ATLAS and LHC

The Large Hadron Collider (LHC) at CERN\(^2\) in Geneva, Switzerland, is the largest accelerator on Earth, colliding counterclockwise accelerated bunches of protons or, in another run mode, lead ions. The first run took place from 2010 to 2013 at and it was operated with a total energy of 7 TeV, rising 8 TeV total energy from 2012. On 2013, the LHC’s first run officially ended, and it was shut down for upgrades. On 2015, the LHC operated with 13 TeV collisions. The expected luminosity during the Run 2 and Run 3 at LHC is of between \(1 - 2 \times 10^{34} \text{cm}^{-2} \text{s}^{-1}\).

Four detectors are located at interaction points around the accelerator ring, named ATLAS, CMS, ALICE, and LHCb.

An essential feature of an accelerator is its integrated luminosity \(L\). Which gives the number of events expected to be detected \(N\) during a particular time \(t\) is given by the product of the cross-section \(\sigma\) times the time-integrated luminosity:

\[
N = \sigma \int L \, dt
\]

The luminosity has the dimensions of events per time per area and is usually expressed in the cgs units of \(\text{cm}^{-2} \text{s}^{-1}\).

A Toroidal LHC ApparatuS (ATLAS) is a general-purpose particle detector, meaning it was designed to be used for a vast area of physics studies. The requirements for the system have been defined according to processes of new phenomena hoped to be observed at the TeV

\(^{2}\)The name stands for Organisation Européenne pour la Recherche Nucléaire (European Organization for Nuclear Research). The acronym originally represented the French words for Conseil Européen pour la Recherche Nucléaire (European Council for Nuclear Research), source wiki.
scale; some of these are confirmations or improved measurements of the Standard Model and possible clues for new physical theories (for a full list see chap1.1 in Ref [13]).

The ATLAS experiment was proposed in 1994 and funded by the CERN member countries in 1995. The Construction was completed in 2008.

Fig. 1.3 ATLAS is 46 m long, 25 m in diameter, and weighing about 7,000 tonnes. The experiment is a collaboration involving approximately 3,000 physicists from over 175 institutions in 38 countries. (ATLAS Experiment ©2013. ATLAS images are under CERN copyright.)

The ATLAS detector consists of the Inner Detector for tracking charged particles, the calorimeters for measuring the energies of interacting particles, the Muon Spectrometer for tracking muons, and two magnet systems that bend charged particles, allowing their momenta to be measured. The detector is "hermetic", meaning it detects all stable particles (apart from neutrinos), with no blind spots. This enables us to calculate the missing transverse energy which is critical for many of the studies conducted.

1.2.1 Magnet system

The magnet system generates the bending power for the momentum measurement of charged particles. It is composed of 3 parts: (1) a Central Solenoid, surrounding the inner tracker with a longitudinal magnetic field; (2) the Barrel Toroid (BT); and (3) the End-Cap Toroid
1.2 ATLAS and LHC

(ECT). At the outer region of the detector, there are three very large air-cored toroids that provide the field for the muon spectrometer, as shown in Fig. 1.4.

The Barrel Toroid provides 1.5 to 5.5 mT of bending power in the pseudorapidity range $0 < |\eta| < 1.4$, and the End-Cap toroids approximately 1 to 7.5 mT in the region $1.6 < |\eta| < 2.7$. The bending power is lower in the transition regions where the two magnets overlap $1.4 < |\eta| < 1.6$. The Central Solenoid provides a 2 T axial field. The layouts preceding the calorimeter were designed to have minimal material budget avoiding energy loss before measurement. The design of Central Solenoid was based on the experience attained from previous projects of similar magnets for best reliability.

1.2.2 Muon Spectrometer

The Muon Spectrometer consist of 1200 chambers measuring with high spatial precision tracks of the passing muons (very few other types of particles are expected to pass beyond the calorimeter) and measures their momentum in the pseudorapidity range $|\eta| < 2.7$. It is also designed to trigger on detected particles in the region $|\eta| < 2.4$. It starts at a radius of 4.25 m
near to the calorimeters and reaching out a full radius of 11 m. It’s intended to measure the momentum of 100 GeV muons with 3% accuracy and of 1 TeV muons with 10% accuracy.

Fig. 1.5 The muon spectrometer possesses four chamber sub-systems: the precision measurement tracking chambers (MDT’s and CSC’s) and the trigger chambers (RPC’s and TGC’s). In the end-cap, the TGC layer (I) is positioned in front of the innermost tracking layer; the subsequent three layers stand in front (M1) and behind (M2 and M3) the second MDT wheel. The first letter (B and E) of the MDT naming scheme indicates the barrel and end-cap chambers, respectively. The second letter of the acronyms refer to the inner, middle, and outer layer. the third letter refers to the large or small sector types. (ATLAS Experiment ©2013. ATLAS images are under CERN copyright.)

The spectrometer is outfitted with a system of fast trigger chambers capable of delivering track information within a few tens of nanoseconds after the passage of the particle. This is executed by the Resistive Plate Chambers (RPC) and at the end-cap by Thin Gap Chambers (TGC), as shown in Fig.1.5.
1.2.3 Calorimeter

The calorimeters (Fig. 1.6) are two types of the inner electromagnetic calorimeter and an outer hadronic calorimeter. They absorb particles for the purpose of measuring their energy. Calorimeters must have a length that provides good containment for electromagnetic and hadronic showers, and must also limit punch-through into the muon system. Both types make use of liquid argon (LAr) as the active detector medium. The selection has fallen on it for its intrinsic linear behavior, its stability of response over time and its intrinsic radiation hardness.

Fig. 1.6 Cut-away view of the ATLAS calorimeter system showing the electromagnetic barrel calorimeter, an Electromagnetic End-cap Calorimeter (EMEC) contained by the two end-cap cryostats, a Hadronic Endcap Calorimeter (HEC), located behind the EMEC, and a Forward Calorimeter (FCal) incorporating the region closest to the beam. (ATLAS Experiment ©2013. ATLAS images are under CERN copyright.)

**EM Calorimeter:** The EM calorimeter is consist of a barrel part (|\(\eta| < 1.475\)) and two end-cap components (1.375 < |\(\eta| < 3.2\)). The EM calorimeter is a lead-LAr detector over its full cover range. The total thickness of the EM calorimeter is < \(22X_0\) (radiation lengths) in the barrel and < \(24X_0\) in the end-caps.
**Hadronic Calorimeter:** The Tile Calorimeter is a sampling calorimeter utilizing steel as the absorber and scintillating tiles as the active material. It is sectioned in three layers, approximately 1.5, 4.1 and 1.8\(\lambda\) (interaction lengths) thick for the barrel and 1.5, 2.6, and 3.3\(\lambda\) for the extended barrel. The full detector thickness at the outer edge of the tile-instrumented region is 9.7\(\lambda\) at \(\eta = 0\).

The Hadronic end-cap Calorimeter (HEC) consists of two wheels per end-cap, located directly behind the end-cap electromagnetic calorimeter. The wheel plates are made of copper interleaved with LAr gaps being the active medium.

The Forward Calorimeter (FCal) is approximately 10\(\lambda\) deep based on a high-density design and consists of three modules in each end-cap: the first is made of copper for electromagnetic measurements, while the other two, made of tungsten, measuring predominantly the energy of hadronic interactions. The LAr is used as a sensitive medium for FCal.

### 1.2.4 Inner Detector

The ID shown in Fig. 1.7 measures the position of charged particles as they traverse the detector operating in a 2 Tesla magnetic field to curve the charged particles trajectory revealing its charge and the degree of curvature reveals its momentum. The ID consists of 4 sub-detectors: the Insertable B-layer (IBL), Pixel Detector, Semi-Conductor Tracker built using silicon sensors technologies and the Transition Radiation Tracker built up of straw drift tubes.

When charged particles cross the silicon sensors, they generate electron-hole pairs that can be collected with an applied electric field. This charge is recorded locally in the sensor, identifying the position of the particle. A similar process occurs in the Transition Radiation Tracker filled with gas that becomes ionized when a charged particle traverses it. Then the liberated electrons are drifted, with an applied electron field, to the wire at the center of the straw, where they are recorded.

Unlike the silicon sensors, in drift tubes, the primary ionization is multiplied before detection and their position resolution is about 200 micrometers while silicon detectors have a resolution of few micrometers.
Fig. 1.7 Sketch of the ATLAS inner detector showing Transition Radiation Tracker, Semi-Conductor Tracker, Pixel Detector including the new insertable B-layer (IBL). The distances to the interaction point are also shown. (ATLAS Experiment ©2013. ATLAS images are under CERN copyright.)
Chapter 2

3D Charge Collection Efficiency and Computed Tomography

2.1 Test-beam experiment

Before installing new sensors into the ATLAS detector, the sensor type needs to be put to a series of tests to acquire experience about its behavior and to prove that the technology at hand will be sufficiently working even at the end of the expected lifetime. The tests are done through laboratory characterization and measurements in test-beam experiments. To achieve a quality attribute of the sensor, special properties of the sensor need to be known, like hit and current Charge Collection Efficiency (CCE). Charge Collection Efficiency is measured during the test-beam experiments, in which incident beam of high energetic particles traverses the investigated sensor to determine the ability of the sensor for detection. For this propose test-beam telescope is used to identify the particle’s trajectory. There are two types of sensors studied during a test-beam period: the new sensor and the irradiated sensor, that is intentionally damaged by energetic protons. The latter is the most interesting as it gives us insight on how the radiation damage caused by the high energetic particles interactions affect the sensor. At CERN, the particle beam for the test-beam experiments is delivered by the Super Proton Synchrotron (SPS). The SPS provides 450 GeV protons to the test site. They are then crashed into a target, creating secondary particles yielding the final particles with the right energy for the tests. The beam used to test ATLAS 3-D pixel sensors at CERN consists of pions, $\pi^+$, with energy around 120 GeV. The telescopes used for this type tests consist of 3-4 plates in front and after the Device Under Test (DUT) where a high-resolution sensor are mounted in each of those plates. The DUT is usually mounted in a fixed position on the
mechanical support table but a rotating table, is available on the Timepix\textsuperscript{1} telescope, see Fig. 2.1 otherwise it can be available by request for the test-beam at CERN. The rotation stages for the rotating arm have an accuracy of a hundredth of a degree, for the one installed on Timepix [26].

Fig. 2.1 Layout of the Timepix Telescope mechanics, pixel planes and scintillators with respect to the beam axis. Figure from [26].

2.2 3D pixel sensor

The 3D pixel detector consists of the front end electronics and a sensitive part, the 3D pixel sensor. It is made of silicon and has cylindrical electrodes (p+ and n+) that penetrate the substrate. The main motivation for using 3D sensors is their high radiation hardness compared with the planar sensors that have the electrodes on the surface of the substrate.

CERN has been investigating two versions of 3D sensors for use in the upcoming upgrade of the ATLAS Inner Detector and for the Insertable B-Layer (IBL), which is now installed in ATLAS and took data during the 2015 LHC run. Mainly the difference between the two

\textsuperscript{1}“The Timepix particle tracking telescope has been developed as part of the LHCb VELO Upgrade project, supported by the Medipix Collaboration and the AIDA framework. It is a primary piece of infrastructure for the VELO Upgrade project and is being used for the development of new sensors and front end technologies for several upcoming LHC trackers and vertexing systems.” quoted from [26]
sensor versions is the depth of the cathodes. For the one produced at CNM\(^2\) the columns do not fully penetrate the substrate, whereas for the one produced in FBK\(^3\) the cathodes fully penetrate the substrate. In addition, there are substantial differences concerning the isolation implantation between the n+ columns at the surface and differences in the design of the edge guard ring; Fig. 2.2 shows details of the 3D layout for the two versions. The CNM version is called Double Side, Double Type Columns (DDTC) or simply called double sided and the FBK version called full 3D, or just single sided.

Fig. 2.2 The 3-D etched columns from the pixel sensor design of FBK (a) and CNM (b) fabrication. We see that the isolation implantation between the n+ columns at the surface differ between the two versions. The p-stops are implanted on the front side of CNM sensors while FBK sensors use p-spray implantations on both sides. Figure from [33].

The accepted 3-D sensors for the use in IBL satisfied the geometrical specifications on Table 2.1. The 3-D detector is divided into small units called pixels. A pixel with three n+ readout cathodes is surrounding by eight p+ cathodes, as shown in Fig. 2.3 in the Top view. The vertical cross section layout of the 3-D sensor is illustrated for simplicity as shown in Fig. 2.3.

