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# On the Theory of Rough Paths, Fractional and Multifractional Brownian Motion

With Applications to Finance

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# On the Theory of Rough Paths, Fractional and Multifractional Brownian motion - with applications to finance.

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## Abstract

In recent years the theory of rough paths has become increasingly popular. The theory gives simple and “free of probability” way of looking at random noise. In this thesis we will give existence and uniqueness results of a differential equation of the form

$$dY_t = f(Y_t)dX_t, Y_0 = y$$

Where  $X_t$  is a rough signal in the sense that  $|X_t - X_s| \lesssim |t - s|^\alpha$  where  $\alpha \in (\frac{1}{4}, \frac{1}{3}]$ . Further we will use rough path theory to study fractional and multifractional Brownian motion, and construct two Itô formulas for different regularities of the respective processes. At last we will apply this theory to a square root process (as used in Heston[8] and CIR[12]), and show existence of solutions to the square root process driven by a multifractional brownian motion with a regularity function  $h$ , when  $h : [0, T] \rightarrow [a, b] \subset (0, 1)$ .

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## Organization of this thesis

We have divided this thesis in to three parts, the first part introduces rough path theory, and extend essential theorems from the book by Hairer and Friz [16] to the case when the  $\alpha$ -Hölder regularity is such that  $\alpha \in (\frac{1}{4}, \frac{1}{3}]$ . The second part will take a closer look at fractional Brownian motion and multifractional Brownian motion as rough paths. We will study their regularity, and introduce two new Itô formula's; one describing the behavior of multifractional Brownian motion with regularity function  $h : [0, T] \rightarrow [a, b] \subset (0, 1)$  and one for fractional Brownian motion with  $H \in (\frac{1}{4}, \frac{1}{2}]$ . The last part will contain a discussion of rough path theory in financial applications, and will show existence of solutions to a “square root process” driven by a multifractional Brownian motion by a simple Wong-Zakai type approximation.

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# 1 Introduction

Integration theory is one of the pillars in mathematical analysis. We know that for sufficiently smooth functions ( $C^1$ ) we may define the line integral

$$\int_0^t x(r)dy(r) := \int_0^t x(r)\dot{y}(r)dr.$$

However, for functions  $x$  and  $y$  which is not  $C^1$ , the integral is not well defined in the same way. L. C. Young showed that the integral still exists if  $x \in \mathcal{C}^\alpha$  and  $y \in \mathcal{C}^\beta$ , as long as  $\alpha + \beta > 1$ . In probability theory, one can do even better. If we let  $\{B_t\}_{t \geq 0}$  be a Brownian motion on a probability space  $(\Omega, \mathcal{F}, P)$ , then the iterated integral with respect to the Brownian motion, given by

$$\mathbb{B}_{0,t} := \int_0^t B_r dB_r \approx \sum_{[u,v] \in \mathcal{P}} B_u (B_v - B_u)$$

for some partition  $\mathcal{P}$  of  $[0, t]$ , and is well defined in a probability sense. That is, the integral is constructed as a limit of simple functions in  $L^2(\Omega)$ . Actually, the iterated integral  $\mathbb{B}_{0,t}$  may be defined in different ways, but the two most common definitions are the Itô integral or the Stratonovich integral. The relationship between the two is given by the following equation,

$$\mathbb{B}_{0,t}^{Itô} = \mathbb{B}_{0,t}^{Strat} + \frac{1}{2}t.$$

Therefore, the difference is equal to half the variance of the Brownian motion  $B_t$ . This stems from the choice of evaluation point in the Riemann sum that constructs the integral.

Probability theory has long been the “go-to” tool for handling random paths of low regularity. However, in recent years rough path theory has risen significantly as it gives an alternative view on how to analyze differential equations, and stochastic processes driven by noise of low Hölder regularity. In the late 1990’s Terry Lyons published a paper introducing differential equations driven by rough signals [14], that is he studied equations on the form

$$dY_t = f(Y_t)dX_t, Y_0 = y,$$

where  $f$  is a sufficiently smooth function and  $X$  is the rough signal controlling  $Y_t$ . In this seminal paper, Lyons discusses the importance of rough path theory and develops framework for the treatment of such differential equations. He shows that if  $X \in \mathcal{C}^\alpha$ , with  $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ , and there exists an iterated integral with respect to the rough noise  $X$ , i.e some object

$$\mathbb{X}_{s,t} = \int_s^t X_{s,r}dX_r,$$

a solution to the differential equation exists. This is not always easy to find, depending on the



choice of  $X$ . The rough path theory is therefore centered around the construction of the iterated integrals, and how they affect solutions to differential equations. Although the theory he uses in his treatment of rough signals is completely without probability theory, he points out the significance of this alternative view on stochastic differential equations as follows,

“A probabilist, interested in stochastic differential equations, might be tempted to believe that this article has little interest for him (except as a theoretical curiosity) because he can do everything that he wanted to do using Itô calculus. So we briefly mention a few situations where we believe that the results we develop here have consequences. The first is conceptual, until now the probabilist’s notion of a solution to an SDE has been as a function defined on path space and lying in some measure class or infinite dimensional Sobolev space. As such, the solution is only defined on an unspecified set of paths of capacity or measure zero. It is never defined at a given path. Given the results below, the solutions to all differential equations can be computed simultaneously for a path with an area satisfying certain Hölder conditions. The set of Brownian paths with their Lévy area satisfying this condition has full measure. Therefore and with probability one, one may simultaneously solve all differential equations over a given driving noise (the content of this remark is in the fact that there are uncountably many different differential equations).” [14] sec. 1.1.7.

Although rough path theory gives an alternative angle on the solution methods of SDE’s, it seems to require higher regularity on the function  $f$  to show existence and uniqueness of solutions than what regular probability theory does. In particular, the lower regularity on  $X_t$  you have, the more regularity you need in the function  $f$ .

In recent years Martin Hairer and Peter Friz has given significant contributions to rough path Theory. With their book first released in 2013 they developed a slightly different and more accessible introduction to, and treatment of, rough path theory. The book covers most subjects relating to rough path theory, including rough differential equations, change of variable formula (Itô formula), and some regularity concepts, but with a  $\alpha$ -Hölder regularity restriction on  $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ . Although Rough Path theory seem to generalize the concept to all paths with some  $\alpha$ -Hölder regularity, such that  $\alpha \in (0, 1)$ , the most simple applications (and maybe most useful) is the fractional Brownian motion. Fractional Brownian motion (fBm) is a stationary centered gaussian process with long range dependence. With long range dependence, or long memory, we mean that the process has some positive/negative autocorrelation, in contrast with the usual Brownian motion. The behavior of the process is determined by what is called a Hurst parameter describing the dependence on the past. The hurst parameter  $H$  lies in  $(0, 1)$ , and one can show that the  $\alpha$ -Hölder regularity of a fBm  $B_t^H$  is such that  $\alpha = H -$ . It also has the property that when  $H < \frac{1}{2}$ , the autocorrelation is negative, and when  $H > \frac{1}{2}$  the autocorrelation is positive. When  $H = \frac{1}{2}$ , the process is just a regular Brownian motion with zero autocorrelation. The standard theory to treat fBm’s has so far

been by the white noise approach. More recently the concept of fBm's has been generalized to a class of processes called multifractional processes. These processes have a hurst parameter function  $h(t)$  which is dependent on time, that makes the process non stationary. However there are a lot of applications which shows evidence of such behavior. For example, when modeling a synthesized mountain one would expect the landscape to have some rougher parts and some smoother parts.

In financial applications the rough path approach to SDE's driven by low regularity noise has become more popular. There has long been a discussion on whether or not log prices and/or log volatility tend to have long memory. If we let prices be driven by long memory processes, arbitrage will arise (see [19]). There have been proposed volatility models driven by fractional Brownian motions (see [7, 5, 9]), which can be argued to still be consistent with the no-arbitrage restrictions, as long as the price process still is a semi-martingale. However, if we model the volatility or log prices with an fBm we are implicitly saying that it has dependence on the past is constant in time. Still there seem to be strong empirical evidence that this assumption is not compatible with financial markets, see [18, 21, 23].

## 1.1 Frequently used notation

Most notation in this thesis, when unclear, will be specified. However, we want to point out that due to sometimes rather lengthy computations when discovering inequalities, we let the multiplicative constant  $C(a, b, c)$  depending on  $a, b$  and  $c$  vary trough the computations. That is, it will sometimes depend on different variables, or combination of different variables through out proofs.

We often make use of the symbol  $\lesssim$ , in the context of  $|X_t - X_s| \lesssim |t - s|^\alpha$ . This means that the left hand side is bounded by the right hand side multiplied by some constant. We define increments of a function  $f : [0, T] \rightarrow V$  as  $f_{s,t} := f(t) - f(s)$ . This is not to be confused with the two variable function  $f : [0, T]^2 \rightarrow V$  defined as  $f_{s,t} := f(t, s)$ . We denote the space of  $\alpha$ -Hölder continuous functions with  $\alpha \in (0, 1)$  by,

$$\mathcal{C}^\alpha := \left\{ f \mid f : [0, T] \rightarrow V \text{ and } \sup_{s,t \in [0, T], s \neq t} \frac{|f_t - f_s|}{|t - s|^\alpha} < \infty \right\}.$$

The space  $\mathcal{C}_2^{2\alpha}$  is defined similarly as follows

$$\mathcal{C}_2^{2\alpha} := \left\{ f \mid f : [0, T]^2 \rightarrow V \text{ and } \sup_{s,t \in [0, T], s \neq t} \frac{|f_{s,t}|}{|t - s|^{2\alpha}} < \infty \right\}.$$

The  $\alpha$ -Hölder semi-norm of  $f$  is denoted by

$$\|f\|_\alpha = \sup_{s,t \in [0, T], s \neq t} \frac{|f_{s,t}|}{|t - s|^\alpha}.$$

We often makes use of functions  $F \in \mathcal{C}^3$ , such that  $F : \mathbb{R}^d \rightarrow \mathbb{R}^n$ . Then, the derivative of the function  $DF : \mathbb{R}^d \rightarrow \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n)$ , where  $\mathcal{L}(\mathbb{R}^d, \mathbb{R}^n)$  is defined as the space of linear functionals from

$\mathbb{R}^d \rightarrow \mathbb{R}^n$ . The second derivative  $D^2F : \mathbb{R}^d \rightarrow \mathcal{L}(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n)) \simeq \mathcal{L}(\mathbb{R}^{d \times d}, \mathbb{R}^n)$ . We will only discuss finite dimensional Banach spaces, and mostly the real  $d$ -dimensional space  $\mathbb{R}^d$ .

We frequently use Taylor approximations, and the remainder terms from Taylor approximations. We use the following notation; Let  $F$  be given as above, and  $x, y \in \mathbb{R}^d$ , then

$$F(x) - F(y) = DF(y)(x - y) + \frac{1}{2}D^2F(y)(x - y)^{\otimes 2} + \bar{R}_{x,y}^F.$$

By the Lagrange remainder term formula, we know that  $\bar{R}_{x,y}^F = \frac{1}{6}D^3F(y^*)(x - y)^{\otimes 3}$  for some  $y^* \in (y, x)$ .

We sometimes write that  $|f_{s,t}| = o(|t - s|)$ . This implies that  $|f_{s,t}| \lesssim |t - s|^\gamma$  for some  $\gamma > 1$ . If the factor  $\gamma$  is of significance for further calculation, we will write  $|f_{s,t}| = o(|t - s|^\gamma)$ .

## Part I

# Rough path theory

In this part, we will give a short introduction to Rough Path theory, following both the notation and arguments from the book of P. Friz and M. Hairer [16]. Although following the book closely, we will extend their results to lower regularity processes. We are interested in the construction of iterated integrals of a process  $X \in \mathcal{C}^\alpha$ . We will show some natural requirements to expect of a iterated integral and how these can be approximated by smooth paths. We will also look at how to construct the integral  $\int Y dX$ , and how to find processes  $Y$  such that the respective integral exists. Then we will investigate differential equations driven by rough signals.

We begin with a basic introduction to the case when the  $\alpha$ -Hölder regularity of the process  $X$  is  $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ .

## 2 Basic introduction to Rough paths

Let  $X^{(1)}$  be a  $\alpha$ -Hölder regular process, i.e  $X^{(1)} \in \mathcal{C}^\alpha$ . An essential question in the theory of rough paths is how to give meaning to integrals of the form

$$X_{s,t}^{(2)} := \int_s^t X_{s,r}^{(1)} dX_r^{(1)}, \quad (1)$$

where  $X_{s,t}^{(1)} := X_t^{(1)} - X_s^{(1)}$ , and when  $X_{s,t}^{(1)}$  is of Hölder regularity  $< \frac{1}{2}$ . When  $X^{(1)} \in \mathcal{C}^1$  then  $X_{s,t}^{(2)}$  is well defined by reading (1) from left to right. If  $X^{(1)} \in \mathcal{C}^\alpha$ ,  $\alpha \in (\frac{1}{2}, 1]$  then  $\int_s^t X_{s,r}^{(1)} dX_r^{(1)}$  is called a Young integral and can be shown to be well defined, and hence, the definition can be read from left to right. The problem arises when  $X^{(1)} \in \mathcal{C}^\beta$ ,  $\beta < \frac{1}{2}$ , then the integral is not, in general, well defined, and we therefore need to construct an object  $X_{s,t}^{(2)}$  such that the definition in (1) can be read from the right to left. In this first section we will give the proper framework to be able to construct such integrals, and analyze them. To give some intuition we will start with a construction when the  $\alpha$ -Hölder regularity of the path  $X^{(1)}$  is in  $[\frac{1}{3}, \frac{1}{2})$ . In section 2 we will show how we can extend the theory to the case when  $\alpha \in [\frac{1}{4}, \frac{1}{3})$ .

### 2.1 The space of $\alpha$ -Hölder rough paths

Let  $\alpha \in (\frac{1}{3}, \frac{1}{2}]$  and let the  $\alpha$ -Hölder semi norm be given by  $\|f\|_\alpha := \sup_{s \neq t \in [0, T]} \frac{|f_{s,t}|}{|t-s|^\alpha}$ . We define the space of  $\alpha$ -Hölder rough paths as the pairs  $(1, X^{(1)}, X^{(2)})$  where  $X^{(1)} : [0, T] \rightarrow V$  and  $X^{(2)} : [0, T]^2 \rightarrow V \otimes V$ , and such that

$$\|X^{(1)}\|_\alpha = \sup_{s \neq t \in [0, T]} \frac{|X_{s,t}^{(1)}|}{|t-s|^\alpha} < \infty, \quad \|X^{(2)}\|_{2\alpha} = \sup_{s \neq t \in [0, T]} \frac{|X_{s,t}^{(2)}|}{|t-s|^\alpha} < \infty$$

and

$$X_{s,t}^{(2)} - X_{s,u}^{(2)} - X_{u,t}^{(2)} = X_{s,u}^{(1)} \otimes X_{u,t}^{(1)}, \quad (\text{Chen's relation}) \quad (2.1)$$

holds for all  $s, u, t \in [0, T]$ . For short we write  $\mathbf{X} := (1, X^{(1)}, X^{(2)}) \in \mathcal{C}^\alpha([0, T], V)$  for a Banach space  $V$ . As we can see the space  $\mathcal{C}^\alpha([0, T], V) \subset C^\alpha \oplus C_2^{2\alpha}$ . We say that if a path  $X^{(1)} \in \mathcal{C}^\alpha$ , the path  $X^{(1)}$  can be *lifted* (or there exists a *lift*) to an element  $(1, X^{(1)}, X^{(2)}) \in \mathcal{C}^\alpha$ , if and only if there exists an object  $X^{(2)} \in C_2^{2\alpha}$ , such that Chen's relation is satisfied.

The space  $\mathcal{C}^\alpha([0, T], V)$  is not a Banach space, as it is not a linear space due to restrictions from Chen's relation.

Chen's relation is an essential piece in the construction of the second iterated integrals, and we will later generalize this to the third iterated integral. We will in general only consider finite dimensional Banach spaces  $V$ , and the tensor product can be thought of as a kind of vector multiplication in the way that, for two vectors  $a, b \in V$ ,  $a \otimes b := ab^T$ . Next we will introduce a suitable metric on the space of rough paths.

**Definition 2.1.** Let  $\mathbf{X}, \mathbf{Y} \in \mathcal{C}^\alpha([0, T], V)$ . We define the metric on  $\mathcal{C}^\alpha([0, T], V)$  by

$$d_\alpha(\mathbf{X}, \mathbf{Y}) = \|X^{(1)} - Y^{(1)}\|_\alpha + \|X^{(2)} - Y^{(2)}\|_{2\alpha}.$$

The metric does not make the space  $\mathcal{C}^\alpha([0, T], V)$  complete, but introducing the initial values of the paths  $X$  and  $Y$  to  $d_\alpha(\mathbf{X}, \mathbf{Y})$  will. That is, by introducing  $X_0$  and  $Y_0$  we can make the space  $\mathcal{C}^\alpha([0, T], V)$  complete under the metric  $\|X_0^{(1)} - Y_0^{(1)}\| + d_\alpha(\mathbf{X}, \mathbf{Y})$ . To show convergence in this metric, we will in this thesis focus on interpolation, described and proved later. Next we will define the notion of a geometric rough path.

### 2.1.1 The space of geometric rough paths.

From "regular" calculus we are familiar with the fact that if  $x$  is a sufficiently smooth path, then

$$\text{sym} \left( \int_s^t x_{s,r} dx_r \right) = \frac{1}{2} x_{s,t} \otimes x_{s,t}. \quad (2)$$

We want to define a space of  $\alpha$ -Hölder rough paths who satisfy (2). That is, for a rough path  $(1, X^{(1)}, X^{(2)})$ , define  $X_{s,t}^{(1)(i)} = e_i(X_{s,t}^{(1)})$  and  $X_{s,t}^{(2)(i,j)} = e_i \otimes e_j(X_{s,t}^{(2)})$ . If  $X^{(2)}$  is such that

$$\begin{aligned} X_{s,t}^{(2)(i,j)} + X_{s,t}^{(2)(j,i)} &= \int_s^t X_{s,r}^{(1),i} dX_r^{(1),j} + \int_s^t X_{s,r}^{(1),j} dX_r^{(1),i} \\ &= \int_s^t X_r^{(1),i} dX_r^{(1),j} + \int_s^t X_r^{(1),j} dX_r^{(1),i} - X_s^{(1),i} X_{s,t}^{(1),j} - X_s^{(1),j} X_{s,t}^{(1),i} \\ &= \int_s^t d(X^{(1),i} X^{(1),j})_r - X_s^{(1),i} X_{s,t}^{(1),j} - X_s^{(1),j} X_{s,t}^{(1),i} = X_{s,t}^{(1),i} X_{s,t}^{(1),j}, \end{aligned}$$

we call the path  $(1, X^{(1)}, X^{(2)})$  a geometric rough path, and we say that  $\text{sym} \left( X_{s,t}^{(2)} \right) = \frac{1}{2} X_{s,t}^{(1)} \otimes X_{s,t}^{(1)}$ . Here,  $\text{sym}$  represents the symmetry operator given by  $\text{sym}(A) = \frac{1}{2}(A + A^T)$  for  $A \in V \otimes V$ . We will give a more formal definition of geometric paths as follows.

**Definition 2.2.** We say that a path  $(1, X^{(1)}, X^{(2)}) \in \mathcal{C}^\alpha$  is geometric if it satisfy the following relation,

$$\text{sym} \left( X_{s,t}^{(2)} \right) = \frac{1}{2} X_{s,t}^{(1)} \otimes X_{s,t}^{(1)}.$$

Formally we write  $(1, X^{(1)}, X^{(2)}) \in \mathcal{C}_g^\alpha$ .

There are two ways to define the space of geometric rough paths. One way is to define the space  $\mathcal{C}_g^\alpha$  as the space with  $\alpha$ -Hölder rough paths, which also satisfy the relation  $X_{s,t}^{(2)(i,j)} + X_{s,t}^{(2)(j,i)} = X_{s,t}^{(1)(i)} \otimes X_{s,t}^{(1)(j)}$  as we did above, or we could define  $\mathcal{C}_g^{0,\alpha}$  as the closure of lifts of smooth paths in  $\mathcal{C}^\alpha$ . We therefore have the relation

$$\mathcal{C}_g^{0,\alpha} \subset \mathcal{C}_g^\alpha \subset \mathcal{C}^\alpha$$

In fact, one can show that the two definitions are equal, see [16] chp. 2.

As one may expect, the rough paths which are geometric can be approximated by smooth paths. Let  $\beta \in (\frac{1}{3}, \frac{1}{2}]$ . For every  $(1, X^{(1)}, X^{(2)}) \in \mathcal{C}_g^\beta([0, T], \mathbb{R}^d)$  there exist a sequence of smooth paths  $X^{(1),n} : [0, T] \rightarrow \mathbb{R}^d$  such that

$$\left( 1, X^{(1),n}, X^{(2),n} \right) := \left( 1, X^{(1),n}, \int_0^\cdot X_{0,r}^{(1),n} dX_r^{(1),n} \right) \rightarrow \left( 1, X^{(1)}, X^{(2)} \right),$$

uniformly on  $[0, T]$ , and with uniform rough bounds  $\sup_n \|X^{(1),n}\|_\beta + \sup_n \|X^{(2),n}\|_{2\beta} < \infty$ . By interpolation convergence holds in  $\alpha$ -Hölder rough paths, with  $\alpha \in (\frac{1}{3}, \beta)$ , namely that

$$\lim_{n \rightarrow \infty} d_\alpha(\mathbf{X}^n, \mathbf{X}) = 0.$$

We will give a lemma where the proof shows a method for interpolation, which will become useful in the rest of this thesis.

**Lemma 2.3.** Let  $(X^{(1),n}, X^{(2),n}) \in \mathcal{C}^\beta$ , for  $\frac{1}{3} < \alpha < \beta$  such that the uniform bounds

$$\sup_n \|X^{(1),n}\|_\beta < \infty \quad \sup_n \|X^{(2),n}\|_{2\beta} < \infty,$$

and with convergence  $X_{s,t}^{(1),n} \rightarrow X_{s,t}^{(1)}$  and  $X_{s,t}^{(2),n} \rightarrow X_{s,t}^{(2)}$  uniformly on  $[0, T]$ . Then  $(X^{(1)}, X^{(2)}) \in \mathcal{C}^\beta$ , and we have convergence in the  $d_\alpha$  metric, that is

$$d_\alpha(\mathbf{X}^n, \mathbf{X}) = \|X^{(1),n} - X^{(1)}\|_\alpha + \|X_{s,t}^{(2),n} - X_{s,t}^{(2)}\|_{2\alpha} \rightarrow 0.$$

*Proof.* Using the uniform convergence and uniform bounds we have that

$$|X_{s,t}^{(1)}| := \lim |X_{s,t}^{(1),n}| \lesssim |t-s|^\beta \quad |X_{s,t}^{(2)}| := \lim |X_{s,t}^{(2),n}| \lesssim |t-s|^{2\beta}$$

and there exist a sequence  $\varepsilon_n$  such that

$$\begin{aligned} |X_{s,t}^{(1)} - X_{s,t}^{(1),n}| &\leq K |t-s|^\beta \quad \text{and} \quad |X_{s,t}^{(1)} - X_{s,t}^{(1),n}| < \varepsilon_n \\ |X_{s,t}^{(2)} - X_{s,t}^{(2),n}| &\leq K |t-s|^{2\beta} \quad \text{and} \quad |X_{s,t}^{(2)} - X_{s,t}^{(2),n}| < \varepsilon_n \end{aligned}$$

uniformly over  $s, t \in [0, T]$ . Using the inequality  $a \wedge b \leq a^{1-\theta} b^\theta$ , where  $\theta \in (0, 1)$ , we combine the two expressions above with  $\theta = \frac{\alpha}{\beta}$  and find

$$\begin{aligned} |X_{s,t}^{(1)} - X_{s,t}^{(1),n}| &\lesssim \varepsilon_n^{1-\frac{\alpha}{\beta}} |t-s|^\alpha \\ |X_{s,t}^{(2)} - X_{s,t}^{(2),n}| &\lesssim \varepsilon_n^{1-\frac{\alpha}{\beta}} |t-s|^{2\alpha} \end{aligned}$$

and the desired  $d_\alpha$  convergence follows.  $\square$

*Remark 2.4.* Given uniform convergence of the paths above, and the fact that both paths is in  $\mathcal{C}^\beta$ , we only get convergence in the  $d_\alpha$  metric, due to the restrictions on the inequality used, still we can choose  $\alpha$  as close to  $\beta$  as we want.