**2.3 2-D Charge Collection Efficiency**

2D Charge Collection Efficiency (2-DCCE) is measured with the telescope using tracks interpolation based on the hits registered to search for a matching hit on the Device Under Test. The number of hits on DUT matching the telescope is divided by the total number of

---

\(^2\)Centro Nacional de Microelectronica (CNM-IMB-CSIC), Campus Universidad Autonoma de Barcelona, 08193 Bellaterra (Barcelona), Spain. See http://www.imbcnm.csic.es.

\(^3\)Fondazione Bruno Kessler (FBK), Via Sommarive 18, 38123 Povo di Trento, Italy. See http://www.fbk.eu.
Table 2.1 General specifications of the 3-D pixels. Table from [33].

<table>
<thead>
<tr>
<th>Item</th>
<th>Sensor Specification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Module type</td>
<td>single</td>
</tr>
<tr>
<td>Number of n⁺ columns per 250 μm pixel</td>
<td>2 (so-called 2E layout)</td>
</tr>
<tr>
<td>Sensor thickness</td>
<td>230 ± 20 μm</td>
</tr>
<tr>
<td>n⁺-p⁻ columns overlap</td>
<td>&gt; 200 μm</td>
</tr>
<tr>
<td>Sensor active area</td>
<td>18860 μm × 20560 μm</td>
</tr>
<tr>
<td>(including scribe line)</td>
<td></td>
</tr>
<tr>
<td>Dead region in Z</td>
<td>&lt; 200 μm guard fence ± 25 μm cut residual</td>
</tr>
<tr>
<td>Wafer bow after processing</td>
<td>&lt; 60 μm</td>
</tr>
<tr>
<td>Front-back alignment</td>
<td>&lt; 5 μm</td>
</tr>
</tbody>
</table>

Fig. 2.3 A Pixel layout is shown in the top view while the dashed lines outline the border for the pixel. The two vertical cross section shows two different 3-D devices, the full 3-D sensor at the left and the double sided on the right. Both devices have three readout electrodes per pixel. The bulk material is white to highlight the electrodes in the two 3-D versions. The red line presents the n⁺ material and the blue presents the p+. Figure from [3], with modification by thesis author.

hits detected by the telescope, gives the percentage 2-DCCE. In the remainder of the thesis each time we use CCE we mean the hit Charge Collection Efficiency. Examples of 2-DCCE are as shown in Fig. 2.4 b,c. The scale represents the percent hit efficiency. We see that the lowest CCE value is about 80% and is measured around the p cathodes where the electric field on the charge cloud is less than the n cathodes. In 2.4 (c) the module is inclined 15° in
2.4 Why 3-D Charge Collection Efficiency

The 2-DCCE measurement allows us to create 2-D maps of charge collection inefficiencies to understand where and how much charge the sensor can collect. However, the information about how the inefficiency change as a function of the distance from the detector surface is absent. The idea behind the 3-DCCE is to make a 3-D mapping of the efficiency using the same principal as the medical x-ray CT. This requires taking a number of 2-D Charge Collection Efficiency measurements at different angles, normally between 180-360. Then they are put together by a computer to form the full spatial map. Taking 360 measurements is a very time-consuming process, especially if we keep in mind that the test-beam experiment is done in CERN twice a year and the time slot available to this experiment is about 20 days.

Fig. 2.4 Cell Efficiency: a) lithography sketch for 3-D CNM sensor, b) the 2-D efficiency map for the CNM sensor with normal incident tracks and c) the 2-D efficiency map for the CNM sensor using 15° inclined tracks. Figure from [33], with modification by thesis author.

the y-z plane according to the orientation shown in Fig. 2.1. For the inclined module, we have substantially less particles passing entirely inside p+ cathodes, resulting in a better CCE. The CCE of the CNM 81 module used in this experiments is 97.5% when the module is exactly perpendicular to the beam line as shown in (b) and is 99% with 15° inclined angle as shown in (c) [33].
In fact, if we are to take 360 projections, also 2-DCCE measurements, for one device we would need 29 days of continuous measurement [17]. Why would we be so interested in this 3-DCCE?

The 3-DCCE would be an addition to the fundamental analysis supporting the TCT\(^4\). We anticipate that it would improve the knowledge of CMOS physics and help to improve simulation of detector response. Let us see some of the primary sources of the inefficiency and what we would like to investigate with 3-DCCE.

**Irradiation damage**  It’s known that irradiation damage in semiconductor sensors alters their response and degrades their performance in many ways such as increased noise level and leakage current. It’s expected that the charge collection homogeneity of an irradiated 3-D pixel sensor to be distorted, resulting in an overall decrease of detection efficiency. 3-DCCE will allow us to see how the distortion changes with depth and can help us to better understand the effect of the annealing process that tends to correct the damage caused by radiation.

**Edge and connection points**  The Edgeless (active edge) silicon pixel detectors have been of great interest due to their capability to minimize the insensitive area of the detector. The way this is implemented causes a local distortion of the electric field at the detector edge. The deformed electric field alters the charge collection and leads to an inaccurate charge interpolation. Using 3-DCCE, we can visualize the influence of active edges techniques on the charge collection.

X-ray spectroscopy has been used to construct the 3-DCCE, and it has been able to confirm that the p-spray isolation method has the benefit of achieving a higher sensitive edge region opposed to the p-stop method [36].

**3-D design comparison**  The electrode depths vary between full or partial penetration for a 3-D pixel detector. The charge collection efficiency will vary with depth of the columns. With 2-DCCE, we see just the average charge efficiency value along the column, and we would like to know if it is more or less constant over the whole length or not. The distance between the columns can have a significant effect on the collection efficiency, and it could be interesting to see how it is compared to the simulations. [12, 19].

**Other interesting use cases**  The choice of Time Over Threshold (TOT) and the applied bias voltage lead to an efficiency characteristic that varies with depth [6]. 3-DCCE can result

\(^4\)Transient current Technique used for Characterization of detectors and to study radiation damage.
in better numerical simulation and model studies. Apart from the silicon sensors, we could use the same technique to study the inefficiency regions of pCVD diamond crystal boundaries and how different pCVD diamond charge collection changes with depth.

### 2.5 Computed Tomography

Computed Tomography (CT) scanning \([23, 15]\) scan is an imaging method that is based on measuring the attenuation in the form of energy deposited by the beam in a medium. Let \(f(x,y)\) be the charge collected at the electrodes from a voxel at position \((x,y)\) in the sensor with a known relation to the energy deposited by the individual particles of the beam. The attenuation is then integrated along the beam particle’s trajectory, forming a 1-D image known as a Radon transform or projection,

\[
p_{\theta}(r) = \int \int f_k(x,y) \, dx \, dy,
\]

where \(r\) is the distance to the origin of the object and \(\theta\) is the angle between the beam trajectory and x-axis. Taking multiple projections along different angles in the interval \(0 – 180^\circ\) gives us a 2D record of efficiency known as a sinogram.

![Sinogram](image.png)

Fig. 2.5 Sinogram at the right and the intensity image at the left. The phantom object on the left consists of 0s except the two spots. The white one with the highest value 1 and is the farthest from the center, the orange spot represents some intensity between 0 and 1 and is the closest spot to the center. The resulting sinogram is obtained by summing up the intensities at each pixel. 180 projection slices were taken over 0 to 180-degree angle, equidistantly sampled (only by coincidence the x-axis marks displacement at -150/150 units). We see two sinusoids in the sinogram, the blue one arise from the orange spot and the yellow arise from the white spot in the phantom.
A sinogram is the superposition of all sinusoids corresponding to the function \( r = x \cos \theta + y \sin \theta \). Each voxel at a position \((x, y)\) contributes to a unique sinusoid to the sinogram, weighted by the value \( f(x, y) \) and with the “amplitude” of the sinusoid being \( \sqrt{x^2 + y^2} \). This is mathematically expressed with the help of Dirac impulse as

\[
p_{\theta_k}(r) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_k(x, y) \delta(x \cos \theta_k + y \sin \theta_k - r_j) \, dx \, dy.
\] (2.2)

There exist several methods for inverting the Radon transform, i.e., for recovering the object physical characteristic \( f(x, y) \). The most basic ones are the direct Fourier reconstruction based on the Fourier-slice theorem, also called Fourier Slice method or Central Slice method, the Convolve Back-project (CBP) method, also called the Filtered Back-projection (FBP) method. The Filtered back-projection is the method that has been most used as it was more practical and easier to implement. In this section, we treat the idealized version of the tomography problem in which the entire continuum of projections \( p_{\theta}(r) \) is available. In practical tomography systems, only a discrete set of projections are available and noise may even be varying with the projection angle \( \theta \).

**Back-projection** is the most used way for recovering a function \( f(x, y) \) from the Randon Transform measurements. This is done by taking the \( p_{\theta_k}(r) \) and “smear” it back into object space along the corresponding measurement angle.

The back-projection results in the following expression:

\[
f_{\theta_k}(x, y) = g(r, \theta_k).
\] (2.3)

The full spatial mapping of charge efficiency will be obtained by integrating the back-projections at each angle:

\[
f(x, y) = \int_0^{\pi} f_{\theta_k}(x, y) \, d\theta.
\] (2.4)

Theoretically, the reconstruction is possible giving an infinite set of projections, but for most applications 360 projections are sufficient to have an exact recovery. In practice, the tomographic image reconstruction usually yields a blurred image. To deal with this problem, we apply a filter to the sinogram \( p_{\theta}(r) \) and then smear back the filtered function \( \hat{p}_{\theta}(r) \).

**Fourier Slice method** makes use of the Fourier Slice theorem that connects the Randon transform of \( f(x, y) \) to its 2D Spacial Fourier Transform. Let

\[
P_{\theta}(\rho) : \text{Continuous Time Fourier Transform } [p_{\theta}(r)]
\]
Fig. 2.6 A demonstration of simple back projection: (a) Experimental setup shows an X-ray source that scans a phantom, consisting of a radiation-absorbing spot in the center, generating the profile/projection as shown. The gray strip in the square illustrates the smeared projection. (b) Four results produced by scanning at various angles. (c) Superposition of four projections. (d) The reconstructed image from multiple projections covering the $[0 - 180^\circ]$ shows the recovered absorption spot and an additional star-like artifact. Source: Wikipedia under “Basic Physics of Nuclear Medicine/X-Ray CT in Nuclear Medicine”, with modification by thesis author.

$$F(u,v) : \text{Continuous Space Fourier Transform} [f(x,y)]$$

Then the Fourier Slice theorem states:

$$P_{\theta}(\rho) = F(\rho \sin \theta, \rho \cos \theta) \quad | \quad u = \rho \sin \theta \text{ and } v = \rho \cos \theta.$$  \hspace{1cm} (2.5)

This means that each CTFT of a projection with an angle $\theta_i$ corresponds to the line in the CSFT of $f(x,y)$ with the same angle $\theta_i$; see Fig. 2.7 for an illustration of how this is implemented.
Fig. 2.7 Illustration of Fourier slice theorem. The Charge Collection Efficiency of a horizontal slice representing a 3-D detector is at the top left of the figure. Once the beam traverses the detector, we measure a 1-D CCE which corresponds to what is called in CT a projection. We show two projections where we have left the DUT fixed and changed the incoming angle of the beam, the orange striped arrows. The projections undergo an FFT and are re-positioned according to the angle of projection. According to the Fourier slice theorem, the Fourier transforms of $P_\theta$ are equal to the lines on the 2-D Fourier transform of the CCE of the DUT at the same polar angle. The square on the lowest part of the figure shows the 2D Fourier transform of the DUT CCE, and it illustrates that superposition of all the Fourier transforms of the projections corresponds to the direct 2-D FFT of the total CCE. Figure from [1], with major modifications by thesis author.

The Fourier Slice Theorem is a bit more complicated to implement than FBP. This is due to the fact that the Fourier transform of the intensity image, $F(u,v)$, has coefficient values spaced in a grid, as it is normal for Cartesian coordinate systems representations. While the $P_\theta(\rho)$ is a representation in the Polar coordinate system with coordinates $(r, \theta)$. Fig2.8 shows the position of the coefficients obtained from the FFT of the projection as the intersection
between the lines through the Origo and the circles. The circles are separated by one unit length. In Cartesian system, the points separated by one unit length are shown as the grid. We see that the intersection of the two coordinate systems rarely corresponds to each other.

Fig. 2.8 A sampling of Fourier space by projections in 2-D. The Fourier transform of each projection samples a slice of Fourier space (black lines); its normal is determined by the tilt angle. The grid unit length is reciprocally proportional to the dimensions of the spatial image. For reconstruction, the Fourier coefficients obtained from the projections (the dots) needs to be approximated to the coefficients in Cartesian coordinates (grid) by an appropriate algorithm before the final IFFT is used to recover the intensity image. The plot from [7].

When we have 360 projections available for the reconstruction, we may get some precision loss when transforming from one coordinate system to the other in the frequency domain. This problem is solved by applying an interpolation algorithm that reconstructs the full frequency space based on the available measurements in the polar coordinate, but when working with extremely few projections, also below 60, the interpolation method does more harm than good, as it becomes a source of additional artifacts. In the last decade, there have been developed efficient algorithms that deal with this problem, such as the Non-uniform Fast Fourier Transform [22]. It has been shown that implementing those algorithms make
the Fourier slice method equally accurate as the Filtered Back-projection method and much faster in obtaining reconstruction with CT or MRI [7].
Chapter 3

Compressed Sensing.

“As our modern technology-driven civilization acquires and exploits ever-increasing amounts of data, “everyone” now knows that most of the data we acquire “can be thrown away” with almost no perceptual loss—witness the broad success of lossy compression formats for sounds, images, and specialized technical data. The phenomenon of ubiquitous compressibility raises very natural questions: why go to so much effort to acquire all the data when most of what we get will be thrown away? Can we not just directly measure the part that will not end up being thrown away.”


The year 2006 witnessed the emergence of Compressed Sensing, as D. L. Donoho published his groundbreaking paper titled simply Compressed Sensing (CS)[2]. The beauty of Compressed Sensing is that it offers us a theory for restoration/recovery of signals that are highly undersampled or from systems that are underdetermined. Compressed Sensing is not about recovering compressed signals, but it is about reconstructing signals that were sensed with far fewer measurements than what is imposed by previous limits, for example by the Shannon sampling theorem. Compressed Sensing is a theory that has the potential to change the way we do experimental science. For isn’t experimental science mostly, about taking measurements to find out some unknown truth, to determine distribution, a particle mass, etc? With Compressed Sensing, we can start being even more efficient, saving time, money, storage space and the amount of data transferred.

Since the emergence of Compressed Sensing, the field has wildly expanded. There are continuously publications about new applications of Compressed Sensing, as in this thesis where we suggested the use of CS for spatial efficiency mapping. Other publications describe new methods for pushing the boundaries of Compressed Sensing toward even better
performance. In this regard, we can state the approximate \( l_0 \) norm minimization algorithms and Compressed Sensing via Spatial Adaptive Filtering, which for instance we favored in this thesis. Another approach that we find quite interesting is the merging of pattern recognition with CS, this can be used for track fitting and may show to be the answer for the tracking problem at high luminosity we are planning to achieve in the upcoming upgrades.

In this Chapter, we will introduce the basics of Compressed Sensing as well as some new approaches, briefly as that was not the scope of this thesis and the time won’t allow for a more detailed presentation neither for a satisfying comparison of them. This chapter is based heavily on Refs. [18, 32].

### 3.1 Linear Inverse Problem

In Inverse Problems [11] we investigate the relationship between a set of measurements and a set of parameters which represent the object under study. Since the measurements are used to deduce information about the object, we call this an Inverse Problem. If the process of obtaining measurements can be expressed as a linear operator we call then the problem of finding some property of the object from its measurements, as Linear Inverse Problem (LIP).

In mathematical terms, the Linear Inverse Problem can be expressed in its simplest form as

\[
y = A \, x,
\]

where \( y \in \mathbb{R}^m \) is the total observed data and \( m \) is the number of measurements. The signal that we aim to recover is \( x \in \mathbb{R}^N \). This signal is, in our case, a 2D intensity image that represents the efficiency of a slice of the 3D pixel sensor. The number of image pixels (picture element) is \( N \). The matrix that models the linear measurement process (linear transformation) is \( A \in \mathbb{R}^{mxN} \).

In linear Algebra \( y \) and \( x \) must be at least of equal size if we are to solve the equation (3.1) and thus recover \( x \) from the available measurements \( y \) when we know the measurement matrix \( A \).

### 3.2 Reconstruction from incomplete spectral

We are naturally confronted in CT reconstruction with Fourier Slice Method, to the problem of regaining the original intensity image from its incomplete Discrete Fourier Transform. This is classically treated by zero-padding the Fourier coefficients of the unobserved frequencies
if the number of projections is equal to or above 360. Better reconstruction algorithms tend to guess the values of the missing Fourier coefficients by interpolation. The interpolation method is essential when the number of projections is around 180 and in our case, it works well until around 120 if we use very powerful interpolation algorithms. Once the number of projections is below 45, the reconstructed intensity image has a very low diagnostic value. For highly undersampled objects the predictions of Fourier coefficients from their neighbors fails totally, due to the global and highly oscillatory character of the Fourier transform.

Emmanuel Candès solved this problem by a strategy based on convex optimization [18]. We consider a discrete signal \( \{ f(x_1, x_2) \in \mathbb{C}^{n \times n} \mid 1 \leq x_1, x_2 \leq n \} \), in our case it is a 2Dimensional intensity measure formed from the set of voxels making up a slice of the 3D pixel sensor. Its observed Discrete Fourier Transform, \( \hat{f}(w_1, w_2) |_{\Omega} \), on a star-shaped frequency domain \( \Omega \). The frequency domain \( \Omega \) is composed of coefficients samples at frequencies \( w = (w_1, w_2) \) situated along radial lines at relatively few angles, giving it what we called star-shape, see Fig3.12 (a-d).

Giving a partial Fourier operator, \( \mathcal{F}_{T \rightarrow \Omega} : f \rightarrow \hat{f}|_{\Omega} \), that transforms the described signal \( f \in T \) to the restricted Fourier Transform \( \hat{f}|_{\Omega} \), the LIP is then expressed as \( \mathcal{F}f = \hat{f}|_{\Omega} \). In case of \( |T| > |\Omega| \) the LIP is then an underdetermined problem with many possible solutions; let us call such a solution \( g(x_1, x_2) \).

The method proposed by Candès recovers \( f^\# \), the solution with minimum complexity to the convex problem expressed as follows:

\[
\min_{g} \|g\|_{TV} \quad \text{subject to} \quad \hat{g}|_{\Omega} = \hat{f}|_{\Omega}.
\] (3.2)

In other words, given a partial observation, \( \hat{f}|_{\Omega} \), we seek a solution with minimum complexity, measured by the Total Variation norm, whose “visible” coefficients match those of the unknown object \( f \). With high probability the reconstruction is exact, meaning \( f^\# = f \), given that we have a sufficient minimal set of measurements.

In the following we will give the relation between the number of measurements needed and the complexity of \( f \) but first let us state some useful definitions.

The cardinality, which we symbolize with \(|.|\), is a count of the number of elements of the set. We will have a need for \( \text{supp}(g) \), which is the support of a some function \( g \). In other words, the set of points where the function is not zero-valued. The Total Variation norm of a 2Dimensional object \( g(x_1, x_2), 1 \leq x_1, x_2 \leq n \), is

\[
\|g\|_{TV} = \sum_{x_1, x_2} \sqrt{|D_1 g(x_1, x_2)|^2 + |D_2 g(x_1, x_2)|^2}.
\] (3.3)
Compressed Sensing.

as shown on [18], where \( D_{\{1,2\}} \) is the finite difference \( D_1 g = g(x_1, x_2) - g(x_1 - 1, x_2) \) and \( D_2 g = g(x_1, x_2) - g(x_1, x_2 - 1) \). Let suppose that \( f \) is a superposition of \(|T|\) spikes and \( T \) is the domain of non-zero intensities\(^1\).

It is proven that a signal \( f \) of length, \( N \), given it is a prime integer, with \( f \) supported on \( T \) and \( \Omega \) being a subset of \( \{1, \ldots, N\} \) such that

\[
|T| \leq |\Omega|/2, \tag{3.4}
\]

then \( f \) can be reconstructed uniquely from \( \hat{f}|\Omega \) [18].

The recovery through TV norm is a method pioneered by Leonid I. Rudin [29]. It is built on the observation that signals with excessive and possibly spurious detail have high total variation. Thus reducing the total variation of the signal result in the removal of unwanted details while preserving the important ones as well as edges. In cases where the signal has a sparse gradient, the TV norm yields the sparsest solution.

### 3.3 Compressed Sensing

Compressed Sensing addresses the recovery problem in more general terms, covering all type of signals, by imposing the recovery condition to be the sparsest of all possible solutions, what is known as \( l_0 \) minimization. The recovery of \( f \) that has \( N \) degrees of freedoms and where only \( N_w = |\Omega| << N \) is measured. This is far fewer measurements than what traditional theory implies. In Compressed Sensing [14, 31, 32] is possible assuming that the signal is sparse, or it can be transformed into a set of basis where the signal can be presented with few components. Mathematically this means solving the combinatorial optimization problem

\[
(P_0) \quad \text{minimize } \|g\|_0 \text{ subject to } \hat{g}|\Omega = \hat{f}|\Omega. \tag{3.5}
\]

Sparsity express the concept that a signal or its linear transform can be expressed as a maximum number of elements \( s = |T| \) that is far less than its bandwidth. We rely as well on compressing the signal if the transformation does not yield a low enough sparsity value. We say that a signal is compressible or weakly sparse with the parameter \( s > 0 \) if most of its coefficients are close to zero and their nonincreasing rearrangement decays according to the

\(^1\)For our sensor, the intensity is inverted so the intensity measure 100% is represented by 0 and charge collection efficiency 60% is represented by the image intensity 0.4 in the phantom and so on. This is done to obtain an object with the highest sparsity as it is crucial for CS that the set of measurements it is applied, is as sparse as possible.
3.3 Compressed Sensing

power law

$$|f_i| \leq C_i^{-1/s} \quad i = \{1, 2, 3..., N\}, \quad (3.6)$$

formula from [32].

We present an example of compressibility where the perceptual loss after compressing is hardly noticeable between the original and the compressed version Fig. 3.1. The frequency spectrum of the image is shown in Fig.3.2 obtained by a wavelet transformation.

![Original Image and Compressed Image](image.jpg)

Fig. 3.1 Image (on left) in gray scale used to illustrate compressing effect as shown on at the right. The reconstruction after 91% compression, at the right. Figure on left from Facebook, with permission from the subject.

We observe, by zooming the spectrum, which the absolute value of coefficients in the frequency domain contains many entries that are with the magnitude less than 30. Compressing the image by zero-padding (set equal zero) the 91.23% of frequencies with the lowest amplitude, result in maintaining just the 26 231 largest coefficients of the original image consisting of 298 951 nonzero coefficients. Then transforming back using only the images 26 231 coefficients to the image domain, we obtain the reconstructed image shown in Fig. 3.1, which is barely distinguishable from the original.

In many applications where we do not have such detailed image, or we are not interested in recovering the minor fluctuations, then Compressed Sensing can, in fact, give us the required result from even fewer coefficients. In practice, sparsity can be a difficult constraint to impose, and we may prefer the weaker concept of compressibility. When working with compressed signals, we would like to measure how close the compressed reconstruction is to the true signal. This is measured by the error of the best s-term approximation $\sigma (f)_p$, according to some norm $l_p$. 

Fig. 3.2 The frequency spectral of the image shown in Figure 3.1, obtained by a wavelet transform and arranged in random order for enhanced visibility. The number of coefficients with values less than 30 is far more than what is observed in this figure. If we get rid of every frequency that has an absolute value less than ten the signal energy will be mostly preserved. As a consequence, such images are highly compressible.
\[ \sigma(f)_p = \inf \{ \| f - g \|_0, \quad g \in \mathbb{C}^N \text{ is } s \text{-sparse} \}, \]  
(3.7)
as proved on [32], where the infimum is achieved by an s-sparse vector \( g \in \mathbb{C}^N \) whose nonzero entries equal the s largest absolute entries of \( f \).

Solving the \( l_0 \)-minimization problem to recover an s-sparse signal of length \( N \) requires a combinatorial search through all \( \binom{N}{s} \) potential solutions and is thus intractable for most practical applications. The remarkable result of CS theory, pioneered by Candès et al. and Donoho is that if one replaces the \( l_0 \) quasi-norm prior in Equation 3.5 with the \( l_1 \) norm, then exact signal recovery is still possible. This is called convex relaxation of \((P_0)\) to \((P_1)\) and the cost of the relaxation is a modest degree of oversampling. Moreover, it’s demonstrated [5] that it is possible to recover a s-sparse vector with \((P_1)\) by \( 3s-5s \) measurements which is about twice what is theoretically necessary for \((P_0)\) as described by Equation. 3.4. This indicates that there is obviously room for improvement for formal CS that is based on \( l_1 \)-minimization. This is exploited in alternative approaches, as we will see later on. The results of those alternative approaches are usually based on the observed result, and their performance is not proven mathematically. This is because those methods were based on ideas that were first tested, and they appear to work better that the \( l_1 \) approach. It looks like there is more interest and more effort made to come with improved methods than to get a formally proven theory.