It turns out that a nice way to view the space  $\mathcal{C}^\alpha([0, T], \mathbb{R}^d)$  is to look at a truncated tensor algebra. The definition is also very suitable for higher order iterated integrals, and will be essential in the construction of Chen's relation in higher order tensor products. We sum it up in the following definition.

**Definition 2.5.** Let the Banach space  $V = \mathbb{R}^d$ ,  $X^{(1)} : [0, T] \rightarrow \mathbb{R}^d$  and  $X^{(2)} : [0, T]^2 \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$  which satisfy Chen's relation 2.1. Then we define

$$\mathbf{X}_{s,t} = \left(1, X_{s,t}^{(1)}, X_{s,t}^{(2)}\right) \in \mathbb{R} \oplus \mathbb{R}^d \oplus \left(\mathbb{R}^d \otimes \mathbb{R}^d\right) =: T^{(2)}(\mathbb{R}^d). \quad (2.2)$$

And we define the truncated tensor algebra multiplication for two elements  $(a, b, c), (\tilde{a}, \tilde{b}, \tilde{c}) \in T^{(2)}(\mathbb{R}^d)$  by,

$$(1, a, b) \otimes (1, \tilde{a}, \tilde{b}) = \left(1, a + \tilde{a}, b + \tilde{b} + a \otimes \tilde{a}\right).$$

We may therefore look at rough paths under this multiplication, and find

$$\mathbf{X}_{s,u} \otimes \mathbf{X}_{u,t} = \left(1, X_{s,u}^{(1)} + X_{u,t}^{(1)}, X_{s,u}^{(2)} + X_{u,t}^{(2)} + X_{s,u}^{(1)} \otimes X_{u,t}^{(1)}\right) =: \mathbf{X}_{s,t}.$$

We see the definition corresponds very nicely to Chen's relation. We also see that  $\mathbf{X}_{s,t} = \mathbf{X}_s^{-1} \otimes \mathbf{X}_t$ , where  $\mathbf{X}_s^{-1} := \left(1, -X_s^{(1)}, -X_{0,s}^{(2)} + X_{0,s}^{(1)} \otimes X_{0,s}^{(1)}\right)$ . The space  $T^{(2)}(\mathbb{R}^d)$  can be generalized

to contain  $n + 1$ -tuples of iterated integrals, in the way that

$$T^{(n)}(\mathbb{R}^d) := \mathbb{R} \oplus \bigoplus_{i=1}^n (\mathbb{R}^d)^{\otimes i}.$$

One would then expect the higher order iterated integrals to obey some kind of Chen's relation in the same way as for rough path tuples. It turns out that given  $\mathbf{X} \in T^{(n)}(\mathbb{R}^d)$ , we find a Chen's relation by postulating that

$$\mathbf{X}_{s,t} := \mathbf{X}_{s,u} \otimes \mathbf{X}_{u,t}$$

Where the tensor multiplication represents multiplication in the truncated algebra sense. In the next section we will take a closer look at the space  $T^{(3)}(\mathbb{R}^d)$ , which will contain the four-tuples necessary to look at regularity problems when  $\alpha \in (\frac{1}{4}, \frac{1}{3}]$ .



### 3 Rough paths of low regularity.

In this thesis, we want to construct a space for geometric rough paths with  $\alpha$ -Hölder coefficient  $\alpha \in (\frac{1}{4}, \frac{1}{3}]$ . We therefore want to define the space  $T^{(3)}(\mathbb{R}^d)$  by

$$T^{(3)}(\mathbb{R}^d) = \mathbb{R} \oplus \bigoplus_{n=1}^3 (\mathbb{R}^d)^{\otimes n}.$$

An element of  $T^{(3)}(\mathbb{R}^d)$  is of the form  $x = (1, a, b, c) \in T^{(3)}(\mathbb{R}^d)$ , and multiplication of two elements  $x$ , and  $y = (1, a', b', c') \in T^{(3)}(\mathbb{R}^d)$  yields

$$x \otimes y := (1, a + a', b + b' + a \otimes a', c + c' + a \otimes b' + b \otimes a').$$

Therefore, for a rough path  $\mathbf{X}$  we have the following relation

$$\begin{aligned} \mathbf{X}_{s,t} &= \mathbf{X}_{s,u} \otimes \mathbf{X}_{u,t} \\ &= \left(1, X_{s,u}^{(1)} + X_{u,t}^{(1)}, X_{s,u}^{(2)} + X_{u,t}^{(2)} + X_{s,u}^{(1)} \otimes X_{u,t}^{(1)}, X_{s,u}^{(3)} + X_{u,t}^{(3)} + X_{s,u}^{(1)} \otimes X_{u,t}^{(2)} + X_{s,u}^{(2)} \otimes X_{u,t}^{(1)}\right), \end{aligned} \quad (3.1)$$

This is what defines a Chen's relation on  $T^{(3)}(\mathbb{R}^d)$ . We see that in addition to the Chen's relation on the second iterated integral, we will require that the third iterated integral satisfy the relation

$$X_{s,t}^{(3)} - X_{s,u}^{(3)} - X_{u,t}^{(3)} = X_{s,u}^{(1)} \otimes X_{u,t}^{(2)} + X_{s,u}^{(2)} \otimes X_{u,t}^{(1)}.$$

Let  $\alpha \in (\frac{1}{4}, \frac{1}{3}]$ , then  $\mathbf{X}_{s,t} = (X_{s,t}^{(1)}, X_{s,t}^{(2)}, X_{s,t}^{(3)}) \in \mathcal{C}^\alpha([0, T], V)$ , where

$$\begin{aligned} X_{s,t}^{(1)} &= X_t^{(1)} - X_s^{(1)} \\ X_{s,t}^{(2)} &= \int_s^t X_{s,r}^{(1)} dX_r^{(1)} \\ X_{s,t}^{(3)} &= \int_s^t X_{s,r}^{(2)} dX_r^{(1)} \end{aligned}$$

As we know, if  $X_{s,t}^{(1)} \in V$  then  $X_{s,t}^{(2)} \in V \otimes V$ , and therefore the third iterated integral  $X^{(3)}$  will take values in  $V^{\otimes 3}$ . Again, remember that the definitions is read from the right to the left; a-priori, we do not have any information about what kind of path  $X^{(1)}$  is, and we have not defined what the second iterated integral should be. One of our goals in rough path theory is to find processes  $X^{(2)}$  and  $X^{(3)}$  such that the Chen's relation and the regularity conditions holds, and then we may define the iterated integrals to be the objects  $X^{(2)}$  and  $X^{(3)}$ .

We introduce the metric on  $\mathcal{C}^\alpha([0, T], V)$  by

$$d_\alpha(\mathbf{X}, \tilde{\mathbf{X}}) := \|X^{(1)} - \tilde{X}^{(1)}\|_\alpha + \|X^{(2)} - \tilde{X}^{(2)}\|_{2\alpha} + \|X^{(3)} - \tilde{X}^{(3)}\|_{3\alpha}$$

Further, we denote the homogenous  $\alpha$ -Hölder norm by

$$\|\mathbf{X}\|_\alpha := \|X^{(1)}\| + \sqrt{\|X^{(2)}\|_{2\alpha}} + \sqrt[3]{\|X^{(3)}\|_{3\alpha}}.$$

In the next section, we will introduce the concept of geometric rough paths. These are the paths where the second and third iterated integrals satisfy basic rules of ordinary calculus, and will play an important part especially in applications, and in rough path theory in general.

### 3.1 Geometric rough paths of lower regularity

The space of geometric rough paths  $\mathcal{C}_g^\alpha$  are defined such that all rough paths in  $\mathcal{C}_g^\alpha$  can be approximated by smooth paths. we would therefore require that the iterated integrals would behave such that they satisfy some basic calculus rules, i.e,

$$\text{sym}\left(X_{s,t}^{(2)}\right) = \frac{1}{2}X_{s,t}^{(1)} \otimes X_{s,t}^{(1)}$$

and

$$\text{sym}\left(X_{s,t}^{(3)}\right) = \frac{1}{6}X_{s,t}^{\otimes 3}.$$

If the rough path obeys the above two conditions, we say that  $\mathbf{X} = (1, X^{(1)}, X^{(2)}, X^{(3)}) \in \mathcal{C}_g^\alpha$ . The space  $\mathcal{C}_g^\alpha$  may be defined in the way done above; by the two conditions, or as a closure of smooth paths in  $\mathcal{C}^\alpha$ . As we have seen before, we can always interpolate smooth paths into the rough path space  $\mathcal{C}^\alpha$ , but naturally they would still satisfy their regular “rules of calculus”.

*Remark 3.1.* A relationship worth noting before we move on is that for  $\alpha < \beta$ ,  $\mathcal{C}^\beta \subset \mathcal{C}^\alpha$ . Indeed, by interpolation, we know that all paths in  $\mathcal{C}^\beta$  can be approximated in  $\mathcal{C}^\alpha$ . And we have the relationship

$$\|X\|_\alpha = \sup_{s,t \in [0,T], s \neq t} \frac{|X_{s,t}|}{|t-s|^\alpha} = \sup_{s,t \in [0,T], s \neq t} \frac{|X_{s,t}|}{|t-s|^\beta} \frac{1}{|t-s|^{\alpha-\beta}} \leq \|X\|_\beta T^{\beta-\alpha},$$

therefore, if  $\|X\|_\beta < \infty$ , then  $X \in \mathcal{C}^\alpha$ , but the converse is not in general true. This can of course be extended to the space  $\mathcal{C}^\alpha$ , in the sense that  $\mathcal{C}^\beta \subset \mathcal{C}^\alpha$  for  $\beta > \alpha$ .

### 3.2 Integration against rough paths

In this section we study chapter four in the book by Hairer, and Friz [16], and extend the results to the case when the  $\alpha$ -Hölder regularity is such that  $\alpha \in (\frac{1}{4}, \frac{1}{3}]$ . The difference between the rough path theory when  $\alpha \in (\frac{1}{3}, \frac{1}{4}]$  and when  $\alpha \in (\frac{1}{3}, \frac{1}{2}]$  is that we need to introduce higher order derivatives of the integrand and higher order iterated integrals from rough paths to be able to define a suitable integral. With suitable integrals we mean integrals of the form  $\int_0^\cdot Y_r dX_r$  where  $Y$  can

be a function of  $X$  or some other path which is integrable with respect to  $X$ . When introducing the higher order derivatives of the integrand  $Y$ , we will get more expressions to handle in our (in)equalities, but we will be able to tractate rougher integrals than shown in the first section.

### 3.2.1 The Sewing Lemma

Now we move on to one of the most important results in the theory of rough paths, namely the Sewing lemma. Before we state the theorem, we will motivate it by an example when  $\alpha > \frac{1}{2}$ . From the theory of Young integration, we want to construct an abstract integration map  $I$ , which works in a way like Rieman-Stieltjes sums. That is, we want to find the function  $Z_{s,t}$  which fully determines the integral of  $Y$  with respect to  $X$ , i.e  $Z_{s,t} = \int_s^t Y_r dX_r$ . If  $X \in \mathcal{C}^\alpha$  and  $Y \in \mathcal{C}^\beta$  such that  $\alpha + \beta > 1$ , Young found that

$$Z_{s,t} = Y_s X_{s,t} + o(|t - s|).$$

That is, the function  $Y_s X_{s,t}$  fully determines  $Z_{s,t}$ . Therefore we want to find  $\psi$ , such that  $Z_{s,t} = I(\psi)_{s,t}$  is a well defined image of  $\psi$  under an abstract integration map  $I$ .

**Definition 3.2.** We define the space  $\mathcal{C}_2^{\alpha,\beta}([0,T]^2, W)$  as functions  $\psi$  from  $[0,T]^2$  into  $W$  s.t  $\psi_{t,t} = 0$  and,

$$\|\psi\|_{\alpha,\beta} := \|\psi\|_\alpha + \|\delta\psi\|_\beta < \infty,$$

where  $\delta\psi_{s,u,t} := \psi_{s,t} - \psi_{s,u} - \psi_{u,t}$ .

This space becomes handy in the proof of the Sewing lemma, and in further applications.

**Lemma 3.3.** (*Sewing Lemma*) Let  $\alpha$  and  $\beta$  be such that  $0 < \alpha \leq 1 < \beta$ . Then there exist a (unique) continuous map  $I : \mathcal{C}_2^{\alpha,\beta}([0,T]^2, W) \rightarrow \mathcal{C}^\alpha([0,T]^2, W)$  such that  $(I\psi)_0 = 0$  and

$$|(I\psi)_{s,t} - \psi_{s,t}| \leq C |t - s|^\beta.$$

Where  $C$  only depends on  $\beta$  and  $\|\delta\psi\|_\beta$ . (The  $\alpha$ -Hölder norm of  $I\psi$  also depends on  $\|\psi\|_\alpha$ , and hence on  $\|\psi\|_{\alpha,\beta}$ ).

*Proof.* We note that  $I$  will be built as a linear map, so that its continuity is a consequence of its boundedness. Uniqueness is immediate; assume, by contradiction that for given  $\psi$  there are two candidates  $F_1$  and  $F_2$  for  $I\psi$ . Define  $F = F_1 - F_2$ . We have that  $F_0 = 0$  and  $|F_{s,t}| \lesssim |t - s|^\beta$  for  $\beta > 1$ , and we know that the only function which satisfy this (i.e higher than Lipschitz regularity), is the function  $F = 0$ . It remains to find the integration map  $I$ . We could guess it would be on

the form

$$(I\psi)_{s,t} = \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} \psi_{u,v}$$

Where  $\mathcal{P}$  denotes a partition on  $[s, t]$  and  $|\mathcal{P}| := \max |u - v|$ ,  $u, v \in \mathcal{P}$ . Friz and Hairer offers two arguments for the proof of the sewing lemma in [16] chp. 4. We will follow the second argument here. The argument that follows is essentially due to Young, which finds that convergence is immediate as  $|\mathcal{P}| \rightarrow 0$ , i.e the same limit is obtained along any sequence  $\mathcal{P}_n$  with  $|\mathcal{P}_n| \rightarrow 0$ . Consider a partition  $\mathcal{P}$  of  $[s, t]$  and let  $r \geq 1$  be the number of intervals in  $\mathcal{P}$ . When  $r \geq 2$  there exist a  $u \in [s, t]$  such that  $[u_-, u], [u, u_+] \in \mathcal{P}$  and

$$|u_+ - u_-| \leq \frac{2}{r-1} |t - s|$$

Since, if we assume otherwise gives the contradiction  $2 |t - s| < \sum_{u \in \mathcal{P}^0} |u_+ - u_-| \leq 2 |t - s|$ . Hence, by removing the point  $u \in \mathcal{P}$  in one integral and look at the difference between the two, we find

$$|\int_{\mathcal{P} \setminus \{u\}} \psi - \int_{\mathcal{P}} \psi| = |\delta\psi_{u_-, u, u_+}| \leq \|\delta\psi\|_\beta \left( \frac{2}{r-1} |t - s| \right)^\beta.$$

Further, we can see that, if there are more than 3 elements in  $\mathcal{P}$ , i.e  $r \geq 3$ , there exist two points  $u, v' \in \mathcal{P}$ , such that

$$\begin{aligned} |\psi_{s,t} - \int_{\mathcal{P}} \psi| &\leq \left| \psi_{s,t} - \int_{\mathcal{P} \setminus \{u\}} \psi \right| + \left| \int_{\mathcal{P} \setminus \{u\}} \psi - \int_{\mathcal{P}} \psi \right| \\ &\leq \left| \psi_{s,t} - \int_{\mathcal{P} \setminus \{u, v\}} \psi \right| + \left| \int_{\mathcal{P} \setminus \{v, v\}} \psi - \int_{\mathcal{P} \setminus \{u\}} \psi \right| + \left| \int_{\mathcal{P} \setminus \{u\}} \psi - \int_{\mathcal{P}} \psi \right|. \end{aligned}$$

By iterating this procedure, selecting a new point  $u$  to remove each time, we get that the difference between  $\psi_{s,t}$  and  $\int_{\mathcal{P}} \psi$  biggest, we see that,

$$\sup_{\mathcal{P} \subset [s, t]} |\psi_{s,t} - \int_{\mathcal{P}} \psi| \leq \|\delta\psi\|_\beta \sum_{i=2}^r \left( \frac{2}{i-1} |t - s| \right)^\beta \leq 2^\beta \|\delta\psi\|_\beta \zeta(\beta) |t - s|^\beta,$$

where  $\zeta(\beta) = \sum_{r=1}^{\infty} \frac{1}{r^\beta}$  is the Riemann zeta function. It then remains to show that

$$\sup_{\mathcal{P} \vee \mathcal{P}' < \varepsilon} |\int_{\mathcal{P}} \psi_{s,t} - \int_{\mathcal{P}'} \psi_{s,t}| \rightarrow 0 \quad \text{as } \varepsilon \searrow 0.$$

Which implies the existence of  $I\psi$  as the limit  $\lim_{|\mathcal{P}| \rightarrow 0} \int_{\mathcal{P}} \psi$ . We may assume without loss of

generality that  $\mathcal{P}'$  refines  $\mathcal{P}$  and therefore  $|\mathcal{P}| \vee |\mathcal{P}'| = |\mathcal{P}|$  and

$$\int_{\mathcal{P}} \psi - \int_{\mathcal{P}'} \psi = \sum_{[u,v] \in \mathcal{P}} \left( \psi_{u,v} - \int_{\mathcal{P}' \cap [u,v]} \psi \right).$$

Then, for  $|\mathcal{P}| < \varepsilon$  we can use the maximal inequality to see that

$$\left| \int_{\mathcal{P}} \psi - \int_{\mathcal{P}'} \psi \right| \leq 2^\beta \|\delta\psi\|_\beta \zeta(\beta) \sum_{[u,v] \in \mathcal{P}} |u - v|^\beta = O(|\mathcal{P}|^\beta) \leq O(\varepsilon^\beta),$$

since  $\beta > 1$ , which concludes the argument.  $\square$

To clarify the impact of the sewing lemma, we will emphasize on the fact that this lemma holds regardless of  $\alpha$ -regularity and construction of  $\psi$ , as long as we choose  $\psi$  such that  $\|\delta\psi\|_\beta < \infty$  for  $\beta > 1$ . Hence, it only depend on the choice of the function  $\psi$ .

We will show the use of the sewing lemma in the context of Young integration. As we discussed earlier, we want to define an integral of the form  $\int_s^t Y_r dX_r = I(\psi)_{s,t}$ , then if we choose  $\psi_{s,t} = Y_s X_{s,t}$  where  $|X_{s,t}| \lesssim |t - s|^\alpha$  and  $|Y_{s,t}| \lesssim |t - s|^\gamma$ , we can see that

$$\delta\psi_{s,u,t} = -Y_{s,u} X_{u,t},$$

and that  $|Y_{s,u} X_{u,t}| \lesssim |t - s|^{\alpha+\gamma}$ . Now, let  $\alpha + \gamma > 1$ . We know from the sewing lemma that  $|I(\psi)_{s,t} - \psi_{s,t}| \lesssim \|\delta\psi\|_\beta |t - s|^\beta$  for  $\beta > 1$ . Hence, we need to check that  $\|\delta\psi\|_\beta < \infty$ . This follows directly from the fact that  $|Y_{s,u} X_{u,t}| \lesssim |t - s|^{\alpha+\gamma}$ , define  $\beta = \alpha + \gamma$ , and we see that the sewing lemma holds, and that  $\psi_{s,t}$  “determines” the integral on small time intervals, in the sense that

$$|I(\psi)_{s,t}| \leq |\psi_{s,t}| + o(|t - s|^\beta).$$

One would expect that when the regularity of  $X$  and  $Y$  is lower, we need to expand the function  $\psi$  to contain higher derivatives on  $Y$  and higher iterated integrals of  $X$ . As we will see, we can construct a function  $\psi$  in the case when  $\alpha \in (\frac{1}{4}, \frac{1}{3}]$  by introducing the first and second derivative of  $Y$  and the rough path  $(1, X^{(1)}, X^{(2)}, X^{(3)}) \in \mathcal{C}^\alpha$ . Actually, the meaning of derivative in this setting is a bit ambiguous, the derivative of  $Y$  is not necessarily unique, we only require that the remainder terms from a Taylor type approximation is finite in the Hölder norm. We shall elaborate on this in definition 3.4.