In the formal Compressed Sensing the LIP is solved by the \( l_1 \)-minimization, expressed as

\[
\minimize_{g \in \mathbb{C}^N} \| g \|_1 \quad \text{subject to } \hat{g}|_{\Omega} = \hat{f}|_{\Omega}.
\]  
(3.8)
It’s proven [18] that given a discrete signal \( f \in \mathbb{C}^N \) supported on an unknown set \( T \), while the measurements are supported on the set \( \Omega \) of size \( |\Omega| = N_\Omega \), letting the frequency samples be choosing uniformly at random radial lines, That the minimizer to the \( l_1 \) problem is unique and is equal to \( f \) with probability at least \( 1 - O(N^{-M}) \), if

\[ |T| \leq C_M (\log N)^{-1} |\Omega|, \]  
(3.9)
as proved on [18], where \( M \) is an accuracy parameter. We notice here that \( |T| \) is of size \( |\Omega| \), modulo a constant and a logarithmic factor. The number of needed frequency coefficients \( |\Omega| \) is proportional to \( \log(N) \) while the probability of exact recovery is proportional to \( N^{-M} \) and doe not depend on the sparsity as long as Equation 3.9 holds. The term \( C_M \) can be approximated by \( C_M = 1/[23(M + 1)] \), which is valid for \( |\Omega| \leq N/4, M \geq 2 \) and \( N \geq 20 \).

A logical question to ask is under what sampling conditions a signal \( x \) can be accurately reconstructed? The answer is to impose a condition on the measurement matrix \( A \) known as
Compressed Sensing.

the Restricted Isometry Property (RIP). Suppose a signal \( \mathbf{x} \) resides on a set of spatial indexes denoted by \( N \). A measurement matrix \( \mathbf{A}(M \times N) \) is said to obey the RIP of order \( k \) with constant \( \delta_k \) if it is the smallest positive constant such that

\[
(1 - \delta_k) \| \mathbf{x} \|_2^2 \leq \| \mathbf{A} \mathbf{x} \|_2^2 \leq (1 + \delta_k) \| \mathbf{x} \|_2^2,
\]

(3.10)
as proved on [32], where \( \delta_k \in [0,1] \) is the restricted isometry constant. Satisfaction of Equation 3.10 assures that no \( s \)-sparse signal is in the null space of \( \mathbf{A} \) and thus inversion of the measurement process is feasible. In other words, RIP ensures that all submatrices of \( \mathbf{A} \) of size \( M \times k \) are close to an isometry, and therefore distance and information preserving. Theoretically, the recovery of an \( s \)-sparse signal via \( l_0 \)-minimization is possible for a value of \( k= 2s \). Unlike the condition imposed by Shannon’s theorem, the signal recovery by \( l_p \)-minimization is possible independently of the dimensionality or bandwidth and instead is linearly proportional to the underlying complexity of the signal.

In most applications, signals are rarely noise-free, and it is desirable to investigate how measurement error affects the robustness of the recovery. The observed partial frequency \( \mathbf{\hat{F}} f |_\Omega \) is only an approximation to \( \mathbf{F} f \) by some error \( e \) according to the expression

\[
\| \mathbf{\hat{F}} f - \mathbf{\hat{F}} |_\Omega \| \leq \eta, \quad \| e \|_2 \leq \eta. \quad (3.11)
\]

Then the approximate recovered signal \( \mathbf{f}^\# \) is within the true signal \( f \) by

\[
\| \mathbf{f} - \mathbf{f}^\# \|_p \leq \frac{C}{s^{1-1/p}} \delta_s(x)_1 + D \eta \frac{1}{s^{1/p-1}}, \quad 1 \leq p \leq 2,
\]

(3.12)
as proved by [32], where \( C = \frac{(1+\rho)^2}{1-\rho} \) and \( D = \frac{(3+\rho)\eta}{(1-\rho)} \) and where \( \rho \in (0,1) \) is a constant added to strengthen the condition on the measurement matrix that insures stability. This is related to the fact that in most realistic scenarios the signal is only close to some \( s \)-sparsity. In fact, the first term of Equation 3.12 comes from stability and the second term is the one imposed by robustness.

### 3.4 Normed spaces

The theory of normed vector spaces was created between the 1920s and 1930s, around the same time as quantum mechanics. This was a relatively new concept that has many applications in Quantum physics and has been applied in other parts of physics, engineering and statistics. The distance expressed in the various \( l_p \) norms with \( p>0 \), is fundamental in the
3.4 Normed spaces

analysis of e.g. continuity, convergence and compactness. In the following we will try to give a more intuitive explanation for why it is preferable to use some norms over others in CS.

A simple technique for choosing a solution $g$ from the infinite set of solution of the underdetermined system, would be to select the one with the minimum energy $l_2$ by solving

$$
\text{(P2)} \quad \text{minimize} \quad \| g \|_2 \quad \text{subject to} \quad \hat{g}|_{\Omega} = \hat{f}|_{\Omega}.
$$

(3.13)

This method based on $l_2$-minimization fails in solving LIP correctly when the system is undetermined. To illustrate this, we will try to reconstruct a simple signal made of 3 sinusoids with three different frequencies and we will compare the number of measurements needed for recovery by the two norms $l_2$ and $l_1$.\(^2\) Let us consider the superposition of sine tones of 50 Hz, 120 Hz, 250 Hz and corresponding amplitudes of 0.7, 1 and 0.5:

$$
X = 0.7 \times \sin(2 \times \pi \times 50 \times t) + \sin(2 \times \pi \times 120 \times t) + 0.5 \times \sin(2 \times \pi \times 250 \times t).
$$

(3.14)

Fig. 3.3 Sinusoidal signal made of 3 sine tones with frequencies 50 Hz, 120 Hz, 250 Hz and corresponding amplitudes of 0.7, 1 and 0.5.

The signal in the time domain is not sparse, so we transform it to the Fourier domain $\hat{X}$. Then we multiply by a matrix with an adjustable number of rows and with a column length equal of that of the signal $\hat{X}$, formed from random scalars, drawn from the standard normal distribution. This process corresponds to what we did call projections in CT and

\(^2\)The code for $l_1$-minimization method is available at http://www.cs.ubc.ca/~mpf/spgl1/
Compressed Sensing.

Fig. 3.4 The spectral reconstruction of the signal in Equation 3.14, with the Basis Pursuit, which is a numerical implementation method for $l_1$-minimization used to solve the underdetermined system of size $30 \times 513$.

Fig. 3.5 The spectral reconstruction of the signal Equation 3.14 with the least squares method to solve as underdetermined system of size $30 \times 513$. 
3.4 Normed spaces

Fig. 3.6 The spectral reconstruction with the Basis Pursuit, which is a numerical implementation method for $l_1$-minimization used to solving the underdetermined system of size $23 \times 513$. We observe a approximate recovery with 75% of the measurements needed for perfect recovery.
Compressed Sensing.

Fig. 3.7 The spectral reconstruction of the signal (3.14), with the least squares method for solving underdetermined system of size $300 \times 513$. We note that the zero values are badly reconstructed and the 3 spikes just are recovered with 1/2 their amplitudes.

Fig. 3.8 The spectral reconstruction of the signal (3.14), with the least squares method for solving an underdetermined system of size $430 \times 513$. We note that the zero values are badly reconstructed, however, the 3 spikes are close to fully recovered.
Fig. 3.9 The spectral reconstruction of the signal (3.14), with the least squares method for solving an underdetermined system of size $490 \times 513$. We note that the zero values never get exact even at a very high number of measurements, however, the 3 spikes are almost fully recovered.
Reconstruction via Least Square Minimization

Fig. 3.10 The spectral reconstruction of the signal (3.14), with the least squares method for solving an underdetermined system of size $500 \times 513$. We note that the zero values never get exact even at a very high number of measurements, however, the 3 spikes are almost fully recovered.
3.4 Normed spaces

A correspond to the Radon Transpose operator. We would like to recover the signal $\hat{X}$ from fewest measurements possible. Each row of $A$ corresponds to the measurements. The recovery by the $l_1$-minimization is implemented with the method known as Basis Pursuit (BP) [32]. The recovery is exact by $l_1$-norm minimizing from only $(30 \times 513)$ measurement matrix $A$, Fig.3.4. While the least squares method, used for selecting a solution with least energy, fails with $A$ of size $(30 \times 513)$ as can be seen in Fig. 3.5, Basis Pursuit succeeds with a perfect reconstruction. Furthermore, we can see that the least squares method is still far from giving a perfect reconstruction even when augmenting the matrix $A$ to size $(300 \times 513)$ and $(490 \times 513)$, as shown in Fig 3.7 and Fig.3.9. To our surprise, we see that $l_1$-minimization method managed to come close to the solution with measurements that is 75% of what is needed for exact recovery (see Fig.3.6). The least squares minimization is not a good choice to recover the least complex of the available solutions even when the system is close to being exact determined. As an example see the result for a measurement matrix of size $(500 \times 513)$ (see Fig.3.10).

Additionally, we propose an illustrative explanation for why $(P2)$ fails to reconstruct the sparse signal and why we can substitute $l_0$ with the $l_1$ norm and still get good recovery. This time, we will think of the signal as a vector of two coordinates, also simply a point $x$. The infinite set of solutions obtained from the underdetermined system will be presented as a line ($l$). To compute the closest point in the line to $x$ using the a unit norm, we can imagine growing a sphere centered on $x$ until it intersects with ($l$). This will be the point $\hat{x}$ that is closest to $x$ corresponding to the used norm. The $l_1$-norm sphere is shown in Fig.3.11 (a) as a dashed rhombus, $l_2$-norm sphere is shown in Fig.3.11 (b) as a dashed circle. We observe from Fig. 3.11 that for $l_2$ the coordinates of $\hat{x}$ does spread out the error more evenly amongst the two coefficients, while $l_1$ norm leads to an error that is more unevenly. Furthermore, we observe that the distance from origo to the line is shorter when $l_2$ norm is used compared to the $l_1$ norm. The point obtained by the $l_2$ norm most often will have two coordinates while for the $l_1$ norm sphere it will have just one coordinate. Thus, the $l_1$-norm result in the least complex solution.

Now we will address the issue of substituting the $l_0$-norm with $l_1$. The $l_0$-norm sphere is a cross along the x and y-axes, so it cannot exactly be considered a sphere, but nerve the less we see easily that $l_1$-sphere has the same intersection point (in the case of unique intersection) as $l_0$ norm. Using the $l_0$ and $l_1$-norm should result in the same solution, but then one might ask why use $l_1$ at all? It is because the $l_1$-norm sphere is a Convex Polygon, giving us the possibility to use convex analysis to find the intersection thus, we know how to implement very efficient algorithms to solve the (P1) problem while the $l_0$-norm is not feasible. By way of illustration, let consider a signal of length $N=10000$, and let us say that its sparsity is
3.5 Approximation to l0 minimization

In our quest to minimize the number of efficiency measurements needed to form a spatial mapping of the 3D sensor, we have been searching for practical solutions that may go beyond the formal CS. We did find a variety of methods that include the software needed to test them and that are very exciting. Methods like recovery via sparsifying learned algorithm via l0-minimization [25], Hybrid between TV norm and l1-norm [35], and recovery via iterative Hard Thresholding [37], just to mention some articles that we find interesting and to gave a taste of the variety of approaches the developments in CS are taking. Those alternative methods are mostly grouped under the category known as F-minimization, a common name for the class of methods that is not based on convex optimization. The most widely used are
3.5 Approximation to $l_0$ minimization

methods [24], that approach $l_0$-norm by assuming general concave priors, resulting in exact reconstruction with fewer samplings than what is needed for (P1) but still a bit more than what is the best possible theoretical minimum.

The $l_p$-norm definition is given for $p \geq 0$ \( p \in \mathbb{Z} \)
\[
\| x \|_p := \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p}.
\] (3.15)

The $l_0$-norm is not a proper norm but it does fit in the general norm definition by defining $0^0 = 0$. In fact, the $l_0$-norm can be rewritten so it might be approximately represented in a numerical implementation, as
\[
\| v \|_0 = \lim_{p \to 0} \sum_{x \in \Omega} |v(x)|^p = \sum_{x \in \Omega} 1(|u(x)| > 0) = \lim_{\sigma \to 0} \sum_{x \in \Omega} \rho(|u(x)|, \sigma),
\] (3.16)

where $\rho$ is any function that satisfies the equation, $\Omega$ is the set that contain the observed Discrete Fourier Transform and $\sigma$ represents a sequence index or scale parameter. The use of the $l_0$ definitions expressed by the use of the limit sequence $\sigma \to 0$ yield the desired unique and exact solution. It exhibits "a sense of" stability for the entire solution path, not only at the sequence limit $\sigma \to 0$.

There exist many possible functionals $\rho$ satisfying Equation 3.16 such as the Laplace function Equation (3.17), the Geman–McClure function Equation 3.18 and the concave logarithmic penalty Equation 3.19; all of those yield exact signal reconstruction for any $\sigma \in [0, \infty)$,
\[
\rho(|u(x)|, \sigma) = 1 - e^{-\frac{|u(x)|}{\sigma}} ,
\] (3.17)
\[
\rho(|u(x)|, \sigma) = \frac{|u(x)|}{|u(x)| + \sigma} ,
\] (3.18)
\[
\rho(|u(x)|, \sigma) = \log \left( \frac{|u(x)|}{\sigma} + 1 \right),
\] (3.19)
as proved in [24].

In fact, it is proven [24] that recovery via any $\rho$ that is nondecreasing and concave over $\mathbb{R}^+$ is guaranteed if a signal can be exactly recovered by $l_1$-minimization. Given that $\rho$ is scaled by a constant factor such that $\rho(1, \sigma) = 1$ for $\sigma \to \infty$ then $\rho(|u|, \sigma) \to |u|$. In other words, $\rho$ interpolates the function space between $l_1$ and $l_0$ across $\sigma[0, \infty)$. The result is recovery from many fewer samples than via $l_1$ based approaches. To show this, we will use the Shepp-Logan Phantom; that is a standard benchmark for reconstruction algorithm performance. In [24], a comparison of $l_1$-minimization versus $l_0$-minimization approximation
based reconstructions of the canonical Shepp–Logan phantom, is made and the result are shown in Fig. 3.12. The first row shows the projections or what we call the observed Fourier coefficients, the ten radial Fourier projections corresponding to roughly 4% of the data while the 18 radial Fourier projections corresponding to roughly 7% of the data. The second row shows the reconstruction based on the \( l_2 \) minimization where the unobserved part of the spectrum is zero-padded. The third row shows the \( l_1 \)-minimization, where the result shows clearly that with 15 projections \( l_1 \) already fails to recover perfectly the phantom. The fourth row shows the reconstructions with the \( l_0 \)-approximation method based on the Laplace function Equation 3.17. The exact reconstruction is obtained with 12 projections with \( l_0 \)-approximation method and 18 for \( l_1 \)-minimization. This shows clearly that the \( l_0 \)-approximation is very close to the theoretical minimum, which should be between 2/3 to 2/5 of the number of projections needed for exact recovery with \( l_1 \) [5]. Also, the minimal number of projections that can still result in exact reconstruction is between 7 and 12. In Chapter 5 we will show a comparison between the result obtained here and the result obtained by, the Reconstruction via Spatial Adaptive Filtering [28].
Fig. 3.12 Sequence of images recovered under radial sampling using (a) 9, (b) 12, (c) 15, and (d) 18 equally spaced Fourier projections corresponding to approximate undersampling rates of 96%, 95%, 94%, and 93%, respectively. (e)–(h) are reconstructions with $l_2$-minimization (The minimum energy reconstruction) obtained by zero-padding. (i)–(l) are the reconstructions obtained via $l_1$-minimization. (m)–(p) are the reconstructions obtained via homotopic $l_0$ minimization using the Laplace error function. Figure from [24]
Chapter 4

Reconstruction via Recursive Spatially Adaptive Filtering

Image reconstruction in Compressed Sensing (conventional Compressed Sensing) is based on the variational principle approach, where the intensity of the reconstruction is minimized according to $l_p$-norm, $0 \leq p \leq 1$. This method selects from the possible reconstructions the sparsest one. In most cases this is subjectively highly similar to the true signal.

The Recursive Spatially Adaptive Filtering method [28] we propose to use here replace the implicit norm-minimizing by explicit filtering. This is an example of a non-parametric regression technique. The filter studied here is called Block Matching 3-D (BM3D). It was first published in 2007 and is currently considered as one of the best denoising filters [20, 27]. The material and notation used in the formulas in this chapter is based heavily on Refs [16, 27, 28].

“...this algorithm (BM3D) has been really hard to beat... it (BM3D) works better than anyone can explain and better than any theoretician would like to acknowledge. More work is needed to understand exactly why it does so well. “

—Prof. Fred Hamprecht, Lecture on Image Analysis, April 2013, available on youtube.

4.1 Block Matching 3 D filter

The Block Matching denoising algorithm [27] is based on locating matching square fragments (blocks) of the image and grouping them into 3-D data arrays, called groups. The BM3D filter makes use of spatial redundancy within each group to improve the image quality.
This type of filtering reveals the finest details shared by the members of the group while conserving the individual features of the fragments or what we call the blocks. In our case, the BM3D algorithm is used to obtain a highly sparse representation of the image intensity to be recovered.

The blocks\(^1\) are fixed sized squares, \(Z_x\), normally of size 16x16 pixels, that can be any part of the image, where \(x\) is the top-left corner coordinate of the block, used as a mean of distinguishing the blocks. The BM3D block size should remain under 25x25 pixels.\(^2\)

The general procedure of BM3D is composed of 2 consecutive stages, see Fig. 4.1. Each of them is divided into 3 steps:

1. **Forming groups:** common for both stages.
2. **Collaborative filtering:**
   - For stage1 this step is called: \(A.\) **Collaborative hard-thresholding.**
   - For stage2 this step is called: \(B.\) **Collaborative Wiener filtering**
3. **Aggregation:** common for both stages

\(^1\)Sometime called patches or macroblocks, the later is frequently used in video compressing.
\(^2\)The studies related to BM3D performance did not include blocks of higher sizes. The minimal size should not be smaller than 4x4 pixels.
4.1 Block Matching 3 D filter

4.1.1 BM3D procedure steps

1. **Forming groups:** A group contains one reference block and several blocks that are mutually similar to the reference block and can be located quite far from each other. The data in groups is processed jointly so we arrive to a non-local estimator with varying adaptive support. This estimation can be seen as a sophisticated high-order generalization of non-local means [8]. The blocks in a group are indexed, where the reference block get the first index and the rest of the blocks are subsequently placed in order of similarity. The reference block is denoted by $Z_{x_R}$, where $Z$ refer to the set of pixels making the reference block and $x_R$ is the coordinate of the top-left corner of the block. In this case $Z_{x_R}$ is the block reference noted by adding R while the other blocks in a group will be denoted as $Z_{x_m}$, where $m$ is the index of the block according to the group it’s in.

![Fig. 4.2 Illustrating selection of blocks to make up a group.](image)

$S_{x_R}$ is a set that contain the coordinates of the blocks similar to $Z_{x_R}$, from all coordinates in our domain $X$ such that the dissimilarity between the reference block and the other blocks making up the image is smaller than a threshold $\tau$, 

$$S_{x_R} = \{ x \in X : d(Z_{x_R}, Z_x) \leq \tau \}.$$  

(4.1)

formula from [27].
The dissimilarity is called d-distance as well, because it is measured by the $l_p$-norm on the difference between two image fragments. In our case we use the $l_2$-norm, which is just the familiar Euclidean distance.

The dissimilarity $d(Z_{x_R}, Z_x)$, (i.e. distance from the reference block) is defined as the squared Euclidean distance between blocks making up the group and some reference where both are denoised beforehand. It is found by using hard-thresholding (Threshold filtering), represented by the operator $\Gamma$, on the frequency domain of the blocks,

$$d(Z_{x_R}, Z_x) = \| T_{2D}^{-1} \Gamma T_{2D} Z_{x_R} - T_{2D}^{-1} \Gamma T_{2D} Z_x \|_2^2,$$

formula from [27].

Hard-thresholding preserves the high-magnitude transform coefficients that convey mostly the true signal and discards the low-magnitude coefficients. Low-magnitudes coefficients originate mostly from noise so excluding them is more convenient in the search for similar blocks.

The lower subscript 2 indicates that this is an Euclidean norm while the upper 2 is an exponent. The wavelet transform, $T$, from 2D to 2D. Wavelet transforms leads to highly sparse representation in the frequency domain and it has high edge detection and preserving property. In case of coding the equation (4.2), the two inverse transforms can be skipped, according to Parseval’s theorem. The blocks that yield $d(Z_{x_R}, Z_x)$ lower than some value $\tau$ form a group.

2. Collaborative filtering: All members of a group undergo a collective improvement. Each member of the group collaborates in the improvement of the other blocks. This is called 3-D denoising or simply group denoising. When denoising a 2-D intensity image the group is of (2+1) dimensions. The third dimension normally is a little different in length for the various groups, meaning in practice that the number of blocks comprising a group is arbitrary.

The set of pixels constituting a block referred to by the pixel coordinate of the top-left corner of the block is $Z_{x_R}$. In a similar way the group $Z_{S_{x_R}}$ is the set of blocks $Z_x$ contained in a specific group (that is often visualized as a box), where $x \in S \subseteq X$ and $x_R$ is the coordinate of the top-left corner of the group’s reference block.

---

3 In fact it’s even normal to just use the word distance when the $l_2$-norm is used as a measure for comparing "the similarity of the blocks".

4 To represent a signal with its major coefficients in the frequency domain is a basic concept in Compressed Sensing, known as sparsity. The sparse representation in the transform domain is used by BM3D in several steps.

5 See this page for a dynamic illustration of block selections: http://www.cs.tut.fi/ foi/GCF-BM3D/
4.1 Block Matching 3 D filter

The denoised group \( Y_{SR} \equiv \{ \hat{Y}_{x_m} \mid x_m \in S_{xR} \} \) is a collection of all the denoised blocks \( \hat{Y}_{x_m} \) where \( x_m \) must be a member of the set \( S_{xR} \). The reason for indexing a denoised block by both its top-left corner coordinate \( x_m \) and by the top-left corner coordinate of the reference block for the set \( S_{xR} \), is because the same block can be a member of several groups. All of the calculations for a given block will taken into account for a final result before patching the blocks back to form the filtered image.

A. Collaborative filtering by hard-thresholding. In Stage1 the 3D filtering of \( Z_{SR} \) is realized by applying first a wavelet transform \( T_{3D} \) from 3-D to 3-D that can represent the blocks sparsely. The shrinkage in 3-D is more profitable than in the 2-D transform domain due to its capability to affect two types of correlations; the intra-fragment correlation which appears between the pixels of each grouped fragment and secondly the inter-fragment correlation which appears between the corresponding pixels of several fragments.

We transform \( Z_{SR} \) to the frequency domain and apply the hard-thresholding operator \( \Gamma \) to attenuate it, producing a sparse representation or what we could call shrinkage of the true fragments.

\[
\hat{Y}_{SR} = T_{3D}^{-1} \Gamma T_{3D} Z_{SR},
\]

as proved in [27].

B. Collaborative Wiener filtering. Stage 1 results in a sparse image representation that is an estimate of the true image. The Wiener filter is then performed within the estimate from stage1. Stage2 follows the same approach as the previous step, with only a few fundamental modifications. The thresholding-based d-distance measure from Stage 1 is replaced with the normalized \( l_2 \)-norm of the difference of two blocks with subtracted means. Hence, the definition of \( S_{xR} \) becomes

\[
S_{xR} = x \in X \mid N_1^{-1} \| (E_{xR} - E_{xR}^c) - (E_{x_m} - E_{x_m}^c) \|_2 < \tau_{match},
\]

as provided on [27], where \( \tau_{match} \) is the maximum distance for which two blocks are considered similar, and \( E_{x_m} \) are the mean values of the blocks \( E_{x_m} \) respectively. The mean subtraction allows for improved matching of blocks with similar structures but different mean values. Then an attenuating coefficient \( W_{SR} \) for the Wiener filter is computed. Finally we filter the 3-D array of noisy observations \( Z_{SR} \) in the \( T_{3D} \)-transform domain by an elementwise multiplication with \( W_{SR} \). The collaborative Wiener filtering is realized according to

\[
\hat{Y}_{SR} = T_{3D}^{-1} (W (T_{3D} Z_{SR})),
\]
formula from [27], where $\hat{Y}_{S_x}$ is comprised of stacked local block estimates of the true image blocks located at the matched locations $x \in S_x$.