The sewing lemma naturally leads to a desire for a space of integrable processes  $Y$  and their derivatives, such that we easily can construct integrals. To motivate the coming definition of such a space, we will discuss the case when  $\alpha \in (\frac{1}{3}, \frac{1}{2}]$  and we want to define the integral of a function  $f(X)$  with respect to  $X$ . If we look at a function  $f \in \mathcal{C}_b^3$  and  $\mathbf{X} \in \mathcal{C}_g^\alpha$ ,  $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ , we have a Taylor

expansion of  $f$  in  $X^{(1)}$  as follows,

$$f(X_t^{(1)}) - f(X_s^{(1)}) = Df(X_s^{(1)})X_{s,t}^{(1)} + R_{s,t}^{f(X^{(1)})},$$

where  $R_{s,t}^{f(X^{(1)})}$  is the remainder term of the Taylor series. The question then becomes what regularity the remainder term  $R_{s,t}^{f(X^{(1)})}$  inherits. From the Lagrange remainder thm. we have that  $|R_{s,t}^{f(X^{(1)})}| \leq |D^2 f|_{\infty} |X_{s,t}^{(1)} \otimes X_{s,t}^{(1)}| \lesssim |t - s|^{2\alpha}$ . Now, if we look at the integral of  $f(X^{(1)})$  with respect to the path  $(1, X^{(1)}, X^{(2)})$ , we see that

$$\int_s^t f(X_r^{(1)})dX_r^{(1)} = \int_s^t f(X_s^{(1)})dX_r^{(1)} + \int_s^t Df(X_s^{(1)})X_{s,r}^{(1)}dX_r^{(1)} + \int_s^t R_{s,t}^{f(X^{(1)})}dX_r.$$

If we can prove that  $\int_s^t R_{s,t}^{f(X^{(1)})}dX_r$  goes to zero when  $s \rightarrow t$  then the integrals  $\int_s^t f(X_s^{(1)})dX_r + \int_s^t Df(X_s^{(1)})X_{s,r}^{(1)}dX_r$  fully determines the integral  $\int_s^t f(X_r^{(1)})dX_r^{(1)}$  in the limit as  $s \rightarrow t$ . Actually, we may calculate these integrals more explicit, as we may move out parts of the integrands as follows,

$$\int_s^t f(X_s^{(1)})dX_r + \int_s^t Df(X_s^{(1)})X_{s,r}^{(1)}dX_r = f(X_s^{(1)}) \int_s^t dX_r^{(1)} + Df(X_s^{(1)}) \int_s^t X_{s,r}^{(1)}dX_r^{(1)}.$$

We have already defined the two integrals on the right hand side of the equation above as follows,  $\int_s^t dX_r^{(1)} := X_{s,t}^{(1)}$  and  $\int_s^t X_{s,r}^{(1)}dX_r^{(1)} := X_{s,t}^{(2)}$ . We are therefore left with the approximation

$$\int_s^t f(X_r^{(1)})dX_r^{(1)} \approx f(X_s^{(1)})X_{s,t}^{(1)} + Df(X_s^{(1)})X_{s,t}^{(2)},$$

under the assumption that  $\int_s^t R_{s,r}^{f(X^{(1)})}dX_r \rightarrow 0$  as  $s \rightarrow t$ . Going back to the sewing lemma, if we choose  $\psi_{s,t} := f(X_s^{(1)})X_{s,t}^{(1)} + Df(X_s^{(1)})X_{s,t}^{(2)}$ , we can obtain that

$$\begin{aligned} \delta\psi_{s,u,t} &= f(X_s^{(1)})X_{s,t}^{(1)} + Df(X_s^{(1)})X_{s,t}^{(2)} - f(X_s^{(1)})X_{s,u}^{(1)} - Df(X_s^{(1)})X_{s,u}^{(2)} - f(X_u^{(1)})X_{u,t}^{(1)} - Df(X_u^{(1)})X_{u,t}^{(2)} \\ &= -f(X^{(1)})_{s,u}X_{u,t}^{(1)} + Df(X_s^{(1)}) \left( X_{u,t}^{(2)} + X_{s,u}^{(1)} \otimes X_{u,t}^{(1)} \right) - Df(X_u^{(1)})X_{u,t}^{(2)} \\ &= -Df(X^{(1)})_{s,u}X_{u,t}^{(2)} - R_{s,u}^{f(X^{(1)})}X_{u,t}^{(1)} \end{aligned}$$

and we see that  $|\delta\psi_{s,u,t}| \lesssim |t - s|^{3\alpha}$ . Therefore, by the sewing lemma we have that the integral given by,

$$\int_s^t f(X_r)d\mathbf{X}_r := \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} \psi_{u,v},$$

exists, and is well defined. To accommodate lower regularities, it seems necessary to do higher order Taylor approximations to get the sufficient regularity on the remainder term  $R^{f(X)}$ . It therefore seem natural to construct the space of integrable processes as functions  $Y : [0, T] \rightarrow \mathbb{R}^m$  such

that the remainder term of a ‘‘Taylor-type’’ expansion is of a certain regularity, depending on the regularity of the rough path. The next definition will propose such a space when  $\alpha \in (\frac{1}{4}, \frac{1}{3}]$ .

### 3.3 The space of controlled rough paths

**Definition 3.4.** Let  $\mathbf{X} \in \mathcal{C}^\alpha([0, T], V)$ , we say that  $Y \in \mathcal{C}^\alpha([0, T], W)$  is controlled by  $\mathbf{X}$  if there exists two functions  $Y' : [0, T] \rightarrow \mathcal{L}(V, W)$  and  $Y'' : [0, T] \rightarrow \mathcal{L}(V, \mathcal{L}(V, W))$  such that the remainder terms  $R^{Y^{(1)}}$ ,  $R^{Y^{(2)}}$ , and  $R^{Y^{(3)}}$ , given implicitly by the relations

$$\begin{aligned} Y_{s,t} &= Y'_s X_{s,t}^{(1)} + Y''_s X_{s,t}^{(2)} + R_{s,t}^{Y^{(3)}} \\ Y'_{s,t} &= Y''_s X_{s,t}^{(1)} + R_{s,t}^{Y^{(2)}} \\ Y''_{s,t} &= R_{s,t}^{Y^{(1)}}, \end{aligned}$$

satisfies  $\|R^{Y^{(3)}}\|_{3\alpha} < \infty$ ,  $\|R^{Y^{(2)}}\|_{2\alpha} < \infty$  and  $\|R^{Y^{(1)}}\|_\alpha < \infty$ . This defines the space of controlled rough paths, which we write formally as

$$\mathcal{D}_X^\alpha := \left\{ (Y, Y', Y'') \in \mathcal{C}^\alpha([0, T], W) : \sum_{i=1}^3 \|R^{Y^{(i)}}\|_{i\alpha} < \infty \right\}.$$

We equip this space with a semi-norm

$$\|(Y, Y', Y'')\|_{\mathcal{D}_X^\alpha} := \|R^{Y^{(1)}}\|_\alpha + \|R^{Y^{(2)}}\|_{2\alpha} + \|R^{Y^{(3)}}\|_{3\alpha}.$$

By including the initial values of  $Y$  and its derivatives, we get the norm

$$(Y, Y', Y'') \rightarrow |Y_0| + |Y'_0| + |Y''_0| + \|(Y, Y', Y'')\|_{\mathcal{D}_X^\alpha}.$$

Under this norm the space  $\mathcal{D}_X^\alpha$  becomes a regular Banach space.

*Remark 3.5.* As a consequence of how we have defined our space, if  $(Y, Y', Y'') \in \mathcal{D}_X^\alpha$  we obtain the following bounds

$$\begin{aligned} \|Y''\|_\alpha &= \|R^{Y^{(1)}}\|_\alpha \\ \|Y'\|_\alpha &\lesssim \|R^{Y^{(1)}}\|_\alpha + \|R^{Y^{(2)}}\|_{2\alpha} \\ \|Y\|_\alpha &\lesssim \|R^{Y^{(1)}}\|_\alpha + \|R^{Y^{(2)}}\|_{2\alpha} + \|R^{Y^{(3)}}\|_{3\alpha}. \end{aligned}$$

Next, we will present some useful estimates.

**Proposition 3.6.** Let  $(Y, Y', Y'') \in \mathcal{D}_X^\alpha$  for some fixed path  $\mathbf{X} \in \mathcal{C}^\alpha([0, T], V)$ , then we have the following three estimate

$$\|Y\|_\alpha \leq |Y'|_\infty \|X^{(1)}\|_\alpha + |Y''|_\infty \|X^{(2)}\|_\alpha + \|R^{Y^{(3)}}\|_{3\alpha} T^{2\alpha}$$

$$\leq C_{T,\alpha} \left( 1 + \|X^{(1)}\|_\alpha + \|X^{(2)}\|_{2\alpha} \right) \left( |Y'_0| + |Y''_0| + \|(Y, Y', Y'')\|_{\mathcal{D}_X^\alpha} \right),$$

$$(2) \quad \|Y'\|_\alpha \leq D_{T,\alpha} \left( 1 + \|X^{(1)}\|_\alpha \right) \left( |Y'_0| + \|(Y, Y', Y'')\|_{\mathcal{D}_X^\alpha} \right)$$

and

$$(3) \quad \|Y\|_\alpha + \|Y'\|_\alpha \leq K_{T,\alpha} \left( 1 + \|X^{(1)}\|_\alpha + \|X^{(2)}\|_{2\alpha} \right) \left( |Y'_0| + |Y''_0| + \|(Y, Y', Y'')\|_{\mathcal{D}_X^\alpha} \right).$$

Where  $C_{T,\alpha}$  is a constant depending only on  $T$  and  $\alpha$  and can be chosen uniformly over  $T$  for  $T \in (0, 1]$ .

*Proof.* The results follows directly by considering that  $Y_{s,t} = Y'_s X_{s,t}^{(1)} + Y''_s X_{s,t}^{(2)} + R_{s,t}^{Y(3)}$ , and  $|Y'|_\infty \leq |Y'_0| + \|Y'\|_\alpha T^\alpha$ , where the same holds for the second derivative of  $Y$ .  $\square$

Knowing this, it is easy to also show that for some suitable constant  $D$  depending on  $T$  and  $\alpha$ . In the next theorem we will use the sewing lemma and the space of controlled paths defined above to show how we can construct the a rough integral when  $\alpha \in (\frac{1}{4}, \frac{1}{3}]$ .

**Theorem 3.7.** (*Lyon's Theorem*) Let  $\mathbf{X} = (1, X^{(1)}, X^{(2)}, X^{(3)}) \in \mathcal{C}^\alpha([0, T], \mathbb{R}^d)$  for  $T > 0$  and  $\alpha \in (\frac{1}{4}, \frac{1}{3}]$ , and let  $(Y, Y', Y'') \in \mathcal{D}_X^\alpha$ . Then the rough integral defined by

$$\int_s^t Y_r d\mathbf{X}_r = \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} \left( Y_u X_{u,v}^{(1)} + Y'_u X_{u,v}^{(2)} + Y''_u X_{u,v}^{(3)} \right)$$

exist and has the bound

$$\left| \int_s^t Y_r d\mathbf{X}_r - Y_s X_{s,t}^{(1)} - Y'_s X_{s,t}^{(2)} - Y''_s X_{s,t}^{(3)} \right| \lesssim \left( \sum_{i=1}^3 \|X^{(4-i)}\|_{(4-i)\alpha} \|R^{Y(i)}\|_{i\alpha} \right) |t - s|^{4\alpha} \quad (3.2)$$

*Proof.* We want to find a function  $\psi$  such that  $\|\delta\psi\|_\beta < \infty$  for some  $\beta > 1$ . If we define the function  $\psi$  such that

$$\psi_{s,t} = Y_s X_{s,t}^{(1)} + Y'_s X_{s,t}^{(2)} + Y''_s X_{s,t}^{(3)},$$

then

$$\begin{aligned} \delta\psi_{s,u,t} &= - \left( Y_{s,u} - Y'_s X_{s,u}^{(1)} - Y''_s X_{s,u}^{(2)} \right) X_{u,t}^{(1)} - \left( Y'_{s,u} - Y''_s X_{s,u}^{(1)} \right) X_{u,t}^{(2)} - Y'''_{s,u} X_{u,t}^{(3)} \\ &= - \sum_{i=1}^3 R_{s,u}^{Y(i)} X_{u,t}^{(4-i)}. \end{aligned}$$



Now, by the sewing lemma, we see that

$$\left| \int_s^t Y_r d\mathbf{X}_r - Y_s X_{s,t}^{(1)} - Y'_s X_{s,t}^{(2)} - Y''_s X_{s,t}^{(3)} \right| \lesssim \left( \sum_{i=1}^3 \|X^{(4-i)}\|_{(4-i)\alpha} \|R^{Y^{(i)}}\|_{i\alpha} \right) |t-s|^{4\alpha},$$

which imply that the rough integral exists. □

*Remark 3.8.* The simplest way of looking at the rough integral is when we look at

$$\int_0^\cdot F(X_s) d\mathbf{X}_s$$

when  $F$  is sufficiently regular. We will give a short argument for how to construct this integral. Let  $F : V \rightarrow \mathcal{L}(V, W)$  be a  $\mathcal{C}_b^3$  function and let  $\mathbf{X} \in \mathcal{C}_g^\alpha$ ,  $\alpha \in (\frac{1}{4}, \frac{1}{2}]$ . Set  $Y_s = F(X_s)$ ,  $Y'_s = DF(X_s)$  and  $Y''_s = D^2F(X_s)$ . The remainder functions are given in the usual manner,  $R_{s,t}^{Y^{(3)}} = Y_{s,t} - Y'_s X_{s,t} - Y''_s X_{s,t}^{(1)}$ ,  $R_{s,t}^{Y^{(2)}} = Y'_s X_{s,t} - Y''_s X_{s,t}^{(1)}$ , and  $R_{s,t}^{Y^{(1)}} = Y''_s X_{s,t}^{(1)}$ . Then we have that

$$\begin{aligned} \|R^{Y^{(1)}}\|_\alpha &\leq \|D^3F\|_\infty \|X\|_\alpha \\ \|R^{Y^{(2)}}\|_{2\alpha} &\leq \frac{1}{2} \|D^3F\|_\infty \|X\|_\alpha^2 \\ \|R^{Y^{(3)}}\|_{3\alpha} &\leq \frac{1}{6} \|D^3F\|_\infty \|X\|_\alpha^3. \end{aligned}$$

Indeed, We know that  $F$  is three times continuously differentiable and bounded, and hence Lipschitz. Therefore, the Three first inequalities can be found the same way; by looking at the Lagrange remainder of a Taylor expansion. To find the bounds for  $R_{s,t}^{Y^{(3)}}$  we look at a Taylor expansion with respect to the Lagrange remainder term,

$$\begin{aligned} R_{s,t}^{Y^{(3)}} &= Y_{s,t} - Y'_s X_{s,t}^{(1)} - Y''_s X_{s,t}^{(2)} = Y_{s,t} - Y'_s X_{s,t}^{(1)} - \frac{1}{2} Y''' X_{s,t}^{(1)} \otimes X_{s,t}^{(1)} \\ &= \frac{1}{6} D^3F \left( X_s^{(1)} + \xi X_{s,t}^{(1)} \right) X_{s,t}^{(1)\otimes 3}, \end{aligned}$$

where  $\xi \in (0, 1)$ , and by the regularity of  $F$  we know that  $D^2F$  is a symmetric operator and since  $\text{Sym}(X_{s,t}^{(2)}) = \frac{1}{2} X_{s,t}^{(1)} \otimes X_{s,t}^{(1)}$  the above holds. Now, the  $\alpha$ -Hölder estimate follow directly, and we see that  $|R_{s,t}^{Y^{(3)}}| \lesssim |t-s|^{3\alpha}$ . The same argument goes for  $R^{Y^{(2)}}$  and  $R^{Y^{(1)}}$ . And hence, by theorem 3.7 the integral  $\int_0^\cdot F(X_s) d\mathbf{X}_s$  is well defined. □

*Remark 3.9.* We want to point out that the bounds found in theorem 3.7 implies that there exist

a continuous map from  $\mathcal{D}_X^\alpha([0, T], \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n))$  into  $\mathcal{D}_X^\alpha([0, T], \mathbb{R}^n)$  such that

$$(Y, Y', Y'') \in \mathcal{D}_X^\alpha([0, T], \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n)) \mapsto \left( \int_0^\cdot Y_r d\mathbf{X}_r, Y, Y' \right) \in \mathcal{D}_X^\alpha([0, T], \mathbb{R}^n).$$

Indeed, if we define  $(Z, Z', Z'') = \left( \int_0^\cdot Y_r d\mathbf{X}_r, Y, Y' \right)$ , and then define the remainder functions  $R^{Z(i)}$  for  $i = 1, 2, 3$  similar as before,

$$\begin{aligned} R_{s,t}^{Z(3)} &= \int_s^t Y_r d\mathbf{X}_r - Y_s X_{s,t}^{(1)} - Y'_s X_{s,t}^{(2)} \\ R_{s,t}^{Z(2)} &= Y_{s,t} - Y'_s X_{s,t}^{(1)} \\ R_{s,t}^{Z(1)} &= Y'_{s,t}. \end{aligned}$$

Then from eq. 3.2 we know that

$$\begin{aligned} \left| R_{s,t}^{Z(3)} \right| &= \left| \int_s^t Y_r d\mathbf{X}_r - Y_s X_{s,t}^{(1)} - Y'_s X_{s,t}^{(2)} \right| \\ &\lesssim |Y''|_\infty \|X^{(3)}\|_{3\alpha} |t-s|^{3\alpha} + \left( \sum_{i=1}^3 \|X^{(4-i)}\|_{(4-i)\alpha} \|R^{Y(i)}\|_{i\alpha} \right) |t-s|^{4\alpha}. \end{aligned}$$

Further, we know  $\|R^{Z(2)}\|_{2\alpha} = \|Y - Y'X^{(1)}\|_{2\alpha} \leq |Y''|_\infty \|X^{(2)}\|_{2\alpha} + \|R^{Y(3)}\|_{3\alpha} < \infty$  and  $\|R^{Z(1)}\|_\alpha = \|Y\|_\alpha \leq |Y''|_\infty \|X^{(1)}\|_\alpha + \|R^{Y(2)}\|_{2\alpha} < \infty$  due to the fact that  $(Y, Y', Y'') \in \mathcal{D}_X^\alpha([0, T], \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n))$ , and hence,  $\left( \int_0^\cdot Y_r d\mathbf{X}_r, Y, Y' \right) \in \mathcal{D}_X^\alpha([0, T], \mathbb{R}^n)$ . That the mapping is continuous will become evident in the next subsection.

The last topic of this section will be on the stability of rough integrals, and we will give some essential bounds which we will use in later sections.

### 3.4 Stability of Rough integration

We will investigate the difference of two controlled paths  $(Y, Y', Y'') \in \mathcal{D}_X^\alpha$  and  $(\tilde{Y}, \tilde{Y}', \tilde{Y}'') \in \mathcal{D}_X^\alpha$ . This could look a little strange, as the spaces  $\mathcal{D}_X^\alpha$  and  $\mathcal{D}_{\tilde{X}}^\alpha$  are two different spaces (because the elements of the spaces are controlled by two different paths). Still, if we are able to define a type of “distance” function between the elements of these spaces, we could use this in, for example approximation. This type of approximation will be elaborated on in part III where we look at financial applications of the rough path theory. We will start this section with a definition of a distance function.

**Definition 3.10.** Let  $(Y, Y', Y'') \in \mathcal{D}_X^\alpha$ , and  $(\tilde{Y}, \tilde{Y}', \tilde{Y}'') \in \mathcal{D}_{\tilde{X}}^\alpha$ , then we define a distance function

between the paths as,

$$d_{X,\tilde{X},\alpha} \left( Y, Y', Y''; \tilde{Y}, \tilde{Y}', \tilde{Y}'' \right) := \sum_{i=1}^3 \| R^{Y^{(i)}} - R^{\tilde{Y}^{(i)}} \|_{i\alpha}.$$

We will follow up with a lemma giving an estimate on the difference between two paths.

**Lemma 3.11.** *Let  $\mathbf{X}, \tilde{\mathbf{X}} \in \mathcal{C}^\alpha$  and  $(Y, Y', Y'') \in \mathcal{D}_X^\alpha, (\tilde{Y}, \tilde{Y}', \tilde{Y}'') \in \mathcal{D}_{\tilde{X}}^\alpha$  with the property that*

$$\begin{aligned} |Y_0| + |Y'_0| + |Y''_0| + \|Y, Y', Y''\|_{\mathcal{D}_X^\alpha} &\leq M \in \mathbb{R} \quad \text{and} \\ d_\alpha(0, \mathbf{X}) := \|X^{(1)}\|_\alpha + \|X^{(2)}\|_{2\alpha} + \|X^{(3)}\|_{3\alpha} &\leq M \end{aligned}$$

*With the same bounds for  $(\tilde{Y}, \tilde{Y}', \tilde{Y}'')$  and  $\tilde{\mathbf{X}}$ . As described in remark 3.9, we can define*

$$(Z, Z', Z'') = \left( \int_0^\cdot Y_s d\mathbf{X}_s, Y, Y' \right),$$

*and in the same way we define  $(\tilde{Z}, \tilde{Z}', \tilde{Z}'')$ . Then we have the following local Lipschitz estimates,*

$$d_{X,\tilde{X},\alpha} \left( Z, Z', Z''; \tilde{Z}, \tilde{Z}', \tilde{Z}'' \right) \leq C_M (d_\alpha(\mathbf{X}, \tilde{\mathbf{X}}) + |Y''_0 - \tilde{Y}''_0| + d_{X,\tilde{X},\alpha} (Y, Y', Y''; \tilde{Y}, \tilde{Y}', \tilde{Y}'') T^\alpha).$$

Where  $C_M := C(T, \alpha, M)$  is uniform in  $T \leq 1$ .

*Proof.* We will use the simple inequality  $|xy - \tilde{x}\tilde{y}| \leq |x| |y - \tilde{y}| + |x - \tilde{x}| |\tilde{y}|$  (\*) extensively through the proof. We look at the difference

$$R_{s,t}^{Z^{(3)}} - R_{s,t}^{\tilde{Z}^{(3)}} = \int_s^t Y_r d\mathbf{X}_r - Y_s X_{s,t}^{(1)} - Y'_s X_{s,t}^{(2)} - \left( \int_s^t \tilde{Y}_r d\tilde{\mathbf{X}}_r - \tilde{Y}_s \tilde{X}_{s,t}^{(1)} - \tilde{Y}'_s \tilde{X}_{s,t}^{(2)} \right)$$

and define  $\psi_{s,t} = Y_s X_{s,t}^{(1)} + Y'_s X_{s,t}^{(2)} + Y''_s X_{s,t}^{(3)}$  and  $\triangle_{s,t} := \psi_{s,t} - \tilde{\psi}_{s,t}$ . We find

$$\begin{aligned} |R_{s,t}^{Z^{(3)}} - R_{s,t}^{\tilde{Z}^{(3)}}| &= |(I\triangle)_{s,t} - \triangle_{s,t} + Y''_s X_{s,t}^{(3)} - \tilde{Y}''_s \tilde{X}_{s,t}^{(3)}| \\ &\leq C \|\delta\triangle\|_{4\alpha} |t-s|^{4\alpha} + |Y''_s X_{s,t}^{(3)} - \tilde{Y}''_s \tilde{X}_{s,t}^{(3)}|, \end{aligned}$$

where

$$\delta\triangle_{s,u,t} = - \sum_{i=1}^3 R_{s,u}^{Y^{(i)}} X_{u,t}^{(4-i)} + \sum_{i=1}^3 R_{s,u}^{\tilde{Y}^{(i)}} \tilde{X}_{u,t}^{(4-i)}.$$

Using the inequality (\*), we obtain the following estimate

$$|Y''_s X_{s,t}^{(3)} - \tilde{Y}''_s \tilde{X}_{s,t}^{(3)}| \leq |Y''_s| |X_{s,t}^{(3)} - \tilde{X}_{s,t}^{(3)}| + |Y''_s - \tilde{Y}''_s| |\tilde{X}_{s,t}^{(3)}|.$$

This implies that

$$\begin{aligned} & \| R^{Z^{(3)}} - R^{\tilde{Z}^{(3)}} \|_{3\alpha} \leq C \| \delta \Delta \|_{4\alpha} T^\alpha + C_M \| X^{(3)} - \tilde{X}^{(3)} \|_{3\alpha} \\ & + C_M \left( |Y_0'' - \tilde{Y}_0''| + T^\alpha \| R^{Y^{(1)}} - R^{\tilde{Y}^{(1)}} \|_\alpha \right). \end{aligned}$$

Similarly we find that

$$\begin{aligned} & \| R^{Z^{(2)}} - R^{\tilde{Z}^{(2)}} \|_{2\alpha} \leq \| R^{Y^{(3)}} - R^{\tilde{Y}^{(3)}} \|_{3\alpha} T^\alpha + C_M \| X^{(2)} - \tilde{X}^{(2)} \|_{2\alpha} \\ & + C_M \left( |Y_0'' - \tilde{Y}_0''| + T^\alpha \| R^{Y^{(1)}} - R^{\tilde{Y}^{(1)}} \|_\alpha \right). \end{aligned}$$

At last, we look at  $R^{Z^{(1)}} - R^{\tilde{Z}^{(1)}}$  and find,

$$\begin{aligned} & \| R^{Z^{(1)}} - R^{\tilde{Z}^{(1)}} \|_\alpha = \| Y' - \tilde{Y}' \|_\alpha = \| Y'' X^{(1)} - \tilde{Y}'' \tilde{X}^{(1)} + R^{Y^{(2)}} - R^{\tilde{Y}^{(2)}} \|_\alpha \\ & \leq C_M \| X^{(2)} - \tilde{X}^{(2)} \|_\alpha + C_M \left( |Y_0'' - \tilde{Y}_0''| + T^\alpha \| R^{Y^{(1)}} - R^{\tilde{Y}^{(1)}} \|_\alpha \right) + \| R^{Y^{(2)}} - R^{\tilde{Y}^{(2)}} \|_\alpha T^\alpha. \end{aligned}$$

Notice that all the terms  $\| R^{Y^{(i)}} - R^{\tilde{Y}^{(i)}} \|_{i\alpha}$  for  $i = 1, 2, 3$  is multiplied by  $T^\alpha$ . Collecting these terms, we obtain the following estimate,

$$\begin{aligned} & d_{X, \tilde{X}, \alpha} \left( Z, Z', Z''; \tilde{Z}, \tilde{Z}', \tilde{Z}' \right) \\ & \leq C_M \left( d_\alpha \left( \mathbf{X}, \tilde{\mathbf{X}} \right) + |Y_0'' - \tilde{Y}_0''| + d_{X, \tilde{X}, \alpha} \left( Y, Y', Y''; \tilde{Y}, \tilde{Y}', \tilde{Y}'' \right) T^\alpha \right), \end{aligned}$$

and we are done. □

## 4 Composition of controlled rough paths with regular functions.

An important result for proving existence and uniqueness of rough differential equations is the composition of regular functions with controlled rough paths, we will in this section give some important estimates, and look at stability of regular functions composed with controlled rough paths.