3. Aggregation: Since the blocks can overlap, the pixels in the image have most probably been a member of several groups, thus for each pixel we obtain many different estimates. Those estimates, for the one specific pixel, have been made independently from each other. As a consequence, there is a necessity for reconciling all those changes before returning the blocks to their original positions. The final intensity value for a given pixel is formed from all those 3-D filterings it participated in. This is obtained by normalizing the weighted sum of each estimation. The weighting factor is the inverse amount of the noise\(^6\) in a group.

$$\hat{y} = \frac{\sum_{x \in X} \sum_{x_m \in S_x} \hat{Y}_{x_m} w_{xR}}{\sum_{x \in X} \sum_{x_m \in S_x} I(x) w_{xR}}$$

The first sum is over all possible reference locations in the domain $X$. The second is over all the pixels in a block that are around the reference location. Then we divide by the normalization term, where $I(x)$ is the indicator function that checks if $x$ is in the blocks around $x_m$.

4.1.2 Measures of reconstruction quality

Comparing reconstructed results requires a measure of image quality [16]. Some of the most widely used objective measures of quality are Mean Squared Error (MSE), Mean Absolute Deviation MAD, and Peak Signal-to-Noise Ratio (PSNR). MSE is computed by subtracting the reference image from the reconstructed or denoised result and then computing the average energy of the difference,

$$MSE = \frac{1}{N^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (C_{ij} - R_{ij})^2,$$

where $N$ is the size of the image, and $C_{ij}$ and $R_{ij}$ are the pixels in the reconstructed/denoised image and the reference image, respectively.

An important term that is closely related to MSE and that we will be referring to later on is MMSE, Bayesian Minimum Mean Squared Error. Simply explained, MMSE is the Bayesian

\(^6\)More precisely the weighting factor is the inverse to the total sample variance of the corresponding block-wise estimates.
approach for estimating the Mean Square Error based on having some prior information about some parameters of the image to be estimated. The MMSE estimator is then defined as the estimator achieving minimal MSE. It is possible to calculate an Upper and a Lower Bound for MMSE.\(^7\) Given a reconstruction or denoising algorithm we can know how good it performs by calculating the PSNR of its result and comparing it to the best possible result under some prior conditions: This is called the upper limit of (MMSE), denoted (MMSE\(_U\)). \(^8\)

The Mean Absolute Difference MAD is a measure of dispersion. A related measure is the Relative Mean Absolute Difference (RMAD), which is the mean absolute difference divided by the arithmetic mean,

\[
MAD = \frac{1}{N^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} |C_{ij} - R_{ij}|
\]

while RMAD is simply

\[
RMAD = \frac{MAD}{\text{arithmetic mean}}.
\]

PSNR is most commonly used to measure the quality of reconstruction,

\[
PSNR = 10 \cdot \log_{10} \left( \frac{\text{MAX}_I^2}{\text{MSE}} \right) = 20 \cdot \log_{10} \left( \frac{\text{MAX}_I}{\sqrt{\text{MSE}}} \right) = 20 \cdot \log_{10} (\text{MAX}_I) - 10 \cdot \log_{10} (\text{MSE})
\]

The MAX\(_I\) is the maximum possible pixel value of the image. PSNR in a way is a correctness measure evaluated in decibels and is inversely proportional to the Mean Squared Error. High PSNR values are produced by best reconstruction/denoising. Values of the PSNR between 40-50 mean that the image is reconstructed correctly to a large extent. Values of the PSNR between 50-60 mean that the image is ”essentially exactly” reconstructed, and the difference to the original lies in minor details that do not harm the ”diagnostic value”. For detector images, we have been using; we could not see the difference between original and reconstruction at PNSR values above 60 or we had to zoom in greatly to see some minor differences.\(^9\)

\(^7\)MMSE\(_U\) is derived using explicit knowledge of the original noise-free image, while MMSE\(_L\) does not involve it.

\(^8\)For more details see Article: Natural Image Denoising: Optimality and Inherent Bounds

\(^9\)This description is just meant as a guide on what to expect. What we did describe as ’a largely correct reconstruction’, is a subjective matter and differs depending on the type of image being studied.
4.1.3 BM3D performance

The most important part of the Recursive Spatially Adaptive filtering is the Filter. That is why I will be stating results from studies that confirm that the BM3D is very hard to beat as a filter.

![Behavior at different noise levels](image)

Fig. 4.3 Comparison on images with various noise levels: the MLP trained for $\sigma = 25$ is competitive for $\sigma = 25$. The results are shown for two types of MLP. MLP (referred to in the study by ‘us’) trained on several noise levels, and MLP trained for $\sigma = 25$. The BM3D is clearly better for general noise level, while still highly competitive with MLP trained on $\sigma = 25$. The two other filter algorithms shown are GSM (Bayes Least Squares with a Gaussian Scale-Mixture) and KSVD an algorithm based on sparse and redundant representations over trained dictionaries. Figure from [20].

Multi-Layer Perceptron (MLP) is a neural network denoising algorithm that maps sets of input data onto a set of appropriate outputs. In the study [20] the BM3D filter was compared to the MLP network that was trained on a large image database. The test images were corrupted by additive white Gaussian noise (AWG). The denoised result was compared with multiple filter types, as a function of the variance of AWG.
The neural network was trained on a total of 362 million samples, retained from 150,000 images with 105 different noise instances, requiring approximately one month of computation time on a GPU (Graphics Processing Units, nVidia’s C2050).

The result as presented in Fig 4.3 shows a comparison against results achieved by GSM, KSVD, and MLP. We see that for $\sigma = 25$ BM3D is very competitive, but when the noise was pre-assumed to be $\sigma = 25$ the performance deteriorates for other noise levels. BM3D was overall clearly better than the MLP. The only time that MLP was better than BM3D, by less than 1dB, was for AWG $\sigma = 25$. It is to be expected for MLP to be strongest at AWG $\sigma = 25$, when it was trained for exactly that variance. The trained filter algorithm at AWG $\sigma = 25$ was able to perform better than BM3D 300 out of 500 times by a PSNR difference less than 1 dB. For the BM3D to be on equal footing with it at $\sigma = 25$ is just amazing.

In Ref.[10], one of the conclusions state: "...for most cases the PSNR values of BM3D are within 0.1dB of the optimal ones, suggesting that the BM3D results are quite close to optimality, for small patch sizes. Thus, future denoising algorithms that use small patches have very little room for improvement over BM3D".

We can conclude that the algorithm of choice in this thesis is a very powerful one. Images with many repeating structures are ideal for BM3D as mentioned in the study comparing MLP network with BM3D: "Images with much repeating structure are ideal for both BM3D and KSVD.". BM3D is then a natural choice for the efficiency pattern of the sensor we want to reconstruct is expected to be formed of repeated structures that are very similar in most cases.

4.2 Recursive Spatially Adaptive Filtering

Reconstruction via Recursive Spatially Adaptive Filtering (RSAF) [9, 28] is used both in noise filtering and in a variety of reconstruction problems. In this thesis we are interested in the use of RSAF in signal reconstruction when the available measurement consists of a small portion of the 2-D Fourier transform spectrum of the intensity that is to be recovered (Radon inversion from sparse projections). The conventional Compressed Sensing can be viewed as a nonlinear filtering method, using parametric optimization to regularize an image and to obtain its corresponding discontinuities. The regularization with a global sparsity penalty often results in inefficient filtering. While this work makes use of spatially adaptive filters sensitive to image features and details, it has been shown [34] that a higher quality can be achieved when the restoration criteria are local and adaptive. When the penalty is constant over the image, details may easily get lost. If on the other hand, the Compressed

10Patches are what we called blocks and the small size mentioned is in the range 4-15 pixels.
Reconstruction via Recursive Spatially Adaptive Filtering

Sensing reconstruction is made adaptive to the local scale and contrast of features in the image, then edges can be easily detected, and boundaries get well localized and preserved. This makes Reconstruction via Spatial Adaptive Filtering an optimal method for our task of reconstructing the efficiency for a highly undersampled 3-D sensor. With this reconstruction method, we can expect a method that is easier to understand, easier to implement and nevertheless more effective than conventional Compressed Sensing.

In mathematical terms, the recovery problem can be expressed in its simplest form as

$$y = Ax,$$  \hspace{1cm} (4.7)

where $$y \in \mathbb{R}^m$$ is the total observed data and $$m$$ is the number of measurements. The signal that we aim to recover is $$x \in \mathbb{R}^N$$. This signal is, in our case, a 2D image that represents the efficiency of a slice of the 3D pixel sensor. The number of image pixels (picture elements) is $$N$$. The matrix that models the linear measurement process (linear transformation) is $$A \in \mathbb{R}^{m \times N}$$.

In linear algebra $$y$$ and $$x$$ must be of equal size if we are to solve Equation 4.7 and thus recover $$x$$ from the available measurements $$y$$ when we know the measurement matrix $$A$$.

When $$m$$ is less than $$N$$, Complex Sensing offers ways of recovering $$x$$ or a sparse representation of it, that we will be calling $$z$$. This can be done if a-priori $$x$$ can be approximated by a sparse image representation $$z$$ and if $$A$$ fulfills the null space property. Then the recovery problem in its simple mathematical form becomes

$$\minimize \|z\|_0 \text{ subject to } y = Az.$$  \hspace{1cm} (4.8)

In other words, this means that we seek to find the vector $$z$$ with the minimum number of entries, with minimal non-zero elements represented by a number $$s$$. This problem can be solved via the basis pursuit method by introducing a parameter $$\lambda \geq 0$$ giving us the equation

$$\|Az - y\|_2^2 + \minimize \|z\|_1.$$  \hspace{1cm} (4.9)

This formulation of the problem is known as parametric modeling. In contrast, reconstruction via SAF is considered a non-parametric regression technique. The reconstruction via SAF is realized by a recursive algorithm. At each iteration, the algorithm is excited by random noise on the unobserved part of the spectrum, while the filter is applied to the whole image to attenuate the noise and reconstruct the features from the incomplete observation.
4.2 Recursive Spatially Adaptive Filtering

4.2.1 Iterative algorithm with stochastic approximation

We start by presenting the notations used to describe the algorithm of the reconstruction via Recursive Spatially Adaptive Filtering and the important formulas given on [28]. The image intensity that we seek to recover will be denoted by $\theta$. The available observation data is transformed to the 2-D Fourier slices and the ensemble of the frequency coefficients are denoted $y_1$. $S$ are the the Euclidean coordinates of $y_1$. $1-S$ is then the position of the frequency coefficients to be recovered that are denoted by $\hat{y}_2$. The hat indicates that this is an estimate. $\hat{y}$ is the full Fourier space so $\hat{y}^{(k)} = y_1 + \hat{y}_2^{(k)}$. The iteration $k \geq 1$ is given in the upper right of $\hat{y}_2$ and $\hat{y}$. The starting value is $\hat{y}_2^{(0)} = 0$. The corresponding $\hat{y}^{(0)}$ is called the zero padded $y_1$. The algorithm applies the 2-D inverse Fourier transform $\mathcal{T}^{-1}$ on $\hat{y}$ to recover the estimated image intensity $\hat{\theta}$ and 2-D Fourier transform $\mathcal{T}$ to get back to the frequency domain. The BM3D filter will be denoted by the operator $\Phi$. We make note of the fact that only the first stage of BM3D filter is used in the RSAF program. A pseudo-random noise $\eta_k$ with zero mean and a constant, finite variance $\sigma^2$ and a sequence of positive step sizes $\gamma_k$ are applied to the unobserved portion of the spectrum $\hat{y}_2$. The later is to insure the convergence of the algorithm according to the stochastic approximation of the Robbins-Monro procedure [21].

The recursive algorithm we are using (4.10) can be treated as a Robbins-Monro procedure. The following is how the developers of RSAF algorithm [28] present it:

\[
\begin{align*}
\hat{y}_2^{(0)} &= 0 \\
\hat{y}_2^{(k)} &= \hat{y}_2^{(k-1)} - \gamma_k (\hat{y}_2^{(k-1)} - (1-S) * \mathcal{T} (\Phi \mathcal{T}^{-1}(\hat{y}_2^{(k-1)})) - (1-S) * \theta_k) \quad k \geq 1
\end{align*}
\] (4.10)

To be more precise, we would say that this method is a filtering approach to a stochastic approximation based on Robbins-Monro procedure. For a better understanding of the logic behind the algorithm, one should study what is known as dynamic programming algorithms. we highly recommend [30] as a step from our RvSAF into the field of dynamic programming algorithms, for those that are interested in gaining more understanding of the underlying principles. However, there is one main difference related to the use of the filter. In Ref.[30] a Kalman filter is used while Robbins-Monro can be considered as a special case where the filter is fixed. In our case the variance of the introduced noise controls the strength of BM3D, affecting the level of smoothing.

We can divided the iteration system, as shown by the flowchart in Fig. 4.4, into 3 main parts; Image-domain estimate filtering; excitation; and the updated estimate.

\footnote{All the formals given in this subsections is from [28] unless else is specified.}
Fig. 4.4 Flowchart of the system based on the reconstruction via Recursive Spatially Adaptive Filtering algorithm. $y_1$ is the measured part of the spectrum of the image intensity, $\mathcal{F}$ is the 2D Fourier transform, $S$ is a mask that covers the available Fourier coefficients from observation data, $\gamma$ is a parameter that ensures convergence to the true image, $\eta$ is the pseudo-random noise (excitation) that speeds up the recovery, $\Phi$ is the filter applied to the image for attenuating the noise, $y_2$ is the part of the spectrum that we are trying to estimate, while $k$ is the count of iterations. Figure from [28].

- **Image-domain estimate:** Here 2 operators are applied on $(y_1 + \hat{y}_2^{(k)})$. First $\mathcal{F}$ to get to the image intensity domain, then the filter $\Phi$ is applied to filter the image after the noise has been introduced in the unobserved part of the spectrum $\hat{y}_2$, in such a way that the new elements in the image domain that coincide with the true image intensity are conserved. Roughly speaking, the recovery is based on that some true portions of the image intensity act as "seeds" from where the recovery spreads while the search direction is driven by the denoising filter.

- **Excitation:** the Robbins-Monro algorithm is based on the assumption that while we cannot directly obtain a function, such in our case $\hat{y}_2$, we can instead use measurements of some random function having the expectation value equal to $\hat{y}_2$ [21]. Adding the random noise $\eta_k$ to $\hat{y}_2$ makes it a random function, thus we are able to use the Robbins-Monro algorithm. The added noise can in a more intuitive basis be seen as the generator of the missing spectral components, while the subsequent iterations attenuate or enhance these components by the action of the filter.

- **The updated estimate,** $y_2^{(k)}$ is obtained as the sum of the estimate we had from the $k - 1$ iterations, and the $\hat{y}_2$ we have obtained from $\mathcal{F}[\Phi \mathcal{F}^{-1}(y_1 + \hat{y}_2^{(k-1)})] + \eta_k$ by a rate of $\gamma_k$ which is referred to as a step size. The parameters $\gamma_k$ governs the rate at which new information is combined with the existing knowledge and it has to fulfill some requirements presented in the Robbins-Monro stochastic approximation. The

---

12 In fact we are using a more advanced version of it, as it was specified earlier.

13 In other communities, it is known by different names, such as learning rate (machine learning), smoothing constant (forecasting) or gain (signal processing).
parameters $\gamma_k$ has to fulfill the conditions:

$$\sum_{k=0}^{\infty} \gamma_k = \infty, \quad \sum_{k=0}^{\infty} \gamma_k^2 < \infty,$$

(4.11)

where the first condition ensures that the process does not stall at an incorrect value and the second condition guarantees the convergence of the sequence and helps with damping the effect of experimental errors and thereby controls the variance of the estimate.\textsuperscript{14}

The values of $\gamma_k$ are found by solving the optimization of the expected value $E$ between the estimate and the true value of $y_2$

$$\min_{\gamma \leq 1} E\{ (\hat{y}_2^{(k)} - y_2)^2 \}.$$

As $k \to \infty$ the estimates $\hat{y}_2^{(k)}$ from the recursive system (4.10) converge in the mean squared sense to a solution $\hat{y}_2$ that represents a very good approximation of the true unknown value $y_2$. Thus the algorithm will cease to work with a weak filter as the filter will be no longer able to affect the $(y_1 + \hat{y}_2)$ and then the second term of our algorithm that introduces the change to the estimate at each iteration will be 0. This is shown mathematically by the following

$$\hat{y}_2 = (1 - S) \ast \Xi(\Phi(\Xi^{-1}(y_1 + \hat{y}_2))) = (1 - S) \ast (y_1 + \hat{y}_2) = \hat{y}_2.$$

The strength of the filter $\Phi$ is evaluated at each iteration and is controlled by evaluating the smoothing effects translated in $y_1$. This is done by comparing $y_1$, the measured part of the spectrum, to $S \ast \Xi(\Phi(\Xi^{-1}(y_1 + \hat{y}_2))) = \hat{y}_1$.

On the other hand, if the filter is too strong, details of the image will be lost. We re-optimized the program by changing the filter strength parameter, the amplitude of the noise and its duration from what was proposed for better reconstruction from a very low number of projections. A comparison between the original program and our optimized version will be shown in the next chapter.

\textsuperscript{14}There are 3 more conditions added to the conditions of $\gamma_k$ in Robbins-Monro algorithm to ensure the convergence. Those are not motioned in the main article of RvSAF, but they are to be found in reference [21].
Chapter 5

Results

In This Chapter, we present the reconstruction results obtained with RSAF and results from our optimization of the RSAF program parameters. Furthermore, we will present our adaptation of the CT for the reconstruction of the 3-DCCE. Since the Charge Collection Efficiency is constrained by the design of the sensor and a simple pattern, compared with a natural image, we managed to adjust the CT to the specific case at hand giving us an amazing outcome.

5.1 RSAF vs reconstitution via $l_0$ and $l_1$-minimization

To start with, we want to know how good the RSAF method is in comparison to the formal Compressed Sensing based on $l_0$ and $l_1$-minimization [24]. This is done by comparing the reconstruction results of the Shepp–Logan phantom (Fig. 5.1). The Shepp–Logan phantom is a standard test image that serves as the model of a human head and is widely used for testing of image reconstruction algorithms. We observe from Fig. 5.2 and Fig. 5.3 that RSAF performs better than $l_1$-minimization method and even better than the approximation to $l_0$-minimization as it manages to reconstruct all the features of the Shepp-Logan phantom from 9 projections when both conventional compressed sensing methods failed. This means that the performance of RSAF is even closer to the theoretical optimal limit of CS than the $l_0$-norm approximation method.
Fig. 5.1 The Shepp–Logan phantom of size \((256 \times 256)\). This phantom is a standard test image that serves as the model of a human head and is widely used for testing of image reconstruction algorithms. Figure frequently used in the medical image-processing community, generated with Matlab using code: phantom(256).
5.1 RSAF vs reconstitution via $l_0$ and $l_1$-minimization

Fig. 5.2 (a) depict minimum energy reconstruction. (b) reconstruction with Compressed Sensing based on $l_1$-norm minimization from 12 projections. (c) reconstruction with Compressed Sensing based on approximate $l_0$-norm minimization from 12 projections. (d) reconstruction via Spatial Adaptive Filtering from 12 projections. Figures (a)-(c) from [24]
Fig. 5.3 Reconstruction from 9 projections. (a) depict minimum energy reconstruction. (b) Reconstruction with Compressed Sensing based on $l_1$-norm minimization from 9 projections. (c) Reconstruction with Compressed Sensing based on approximate $l_0$-norm minimization from 9 projections. (d) Reconstruction via Spatial Adaptive Filtering from 9 projections. Figures (a)-(c) from [24].
5.2 Our optimization of RSAF

In our attempt to reduce the number of projections needed for the 3DCCE, we managed to improve the RSAF program [28]. This was obtained by reducing the injected random noise in the Fourier space to 10% and the filter to 75% of the original value. In addition, the duration of noise was extended.

Fig. 5.4 Reconstruction from 8 projections. (a) depicts minimum energy reconstruction. (b) Reconstruction with the original RSAF. (c) Reconstruction with our optimized RSA. (d) Shepp-Logan phantom we seek to reconstruct.
The reconstruction from 8 projections Fig. 5.4, shows a clear improvement in the reconstruction with our optimized RSAF. We see that the reconstruction with the original program for RSAF has a very low diagnostic value Fig. 5.4 (b), while our optimized version of RSAF (c), reconstruct all the features of the phantom. Our reconstruction result in a blurred image but all ellipses are distinguishable and they do correspond to the original size and form.

Fig. 5.5 Reconstruction from 7 projections. (a) depict minimum energy reconstruction. (b) Reconstruction with the original RSAF. (c) Reconstruction with our optimized RSAF. (d) Shepp-Logan phantom we seek to reconstruct.
The reconstruction from 7 projections Fig. 5.5 (c), shows a reconstruction with optimized RSAF, that is of the same nature as the one obtained from 8 projections, while the original RSAF result in even more degraded reconstruction, Fig. 5.5 (b). Our optimized RSAF fail first at reconstruction with 6 projections as it can be seen from Fig. 5.6. This is shows that the optimized RSAF managed to reconstruct the Shepp-Logan phantom with approximately 25% less projections compared to the original RSAF. In fact, the best exact reconstruction obtained of a Shepp-Logan phantom that we know of, is from 8 projection applying a new CS method, from 2015, based on minimizing the mixed $l_1$ and $l_0$ norm [38].

Fig. 5.6 Reconstruction from 6 projections.(a) depict minimum energy reconstruction. (b) Reconstruction with the original RSAF from 6 projections. (c) Reconstruction with our optimized RSAF from 6 projections. (d) Shepp-Logan phantom we are seeking to reconstruct.
5.3 3-DCCE Phantoms

The Phantoms used to represent the 3-DCCE are mainly the ones used by the 3DCCE CERN group\(^1\). The phantom shows columns penetrating the substrate from both sides, surrounded by layers of augmenting efficiency. The layers surrounding the cathodes are different for each cathode, each phantoms can show a 5 varieties of efficiency pattern, giving us a better idea of the performance of the reconstruction methods.

The phantoms with 3-D columns suggested by 3DCCE CERN group (Fig. 5.7), we will call Dobos phantoms. They have two contrast ranges one with CCE between 60% and 100% and the second with CCE between 90% and 100%. The phantoms size is \(284 \times 300\), which equals a 3-D pixel sensor area of \(284 \times 300 \, \mu m\). The phantoms are rasterized with different telescope resolutions to estimate resolution of the reconstruction.

In order to study the reconstruction in case of efficiency pattern that is more sophisticated than what is presented in the Dobos phantoms and that has larger coefficients spread in the Fourier domain, we made what we call the Spherical phantom with spherical shaped cathodes, shown on Fig. 5.8. One of the cathodes has an extra challenging efficiency layout. The challenging efficiency consists first of the outer layer that has a very low contrast to the bulk, which makes it very hard to reconstruct. In addition, the shape of this layer is random and do not fit any traditional geometrical form. The other part of the challenging cathode efficiency, contain circular shaped efficiency patterns. This is done to test the ability of the methods to handle the circular charge efficiency patterns.

As an ultimate test, we modified one of the most difficult Dobos phantoms Fig. 5.7 (f), and added patterns that should represent highly irregular charge collection conduct and made the Highly-Irregular phantom, Fig. 5.9. The purpose of this phantom is to test the limit of the reconstruction method according to the phantom complexity. We wanted to see if our method will manage to make a trustworthy reconstruction for far complex charge efficiency pattern than we could expect of a 3-D sensor, or will it totally lose control and give us some reconstruction that we can’t relate to the original, also a reconstruction with very low diagnostic value.

5.4 3-DCCE reconstruction with classical methods

The 3DCCE group developed a simple and basic custom simulation software, based on C++ and Root, in order to understand the analysis chain and the requirements needed for the task. It was found that the simulation of measurements (simulating 2-DCCE taken at
\(^1\)The twiki website of 3DCCE group: //twiki.cern.ch/twiki/bin/view/Main/3DCCE
Fig. 5.7 Implementation of the two types of phantoms defined by 3DCCE CERN group, we call Dobos phantoms, with their respective efficiency contrast ranges shown above each group. The phantoms are rasterized with different telescope resolutions to estimate resolution of the reconstruction. The original phantoms used by 3DCCE were downloaded as images and read in by matlab, resulting in the phantoms you see in this figure. The original phantoms are to be found at [17].
Fig. 5.8 3-DCCE phantom with spherical shaped CCE.
Fig. 5.9 3-DCCE phantom with highly irregular CCE pattern.
different angles), are computationally intensive and the reconstruction based on filtered back-projection (which they had chosen as the reconstruction algorithm) is very inefficient. A concluding sentence in [4] reads as: "...especially the reconstruction part demands a much deeper examination than was achievable in the scope of this thesis. Thus, the results presented here serve as premature studies that give hints on the required statistics.". The 3DCCE group seemly did not try CS nor did they consider it. Their goal was to study the different filters used in the filtered back-projections method. In this thesis, we concentrated our effort solely in the reconstruction part. We started by writing the reconstruction program based on the Fourier Slice method and used the extensive library of Matlab that enables the reconstruction with most known filters for the filtered back-projections and we couldn’t notice any significant difference. Some filters can help reduce the needed measurements with few projections but still we couldn’t get any exact recovery with 360 projections - there were always many artefacts reducing the diagnostic usefulness of the result.