### 4.1 Composition with regular functions

Let  $f \in \mathcal{C}_b^3(\mathbb{R}^m, \mathbb{R}^n)$ . We want to show that if  $(Y, Y', Y'') \in \mathcal{D}_X^\alpha([0, T], \mathbb{R}^m)$  such that  $Y : [0, T] \rightarrow \mathbb{R}^m$ ,  $Y' : [0, T] \rightarrow \mathcal{L}(\mathbb{R}^d, \mathbb{R}^m)$  and  $Y'' : [0, T] \rightarrow \mathcal{L}(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^m)) \simeq \mathcal{L}(\mathbb{R}^{d \times d}, \mathbb{R}^m)$  for some  $\mathbf{X} \in \mathcal{C}_g^\alpha(\mathbb{R}^d)$ , then  $(f(Y), f(Y)', f(Y'')) \in \mathcal{D}_X^\alpha$ . We therefore need to find suitable derivatives of  $f$  to obtain the regularity needed on the remainder terms  $R^{f(Y)^{(i)}}$ ,  $i = 1, 2, 3$ . That is, we are interested in finding derivatives  $f' : \mathbb{R}^m \rightarrow \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n)$  and  $f'' : \mathbb{R}^m \rightarrow \mathcal{L}(\mathbb{R}^{d \times d}, \mathbb{R}^n)$  such that the remainder term  $R_{s,t}^{f(Y)^{(3)}}$  given by the relation

$$f(Y)_{s,t} = f(Y_s)' X_{s,t}^{(1)} + f(Y_s)'' X_{s,t}^{(2)} + R_{s,t}^{f(Y)^{(3)}},$$

is finite in the  $3\alpha$  norm, and similar for the two other remainder terms. If we do a Taylor approximation of  $f(Y_t)$ , we find

$$f(Y_t) - f(Y_s) = Df(Y_s)Y_{s,t} + \frac{1}{2}D^2f(Y_s)Y_{s,t}^{\otimes 2} + \bar{R}_{s,t}^f.$$

We know by the Lagrange remainder term theorem and  $f \in \mathcal{C}_b^3$ , that  $\|\bar{R}_{s,t}^f\|_{3\alpha} < \infty$ . Further, we know that  $Y_{s,t} = Y_s' X_{s,t}^{(1)} + Y_s'' X_{s,t}^{(2)} + R_{s,t}^{Y^{(3)}}$ , and we can insert this in the above equation, and find

$$\begin{aligned} f(Y_t) - f(Y_s) &= Df(Y_s) \left( Y_s' X_{s,t}^{(1)} + Y_s'' X_{s,t}^{(2)} + R_{s,t}^{Y^{(3)}} \right) \\ &\quad + \frac{1}{2}D^2f(Y_s) \left( Y_s' X_{s,t}^{(1)} + Y_s'' X_{s,t}^{(2)} + R_{s,t}^{Y^{(3)}} \right)^{\otimes 2} + \bar{R}_{s,t}^f. \end{aligned}$$

From that equation, we can calculate that

$$\begin{aligned} f(Y_t) - f(Y_s) &= Df(Y_s)Y_s' X_{s,t}^{(1)} + Df(Y_s)Y_s'' X_{s,t}^{(2)} \\ &\quad + \frac{1}{2}D^2f(Y_s) \left( Y_s' X_{s,t}^{(1)} \right)^{\otimes 2} + R_{s,t}^{Y^{(3)}} + \bar{R}_{s,t}^f + o(|t-s|). \end{aligned}$$

As we know  $\left| R_{s,t}^{Y^{(3)}} + \bar{R}_{s,t}^f + o(|t-s|) \right| \lesssim |t-s|^{3\alpha}$ , we want to construct the derivatives of  $f$  from the remaining terms when we remove  $R_{s,t}^{Y^{(3)}} + \bar{R}_{s,t}^f + o(|t-s|)$ . It seems natural to choose  $f(Y_s)' = Df(Y_s)Y_s'$ . Then we are left with the terms

$$Df(Y_s)Y_s'' X_{s,t}^{(2)} + \frac{1}{2}D^2f(Y_s) \left( Y_s' X_{s,t}^{(1)} \right)^{\otimes 2}.$$

Notice here that  $\left(Y'_s X_{s,t}^{(1)}\right)^{\otimes 2} \in \mathbb{R}^{m \times m}$ , and  $D^2 f \in \mathcal{L}(\mathbb{R}^{m \times m}, \mathbb{R}^n)$  and symmetric. Therefore if  $\text{sym}(Y'_s X_{s,t}^{(2)} Y'_s) = \frac{1}{2} \left(Y'_s X_{s,t}^{(1)}\right)^{\otimes 2}$ , we would have a suitable candidate for the second derivative, namely  $f(Y)'' = Df(Y)Y'' + D^2 f(Y)(Y')^{\otimes 2}$  where for any  $x \in \mathbb{R}^{d \times d}$ ,  $x \mapsto Df(Y)Y''x + D^2 f(Y)Y'xY'$ , and  $f(Y)'' : \mathbb{R}^m \rightarrow \mathcal{L}(\mathbb{R}^{d \times d}, \mathbb{R}^n)$ . It is not difficult to show that  $\text{sym}(Y'_s X_{s,t}^{(2)} Y'_s) = \frac{1}{2} \left(Y'_s X_{s,t}^{(1)}\right)^{\otimes 2}$ . Indeed, by the fact that  $\mathbf{X}$  is geometric, we can check that

$$\begin{aligned} \text{sym}(Y'_s X_{s,t}^{(2)} Y'_s) &= \frac{1}{2} \left( Y'_s X_{s,t}^{(2)} Y'_s + \left( Y'_s X_{s,t}^{(2)} Y'_s \right)^T \right) = Y'_s \text{Sym}(X_{s,t}^{(2)}) Y'_s \\ &= Y'_s \frac{1}{2} X_{s,t}^{(1)} \otimes X_{s,t}^{(1)} Y'_s = \frac{1}{2} \left( Y'_s X_{s,t}^{(1)} \right)^{\otimes 2}. \end{aligned}$$

Therefore, the candidates  $f(Y)' = Df(Y)Y'$  and  $f(Y)'' = Df(Y)Y'' + D^2 f(Y)(Y')^{\otimes 2}$  seems to be suitable. We get that  $R^{f(Y)^{(3)}}$  is given implicitly by the relation

$$f(Y)_{s,t} = Df(Y_s)Y'_s X_{s,t}^{(1)} + (D^2 f(Y_s)(Y'_s)^{\otimes 2} + Df(Y_s)Y''_s)X_{s,t}^{(2)} + R_{s,t}^{f(Y)^{(3)}},$$

as this choice satisfy  $\left\| R^{f(Y)^{(3)}} \right\|_{3\alpha} < \infty$ . This follows from the fact that  $R^{f(Y)^{(3)}} = f(Y)_{s,t} - f(Y)'_s X_{s,t}^{(1)} - f(Y)''_s X_{s,t}^{(2)}$ , therefore by inserting the derivatives we have chosen above in this relation, and the calculations we have done, we find that

$$R^{f(Y)^{(3)}} = R_{s,t}^{Y^{(3)}} + \bar{R}_{s,t}^f + o(|t-s|)$$

Then we use the Lagrange remainder term formula to find  $\left| \bar{R}_{s,t}^f \right| \leq C |Y_{s,t}|^3$ .

$$\left\| R^{f(Y)^{(3)}} \right\|_{3\alpha} \leq C \left( \left\| Y \right\|_{\alpha}^3 + \left\| R^{Y^{(3)}} \right\|_{3\alpha} \right),$$

where  $C$  depends on  $\|f\|_{C_b^3}$ ,  $T$ , and  $\alpha$  from calculations above.

Now we need to check that the terms  $R^{f(Y)^{(1)}}$  and  $R^{f(Y)^{(2)}}$  are bounded in the  $\alpha$  and  $2\alpha$  norm respectively, under the choice of  $f(Y)'$  and  $f(Y)''$ . Let us start to look at  $R^{f(Y)^{(2)}}$ ,

$$R_{s,t}^{f(Y)^{(2)}} = (Df(Y)Y')_{s,t} - \left( Df(Y_s)Y''_s + D^2 f(Y_s)(Y'_s)^{\otimes 2} \right) X_{s,t}^{(1)}.$$

Through some calculation, we can find,

$$R_{s,t}^{f(Y)^{(2)}} = Df(Y_t)Y'_s + Df(Y)_{s,t}Y'_s - (D^2 f(Y_s)(Y'_s)^{\otimes 2} + Df(Y_s)Y''_s) X_{s,t}^{(1)}.$$

Using the fact that  $Y'_{s,t} = Y''_s X_{s,t}^{(1)} + R_{s,t}^{Y^{(2)}}$ , we get the following expression,

$$R_{s,t}^{f(Y)^{(2)}} = Df(Y)_{s,t}Y''_s X_{s,t}^{(1)} + (Df(Y)_{s,t} - D^2 f(Y_s)Y'_s X_{s,t}^{(1)})Y'_s + Df(Y_s)R_{s,t}^{Y^{(2)}}.$$

Where we have used the fact that the second derivative of  $f$  we constructed is such that  $X_{s,t}^{(1)} \mapsto Df(Y_s)Y_s''X_{s,t}^{(1)} + D^2f(Y_s)Y_s'X_{s,t}^{(1)}Y_s'$ . If we now do a first order Taylor approximation of  $Df(Y)$ , i.e  $Df(Y)_{s,t} = D^2f(Y_s)Y_{s,t} + \bar{R}_{s,t}^{Df}$ . Then we know by the Lagrange reminder thm. that  $\|\bar{R}_{s,t}^{Df(Y)}\|_{2\alpha} < \infty$  since  $|Y_{s,t}| \lesssim |t-s|^\alpha$ . Substituting this above, we find

$$\begin{aligned} R_{s,t}^{f(Y)^{(2)}} &= Df(Y)_{s,t}Y_s''X_{s,t}^{(1)} + D^2f(Y)_s(Y_{s,t} - Y_s'X_{s,t}^{(1)})Y_s' + Df(Y_s)R_{s,t}^{Y^{(2)}} + \bar{R}_{s,t}^{Df}Y_s' \\ &= Df(Y)_{s,t}Y_s''X_{s,t}^{(1)} + D^2f(Y)_s(Y_s''X_{s,t}^{(2)} + R_{s,t}^{Y^{(3)}} + \bar{R}_{s,t}^{Df})Y_s' + Df(Y_s)R_{s,t}^{Y^{(2)}}. \end{aligned}$$

Now, all these terms are bounded in the  $2\alpha$  norm. That is, We can see that

$$\left| Df(Y)_{s,t}Y_s''X_{s,t}^{(1)} \right| \leq |Df|_\infty |Y''|_\infty |Y_{s,t}| \left| X_{s,t}^{(1)} \right| \lesssim |t-s|^{2\alpha},$$

and

$$\left| D^2f(Y)_s(Y_s''X_{s,t}^{(2)} + R_{s,t}^{Y^{(3)}})Y_s' \right| \leq |D^2f|_\infty |Y'|_\infty \left( |Y''|_\infty \left| X_{s,t}^{(2)} \right| + \left| R_{s,t}^{Y^{(3)}} \right| + \left| \bar{R}_{s,t}^{Df} \right| \right) \lesssim |t-s|^{2\alpha},$$

and lastly, of course  $\left| Df(Y_s)R_{s,t}^{Y^{(2)}} \right| \leq |Df(Y)|_\infty \left| R_{s,t}^{Y^{(2)}} \right| \lesssim |t-s|^{2\alpha}$ . Therefore, if we choose the first and second derivative of  $f$  to be such that

$$f(Y)_t = f(Y_t), \quad f(Y)'_t = Df(Y_t)Y'_t, \quad f(Y)''_t = D^2f(Y_t)(Y'_t)^{\otimes 2} + Df(Y_t)Y''_t,$$

we have that all remainder terms, given implicitly by the usual relations (see definition 3.4) are bounded in their respective  $\alpha$  metric. Next, we will give a theorem giving a suitable bound for the  $\mathcal{D}_X^\alpha$  semi-norm of a regular function composed with a controlled path, which we will later use in showing existence of RDE's.

**Theorem 4.1.** *Let  $f \in \mathcal{C}_b^3(\mathbb{R}^m, \mathbb{R}^n)$ ,  $(Y, Y', Y'') \in \mathcal{D}_X^\alpha([0, T], \mathbb{R}^m)$  for some  $\mathbf{X} \in \mathcal{C}_g^\alpha([0, T], \mathbb{R}^d)$ , with  $\alpha \in (\frac{1}{4}, \frac{1}{3}]$ . Then the controlled rough path  $(f(Y), f(Y)', f(Y)'') \in \mathcal{D}_X^\alpha([0, T], \mathbb{R}^n)$  where  $f(Y)'$  and  $f(Y)''$  is given by the derivatives above. Assume that*

$$|Y_0| + |Y'_0| + |Y''_0| + \|Y, Y', Y''\|_{\mathcal{D}_X^\alpha} \leq M \in [1, \infty),$$

*and the same bound for  $d_\alpha(0, \mathbf{X})$ . Then there exists a constant  $C_M$  depending on  $T > 0$ ,  $M$ ,  $\|f\|_{\mathcal{C}_b^3}$ , and  $\alpha$ , such that*

$$\|f(Y), f(Y)', f(Y)''\|_{\mathcal{D}_X^\alpha}$$

$$\leq C_M \left( 1 + \|X^{(1)}\|_\alpha + \|X^{(2)}\|_{2\alpha} \right) (1 + |Y'_0| + |Y''_0| + \|Y, Y', Y''\|_{\mathcal{D}_X^\alpha}).$$

*Proof.* We have established bounds for the three remainder terms above theorem 4.1. We therefore only need to combine them to find a suitable bound. We start to look at  $R^{f(Y)^{(3)}}$ . By adding positive terms, and make use of the bound  $|Y'_0| + |Y''_0| + \|Y, Y', Y''\|_{\mathcal{D}_X^\alpha} \leq M \in [1, \infty)$ , and find that

$$\begin{aligned} \|R^{f(Y)^{(3)}}\|_{3\alpha} &\leq C \left( 1 + \|Y\|_\alpha^3 + \|R^{Y^{(3)}}\|_{3\alpha} \right) \\ &\leq C_M \left( 1 + \|X^{(1)}\|_\alpha + \|X^{(2)}\|_{2\alpha} \right) (1 + |Y'_0| + |Y''_0| + \|Y, Y', Y''\|_{\mathcal{D}_X^\alpha}). \end{aligned}$$

Where we have used that

$$\|Y\|_\alpha \leq \left( 1 + \|X^{(1)}\|_\alpha + \|X^{(2)}\|_{2\alpha} \right) (|Y'_0| + |Y''_0| + \|Y, Y', Y''\|_{\mathcal{D}_X^\alpha}).$$

Next we see that  $R^{f(Y)^{(2)}}$  can be bounded the same way. From the calculations done above the theorem, we can see that,

$$\begin{aligned} \|R^{f(Y)^{(2)}}\|_{2\alpha} &\leq C_M (\|Y\|_\alpha \|Y'\|_\alpha + \|R^{Y^{(2)}}\|_{2\alpha} \\ &+ \left( \|R^{Y^{(3)}}\|_{3\alpha} T^\alpha + \left( |Y''_0| + \|R^{Y^{(1)}}\|_\alpha T^\alpha \right) \|X^{(2)}\|_{2\alpha} + \|\bar{R}^{Df(Y)}\|_{2\alpha} \right) (|Y'_0| + \|Y'\|_\alpha T^\alpha)). \end{aligned}$$

Where  $\bar{R}^{Df(Y)}$  is the remainder term from a first order taylor expansion of  $Df(Y)$ . Then by using the bounds established in proposition 3.6 we see that,

$$\|R^{f(Y)^{(2)}}\|_{2\alpha} \leq C_M \left( 1 + \|X^{(1)}\|_\alpha + \|X^{(2)}\|_{2\alpha} \right) (1 + |Y'_0| + |Y''_0| + \|Y, Y', Y''\|_{\mathcal{D}_X^\alpha}).$$

And at last, using the estimate above for  $R^{f(Y)^{(1)}}$  and proposition 3.6, we find

$$\begin{aligned} \|R^{f(Y)^{(1)}}\|_\alpha &\leq \|f\|_{\mathcal{C}_b^3} \left( \|Y'\|_\alpha (|Y'_0| + \|Y'\|_\alpha T^\alpha) + \|R^{Y^{(1)}}\|_\alpha \right) \\ &\leq C_M \left( 1 + \|X^{(1)}\|_\alpha + \|X^{(2)}\|_{2\alpha} \right) (1 + |Y'_0| + |Y''_0| + \|Y, Y', Y''\|_{\mathcal{D}_X^\alpha}). \end{aligned}$$

Combining the above estimates, we get

$$\begin{aligned} \|f(Y), f(Y)', f(Y)''\|_{\mathcal{D}_X^\alpha} &= \sum_{i=1}^3 \left\| R^{f(Y)^{(i)}} \right\|_{i\alpha} \\ &\leq C_M \left( 1 + \|X^{(1)}\|_\alpha + \|X^{(2)}\|_{2\alpha} \right) (1 + |Y'_0| + |Y''_0| + \|Y, Y', Y''\|_{\mathcal{D}_X^\alpha}), \end{aligned}$$

and we are done.  $\square$



In the next section, we will take a closer look on how the functions composed with controlled paths behave, and show an estimate for the distance between a function composed with two different controlled paths.

## 4.2 Stability of regular functions of controlled paths.

We will present a stability result for regular functions composed with controlled rough paths similar to the one given in lemma 3.11. The idea is the same, namely to be able to say something about how far apart two controlled paths are, when composed with a regular function.

**Lemma 4.2.** *Let  $\mathbf{X}, \tilde{\mathbf{X}} \in \mathcal{C}_g^\alpha([0, T], \mathbb{R}^d)$ , and  $(Y, Y', Y'') \in \mathcal{D}_X^\alpha([0, T], \mathbb{R}^m)$ ,  $(\tilde{Y}, \tilde{Y}', \tilde{Y}'') \in \mathcal{D}_{\tilde{X}}^\alpha([0, T], \mathbb{R}^m)$  respectively. For a function  $f \in \mathcal{C}_b^4(\mathbb{R}^m, \mathbb{R}^n)$ , define*

$$(Z, Z', Z'') := \left( f(Y), Df(Y)Y', D^2f(Y)(Y')^{\otimes 2} + Df(Y)Y'' \right) \in \mathcal{D}_X^\alpha$$

*is such that  $|Y_0| + |Y'_0| + |Y''_0| + \|Y, Y', Y''\|_{\mathcal{D}_X^\alpha} \leq M < \infty$  and  $\delta_\alpha(\mathbf{X}, 0) \leq M$ . The triple  $(\tilde{Z}, \tilde{Z}', \tilde{Z}'') \in \mathcal{D}_{\tilde{X}}^\alpha$  is constructed the same way. Then we have the following local Lipschitz estimates,*

$$\begin{aligned} & d_{X, \tilde{X}, \alpha}(Z, Z', Z''; \tilde{Z}, \tilde{Z}', \tilde{Z}'') \\ & \leq C_M \left( d_\alpha(\mathbf{X}, \tilde{\mathbf{X}}) + |Y_0 - \tilde{Y}_0| + |Y'_0 - \tilde{Y}'_0| + |Y''_0 - \tilde{Y}''_0| + d_{X, \tilde{X}, \alpha}(Y, Y', Y''; \tilde{Y}, \tilde{Y}', \tilde{Y}'') \right) \end{aligned}$$

Where  $C_M := C(T, \alpha, f, M, Y_0)$ .