From various 3DCCE simulation studies available at 3DCCE twiki page, we know that the group has been using the sine filter. This is confirmed from the simulation code of various members of the group we had access to. We show in Fig. 5.10 and Fig. 5.12 the reconstruction of Dobos phantoms using the sine filtered back-projections method available on Matlab from 360 and 180 projections respectively. We can clearly see that even 360 projections are not sufficient for exact reconstruction. The reconstruction from simulation confirms that, even in the theoretical case study, 360 projections are not enough to fully reconstruct the spatial CCE [17].

5.5 3-DCCE reconstruction with Compressed Sensing

It’s not difficult to show that reconstruction with Compressed Sensing is superior to classical CT methods in case we seek to minimize the measurement time. Therefore, we will dedicate the rest of this chapter to clarify the improvements we did to the formidable CS method, reconstruction via Recursive Spatial Adaptive Filtering.

The software used in [9], we will call original RSAF. The version with the optimized parameters as explained in Chapter 4, we call optimized RSAF. In addition, we did change the way projections are taken. We have chosen to have more concentrated measurements along the 0 and 90 degrees. This choice of projections was motivated by the distinctive 2-D Fourier Transform of the phantoms representing Double Sided sensor Charge Collection Efficiency, Fig. 5.13. The 2-DFT have the most significant coefficients concentrated along the y and x-axes and around the origin. By taking more measurements closer to the two axes we hoped to ease the task of RSAF as it will have fewer of the most significant coefficients to
Fig. 5.10 Dobus phantom of 2 and 3 μm resolution reconstructed with filtered Back-projection using 360 projections. The two types of Dobos phantom (a) is the one with efficiency 60-100% and 2μm resolution (b) is the one with efficiency 60-100% and 3μm resolution. (c) is the one with efficiency 90-100% and 2μm resolution (d) is the one with efficiency 90-100% and 3μm resolution.
Fig. 5.11 The two types of Dobos phantom of 2 and 3 um resolution reconstructed with filtered Back-projection using 180 projections. (a) and (d) reconstruction from 360. (b) and (e) reconstruction from 180. (c) and (f) reconstruction from 36.
Fig. 5.12 Challenging phantoms reconstructed with filtered Back-projection using 360, 180 and 36 projections. (a) is the one with efficiency 60-100% and 2μm resolution (b) is the one with efficiency 60-100% and 3μm resolution. (c) is the one with efficiency 90-100% and 2μm resolution (d) is the one with efficiency 90-100% and 2μm resolution.
Results reveal. The 2-D plot of the radon transform for this case does not show sine curve, we think the overall structure looks similar to a diamond cut. For distinction, we exchange sinogram with diamogram and we call the reconstruction based on this method Symmetric Diamogram RSAF.

![2-D Fourier Transform of Dobos phantom 3 um rasterized with contrast range between 90% and 100%](image)

Fig. 5.13 2-D Fourier Transform of Dobos phantom 3 um rasterized with contrast range between 90% and 100%.

In Compressed Sensing it's advantageous to have random measurements, this is why we have been tempted to break the projection symmetry rendering the projections what we call quasi-random.

The reconstructions of Dobos phantom rasterized 3um 90 to 100% CCE (3um90), from Fig. 5.14 shows that optimized is a little bit better than original CS but they are still not able to make an exact recovery while the diamogram-based reconstruction has no problems reconstructing exactly the original phantom down to the most minor details.

The Diamo RSAF reconstruction manages the exact recovery even from 18 projections, as shown in Fig. 5.15. First at reconstruction from 12 projections Fig. 5.16, we can start to see some deviation from the original 3um90 phantom. So far we can’t confirm a distinct reconstruction capability between the two Diamogram based methods. In the case of reconstruction from 12 projection the Quasi-random Diamogram RSAF had it "misconstruction"
Fig. 5.14 Dobos phantom 3um rasterized reconstruction from 38 projections with optimized RSAF (a), original RSAF(b), and quasi-random diamo RSAF(c), (d) shows the projections used for plot (c).
more visible than Symmetric Diamogram but a careful investigations shows that the later have more overall deviations. The Symmetric and the Quasi-random methods shows a very satisfactory reconstruction even at 12 projections.

Fig. 5.15 Dobos phantom 3um rasterized reconstruction from 18 projections with quasi-random diamo RSAF and symmetric diam RSAF.(a) is the reconstruction with quasi-random Diamogram RSAF and (c) is it projections. (b) the reconstruction with symmetric Diamogram RSAF and (d) is it projections.
Fig. 5.16 Dobos phantom 3um rasterized reconstruction from 12 projections with quasi-random Diamogram RSAF and symmetric diam RSAF. (a) is the reconstruction with quasi-random diamo RSAF and (c) is it projections. (b) the reconstruction with symmetric Diamogram RSAF and (d) is it projections.
5.6 3-DCCE of challenging phantoms

We have seen so far that RSAF once optimized and adapted to the Double Sided type sensor managed to lower the number of projections needed from around 40 to 18 while producing exact recovery. We have seen that with so few as 12 projection the reconstruction is of a high diagnostic value. But we can’t help wondering about the performance of our final RSAF to reconstruct efficiency that deviates largely from the suggested Dobos phantoms. It’s totally possible that we have adapted RSAF to optimally reconstruct phantoms with rectangular charge efficiency pattern, which may result in a weakened performance on reconstructing spherical shapes. To test the performance of Diamogram RSAF we did reconstruct the Spherical phantom with no rectangular shapes at all. In addition, the charge efficiency pattern is slightly rotated to simulate that the cathode of a real 3-D sensor may deviate up to 3-5 degrees from an absolute vertical alignment.

In the end of the master thesis, we began to investigate the effect of abstaining from taking measurements along the x-axis and tried to reconstruct the phantom in various ways in hope of minimizing the consequences of this considerable loss of major coefficients in the Fourier domain. This case can be important if the quality of the test-beam data is disturbed by additional material surrounding the sensor. The reconstruction with 19 projections as shown in Fig. 5.17 where we used an approach known as limited angle projections. The result shows considerable loss of quality and any type of projections ends up with similar reconstruction. In our attempt to overcome this challenge we started to investigate a new way to optimize the algorithm. This new optimization consists of making a simple but close estimation of the 2D Fourier spectrum of the original phantom from the available projections to guess the empty part of the 2DFT. The original implementation software start from the minimal energy state (zero pads the unobserved part of the spectrum), however, our first estimation of the unobserved part did not give a noticeable improvement that we could report here. We suspect strongly that a better first estimate, based for example on a merging of information from the Fourier spectrum and a back-projection, could give a better result, but this is a work for the future.

Finally we perform the ultimate test of our improved RSAF. The next test does not aim to get a total exact recovery but rather to confirm or disprove that 18 projections should theoretically be enough to get a reconstruction of very high diagnostic value even for some of the most unthinkable charge efficiency behaviors.

Now we have clearly confirmed that our optimization works, as seen in Fig. 5.18 and Fig. 5.19. The choice of concentrated projections along x and y-axes gives a better result and we see that we can trust the performance to be optimal with the changes we did to RSAF even in the worst thinkable cases. The advantage of the Quasi-random Diamogram RSAF
Fig. 5.17 Reconstruction Dobos phantom 3um rasterized applying partial limited angle projections. (a) is the reconstruction prior to the use of RSAF. (b) is the final reconstruction. (c) the projections used. (d) the original Dobos Phantom.
Fig. 5.18 Reconstruction of the irregular phantom. (a) the original phantom. (b) reconstruction with quasi-random diamo RSAF from 18 projection. (c) reconstruction with optimized RSAF from 18 projection. (d) reconstruction with quasi-random diamo RSAF from 36 projection. (e) reconstruction with symmetric diamo RSAF from 18 projection. (f) reconstructions with original RSAF from 18 projection.
Fig. 5.19 Reconstruction of the Spherical phantom. (a) the original phantom. (b) reconstruction with quasi-random diamo RSAF from 18 projection. (c) reconstruction with optimized RSAF from 18 projection. (d) reconstruction with quasi-random diamo RSAF from 36 projection. (e) reconstruction with symmetric diamo RSAF from 18 projection. (f) reconstructions with original RSAF from 18 projection.
over normal RSAF was also made clear. With this we conclude this section by saying that we can trust and apply all our improvements to all cases we may encounter of CCE patterns for the 3-D sensors.
Chapter 6

Conclusion

I conclude this thesis starting with a short recapitulation of the work that led up to the material included in the thesis. We have been on an investigational journey from the basics of CT to the cutting edge reconstruction algorithms of CS. We wanted to work on the challenge of rendering an extremely time-consuming 3-DCCE affordable. I had no prior experience with CT or CS whatsoever. My supervisor believed that this almost unrealistic task might be done relying on two main things. First, we wanted to find a method that could take advantage of the repetitiveness of the sensor structure. Secondly, that we could rely on the "magic" of CS to reduce our need for measurements in the way it did for CT. We did not know at that time fully how it worked or how to implement it but the potential was very obvious.

This process of starting all the way from the basics did enrich our experience base, but it consumed a lot of time, so we had to let it be unmentioned in the main chapters of the thesis.

At first we tried to lower the number of projections needed just using the filtered Back-projection. It seems as though that is still considered by some of the 3DCCE team members as a viable strategy, as can be seen from the 2015 Ph.D. thesis [4]. The thought of improving the performance by the right choice of the filter on filtered back-projection can prove to be a total waste of time. In this regard, we may say that we tried every possible filter available in Matlab. We have seen in the literature that the scientific community in this field makes extensive use of the large library of methods related to filtered back-projection in Matlab. The performances of all the filters are so close to each other that even if they manage the task of an exact recovery of a Dobos phantom from 360 projections we don’t see how they can be used to reduce the number of projections below this.

If the aim is to reduce the number of projection then the Fourier Slice Method shows a clear advantage to filtered Back-projection: It works well between 180 down to around 90 projections. Unfortunately, various artifacts drown the intensity image below 90 projections. Not even the most capable interpolation algorithm could help.
We have learned greatly from our treatment of filtered back-projection and Fourier Slice Method and we can firmly say that CS is revolutionizing the reconstruction part of CT. But still when we started to gain enough understanding of the very complex theory that forms the basis of CS and it amazing power became clear, we weren’t fully satisfied yet, for various simple reasons: CS is a theory of advanced mathematics making it difficult to start with. Our intensity image and its Fourier transform don’t show directly the sparsity required by CS to reach the improvement we were hoping for. We would have to perform a basis transformation that would complicate the already complicated CS. Finally, we weren’t fully happy with the fact that conventional CS can’t directly take into consideration the distinct features of our sensors and the apriori knowledge we thought could come handy in a reconstruction.

Four months before the deadline of this thesis we came across RSAF and we could immediately see that this was exactly the thing we had been looking for from the beginning.

We then embarked on understanding the core of RSAF, the BM3D filtering, which makes use of the similarities in the image to regain an ultimate reconstruction. We succeeded to optimize RSAF parameters so we could reconstruct phantoms from even fewer projections with the cost of some blurring that doesn’t harm the diagnostic value of the recovery. The second breakthrough in this thesis came with concentrating the projections more along the x and y-axes. This idea makes use of the knowledge of the sensor structure. Our knowledge of the basics of formal CS also served us well, as we attempted to break any similarity on the projection radials using quasi-randomness to improve the results. The improvements becomes more obvious as the original intensity pattern gets more complicated.

A major result in this thesis is that even if the intensity pattern should be highly more irregular than what we expect, 18 projections are theoretically enough to give us an accurate picture of what is going on in the core of the sensor. It seems that RSAF is an ultimate CS method matched to our purpose. By adapting it to our Double Sided sensor type we could get out of it far more that any theoretical CS method could promise and we probably haven’t reached the bottom yet.

For additional improvement, we think that we can use both back-projection and Fourier methods jointly with RSAF for an ultimate performance. We started to exploit this idea in the case where we abstain from taking measurements along the x-axis, in case the quality of the test-beam data is disturbed by additional material surrounding the sensor. Another very exciting idea is to rerun the RSAF on the complete 3DCCE reconstruction but, this time, the sensor would be rotated about an orthogonal axis. There is a different form of repetitive structure along other dimensions of the sensor so maybe a small number of additional RSAF runs on the 3-D data could improve the reconstruction even further.
References


References


[26] K. Akiba, R. Plackett, et al. The timepix telescope for high performance particle track-


[29] L.I. Rudin, S. Osher and E. Fatemi . Nonlinear total variation based noise removal al-


[31] R. Baraniuk, A. Davenport, F. Duarte and H. Chinmay . An introduction to compres-
sive sensing. URL http://legacy.cnx.org/content/col11133/1.5/.