*Proof.* Following the proof from Friz and Hairer[16] and modifying it to the lower regularity, we need to find a bound for the expression,

$$\begin{aligned} & d_{X, \tilde{X}, \alpha}(Z, Z', Z''; \tilde{Z}, \tilde{Z}', \tilde{Z}'') = \\ & + \|R^{Z^{(1)}} - R^{\tilde{Z}^{(1)}}\|_\alpha + \|R^{Z^{(2)}} - R^{\tilde{Z}^{(2)}}\|_{2\alpha} + \|R^{Z^{(3)}} - R^{\tilde{Z}^{(3)}}\|_{3\alpha}. \end{aligned}$$

To shorten notation, we will define the following variables  $\varepsilon_X = d_\alpha(\mathbf{X}, \tilde{\mathbf{X}})$ ,  $\varepsilon_0 = |Y_0 - \tilde{Y}_0|$ ,  $\varepsilon'_0 = |Y'_0 - \tilde{Y}'_0|$ ,  $\varepsilon''_0 = |Y''_0 - \tilde{Y}''_0|$ , and  $\varepsilon = d_{X, \tilde{X}, \alpha}(Y, Y', Y''; \tilde{Y}, \tilde{Y}', \tilde{Y}'')$  such that

$$d_{X, \tilde{X}, \alpha}(Z, Z', Z''; \tilde{Z}, \tilde{Z}', \tilde{Z}'') \lesssim \varepsilon_X + \varepsilon_0 + \varepsilon'_0 + \varepsilon''_0 + \varepsilon.$$

We will use the inequality  $|ab - \tilde{a}\tilde{b}| \leq |a| |b - \tilde{b}| + |a - \tilde{a}| |\tilde{b}|$  (\*) extensively through out the proof, and refer to as (\*). We will also consider three inequalities, based on the (in)equalities established

in proposition 3.6,

$$\begin{aligned} \|Y'' - \tilde{Y}''\|_\alpha &= \|R^{Y^{(1)}} - R^{\tilde{Y}^{(1)}}\|_\alpha \\ \|Y' - \tilde{Y}'\|_\alpha &\leq C_M \left( d_\alpha(\mathbf{X}, \tilde{\mathbf{X}}) + |Y_0'' - \tilde{Y}_0''| + d_{X, \tilde{X}, \alpha}(Y, Y', Y''; \tilde{Y}, \tilde{Y}', \tilde{Y}'') \right) \\ \|Y - \tilde{Y}\|_\alpha &\leq C_M \left( d_\alpha(\mathbf{X}, \tilde{\mathbf{X}}) + |Y_0' - \tilde{Y}_0'| + |Y_0'' - \tilde{Y}_0''| + d_{X, \tilde{X}, \alpha}(Y, Y', Y''; \tilde{Y}, \tilde{Y}', \tilde{Y}'') \right). \end{aligned}$$

We will start to investigate  $R_{s,t}^{Z^{(1)}} - R_{s,t}^{\tilde{Z}^{(1)}} = f(Y)''_{s,t} - f(\tilde{Y})''_{s,t}$ . We look at

$$\left| R_{s,t}^{Z^{(1)}} - R_{s,t}^{\tilde{Z}^{(1)}} \right| \leq |D^2 f(Y) (Y')_{s,t}^{\otimes 2} - D^2 f(\tilde{Y}) (\tilde{Y}')_{s,t}^{\otimes 2}| + |Df(Y)Y''_{s,t} - Df(\tilde{Y})\tilde{Y}''_{s,t}|.$$

We take a closer look at the first term, and point out that

$$D^2 f(Y) (Y')_{s,t}^{\otimes 2} = D^2 f(Y_t) (Y'_t)^{\otimes 2} - D^2 f(Y_s) (Y'_s)^{\otimes 2},$$

and the same for  $\tilde{Y}$ . Therefore, by adding and subtracting  $D^2 f(Y_t) (Y'_t)^{\otimes 2}$  (and the same for  $\tilde{Y}$ ) and use inequality (\*), we see that

$$\begin{aligned} &|D^2 f(Y) (Y')_{s,t}^{\otimes 2} - D^2 f(\tilde{Y}) (\tilde{Y}')_{s,t}^{\otimes 2}| \leq |D^2 f(Y_t) (Y'_t)^{\otimes 2} - (Y')_{s,t}^{\otimes 2}| \\ &+ |D^2 f(Y_t) - D^2 f(\tilde{Y}_t)| (Y'_t)^{\otimes 2} + |D^2 f(Y)_{s,t} - D^2 f(\tilde{Y})_{s,t}| |(Y'_s)^{\otimes 2}| \\ &+ |(Y'_s)^{\otimes 2} - (\tilde{Y}'_s)^{\otimes 2}| |D^2 f(\tilde{Y})_{s,t}|. \end{aligned}$$

Using the fact that  $|D^2 f(\tilde{Y})_{s,t}| \leq |D^2 f|_\infty |\tilde{Y}_{s,t}|$ , and by adding and subtracting  $Y'_t \otimes Y'_s$  and similar for  $\tilde{Y}'$ , and by using (\*) we find

$$\left\| (Y')^{\otimes 2} - (\tilde{Y}')^{\otimes 2} \right\|_\alpha \leq C_M \left( |Y'_0 - \tilde{Y}'_0| + \|Y' - \tilde{Y}'\|_\alpha \right),$$

and

$$|(Y'_s)^{\otimes 2} - (\tilde{Y}'_s)^{\otimes 2}| \leq |(Y'_0)^{\otimes 2} - (\tilde{Y}'_0)^{\otimes 2}| + C_M \left( |Y'_0 - \tilde{Y}'_0| + \|Y' - \tilde{Y}'\|_\alpha \right),$$

where  $C_M$  depends on  $M$ ,  $T$ , and  $\alpha$ . Combining the expressions above and using the inequalities established in the beginning of the proof, we find,

$$\|D^2 f(Y) (Y')^{\otimes 2} - D^2 f(\tilde{Y}) (\tilde{Y}')^{\otimes 2}\|_\alpha \leq C_M (\varepsilon_X + \varepsilon_0 + \varepsilon'_0 + \varepsilon''_0 + \varepsilon),$$

where  $C_M$  depends on  $T$ ,  $\alpha$ ,  $\|f\|_{C_b^2}$ , and  $\left| (\tilde{Y}')^{\otimes 2} \right|_\infty = \sup_{s \in [0, T]} \left| (\tilde{Y}'_s)^{\otimes 2} \right|$ . Next by using (\*) and

the estimates from the beginning of the proof, we find,

$$\| Df(Y)Y'' - Df(\tilde{Y})\tilde{Y}'' \|_{\alpha} \leq C_M (\varepsilon_X + \varepsilon_0 + \varepsilon'_0 + \varepsilon''_0 + \varepsilon).$$

Combining the two inequalities above, we get that

$$\| R^{Z^{(1)}} - R^{\tilde{Z}^{(1)}} \|_{\alpha} \leq C_M (\varepsilon_X + \varepsilon_0 + \varepsilon'_0 + \varepsilon''_0 + \varepsilon).$$

Further we want to look at the difference

$$\begin{aligned} R_{s,t}^{Z^{(2)}} - R_{s,t}^{\tilde{Z}^{(2)}} &= (Df(Y)Y')_{s,t} - \left( D^2f(Y_s) (Y'_s)^{\otimes 2} + Df(Y_s)Y''_s \right) X_{s,t}^{(1)} \\ &\quad - \left( (Df(\tilde{Y})\tilde{Y}')_{s,t} - \left( D^2f(\tilde{Y}_s) (\tilde{Y}'_s)^{\otimes 2} + Df(\tilde{Y}_s)\tilde{Y}''_s \right) X_{s,t}^{(1)} \right). \end{aligned}$$

We want to show that

$$| R_{s,t}^{Z^{(2)}} - R_{s,t}^{\tilde{Z}^{(2)}} | \leq | P_1 | + | P_2 | + | P_3 | \lesssim (\varepsilon_X + \varepsilon_0 + \varepsilon'_0 + \varepsilon''_0 + \varepsilon) |t - s|^{2\alpha}.$$

First, for  $P_1 = (Df(Y)Y')_{s,t} - (Df(\tilde{Y})\tilde{Y}')_{s,t}$  we add and subtract  $Df(Y_t)Y'_s$  and the same for  $\tilde{Y}$ , and look at the bounds for  $P_1$ ,

$$\begin{aligned} &\left| (Df(Y)Y')_{s,t} - (Df(\tilde{Y})\tilde{Y}')_{s,t} \right| \\ &\leq | Df(Y_t) | | Y'_{s,t} - \tilde{Y}'_{s,t} | + | Df(Y_t) - Df(\tilde{Y}_t) | | \tilde{Y}'_{s,t} | \\ &\quad + | Df(Y)_{s,t} | | Y'_s - \tilde{Y}'_s | + | Df(Y)_{s,t} - Df(\tilde{Y})_{s,t} | | \tilde{Y}'_s |. \end{aligned}$$

We know that  $| Y'_s - \tilde{Y}'_s | \leq | Y'_0 - \tilde{Y}'_0 | + \| Y' - \tilde{Y}' \|_{\alpha} T^{\alpha}$  and  $| Df(Y_t) - Df(\tilde{Y}_t) | \leq \| D^2f \|_{\infty} | Y_t - \tilde{Y}_t |$ , and therefore, just like in above calculations,

$$| P_1 | \lesssim (\varepsilon_X + \varepsilon_0 + \varepsilon'_0 + \varepsilon''_0 + \varepsilon) |t - s|^{2\alpha}$$

Secondly, we look at  $P_2$ ,

$$P_2 = D^2f(Y_s) (Y'_s)^{\otimes 2} X_{s,t}^{(1)} - D^2f(\tilde{Y}_s) (\tilde{Y}'_s)^{\otimes 2} \tilde{X}_{s,t}^{(1)}.$$

Using the inequality (\*) twice, and the previously stated estimates, we find that

$$| P_2 | \lesssim (\varepsilon_X + \varepsilon_0 + \varepsilon'_0 + \varepsilon''_0 + \varepsilon) |t - s|^{2\alpha}.$$

Next,  $P_3$  is given as follows,

$$P_3 = Df(Y_s)Y_s''X_{s,t}^{(1)} - Df(\tilde{Y}_s)\tilde{Y}_s''X_{s,t}^{(1)}.$$

We use the same inequality as before, and obtain the same type of results,

$$|P_3| \lesssim (\varepsilon_X + \varepsilon_0 + \varepsilon'_0 + \varepsilon''_0 + \varepsilon) |t - s|^{2\alpha}.$$

Therefore we have that

$$|R_{s,t}^{Z(2)} - R_{s,t}^{\tilde{Z}(2)}| \lesssim (\varepsilon_X + \varepsilon_0 + \varepsilon'_0 + \varepsilon''_0 + \varepsilon) |t - s|^{2\alpha}.$$

At last we need to look at  $R^{Z(3)} - R^{\tilde{Z}(3)}$ , which we know is given by

$$\begin{aligned} & R_{s,t}^{Z(3)} - R_{s,t}^{\tilde{Z}(3)} \\ &= f(Y)_{s,t} - Df(Y_s)Y_s'X_{s,t}^{(1)} - \left( D^2f(Y_s)(Y_s')^{\otimes 2} + Df(Y_s)Y_s'' \right) X_{s,t}^{(2)} \\ & - \left( f(\tilde{Y})_{s,t} - Df(\tilde{Y}_s)\tilde{Y}_s'X_{s,t}^{(1)} - \left( D^2f(\tilde{Y}_s)(\tilde{Y}_s')^{\otimes 2} + Df(\tilde{Y}_s)\tilde{Y}_s'' \right) X_{s,t}^{(2)} \right) \end{aligned}$$

In section 4.1 we found an equality for  $R^{Z(3)}$  such that

$$R_{s,t}^{Z(3)} = R_{s,t}^{Y(3)} + \bar{R}_{s,t}^{f(Y)} + o(|t - s|),$$

and then same relation with respect to  $\tilde{Z}$ , where  $\bar{R}_{s,t}^{f(Y)}$  is the remainder of a third degree taylor approximation of  $f(Y_t)$  around  $Y_s$ . Therefore,  $R_{s,t}^{Z(3)} - R_{s,t}^{\tilde{Z}(3)}$  can be expressed as follows,

$$R_{s,t}^{Z(3)} - R_{s,t}^{\tilde{Z}(3)} = R_{s,t}^{Y(3)} - R_{s,t}^{\tilde{Y}(3)} + \bar{R}_{s,t}^{f(Y)} - \bar{R}_{s,t}^{f(\tilde{Y})} + o(|t - s|).$$

We can find explicit representation of the expression  $\bar{R}_{s,t}^{f(Y)} - \bar{R}_{s,t}^{f(\tilde{Y})}$  by using the Lagrange remainder term formula, we have for  $\theta \in [0, 1]$  that  $\bar{R}_{s,t}^{f(Y)} = \frac{1}{6} D^3 f(Y_s + \theta Y_{s,t})(Y_{s,t})^{\otimes 3}$  and similar for  $\bar{R}_{s,t}^{f(\tilde{Y})}$ . Looking at the difference of the two, we find

$$\left\| \bar{R}_{s,t}^{f(Y)} - \bar{R}_{s,t}^{f(\tilde{Y})} \right\|_{3\alpha} \leq \frac{1}{6} |D^3 f|_{\infty} \left\| (Y_{s,t})^{\otimes 3} - (\tilde{Y}_{s,t})^{\otimes 3} \right\|_{3\alpha} \leq C 2M^3$$

We can therefore see that

$$\left\| R_{s,t}^{Z(3)} - R_{s,t}^{\tilde{Z}(3)} \right\|_{3\alpha} \leq C \left\| R_{s,t}^{Y(3)} - R_{s,t}^{\tilde{Y}(3)} \right\|_{\alpha} \lesssim (\varepsilon_X + \varepsilon_0 + \varepsilon'_0 + \varepsilon''_0 + \varepsilon)$$

Combining all these inequalities, we obtain our desired bound, namely that

$$\begin{aligned} & d_{X, \tilde{X}, \alpha} \left( Z, Z', Z''; \tilde{Z}, \tilde{Z}', \tilde{Z}'' \right) \\ & \leq C_M \left( d_\alpha \left( \mathbf{X}, \tilde{\mathbf{X}} \right) + |Y_0 - \tilde{Y}_0| + |Y'_0 - \tilde{Y}'_0| + |Y''_0 - \tilde{Y}''_0| + d_{X, \tilde{X}, \alpha} \left( Y, Y', Y''; \tilde{Y}, \tilde{Y}', \tilde{Y}'' \right) \right). \end{aligned}$$

□

*Remark 4.3.* The observant reader may notice that we now have obtained two local Lipschitz estimates (lemma 3.11 and 4.2) which may be somewhat used in a circle. If we look at a path  $(Y, Y', Y'') \in \mathcal{D}_X^\alpha$ , and let  $(f(Y), f(Y)', f(Y)'') \in \mathcal{D}_X^\alpha$  be composed from  $Y$ . Then, by remark 3.9 we know that  $(Z, Z', Z'') := (\int_0^\cdot f(Y) d\mathbf{X}, f(Y), f(Y)') \in \mathcal{D}_X^\alpha$ . Define the same relations for  $(\tilde{Y}, \tilde{Y}', \tilde{Y}'') \in \mathcal{D}_{\tilde{X}}^\alpha$ , and let  $Y_0 = \tilde{Y}_0$ ,  $Y'_0 = \tilde{Y}'_0$ , and  $Y''_0 = \tilde{Y}''_0$ . Then we know from lemma 3.11 and lemma 4.2 that

$$\begin{aligned} d_{X, \tilde{X}, \alpha} \left( Z, Z', Z'' : \tilde{Z}, \tilde{Z}', \tilde{Z}'' \right) & \lesssim d_\alpha \left( \mathbf{X}, \tilde{\mathbf{X}} \right) + d_{X, \tilde{X}, \alpha} \left( f(Y), f(Y)', f(Y)'' : f(\tilde{Y}), f(\tilde{Y})', f(\tilde{Y})'' \right) T^\alpha \\ & \lesssim d_\alpha \left( \mathbf{X}, \tilde{\mathbf{X}} \right) (1 + T^\alpha) + d_{X, \tilde{X}, \alpha} \left( Y, Y', Y'' : \tilde{Y}, \tilde{Y}', \tilde{Y}'' \right) T^\alpha. \end{aligned}$$

Now, in light of differential equations, what if  $(Y, Y', Y'') = (Y_0 + \int_0^\cdot f(Y) d\mathbf{X}, f(Y), f(Y)')$  then we would have an estimate stating

$$d_{X, \tilde{X}, \alpha} \left( Y, Y', Y'' : \tilde{Y}, \tilde{Y}', \tilde{Y}'' \right) \lesssim d_\alpha \left( \mathbf{X}, \tilde{\mathbf{X}} \right) (1 + T^\alpha) + d_{X, \tilde{X}, \alpha} \left( Y, Y', Y'' : \tilde{Y}, \tilde{Y}', \tilde{Y}'' \right) T^\alpha.$$

As we understand the two stability results, will come in handy when proving stability of differential equations. The next section will show existence and uniqueness of differential equations driven by rough paths, and then give a more rigorous proof of the stability of the solutions.

## 5 Solutions to rough differential equations driven by rough paths.

This section extends chapter 8 in the book by Friz and Hairer [16] to the case when the  $\alpha$ -Hölder regularity on the rough paths is such that  $\alpha \in (\frac{1}{4}, \frac{1}{3}]$ . Our goal is to establish a method to prove existence of solutions of differential equations on the form

$$dY_t = f(Y_t)dX_t \quad Y_0 = x \in \mathbb{R}^d ,$$

where  $X : [0, T] \rightarrow \mathbb{R}^d$  is  $\alpha$ -Hölder rough, with  $\alpha \in (\frac{1}{4}, \frac{1}{3}]$ , and  $Y : [0, T] \rightarrow \mathbb{R}^n$  is the “output” of the system, and  $f$  is a sufficiently nice function. From the theory of classical ODE’s we are familiar with such equations when  $X \in \mathcal{C}^1$  and we can write

$$dY_t = f(Y_t)\dot{X}_t dt \quad Y_0 = x \in \mathbb{R}^d ,$$

and a solution is easily found. However, when  $X$  is rough, it is not trivial to find such solutions, or even to prove existence. The solution method used in this section is based on a Picard iteration, and our proof is similar to that of [16], but the lower regularity requires some extra computations.

### 5.1 Existence of solutions to geometric RDE’s with $\alpha \in (\frac{1}{4}, \frac{1}{3}]$ .

The aim of this section is to prove the existence and uniqueness of solutions to geometric RDE’s of the form

$$dY_t = f(Y)d\mathbf{X}_t,$$

where  $\mathbf{X} \in \mathcal{C}_g^\alpha([0, T], \mathbb{R}^d)$  and  $f$  is a sufficiently smooth function  $f : \mathbb{R}^m \rightarrow \mathcal{L}(\mathbb{R}^d, \mathbb{R}^m)$ . We will establish the result by showing that there exist a fixed point  $Y$  on a small enough interval  $[0, T_0]$ , then one can iterate the procedure on the interval  $[T_0, 2T_0]$  and let  $Y_0 = f(T_0)$  on that interval, and so on.

**Theorem 5.1.** *Let  $\frac{1}{4} < \alpha < \beta < \frac{1}{3}$  and  $[0, T] \subset [0, 1]$ , and let  $\mathbf{X} = (X^{(1)}, X^{(2)}, X^{(3)}) \in \mathcal{C}_g^\beta([0, T], \mathbb{R}^d)$ , with  $d_\beta(0, \mathbf{X}) \leq M$ . Further, Let  $f \in C_b^4(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^m))$  and  $x \in \mathbb{R}^m$ . Then there exist a unique element  $(Y, Y', Y'') \in \mathcal{D}_X^\alpha([0, T], \mathbb{R}^m)$  such that*

$$Y_t = x + \int_0^t f(Y_s)d\mathbf{X}_s.$$

Where  $t < T$ , and the integral  $\int_0^t f(Y_s)d\mathbf{X}_s$  is interpreted as a rough integral, in the sense of theorem 3.7 and  $(f(Y), f(Y)', f(Y)'') \in \mathcal{D}_X^\alpha$  is built from  $Y$  in the sense of section 4.1.

*Proof.* We will construct the solution as a fixed point of in the space  $\mathcal{D}_X^\alpha$  where  $\mathbf{X} \in \mathcal{C}_g^\beta$ . The reason is that this will simplify the proof quite a bit, and it turns out that the solution actually is in  $\mathcal{D}_X^\beta$ , due to the fact that  $|Y_{s,t}| \leq |Y'|_\infty |X_{s,t}^{(1)}| + |Y''|_\infty |X_{s,t}^{(2)}| + |R_{s,t}^Y|$ . From section 4.1 we know that for all  $Y$ , such that  $(Y, Y', Y'') \in \mathcal{D}_X^\alpha$  and some function  $f \in \mathcal{C}_b^4(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^m))$ , we can define a controlled path

$$(f(Y), f(Y)', f(Y'')) \in \mathcal{D}_X^\alpha$$

where  $f(Y)' = Df(Y)Y'$  and  $f(Y)'' = D^2f(Y)(Y')^{\otimes 2} + Df(Y)Y''$ , as we discussed in section 4.1. We then define the map

$$\mathcal{M}_T(Y, Y', Y'') = \left( x + \int_0^\cdot f(Y)_s d\mathbf{X}_s, f(Y), f(Y)' \right) \in \mathcal{D}_X^\alpha$$

Which is called the Itô-Lyons map. We want to construct a unit ball  $\mathcal{B}_T$  in  $\mathcal{D}_X^\alpha$  such that, when choosing  $T$  small enough we leave the unit ball invariant under the mapping  $\mathcal{M}_T$ . To do this, we need to define a center of such a ball  $\mathcal{B}_T$ . The intuitive point  $(x, f(x), f(x)')$  is not in general in  $\mathcal{D}_X^\alpha$ , and we therefore need to pick another point as our center which is in some sense dependent on the path  $X^{(1)}$ . It is straight forward to check that

$$t \mapsto \left( x + f(x)X_{0,t}^{(1)}, f(x), 0 \right) \in \mathcal{D}_X^\alpha.$$

Indeed, we can see that  $\sum_{i=1}^3 \|R^{x+f(x)X^{(1)}}\|_{i\alpha} < \infty$  as all the remainder terms is essentially zero. And therefore this seems to be a good center of the ball  $\mathcal{B}_T$ . Remember here that the path above is a path evaluated in one variable, i.e we may define  $\hat{Y}_s = x + f(x)X_{0,s}^{(1)}$ , and the increments of  $\hat{Y}$  is given by  $\hat{Y}_{s,t} = f(x)X_{s,t}^{(1)}$ , further  $\hat{Y}'_s = f(x)$ , and therefore  $\hat{Y}'_{s,t} = 0$ . To shorten notation, we define the center

$$(\hat{Y}, \hat{Y}', \hat{Y}'') = \left( x + f(x)X_{0,\cdot}^{(1)}, f(x), 0 \right).$$

Considering the norm  $|Y_0| + |Y'_0| + |Y''_0| + \|Y, Y', Y''\|_{\mathcal{D}_X^\alpha}$ , and  $(Y, Y', Y'')$ , such that  $Y_0 = x = \hat{Y}_0$ ,  $Y'_0 = f(x) = \hat{Y}'_0$  and  $Y''_0 = 0 = \hat{Y}''_0$ . Then the unit ball  $\mathcal{B}_T$  is defined such that

$$\begin{aligned} & |Y_0 - x| + |Y'_0 - f(x)| + |Y''_0 - 0| + \|Y - \hat{Y}, Y' - \hat{Y}', Y'' - \hat{Y}''\|_{\mathcal{D}_X^\alpha} \\ &= \|Y - \hat{Y}, Y' - \hat{Y}', Y'' - \hat{Y}''\|_{\mathcal{D}_X^\alpha} \leq 1. \end{aligned}$$

Actually, we can show that

$$\|Y - \hat{Y}, Y' - \hat{Y}', Y'' - \hat{Y}''\|_{\mathcal{D}_X^\alpha} = \|Y, Y', Y''\|_{\mathcal{D}_X^\alpha}.$$

Which is due to the fact that

$$\begin{aligned} R_{s,t}^{(Y-\hat{Y})^{(3)}} &= Y_{s,t} - \hat{Y}_{s,t} - \left( Y'_s X_{s,t}^{(1)} - \hat{Y}'_s X_{s,t}^{(1)} \right) - \left( Y''_s X_{s,t}^{(2)} - \hat{Y}''_s X_{s,t}^{(2)} \right) \\ &= Y_{s,t} - Y'_s X_{s,t}^{(1)} - Y''_s X_{s,t}^{(2)} = R_{s,t}^{Y^{(3)}} \end{aligned}$$

and hence  $\| R^{(Y-\hat{Y})^{(3)}} \|_{3\alpha} \| R^{Y^{(3)}} \|_{3\alpha}$ . Next, looking at  $R^{(Y-\hat{Y})^{(2)}}$  we see that

$$R_{s,t}^{(Y-\hat{Y})^{(2)}} = Y'_{s,t} - \hat{Y}'_{s,t} - \left( Y''_s X_{s,t}^{(1)} - \hat{Y}''_s X_{s,t}^{(1)} \right) = Y'_{s,t} - Y''_s X_{s,t}^{(1)}.$$

And therefore  $\| R^{(Y-\hat{Y})^{(2)}} \|_{2\alpha} = \| R^{Y^{(2)}} \|_{2\alpha}$ . The last to check is  $R^{(Y-\hat{Y})^{(1)}}$ , where it is easy to see that  $\| R^{(Y-\hat{Y})^{(1)}} \|_{\alpha} = \| R^{Y^{(1)}} \|_{\alpha}$ . The ball  $B_T$  is therefore defined as follows,

$$B_T = \{ (Y, Y', Y'') \in \mathcal{D}_X^\alpha : Y_0 = x, Y'_0 = f(x), Y''_0 = Df(x)f(x); \| Y, Y', Y'' \|_{\mathcal{D}_X^\alpha} \leq 1 \}.$$

We also have for all  $(Y, Y', Y'') \in B_T$ ,

$$| Y'_0 | + | Y''_0 | + \| Y, Y', Y'' \|_{\mathcal{D}_X^\alpha} \leq \| f \|_{C_b^3} + 1.$$

Our goal is now to find a  $T$  such that the ball  $B_T$  is left invariant under the mapping  $(Y, Y', Y'') \mapsto \mathcal{M}_T(Y, Y', Y'')$ . From theorem 4.1 we have that

$$\| f(Y), f(Y)', f(Y)'' \|_{\mathcal{D}_X^\alpha} \leq C_M \left( 1 + \| X^{(1)} \|_{\alpha} + \| X^{(2)} \|_{2\alpha} \right) (1 + | Y'_0 | + | Y''_0 | + \| Y, Y', Y'' \|_{\mathcal{D}_X^\alpha})$$

To simplify the calculations, we are letting the constant  $C$  vary and depend on  $T, \alpha, \beta, \| f \|_{C_b^3}, X^{(1)}$  and  $X^{(2)}$ , and we find

$$\begin{aligned} & \left\| \int_0^\cdot f(Y) d\mathbf{X}_s, f(Y), f(Y)' \|_{\mathcal{D}_X^\alpha} \leq \| f(Y)' \|_{\alpha} + \| f(Y) - f(Y)' X_{\cdot}^{(1)} \|_{2\alpha} \\ & + \left\| \int_0^\cdot f(Y) d\mathbf{X}_s - f(Y) X^{(1)} - f(Y)' X^{(2)} \|_{3\alpha} \\ & \leq \left( | f(Y_0)'' | + \| R^{f(Y)^{(1)}} \|_{\alpha} T^\alpha \right) \left( \sum_{i=1}^3 \| X^{(i)} \|_{i\alpha} \right) + \| R^{f(Y)^{(2)}} \|_{2\alpha} T^\alpha + \| R^{f(Y)^{(3)}} \|_{3\alpha} T^\alpha \\ & + C T^\alpha \left( \sum_{i=1}^3 \| X^{(4-i)} \|_{(4-i)\alpha} \| R^{f(Y)^{(i)}} \|_{i\alpha} \right) \end{aligned}$$

Where we have used inequalities we have established in 3.7 and 4.1. calculating further, we find



that

$$\begin{aligned} &\leq T^\alpha (|f(Y_0)'| + |f(Y_0)''| + \|f(Y), f(Y)', f(Y)''\|_{\mathcal{D}_X^\alpha}) \left(1 + \sum_{i=1}^3 \|X^{(i)}\|_{i\alpha}\right) \\ &\quad + C (|f(Y_0)'| + |f(Y_0)''| + \|f(Y), f(Y)', f(Y)''\|_{\mathcal{D}_X^\alpha}) \left(\sum_{i=1}^3 \|X^{(i)}\|_{i\alpha}\right). \end{aligned}$$

We then use that  $\|X^{(i)}\|_{i\alpha} \leq \|X^{(i)}\|_{i\beta} T^{i(\beta-\alpha)}$  for  $i = 1, 2, 3$ , and see that

$$\left\| \int_0^\cdot f(Y)d\mathbf{X}_s, f(Y), f(Y)' \right\|_{\mathcal{D}_X^\alpha} \leq C (|f(Y_0)'| + |f(Y_0)''| + \|f(Y), f(Y)', f(Y)''\|_{\mathcal{D}_X^\alpha}) \times T^{\beta-\alpha},$$

where  $C = C(T, \alpha \|X\|_\beta^{(1)}, \|X\|_{2\beta}^{(2)})$ . The next step is to look at the Itô-Lyons map in the  $\mathcal{D}_X^\alpha$  norm, and find sufficient bounds. We want the bounds to be multiplied by a factor  $T^\gamma$ , for  $\gamma \in \mathbb{R}_+$ , such that when  $T$  gets smaller, the whole right hand side of our inequalities will get smaller. In that way, we can obtain a  $T_0$  such that  $\|\mathcal{M}_T(Y, Y', Y'')\|_{\mathcal{D}_X^\alpha} \leq 1$ .

$$\begin{aligned} &\|\mathcal{M}_T(Y, Y', Y'')\|_{\mathcal{D}_X^\alpha} = \left\| \int_0^\cdot f(Y_s)d\mathbf{X}_s, f(Y), f(Y)' \right\|_{\mathcal{D}_X^\alpha} \\ &\leq C (|f(Y_0)'| + |f(Y_0)''| + \|f(Y), f(Y)', f(Y)''\|_{\mathcal{D}_X^\alpha}) \times T^{\beta-\alpha} \\ &\leq C (|f(Y_0)'| + |f(Y_0)''| + C_M (1 + \|X^{(1)}\|_\alpha + \|X^{(2)}\|_{2\alpha}) (1 + |Y_0'| + |Y_0''| + \|Y, Y', Y''\|_{\mathcal{D}_X^\alpha})) \times T^{\beta-\alpha} \end{aligned}$$

Using the fact that  $|Y_0'| + |Y_0''| + \|Y, Y', Y''\|_{\mathcal{D}_X^\alpha} \leq \|f\|_{C_b^3} + 1$ , we find that

$$\begin{aligned} &\leq C \left( \|f\|_{C_b^3} + C(M+1)(\|f\|_{C_b^3} + 2) \right) \times T^{\beta-\alpha} \\ &= o(T^{\beta-\alpha}) \end{aligned}$$

Hence, we have found that

$$\|\mathcal{M}_T(Y, Y', Y'')\|_{\mathcal{D}_X^\alpha; [0, T]} \leq o(T^{\beta-\alpha})$$

therefore, we can find a small enough  $T_0$  such that

$$\|\mathcal{M}_{T_0}(Y, Y', Y'')\|_{\mathcal{D}_X^\alpha; [0, T_0]} \leq 1$$

which shows that  $\mathcal{M}_{T_0}(B_{T_0}) \subset B_{T_0}$ , and therefore leaves  $B_{T_0}$  invariant.

Now we continue by showing the contraction property of  $\mathcal{M}_T$  which let us know there exist a fixed point, and therefore uniqueness of the solution from Banach's fixed point theorem. We are

interested in looking at two different controlled rough paths  $(Y, Y', Y'')$  and  $(\tilde{Y}, \tilde{Y}', \tilde{Y}'')$  controlled by same  $\mathbf{X}$ , which have the same initial values, i.e  $Y_0 = \tilde{Y}_0 = x$ , and so on. We define an increment function  $\Delta_s = f(Y_s) - f(\tilde{Y}_s)$  and  $\Delta'_s = f(Y_s)' - f(\tilde{Y}_s)'$  and  $\Delta''_s = f(Y_s)'' - f(\tilde{Y}_s)''$ , and look at the difference in the Itô-Lyons map evaluated in the two different controlled paths,

$$\begin{aligned} & \| \mathcal{M}_T(Y, Y', Y'') - \mathcal{M}_T(\tilde{Y}, \tilde{Y}', \tilde{Y}'') \|_{\mathcal{D}_X^\alpha} \\ &= \| \int_0^\cdot \Delta_s d\mathbf{X}_s, \Delta, \Delta' \|_{\mathcal{D}_X^\alpha} \leq C (| \Delta'_0 | + | \Delta''_0 | + \| \Delta, \Delta', \Delta'' \|_{\mathcal{D}_X^\alpha}) \times T^{\beta-\alpha}. \end{aligned}$$

Therefore we want to show that  $\| \Delta, \Delta', \Delta'' \|_{\mathcal{D}_X^\alpha} \lesssim \| Y - \tilde{Y}, Y' - \tilde{Y}', Y'' - \tilde{Y}'' \|_{\mathcal{D}_X^\alpha}$ . Using that  $f \in \mathcal{C}_b^4$ , there exists functions

$$G_s := g(Y_s, \tilde{Y}_s)$$

$$H_s := Y_s - \tilde{Y}_s$$

such that  $\Delta_s = G_s H_s$ , where the function  $g$  is given by,

$$g(x, y) := \int_0^1 Df(xt + (1-t)y)dt$$

We see that  $g \in \mathcal{C}_b^3$  in both the  $x$  and  $y$  variable, with  $\| g \|_{\mathcal{C}_b^3} \leq C \| f \|_{\mathcal{C}_b^4}$ . Further, we see that,

$$D_x g(x, y) = \int_0^1 D^2 f(xt + (1-t)y)tdt \quad \text{and} \quad D_y g(x, y) = \int_0^1 D^2 f(xt + (1-t)y)(1-t)dt.$$

From the section about composition of controlled paths with regular functions 4.1, we know that

$$\begin{aligned} & (G, G', G'') = \\ & (G, D_x G Y' + D_y G \tilde{Y}', \\ & (D_x^2 G (Y')^{\otimes 2} + D_x G Y'') + (D_y^2 G (\tilde{Y}')^{\otimes 2} + D_y G \tilde{Y}'')) \in \mathcal{D}_X^\alpha. \end{aligned}$$

From there, we easily obtain the bound

$$\| G, G', G'' \|_{\mathcal{D}_X^\alpha} \leq C \| f \|_{\mathcal{C}_b^4},$$

where  $C = C(T, \alpha \| X \|_\alpha^{(1)}, \| X \|_{2\alpha}^{(2)})$  which is uniform over  $(Y, Y', Y''), (\tilde{Y}, \tilde{Y}', \tilde{Y}'') \in \mathcal{B}_T$  for  $T \leq 1$ . We continue by looking at the path constructed by  $G$  and  $H$  as follows,  $(GH, (GH)', (GH)'') \in \mathcal{D}_X^\alpha$ . Where the derivatives is defined as follows  $(GH)' = G'H + GH'$  and  $(GH)'' = G''H + 2G'H' +$

$GH''$ . We find the estimate, which is a long calculation, but quite straight forward,

$$\begin{aligned} & \| GH, (GH)', (GH)'' \|_{\mathcal{D}_X^\alpha} \leq \\ & C ( \| G_0 \| + \| G'_0 \| + \| G''_0 \| + \| G, G', G'' \|_{\mathcal{D}_X^\alpha} ) \\ & \times ( \| H_0 \| + \| H'_0 \| + \| H''_0 \| + \| H, H', H'' \|_{\mathcal{D}_X^\alpha} ). \end{aligned}$$

The calculations are obtained, knowing the fact that  $G'$  is a symmetric operator. Further, knowing that for all  $(Y, Y', Y''), (\tilde{Y}, \tilde{Y}', \tilde{Y}'') \in \mathcal{B}_T$ , we have  $H_0 = Y_0 - \tilde{Y}_0 = 0$ ,  $H'_0 = 0$  and  $H''_0 = 0$  it follows that

$$\begin{aligned} & \| \Delta, \Delta', \Delta'' \|_{\mathcal{D}_X^\alpha} \\ & \leq C ( \| G_0 \| + \| G'_0 \| + \| G''_0 \| + \| G, G', G'' \|_{\mathcal{D}_X^\alpha} ) ( \| H, H', H'' \|_{\mathcal{D}_X^\alpha} ) \\ & \leq D ( \| g \|_\infty + \| g' \|_\infty ( \| Y'_0 \| + \| \tilde{Y}'_0 \| ) \\ & + \| g'' \|_{\mathcal{C}_b^2} ( \| (Y'_0)^{\otimes 2} \| + \| Y''_0 \| + \| \tilde{Y}''_0 \| + \| (\tilde{Y}'_0)^{\otimes 2} \| ) + C \| f \|_{\mathcal{C}_b^4} ) \\ & \times ( \| Y - \tilde{Y}, Y' - \tilde{Y}', Y'' - \tilde{Y}'' \|_{\mathcal{D}_X^\alpha} ) \end{aligned}$$

Using previously stated estimates, we can see that  $\| g \|_\infty, \| g' \|_\infty, \| g'' \|_{\mathcal{C}_b^2} \leq K \| f \|_{\mathcal{C}_b^4}$  and  $\| (Y'_0)^{\otimes 2} \| + \| Y''_0 \| + \| \tilde{Y}''_0 \| + \| (\tilde{Y}'_0)^{\otimes 2} \| \leq K_2 ( \| f \|_{\mathcal{C}_b^4} + \| f \|_{\mathcal{C}_b^4}^2 )$ , and hence we see that

$$\| \Delta, \Delta', \Delta'' \|_{\mathcal{D}_X^{2\alpha, 3\alpha}} \leq C \| Y - \tilde{Y}, Y' - \tilde{Y}', Y'' - \tilde{Y}'' \|_{\mathcal{D}_X^\alpha}.$$

Therefore, we have that

$$\begin{aligned} & \| \mathcal{M}_T(Y, Y', Y'') - \mathcal{M}_T(\tilde{Y}, \tilde{Y}', \tilde{Y}'') \|_{\mathcal{D}_X^\alpha} \\ & \leq C ( \| Y - \tilde{Y}, Y' - \tilde{Y}', Y'' - \tilde{Y}'' \|_{\mathcal{D}_X^\alpha} ) \times T^{\beta-\alpha} \end{aligned}$$

Hence, by the inequalities obtained here, and the results obtained when proving invariance of the ball  $\mathcal{B}_{T_0}$  i.e  $\mathcal{M}(\mathcal{B}_{T_0}) \subset \mathcal{B}_{T_0}$  for small enough  $T_0$ , there exist a  $q \in [0, 1)$  such that

$$\| \mathcal{M}_{T_0}(Y, Y', Y'') - \mathcal{M}_{T_0}(\tilde{Y}, \tilde{Y}', \tilde{Y}'') \|_{\mathcal{D}_X^\alpha} \leq q \| Y - \tilde{Y}, Y' - \tilde{Y}', Y'' - \tilde{Y}'' \|_{\mathcal{D}_X^\alpha}.$$

The result now follow from Banach's fixed point theorem, and we can construct the solution iteratively on  $[0, 1]$  as described earlier.  $\square$

We are often interested in rough differential equations with a drift term, as we come across this in the theory of stochastic differential equations, and financial mathematics. That is, we want to

look at differential equations of the form

$$dY_t = b(Y_t)dt + \sigma(Y_t)d\mathbf{X}_t$$

Where  $b$  is a sufficiently smooth function. It turns out that to do this, we can define a space-time extension of  $\mathbf{X}$ , and let  $f = (b, \sigma)$ . We will give a proposition for the construction of such solutions.

**Proposition 5.2.** *Assume  $b \in \mathcal{C}_b^4(\mathbb{R}^m, \mathcal{L}(\mathbb{R}, \mathbb{R}^m))$ , and  $\sigma \in \mathcal{C}_b^4(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^m))$ , and let  $(Y, Y', Y'') \in \mathcal{D}_X^\alpha([0, T], \mathbb{R}^m)$ , s.t.  $Y_0 = x$ . Let  $\mathbf{X} \in \mathcal{C}_g^\alpha([0, T], \mathbb{R}^d)$ , then the differential equation given by*

$$dY_t = b(Y_t)dt + \sigma(Y_t)d\mathbf{X}_t, Y_0 = x$$

*exists and is unique.*

*Proof.* Define  $f : \mathbb{R}^m \rightarrow \mathcal{L}(\mathbb{R}^k, \mathbb{R}^m)$  by  $f(Y_t) = (b(Y_t), \sigma(Y_t))$  where  $k = d + 1$ , and it's derivatives is constructed in the familiar way, and define the space time extension  $\tilde{\mathbf{X}}$  of  $\mathbf{X}$  such that  $\tilde{\mathbf{X}}$  is built from the function  $\tilde{X}^{(1)} = (t - s, X^{(1)}) \in \mathbb{R}^k$ . In this way, we see that  $\tilde{X}^{(1)} : [0, T] \rightarrow \mathbb{R}^k$ . The second iterated integral  $\tilde{X}^{(2)}$  would be a  $k \times k$  dimensional matrix, containing the four elements  $\frac{1}{2}(t - s)^2$ ,  $X_{s,t}^{(2)}$ ,  $\int_s^t X_{s,r} dr$  and  $\int_s^t (r - s)dX_r$ . In a way, one may visualize the matrix to look like

$$\tilde{X}_{s,t}^{(2)} = \begin{pmatrix} \frac{1}{2}(t - s)^2 & \int_s^t X_{s,r}^{(1),1} dr & \dots & \int_s^t X_{s,r}^{(1),d} dr \\ \int_s^t (r - s)dX_r^{(1),1} & X_{s,t}^{(2),1,1} & \dots & X_{s,t}^{(2),1,d} \\ \vdots & \vdots & \ddots & \vdots \\ \int_s^t (r - s)dX_r^{(1),d} & X_{s,t}^{(2),d,1} & \dots & X_{s,t}^{(2),d,d} \end{pmatrix}.$$

The “cross-integrals” comes naturally as a consequence of the requirement of Chen's relation, i.e  $\tilde{\mathbf{X}}_{s,u} \otimes \tilde{\mathbf{X}}_{u,t} = \tilde{\mathbf{X}}_{s,t}$ . The third iterated integral  $\tilde{X}_{s,t}^{(3)}$  will be a cube with dimension  $k \times k \times k$ . That is  $\tilde{X}^{(3)} : [0, T]^2 \rightarrow \mathbb{R}^{k \times k \times k}$ , and will contain 6 different integral elements, namely  $\frac{1}{6}(t - s)^3$ ,  $X_{s,t}^{(3),i,j,k}$ ,  $\int_s^t \int_s^{t_1} X_{s,r}^{(1),i} dr dt_1$ ,  $\int_s^t \int_s^{t_1} (r - s)dX_r^{(1),i} dt_1$ ,  $\int_s^t X_{s,r}^{(2),i,j} dr$ , and  $\int_s^t \int_s^{t_1} (r - s)dr dX_{t_1}^{(1),i}$  where  $i, j, k \in \{1, 2, \dots, d\}$ . What is important is that when  $f$  acts on  $\tilde{X}^{(3)}$ , the operator  $b''$  acts on the terms  $\frac{1}{6}(t - s)^3$ ,  $\int_s^t \int_s^{t_1} X_{s,r}^{(1),i} dr dt_1$ ,  $\int_s^t \int_s^{t_1} (r - s)dX_r^{(1),i} dt_1$  again for  $i \in \{1, 2, \dots, d\}$ , and  $\sigma''$  acts on the rest of the integral terms in the cube  $\tilde{X}^{(3)}$ . Then  $\tilde{\mathbf{X}} \in \mathcal{C}_g^\alpha([0, T], \mathbb{R}^{d+1})$ , and then the solution

$$Y_t = x + \int_0^t f(Y_s)d\tilde{\mathbf{X}}_s$$

exists and are unique for  $0 \leq t < T$ . The integral is interpreted as the rough integral

$$\begin{aligned} \int_0^t f(Y_s)d\tilde{\mathbf{X}}_s &= \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} f(Y_u)\tilde{X}_{u,v}^{(1)} + f(Y_u)'\tilde{X}_{u,v}^{(2)} + f(Y_u)\tilde{X}_{u,v}^{(3)} \\ &= \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} b(Y_u)(v - u) + \sigma(Y_u)X_{u,v}^{(1)} + \frac{1}{2}b(Y_u)'(v - u)^2 + \sigma(Y_u)'X_{u,v}^{(2)} + \end{aligned}$$

$$\begin{aligned}
& +b(Y_u)' \int_u^v X_{u,r}^{(1)} dr + \sigma(Y_u)' \int_u^v (r-u) dX_r^{(1)} + \frac{1}{6} b(Y_u)'' (u-v)^3 + b(Y_u)'' \int_u^v \int_u^{t_1} X_{s,r} dr dt_1 \\
& + b(Y_u)'' \int_u^v \int_u^{t_1} (u-r) dX_r^{(1)} dt_1 + \sigma(Y_u)'' \int_u^v \int_u^{t_1} (u-r) dX_r^{(1)} dt_1 + \sigma(Y_u)'' \int_u^v X_{u,r}^{(2)} dr \\
& + \sigma(Y_u)'' X_{u,v}^{(3)}.
\end{aligned}$$

We will show that all the higher order cross integration terms will disappear as the length of the intervals in the partition goes to zero, and therefore we will be left with

$$dY_t = b(Y_t)dt + \sigma(Y_t)d\mathbf{X}_t.$$

We have three different types of integrals in the sum we have shown above. We have the integrals which are well defined from regular calculus, i.e.  $\int_s^t (r-s)dr$  and  $\int_s^t \int_s^{t_1} (r-s)dr dt_1$  which are of order  $o(|v-u|^2)$  and  $o(|v-u|^3)$  respectively. We also have the rough path integrals from the path  $(X^{(1)}, X^{(2)}, X^{(3)}) \in \mathcal{C}_g^\alpha$  which are well defined by assumption. At last we have the cross integrals  $\int_s^t X_{s,r} dr$ ,  $\int_s^t (r-s) dX_r$ ,  $\int_s^t \int_s^{t_1} X_{s,r} dr dt_1$  and so on, which are well defined from Young theory. Indeed, as the cross integral terms can be expressed as an integral where either the integrand is  $\mathcal{C}^1$  or the function we integrate against is  $\mathcal{C}^1$ , then the Sewing lemma 3.2.1 tells us that the cross integrals are of order  $o(|u-v|^{1+\alpha})$ . Therefore if we let  $\tilde{Z}_{u,v}^i$  where  $i = 1, 2, \dots, 8$  represent all the integrals of order  $o(|v-u|^\beta)$  for  $\beta > 1$ , we may write,

$$\begin{aligned}
\int_0^t f(Y_s) d\tilde{\mathbf{X}}_s &= \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} f(Y_u) \tilde{X}_{u,v}^{(1)} + f(Y_u)' \tilde{X}_{u,v}^{(2)} + f(Y_u)'' \tilde{X}_{u,v}^{(3)} \\
&= \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} b(Y_u)(v-u) + \sigma(Y_u) X_{u,v}^{(1)} + \sigma(Y_u) X_{u,v}^{(2)} + \sigma(Y_u) X_{u,v}^{(3)} \\
&\quad + \sum_{i=1}^8 \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} f(Y_u)^k \tilde{Z}_{u,v}^i \\
&= \int_0^t b(Y_s) ds + \int_0^t \sigma(Y_s) d\mathbf{X}_s + \sum_{i=1}^8 \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} f(Y_u)^k \tilde{Z}_{u,v}^i,
\end{aligned}$$

where  $\int_0^t b(Y_s) ds$  is well defined as a Young integral. Looking at the last sum, we find

$$\left| \sum_{i=1}^8 \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} f(Y_u)^k \tilde{Z}_{u,v}^i \right| \leq \sum_i \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} \left| f(Y_u)^k \right|_\infty \left| \tilde{Z}_{u,v}^i \right|,$$

where we know that  $\left| \tilde{Z}_{u,v}^i \right| \leq C |v - u|^\beta$ , for  $\beta > 1$ . From this, we conclude that

$$\sum_i \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} f(Y_u)^k \tilde{Z}_{u,v}^i \rightarrow 0,$$

and hence that

$$Y_t = x + \int_0^t f(Y_s) d\tilde{\mathbf{X}}_s = x + \int_0^t b(Y_s) ds + \int_0^t \sigma(Y_s) d\mathbf{X}_s.$$

□

In the rest of this thesis, we will for simplicity look at differential equations of the form

$$dY_t = f(Y_t) d\mathbf{X}_t,$$

if not stated otherwise. We may still think of this as an equation with drift, as shown above.

## 5.2 Stability of solutions to RDE's

In previous sections, we have established stability results with respect to rough integration, and composition of controlled paths with regular functions. In this subsection we will look closer at the stability of solutions to RDE's on the form

$$dY_t = f(Y_t) d\mathbf{X}_t, Y_0 = x$$

when  $f$  is sufficiently smooth. We have already shown in a way how we may use the stability results we have already established, to look at the stability of differential equations, see remark 4.3. However, we will in the next theorem give a more rigorous proof of the result.

**Theorem 5.3.** *Let  $f \in \mathcal{C}_b^4(\mathbb{R}, \mathbb{R})$  and  $\tilde{\mathbf{X}}, \mathbf{X} \in \mathcal{C}_g^\beta$ , and  $\frac{1}{4} < \alpha < \beta \leq \frac{1}{3}$ . Let  $(Y, f(Y), f(Y)') \in \mathcal{D}_X^\alpha$  be the unique RDE solution to*

$$dY_t = f(Y_t) d\mathbf{X}_t \quad Y_0 = x \in \mathbb{R},$$

*and let  $(\tilde{Y}, f(\tilde{Y}), f(\tilde{Y})') \in \mathcal{D}_{\tilde{X}}^\alpha$  be a solution driven by  $\tilde{\mathbf{X}}$  and started at  $y \in \mathbb{R}$ . Assuming that*

$$\begin{aligned} &|f(Y_0)'| + |f(Y_0)''| + \|f(Y), f(Y)', f(Y)''\|_{\mathcal{D}_X^\alpha} \leq M \in \mathbb{R} \quad \text{and} \\ &d_\alpha(0, \mathbf{X}) = \|X^{(1)}\|_\alpha + \|X^{(2)}\|_{2\alpha} + \|X^{(3)}\|_{3\alpha} \leq M \end{aligned}$$

*With the same bounds for  $(\tilde{Y}, \tilde{Y}', \tilde{Y}'')$  and  $\tilde{\mathbf{X}}$ . Then, there exist a constant  $C_M$  depending on  $T, \alpha, \beta$  and  $f$  such that*

$$d_{X, \tilde{X}, \alpha} \left( Y, f(Y), f(Y)'; \tilde{Y}, f(\tilde{Y}), f(\tilde{Y})' \right)$$

$$\leq C_M \left( d_\beta \left( \mathbf{X}, \tilde{\mathbf{X}} \right) + |Y_0 - \tilde{Y}_0| + |Y'_0 - \tilde{Y}'_0| + |Y''_0 - \tilde{Y}''_0| \right).$$

*Proof.* In section 5 we established that for a given  $\mathbf{X} \in \mathcal{C}_g^\beta$  the RDE solution  $(Y, f(Y), f(Y)')$  was constructed as a fixed point of the map

$$\mathcal{M}_T(Y, f(Y), f(Y)') := (Z, Z', Z'') := \left( \int_0^\cdot f(Y_s) d\mathbf{X}_s, f(Y), Df(Y)Y' \right) \in \mathcal{D}_{\tilde{X}}^\alpha$$

and in the same way for  $\mathcal{M}_T(\tilde{Y}, f(\tilde{Y}), f(\tilde{Y})') \in \mathcal{D}_{\tilde{X}}^\alpha$ . Then, by the fixed point property, we may write

$$(Y, f(Y), f(Y)') = (Y, Y', Y'') = (Z, Z', Z'') = (Z, f(Y), f(Y)')$$

and in the same way for  $(\tilde{Y}, f(\tilde{Y}), f(\tilde{Y})')$ . From the section about rough integration, lemma 3.11, and lemma 4.2 (Where the results are uniform in  $T \leq 1$ ) we have obtained inequalities dealing with stability of regular functions composed with controlled paths and stability of rough integration. One would think that combining the two inequalities would get us far in proving the desired inequality for RDE's. We start to use the result from lemma 3.11, we have that

$$d_{X, \tilde{X}, \alpha} \left( Z, f(Y), f(Y)'; \tilde{Z}, f(\tilde{Y}), f(\tilde{Y})' \right) \leq$$

$$C_M(d_\alpha(\mathbf{X}, \tilde{\mathbf{X}}) + |Y''_0 - \tilde{Y}''_0| + d_{X, \tilde{X}, \alpha} \left( f(Y), f(Y)', f(Y)''; f(\tilde{Y}), f(\tilde{Y})', f(\tilde{Y})'' \right) T^\alpha)$$

Further, we know from stability of composition of regular functions with controlled rough paths lemma 4.2 that

$$\begin{aligned} & d_{X, \tilde{X}, \alpha} \left( f(Y), f(Y)', f(Y)''; f(\tilde{Y}), f(\tilde{Y})', f(\tilde{Y})'' \right) \\ & \leq C_M \left( d_\alpha(\mathbf{X}, \tilde{\mathbf{X}}) + |Y_0 - \tilde{Y}_0| + |Y'_0 - \tilde{Y}'_0| + |Y''_0 - \tilde{Y}''_0| + d_{X, \tilde{X}, \alpha} \left( Y, Y', Y''; \tilde{Y}, \tilde{Y}', \tilde{Y}'' \right) \right). \end{aligned}$$

Combining the two inequalities, we see that

$$\begin{aligned} & d_{X, \tilde{X}, \alpha} \left( Y, Y', Y''; \tilde{Y}, \tilde{Y}', \tilde{Y}'' \right) \\ & \leq C_M \left( d_\alpha(\mathbf{X}, \tilde{\mathbf{X}}) + |Y_0 - \tilde{Y}_0| + |Y'_0 - \tilde{Y}'_0| + |Y''_0 - \tilde{Y}''_0| + d_{X, \tilde{X}, \alpha} \left( Y, Y', Y''; \tilde{Y}, \tilde{Y}', \tilde{Y}'' \right) T^\alpha \right). \end{aligned}$$

As we see that  $T^\alpha$  is multiplied by  $d_{X, \tilde{X}, \alpha} \left( Y, Y', Y''; \tilde{Y}, \tilde{Y}', \tilde{Y}'' \right)$  on the right hand side of the inequality, we may choose  $T_0 = T(C_M, \alpha)$  small enough, such that  $C_M T_0 \leq \frac{1}{2}$ . Then we obtain our desired result,

$$d_{X, \tilde{X}, \alpha} \left( Y, Y', Y''; \tilde{Y}, \tilde{Y}', \tilde{Y}'' \right) \leq 2C_M \left( d_\alpha(\mathbf{X}, \tilde{\mathbf{X}}) + |Y_0 - \tilde{Y}_0| + |Y'_0 - \tilde{Y}'_0| + |Y''_0 - \tilde{Y}''_0| \right).$$

And we are done.  $\square$

This concludes the part directly concerning the rough path theory. However, the results will

become very useful in treatment of fractional and multifractional Brownian motions, as well as in financial applications. As we have seen, to prove existence and uniqueness of RDE's , we require relatively high regularity of the function  $f$  in  $dY_t = f(Y_t)d\mathbf{X}_t$  (i.e ,  $f \in \mathcal{C}_b^4$ ) . As we are familiar with, the theory of SDE's or regular ODE's require only Lipschitz function, and we may conclude that with lower regularity on the driving signal require more regularity on the integrand function, at least from a rough-path-point of view. Next we will take a closer look at fractional and multifractional Brownian motions in a rough path setting, and show two Itô formulas for the respective rough processes.



## Part II

# Fractional brownian motion and Multifractional Brownian motion as Rough paths.

In this part we will take a closer look on the construction of fractional and multifractional brownian motion (fBm and mBm, respectively) as rough paths. In particular, we will study fractional and multifractional brownian motion as a geometric rough path, and then perturbate it with a function to be able to look at more general classes of fractional and multifractional brownian motions.

Essential for this section is the ability to split iterated integrals into a symmetric and an anti symmetric part, in the sense that

$$\begin{aligned} B_{s,t}^{(2)} &= \text{sym}(B_{s,t}^{(2)}) + \text{anti}(B_{s,t}^{(2)}) := S_{s,t}^{(2)} + A_{s,t}^{(2)} \quad \text{and} \\ B_{s,t}^{(3)} &= \text{sym}(B_{s,t}^{(3)}) + \text{anti}(B_{s,t}^{(3)}) := S_{s,t}^{(3)} + A_{s,t}^{(3)}. \end{aligned}$$

Where the symmetry operator  $\text{sym}$  on a matrix  $X \in \mathbb{R}^{d \times d}$  is given by  $\text{sym}(X) = \frac{1}{2}(X + X^T)$  and the anti symmetric operator  $\text{anti}$  is given by  $\text{anti}(X) = \frac{1}{2}(X - X^T)$ . The symmetry operators for a cube, i.e for an element  $X \in \mathbb{R}^{d \times d \times d}$ , is found in a similar way, by adding the transposition of the six sides of the cube, and divide by 6. The concept is difficult to visualize, but it will become more evident in the coming sections how we use this operation. Further, we can quite easily see that if we perturbate the second iterated integral from a rough path in  $T^{(2)}(\mathbb{R}^d)$ , i.e  $f_{s,t} \mapsto (B_{s,t}^{(1)}, B_{s,t}^{(2)} + f_{s,t})$  with a  $2\alpha$  regular function  $f$ , such that  $f_{s,t} = f(t) - f(s)$ , we get a new rough path, satisfying Chen's relation. This follows from the fact that  $f_{s,t} - f_{s,u} - f_{u,t} = 0$ . We will show how to continue this idea to higher order iterated integrals, and show how to choose the third iterated integral, when the second iterated integral is perturbed with a function  $f$ .

## 6 Fractional and multifractional Brownian motion

Fractional Brownian motion was first introduced - in its modern sense - in 1968 by Mandelbrot and Van Ness [3]. They proposed a modification of the brownian motion where the sample path was self similar. That is, if we let  $B_t^H$  be a fBm with Hürst parameter  $H$ , then  $\{B_t^H; 0 \leq t \leq T\} \sim \{a^{-H} B_{at}^H; 0 \leq t \leq T\}$ . Mandelbrot later argued the use of this kind of process in finance and economics, arguing as financial practitioners long before him, that financial markets are trending.

### 6.1 Fractional brownian motion.

We will introduce some essential concepts from the theory of fractional brownian motion. The focus will be on fBm's with  $H < \frac{1}{2}$ , to best accommodate later sections. In this thesis, the choice of representation of the fBm is irrelevant. We will use a rough path method to construct iterated

integrals, which turns out to only depend on the covariance function of the fBm, which by definition is equal regardless of representation. We start with a formal definition of the fBm.

**Definition 6.1.** A  $d$ - dimensional fractional Brownian motion  $\{B_t^H; t \geq 0\}$  with hurst parameter  $H \in (0, 1)$  is centered Gaussian process with the covariance function

$$E [B_t^H B_s^H] = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}) I^{d \times d}.$$

It follows that  $\{a^{-H} B_t^H; t \geq 0\}$  and  $\{B_t^H; t \geq 0\}$  has the same distribution. Further, we define the covariance of increments of fractional Brownian motions by a function  $R : [0, T]^4 \rightarrow \mathbb{R}^{d \times d}$  by

$$R \begin{pmatrix} s, & t \\ u, & v \end{pmatrix} = E [B_{s,t}^H B_{u,v}^H].$$

There are three very different scenarios to consider with respect to fractional Brownian motion, one is when  $H > \frac{1}{2}$ , one when  $H = \frac{1}{2}$ , and one where  $H < \frac{1}{2}$ . When  $H > \frac{1}{2}$  the previously mentioned Young theory lets us integrate functions of sufficient regularity, or the fBm itself, with respect to an fBm. In the case that  $H = \frac{1}{2}$ ,  $B_t^H$  is just a regular brownian motion, and the integral may be constructed using probability theory. When  $H < \frac{1}{2}$  it is more difficult to define an integral with respect to  $B_t^H$ , but it seems that rough path theory can help us with that respect. As we have seen, to be able to define the rough integral with respect to a controlled function, we need to have existing iterated integrals.

Before moving on to the construction of the iterated integrals, we want to show some nice properties of the covariance of the fBm. Let  $\frac{1}{3} < H \leq \frac{1}{2}$ , we define the function  $\sigma^2(u) := E [(B_{t,t+u}^H)^2] = u^{2H}$ , then  $\sigma^2(u)$  is concave on an interval  $[0, p]$ ,  $p > 0$ . Indeed, we may look at the second derivative and see if  $\frac{d\sigma^2(u)}{du} \leq 0$  for all  $u \in (0, p]$ . We see that  $\frac{d\sigma^2(u)}{du} = 2H(2H - 1)u^{2H-2} \leq 0$  for all  $\frac{1}{3} < H \leq \frac{1}{2}$ . We can also see that  $\sigma^2$  is non-decreasing, since we have that  $\sigma^2(t) \leq \sigma^2(s)$  for all  $t \leq s$ . Using these properties, we can find some interesting results regarding the covariance function of the fBm.

**Theorem 6.2.** Let  $\sigma^2(u) := E [(B_{t,t+u}^H)^2]$ . Then one has non-positive correlation of non-overlapping increments of the  $B^H$ , in the sense that for  $0 \leq s \leq t \leq u \leq v \leq p$ ,

$$E [B_{s,t}^H B_{u,v}^H] \leq 0.$$

In addition, for overlapping increments such that  $0 \leq s \leq u \leq v \leq t \leq p$ , we have that,

$$0 \leq E [B_{s,t}^H B_{u,v}^H] = |E [B_{s,t}^H B_{u,v}^H]| \leq E [(B_{u,v}^H)^2] = \sigma^2(u - v).$$

*Proof.* We prove the first claim first. By using the identity  $2ac = (a+b+c)^2 + b^2 - (b+c)^2 - (a+b)^2$ , and

set  $a = B_{s,t}^H, b = B_{t,u}^H, c = B_{u,v}^H$ , and we find that

$$\begin{aligned} 2E[B_{s,t}^H B_{u,v}^H] &= E[(B_{s,v}^H)^2] + E[(B_{t,u}^H)^2] - E[(B_{t,v}^H)^2] - E[(B_{s,u}^H)^2] \\ &= \frac{1}{2}((v-s)^{2H} + (u-t)^{2H} - (v-t)^{2H} - (u-s)^{2H}) \leq 0. \end{aligned}$$

This follows as we see that  $v-s \geq v-t$  and  $u-t \leq u-s$ , and the midpoint of the line interval  $[(v-t)^{2H}, (v-s)^{2H}]$  has the same coordinate as the midpoint of the interval  $[(u-t)^{2H}, (u-s)^{2H}]$ , then the result follow from concavity (see [15] lemma 7.2.7).

Now we want to show that for  $0 \leq s \leq u \leq v \leq t \leq p$ ,

$$0 \leq E[B_{s,t}^H B_{u,v}^H] = |E[B_{s,t}^H B_{u,v}^H]| \leq E[(B_{u,v}^H)^2] = \sigma^2(u-v).$$

We start by considering the identity  $2(a+b+c)b = (a+b)^2 - a^2 + (c+b)^2 - c^2$ , an let  $a = B_{s,u}^H, b = B_{u,v}^H$  and  $c = B_{v,t}^H$ , then

$$\begin{aligned} 2E[B_{s,t}^H B_{u,v}^H] &= E[(B_{s,v}^H)^2] - E[(B_{s,u}^H)^2] + E[(B_{u,t}^H)^2] - E[(B_{v,t}^H)^2] \\ &= ((v-s)^{2H} - (u-s)^{2H}) + ((t-u)^{2H} - (t-v)^{2H}) \geq 0. \end{aligned}$$

Now, using the identity  $(a+b+c)b = ab + b^2 + cb$ , with the same choice for  $a, b$ , and  $c$  as above, we see

$$E[B_{s,t}^H B_{u,v}^H] = E[B_{s,u} B_{u,v}] + E[(B_{u,v}^H)^2] + E[B_{v,t}^H B_{u,v}^H] \leq E[(B_{u,v}^H)^2] = (v-u)^{2H}.$$

Where we have used that the non-overlapping increments has  $\leq 0$  covariance, as proved in the beginning.  $\square$

The properties of the covariance function for fBm's allows for a nice construction of the iterated integrals using rough path theory.

To simplify notation, we will let  $X$  be a  $d$ -dimensional fractional Brownian motion of Hürst parameter  $H \in (\frac{1}{3}, \frac{1}{2}]$ , with independent components, i.e  $E[X_{s,t}^i X_{s,t}^j] = 0$ . By the covariance function  $R$  of  $X$  we are able to construct the iterated integrals of an fBm. From the properties discovered in the last lemma, we see that  $R\left(\begin{smallmatrix} s & t \\ u & v \end{smallmatrix}\right) \leq R\left(\begin{smallmatrix} u & v \\ u & v \end{smallmatrix}\right)$  for all  $0 \leq s \leq u \leq v \leq t \leq p$  for some  $p > 0$ . In [16] chp. 10, the authors shows that under some assumptions on  $R$ , we are able to construct the iterated integral for a general centered Gaussian process with stationary increments. We will apply this method to fBm's and show existence of the second iterated integral  $X_{s,t}^{(2)}$ . We start with a definition of the variation norm of the covariance  $R$ .

**Definition 6.3.** Let  $X$  be a  $d$ -dimensional fBm with covariance function  $R$  as defined above. We

define the  $\rho$  variation norm of the covariance function as follows,

$$\| R \|_{I \times I'; \rho} := \left( \sup_{\substack{\mathcal{P} \subset I \\ \mathcal{P}' \subset I'}} \sum_{\substack{[u, v] \in I \\ [u', v'] \in I'}} \left| R \begin{pmatrix} u & v \\ u' & v' \end{pmatrix} \right|^\rho \right)^{\frac{1}{\rho}}.$$

Where  $I$  and  $I'$  is intervals of  $[0, T]$ .

The  $\rho$  variation norm plays an important role, and we will later see that if the  $\rho$  variation norm is finite, we are able to show that the iterated integrals of  $X$  is bounded by this norm. We will therefore show that the fBm has a covariance function of finite  $\rho$  variation.

**Lemma 6.4.** *Let  $X$  be a fBm with  $H \in (\frac{1}{3}, \frac{1}{2}]$ , and let  $R$  be the covariance function of  $X$  defined above. Let  $\rho = \frac{1}{2H}$ , then*

$$\| R \|_{[s, t]^2; \rho} \leq M |t - s|^{\frac{1}{\rho}}$$

for  $t - s \leq p$ , and  $p > 0$ , and  $M := M(\rho)$ .

*Proof.* Let  $[s, t]$  be some interval such that  $t - s \leq p$ , for  $p > 0$ . We then construct two dissections of  $[s, t]$ , such that  $D_1 = \{t_i\}_{i=1}^n$  and  $D_2 = \{t_j\}_{j=1}^m$ . For a fixed  $i$ ,  $1 \leq i \leq n$ , we have that

$$\begin{aligned} \sum_{t_j \in D_2} |E[X_{t_i, t_{i+1}} X_{t_j, t_{j+1}}]|^\rho &\leq \|E[X_{t_i, t_{i+1}} X_{\cdot, \cdot}]\|_{[s, t]; \rho}^\rho \\ &\leq 3^{\rho-1} \left( \|E[X_{t_i, t_{i+1}} X_{\cdot, \cdot}]\|_{[s, t_i]; \rho}^\rho + \|E[X_{t_i, t_{i+1}} X_{\cdot, \cdot}]\|_{[t_i, t_{i+1}]; \rho}^\rho + \|E[X_{t_i, t_{i+1}} X_{\cdot, \cdot}]\|_{[t_{i+1}, t]; \rho}^\rho \right), \end{aligned}$$

where we have used that  $(a + b + c)^\rho \leq 3^{\rho-1} (a^\rho + b^\rho + c^\rho)$ , for  $\rho \geq 1$ . Let us first look at  $\|E[X_{t_i, t_{i+1}} X_{\cdot, \cdot}]\|_{[s, t_i]; \rho}$ . We see that

$$\begin{aligned} \|E[X_{t_i, t_{i+1}} X_{\cdot, \cdot}]\|_{[s, t_i]; \rho}^\rho &\leq |E[X_{t_i, t_{i+1}} X_{s, t_i}]|^\rho \leq \left( |E[X_{t_i, t_{i+1}} X_{s, t_{i+1}}]| + |E[(X_{t_i, t_{i+1}})^2]| \right)^\rho \\ &\leq (t_{i+1} - t_i)^{2H\rho} = (t_{i+1} - t_i). \end{aligned}$$

Where we have used the properties of the fBm from theorem 6.2 and  $\rho = \frac{1}{2H}$ . The same way as above we will find that

$$\|E[X_{t_i, t_{i+1}} X_{\cdot, \cdot}]\|_{[t_{i+1}, t]; \rho}^\rho \leq (t_{i+1} - t_i).$$

Next, we look at  $\|E[X_{t_i, t_{i+1}} X_{\cdot, \cdot}]\|_{[t_i, t_{i+1}]; \rho}$ . Let  $D_3$  be the dissection  $D_2 \cap [t_i, t_{i+1}]$ , then

$$\|E[X_{t_i, t_{i+1}} X_{\cdot, \cdot}]\|_{[t_i, t_{i+1}]; \rho}^\rho \leq \sup_{D_3} \sum_{t_j \in D_3} |E[X_{t_i, t_{i+1}} X_{t_j, t_{j+1}}]|^\rho$$

$$\leq \sup_{D_3} \sum_{t_j \in D_3} \left| (t_{j+1} - t_j)^{2H} \right|^\rho \leq (t_{i+1} - t_i).$$

Where we have used the result of the covariance of overlapping increments in theorem 6.2, and  $\rho = \frac{1}{2H}$ . Putting the three estimates together, we see that

$$\sup_{D_2} \sum_{t_j \in D_2} |E [X_{t_i, t_{i+1}} X_{t_j, t_{j+1}}]|^\rho \leq M(t_{i+1} - t_i).$$

Then by taking the supremum over partitions of  $\{t_i\}$ , we find that

$$\sup_{D_1, D_2} \sum_{t_i \in D_1, t_j \in D_2} |E [X_{t_i, t_{i+1}} X_{t_j, t_{j+1}}]|^\rho \leq M |t - s|.$$

Hence,  $\|R\|_{[s, t]^2; \rho} \leq M |t - s|^{\frac{1}{\rho}}$  with  $\rho = \frac{1}{2H}$ .  $\square$

As we can see,  $\|R\|_{[s, t]^2; \rho} \lesssim |t - s|^{2H}$  due to the concavity and non decreasing nature of the covariance of the fBm. Using this, we are able to define an iterated integral. In the sense that if we let  $X$  and  $\tilde{X}$  be two independent fBm's, define the integral over a partition  $\mathcal{P}$  of  $[0, T]$  by

$$\int_{\mathcal{P}} X_{0,r} d\tilde{X}_r := \sum_{[u,v] \in \mathcal{P}} X_{0,u} \tilde{X}_{u,v},$$

then the  $L^2$  norm of this object is conveniently bounded by the variation norm of the covariance function. We will prove this in the next lemma.

**Lemma 6.5.** *Let  $X$  and  $\tilde{X}$  be two fBm's with  $H \in (\frac{1}{3}, \frac{1}{2}]$ , with respective covariance functions  $R$  and  $\tilde{R}$  of finite  $\rho$  variation. We define the integral of  $X$  with respect to  $\tilde{X}$  as*

$$\int_{\mathcal{P}} X_{0,r} d\tilde{X}_r := \sum_{[s,t] \in \mathcal{P}} X_{0,s} \tilde{X}_{s,t}.$$

Then,

$$\sup_{\mathcal{P} \subset [0,1]} E \left[ \left( \int_{\mathcal{P}} X_{0,r} d\tilde{X}_r \right)^2 \right] \leq C \|R\|_{\rho; [0,1]^2} \|\tilde{R}\|_{\rho; [0,1]^2}$$

Where  $C$  is a constant depending on  $\rho$ .

*Proof.* We see that

$$\begin{aligned} E \left[ \left( \int_{\mathcal{P}} X_{0,r} d\tilde{X}_r \right)^2 \right] &= E \left[ \left( \sum_{[s,t] \in \mathcal{P}} X_{0,s} \tilde{X}_{s,t} \right)^2 \right] = E \left[ \sum_{[s,t], [s',t'] \in \mathcal{P}} X_{0,s} X_{0,s'} \tilde{X}_{s,t} \tilde{X}_{s',t'} \right] \\ &= \sum_{[s,t], [s',t'] \in \mathcal{P}} E [X_{0,s} X_{0,s'}] E [\tilde{X}_{s,t} \tilde{X}_{s',t'}] = \sum_{[s,t], [s',t'] \in \mathcal{P}} R \begin{pmatrix} 0 & s \\ 0 & s' \end{pmatrix} \tilde{R} \begin{pmatrix} s & t \\ s' & t' \end{pmatrix}. \end{aligned}$$

By the generalization of Young's inequality to multidimensional processes given in [25], we have that

$$\sup_{\mathcal{P} \subset [0,1]} \left| \sum_{[s,t],[s',t'] \in \mathcal{P}} R \begin{pmatrix} 0, & s \\ 0, & s' \end{pmatrix} \tilde{R} \begin{pmatrix} s, & t \\ s', & t' \end{pmatrix} \right| \leq C \|R\|_{\rho;[0,1]^2} \|\tilde{R}\|_{\rho;[0,1]^2}.$$

And we are done.  $\square$

Now that we have a sense of an integral over a partition, we want to see if this converges as the length of the intervals in  $\mathcal{P}$  tends to 0. Under the construction from the lemma above, we are able to define a iterated integral of the process in the following way, on an interval  $[0, T] \subset [0, 1]$ .

**Proposition 6.6.** *Under the same assumptions as in lemma 6.5 and assume  $\rho = \frac{1}{2H}$ , we have that*

$$\lim_{\varepsilon \rightarrow 0} \sup_{\substack{\mathcal{P}, \mathcal{P}' \subset [0,1] \\ |\mathcal{P}| \vee |\mathcal{P}'| < \varepsilon}} E \left[ \left( \int_{\mathcal{P}} X_{0,r} d\tilde{X}_r - \int_{\mathcal{P}'} X_{0,r} d\tilde{X}_r \right)^2 \right] = 0.$$

Therefore,  $\int_0^1 X_{0,r} d\tilde{X}_r$  exists as the  $L^2$  limit of  $\int_{\mathcal{P}} X_{0,r} d\tilde{X}_r$  as  $|\mathcal{P}| \downarrow 0$ , and

$$E \left[ \left( \int_0^1 X_{0,r} d\tilde{X}_r \right)^2 \right] \leq C \|R\|_{\rho;[0,1]^2} \|\tilde{R}\|_{\rho;[0,1]^2}.$$

*Proof.* Let  $\mathcal{P}_1$  and  $\mathcal{P}_2$  be two partitions on  $[0, 1]$  and assume  $\mathcal{P}_1$  refines  $\mathcal{P}_2$ . Then for an interval  $[u, v] \subset [0, 1]$ , define  $\mathcal{P} \cap [u, v] = \left\{ [u, v] \cap [u', v'] : [u', v'] \in \mathcal{P} \text{ and } [u, v] \cap [u', v'] \neq \emptyset \right\}$ . We have that

$$\int_{\mathcal{P}_1} X_{0,r} d\tilde{X}_r - \int_{\mathcal{P}_2} X_{0,r} d\tilde{X}_r = \sum_{[u,v] \in \mathcal{P}_2} \int_{\mathcal{P} \cap [u,v]} X_{u,r} d\tilde{X}_r =: I$$

We want to show that  $I \rightarrow 0$  in  $L^2$  as  $|\mathcal{P}_2| = |\mathcal{P}_1| \vee |\mathcal{P}_2| \rightarrow 0$ . We look at  $I$  in the  $L^2$  norm,

$$\begin{aligned} E \left[ (I)^2 \right] &= \sum_{\substack{[u,v] \in \mathcal{P}_2 \\ [u',v'] \in \mathcal{P}_2}} E \left[ \int_{\mathcal{P} \cap [u,v]} X_{u,r} d\tilde{X}_r \int_{\mathcal{P} \cap [u',v']} X_{u',r} d\tilde{X}_r \right] \\ &= \sum_{\substack{[u,v] \in \mathcal{P}_2 \\ [u',v'] \in \mathcal{P}_2}} \sum_{\substack{[s,t] \in \mathcal{P}_1 \cap [u,v] \\ [s',t'] \in \mathcal{P}_1 \cap [u',v']}} R \begin{pmatrix} u, & s \\ u', & s' \end{pmatrix} \tilde{R} \begin{pmatrix} s, & t \\ s, & t \end{pmatrix} \\ &\leq \sum_{\substack{[u,v] \in \mathcal{P}_2 \\ [u',v'] \in \mathcal{P}_2}} C(\rho) \|R\|_{[u,v] \times [u',v']; \rho} \|\tilde{R}\|_{[u,v] \times [u',v']; \rho}. \end{aligned}$$

Where we have used the generalization of Young's maximal inequality due to Towghi [25] in the last inequality. Since  $X$  and  $\tilde{X}$  are fBm's we know that  $\|R\|_{[s,t]^2;\rho} \leq M |t-s|^{\frac{1}{\rho}}$ . Assume, without loss of generality, that  $|v-u| \geq |v'-u'|$ . Then  $\|R\|_{[u,v] \times [u',v'];\rho} \leq \|R\|_{[u,v]^2;\rho}$ , and we find that

$$\begin{aligned} \sum_{\substack{[u,v] \in \mathcal{P}_2 \\ [u',v'] \in \mathcal{P}_2}} \|R\|_{[u,v] \times [u',v'];\rho} \|\tilde{R}\|_{[u,v] \times [u',v'];\rho} &\leq \sum_{\substack{[u,v] \in \mathcal{P}_2 \\ [u',v'] \in \mathcal{P}_2}} \|R\|_{[u,v]^2;\rho} \|\tilde{R}\|_{[u,v]^2;\rho} \\ &\leq \sum_{\substack{[u,v] \in \mathcal{P}_2 \\ [u',v'] \in \mathcal{P}_2}} M^2 (v-u)^{\frac{2}{\rho}}. \end{aligned}$$

Now, by assumption we know  $\rho = \frac{1}{2H}$ , and hence,

$$\lim_{\varepsilon \rightarrow 0} \sup_{\substack{\mathcal{P}_1, \mathcal{P}_2 \subset [0,1] \\ |\mathcal{P}_1| \vee |\mathcal{P}_2| < \varepsilon}} E \left[ (I)^2 \right] \leq \lim_{\varepsilon \rightarrow 0} \sup_{\substack{\mathcal{P}_1, \mathcal{P}_2 \subset [0,1] \\ |\mathcal{P}_1| \vee |\mathcal{P}_2| < \varepsilon}} \sum_{\substack{[u,v] \in \mathcal{P}_2 \\ [u',v'] \in \mathcal{P}_2}} M^2 (v-u)^{4H} = 0,$$

as long as  $H > \frac{1}{4}$ . The last inequality in the theorem follows from the results in lemma 6.5.  $\square$

Proposition 6.6 shows that the integral of a fractional Brownian motion with respect to another fractional Brownian motion exists on  $[0, 1]$  and is equal regardless of the chosen partition  $\mathcal{P}$  of  $[0, 1]$ . The same result holds for an fBm on  $[s, t]$  with  $t-s \leq 1$ , since by reparametrization of the fBm, we see that  $(B_s : 0 \leq s \leq 1) \sim (B_{s+\theta(t-s)} : 0 \leq \theta \leq 1)$ . As lemma 6.5 shows, we have chosen the left point evaluation in our Riemann sum, but the same result would hold for right point or mid point evaluation. The result gives rise to the next theorem, which will give us a canonical way of determining the second iterated integral of a fBm.

**Theorem 6.7.** *Let  $\{X_t\}_{0 \leq t \leq T}$  be a  $d$ -dimensional fBm with  $H \in (\frac{1}{3}, \frac{1}{2}]$ , such that each component  $X_t^i$  and  $X_t^j$  are independent. Define, for  $1 \leq i \leq j \leq d$  and  $0 \leq s \leq t \leq T$ , in  $L^2$  sense*

$$X_{s,t}^{(2),i,j} = \lim_{|\mathcal{P}| \rightarrow 0} \int_{\mathcal{P}} (X_r^i - X_s^i) dX_r^j.$$

Then

$$X_{s,t}^{(2),i,i} = \frac{1}{2} (X_{s,t}^i)^2 \quad X_{s,t}^{(2),i,j} = -X_{s,t}^{(2),j,i} + X_{s,t}^i X_{s,t}^j,$$

and the following properties hold: a) for every  $q \in [1, \infty)$  there exists  $C_1 = C_1(q, \rho, d, T)$  such that for all  $0 \leq s \leq t \leq T$

$$E \left[ |X_{s,t}|^{2q} + |X_{s,t}^{(2)}|^q \right] \leq C_1 M^q |t-s|^{\frac{q}{\rho}}.$$

b) There exists a continuous modification of  $X^{(2)}$ , and for any  $\alpha < \frac{1}{2\rho}$  and  $q \in [1, \infty)$  there exists a  $C_2 = C_2(q, \rho, d, \alpha)$  s.t

$$E \left[ \|X\|_\alpha^{2q} + \|X^{(2)}\|_{2\alpha}^q \right] \leq C_2 M^q.$$

We will always consider this continuous version of  $X^{(2)}$ .

c) For any  $\alpha < \frac{1}{2\rho}$ , with probability one, the pair  $(X^{(1)}, X^{(2)})$  satisfies Chen's relation 2.1, the analytical conditions, and geometric conditions. In particular, for  $\rho \in [1, \frac{3}{2})$  and any  $\alpha \in (\frac{1}{3}, \frac{1}{2\rho})$  we have  $(X^{(1)}, X^{(2)}) \in \mathcal{C}_g^\alpha$  almost surely.

*Proof.* We know that the definition  $X_{s,t}^{(2),i,j} = \lim_{|\mathcal{P}| \rightarrow 0} \int_{\mathcal{P}} (X_r^i - X_s^i) dX_r^j$  is well defined by proposition 6.6. Then, the choice  $X_{s,t}^{(2),i,i} = \frac{1}{2}(X_{s,t}^i)^2$  and  $X_{s,t}^{(2),i,j} = -X_{s,t}^{(2),j,i} + X_{s,t}^i X_{s,t}^j$  is the only (canonical) choice that satisfies Chen's relation. Point a) and b) now follows naturally from the fact that  $E \left[ (X_{s,t})^2 \right] = \frac{1}{2}(t-s)^{2H}$  and

$$E \left[ \left( \int_s^t X_{s,r} dX_r \right)^2 \right] \leq C \|R\|_{\rho;[s,t]^2}^2 \leq CM(t-s)^{4H},$$

and by the  $L^2$ - $L^q$  equivalence of norm on the second Wiener-Itô chaos, and then Kolmogorov's continuity theorem. The lift to a geometric path follows from the definition of the second iterated integral, and the fact that this choice satisfies Chen's relation, and the analytical bounds. We see that  $\text{Sym} \left( X_{s,t}^{(2)} \right) = \frac{1}{2} X_{s,t} \otimes X_{s,t}$ , and hence Geometric.  $\square$

*Remark 6.8.* It now follows that we have a canonical construction of the second iterated integral of a fractional Brownian motion. The construction is done under the fact that  $B_{s,t}^H$  is of  $H$ -regularity when  $H \in (\frac{1}{3}, \frac{1}{2}]$ . When  $H \in (\frac{1}{4}, \frac{1}{3}]$  there exists a canonical choice of  $X^{(2)}$  and  $X^{(3)}$  such that the lift  $(X^{(1)}, X^{(2)}, X^{(3)}) \in \mathcal{C}_g^\alpha$ , see the book by P. Friz, And N. Victoir, [17] chp. 15.3. In later sections when developing an Itô formula for fBm's with  $H \in (\frac{1}{4}, \frac{1}{3}]$  we will use this geometric lift.

We will now move on to construction of a multifractional Brownian motion, which is similar to the fBm, but with a Hürst parameter which is dependent on time.

## 6.2 Multifractional Brownian motion

We will construct the multifractional brownian motion a little different than how we constructed the fBm. The reason is that we desire the mBm to be a kind of fBm which need to be continuous with respect to  $H$  on the compact interval  $[a, b] \subset (0, 1)$ . As there exist many representations of fBm's where this is not the case, we need to be more careful with the choice of representation. We will base this introduction on the pioneering works of Lévy Véhel and Peltier [20]. However, we have observed that in [10] the authors use a Volterra type of fBm representation and claims result is valid for all  $H$  in a compact subinterval  $[a, b] \subset (0, 1)$ , but the representation used seem to be tend to infinity as  $H$  goes below  $\frac{1}{2}$ . Therefore, we will use the representation first introduced by



Mandelbrot and Van Ness (1968) [3] for fractional Brownian motion,

$$B_H(t) :=$$

$$\frac{1}{\Gamma(H + \frac{1}{2})} \left\{ \int_{-\infty}^0 \left[ (t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}} \right] dW(s) + \int_0^t (t-s)^{H-\frac{1}{2}} dW(s) \right\},$$

where  $W$  represents a Wiener process on  $\mathbb{R}^d$ , and  $\Gamma$  represents the gamma function. We extend this, and let  $h : [0, T] \rightarrow [a, b] \subset (0, 1)$  be a  $\beta$  regular function, i.e  $h \in \mathcal{C}^\beta$ . We denote the multifractional Brownian motion by

$$B_t^h = B(t, h(t)) :=$$

$$\frac{1}{\Gamma(h(t) + \frac{1}{2})} \left\{ \int_{-\infty}^0 \left[ (t-s)^{h(t)-\frac{1}{2}} - (-s)^{h(t)-\frac{1}{2}} \right] dW(s) + \int_0^t (t-s)^{h(t)-\frac{1}{2}} dW(s) \right\}.$$

For applications, we are also interested in the covariance of a multifractional Brownian motion. In the paper by Ayache et.al. [1], the authors gives an explicit expression for this function. We will sum it up in a proposition as follows.

**Proposition 6.9.** *Let  $B_t^h$  be a standard multifractional Brownian motion (i.e s.t.  $\text{Var}(B_1^h) = 1$ ) with time varying hurst parameter  $h : [0, T] \rightarrow [a, b] \subset (0, 1)$ . Then the covariance is given by*

$$E \left[ B_t^h B_s^h \right] = D(t, s) \left( |t|^{h(t)+h(s)} + |s|^{h(t)+h(s)} - |t-s|^{h(t)+h(s)} \right).$$

Where  $D : [0, T]^2 \rightarrow \mathbb{R}$  is a deterministic function.

*Proof.* The proof from [1] is given by a complex representation of the mBm. That is, we may represent an mBm  $B_t^{h(t)}$  as

$$\tilde{B}_t^{h(t)} := \frac{1}{C(h(t))} \int_{\mathbb{R}} \frac{e^{it\xi-1}}{|\xi|^{h(t)+\frac{1}{2}}} dB(\xi),$$

where  $B(\xi)$  is a complex valued Brownian motion and  $C(h(t)) = \sqrt{\frac{\pi}{h(t)\Gamma(2h(t)+1)\sin(\pi h(t))}}$ . The two representations  $B_t^h$  and  $\tilde{B}_t^h$  are equal in law up to a multiplicative deterministic function, see [6] Thm. 1. Therefore, we may conduct the proof under the complex-representation given by  $\tilde{B}_t^{h(t)}$  and scale our result by a deterministic function. By definition of the covariance, we have that

$$\begin{aligned} E \left[ \tilde{B}_t^{h(t)} \tilde{B}_s^{h(s)} \right] &= \frac{1}{C(h(t))C(h(s))} E \left[ \int_{\mathbb{R}} \frac{e^{it\xi-1}}{|\xi|^{h(t)+\frac{1}{2}}} dB(\xi) \int_{\mathbb{R}} \frac{e^{is\xi-1}}{|\xi|^{h(s)+\frac{1}{2}}} dB(\xi) \right] \\ &= \frac{1}{C(h(t))C(h(s))} \int_{\mathbb{R}} E \left[ \frac{(e^{it\xi-1})(e^{is\xi-1})}{|\xi|^{h(t)+h(s)+1}} \right] d\xi. \end{aligned}$$

Fix  $t, s$  and let  $B_t^H$  be a standard fractional brownian motion with complex representation, as above.

Then we know

$$\begin{aligned} E [B_t^H B_s^H] &= \frac{1}{C(H)C(H)} \int_{\mathbb{R}} E \left[ \frac{(e^{it\xi}-1)(e^{is\xi}-1)}{|\xi|^{2H+1}} \right] d\xi \\ &= \frac{1}{2} (|t|^{2H} + |s|^{2H} - |t-s|^{2H}). \end{aligned}$$

Now, if we choose  $H = \frac{1}{2}(h(t) + h(s))$  we see that,

$$\begin{aligned} C(\frac{1}{2}(h(t) + h(s)))^2 E [B_t^H B_s^H] &= \int_{\mathbb{R}} E \left[ \frac{(e^{it\xi}-1)(e^{is\xi}-1)}{|\xi|^{h(t)+h(s)+1}} \right] d\xi \\ &= C(h(t))C(h(s)) E [\tilde{B}_t^{h(t)} \tilde{B}_s^{h(s)}] \end{aligned}$$

which implies that

$$E [\tilde{B}_t^{h(t)} \tilde{B}_s^{h(s)}] = \frac{C(\frac{1}{2}(h(t) + h(s)))^2}{C(h(t))C(h(s))} (|t|^{h(t)+h(s)} + |s|^{h(t)+h(s)} - |t-s|^{h(t)+h(s)}).$$

Then, as already mentioned, there exists a deterministic function  $g(t, s)$  such that

$$E [B_t^h B_s^h] = g(t, s) E [\tilde{B}_t^{h(t)} \tilde{B}_s^{h(s)}],$$

therefore  $D(t, s) = g(t, s) \frac{C(\frac{1}{2}(h(t)+h(s)))^2}{C(h(t))C(h(s))}$ , and we are done.  $\square$

The mBm is a non-stationary process, as the Hürst parameter, describing the auto covariance is dependent on time. We are interested in finding the regularity of the mBm, such that we can use rough path theory. The next lemma will give a regularity estimate, and is based on the proof of Peltier and Lévy Vehél in [20], where we use the Mandelbrot- Van Ness representation.

**Lemma 6.10.** *Let  $B_t^h$  be given as defined above. Let  $[a, b] \subset (0, 1)$ , s.t  $b - a < 1$ , and let  $h : [0, T] \rightarrow [a, b]$  be a  $\beta$ -Hölder continuous function for all two times  $(t, t') \in [0, T]^2$ , i.e  $|h(t) - h(t')| \lesssim |t - t'|^\beta$ . Let  $B_t^h = \frac{1}{\Gamma(h(t)+\frac{1}{2})} \{P_1(t) + P_2(t)\}$ , where  $P_1(t) = \int_{-\infty}^0 f(t, s) dW(s)$ , and  $P_2(t) = \int_0^t g(t, s) dW(s)$ , for  $g(t, s) = (t - s)^{h(t)-\frac{1}{2}}$ . Define  $f(t, s) = g(t, s) - (-s)^{h(t)-\frac{1}{2}}$  for each  $s < 0 < t$ . Then there exist  $\gamma_1, \gamma_2 > 0$  such that*

$$\begin{aligned} E [(P_1(t) - P_1(t'))^p] &\leq \gamma_1 |t - s|^{p\beta} & \text{for } p > \frac{1}{\beta} \\ E [(P_2(t) - P_2(t'))^p] &\leq \gamma_2 |t - s|^{p \min(\frac{1}{2}, \beta, a)} & \text{for } p > \frac{1}{\min(\frac{1}{2}, \beta, a)}, \end{aligned}$$

and hence,  $B_t^h \in \mathcal{C}^{\min(\frac{1}{2}, \beta, a)-}$ .

*Proof.* We present here just the main ideas behind the rather long proof from [20]. The crucial elements lies in the following facts: We have that  $P_1(t) \sim N(\mu, \sigma^2)$  and  $P_2(t) \sim N(\tilde{\mu}, \tilde{\sigma}^2)$ . Let  $X$  be centered normal random variable i.e  $X \sim N(\mu, \sigma^2)$ , Then for  $1 \leq n \in \mathbb{N}$ , the following inequality

holds:

$$E[|X|^n] \leq \sigma^n \frac{2^{n/2} \Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}} \Phi\left(-\frac{n}{2}, \frac{1}{2}; -\frac{\mu^2}{2\sigma^2}\right)$$

Where  $\Phi$  denotes Kummer's confluent hypergeometric functions

$$\Phi(x, y, z) := \sum_{n=0}^{\infty} \frac{\frac{\Gamma(x+n)}{\Gamma(x)} z^n}{\frac{\Gamma(y+n)}{\Gamma(y)} n!}.$$

(see [26] for full proof). From this, we may use Kolmogorov's continuity theorem, and therefore the continuous version exists.  $\square$

As we know that the process has a Hürst parameter which is changing in time, we are not only interested in the global regularity of the mBm, but locally we may obtain a regularity estimate, and see that the regularity is a function in time. We give the following definition, and proposition.

**Definition 6.11.** We define the local Hölder exponent of a stochastic process  $X_t(\omega)$  at time  $t$  is given by

$$\theta_X(s, \omega) = \sup \left\{ \theta : \lim_{h \rightarrow 0} \frac{|X_{t+h}(\omega) - X_t(\omega)|}{|h|^\theta} = 0 \right\}.$$

**Proposition 6.12.** Let  $h$  be of  $\beta$  regularity, and  $h : [0, T] \rightarrow [a, b] \subset (0, \min(1, \beta))$ , and  $B_t^h$  be a mBm. For each  $t_0 \in [0, T]$  we have with probability one that

$$\theta_{B^h}(t_0, \omega) = h(t_0).$$

*Proof.* See [20] for proof.  $\square$

This tells us that at each point the mBm is of a certain regularity. If  $\theta_{B^h}(t_0, \omega)$  is above  $\frac{1}{2}$  then the mBm behaves more regular than a usual Brownian motion, and positively autocorrelated sample paths. The opposite is true when  $\theta_{B^h}(t_0, \omega)$  is less than  $\frac{1}{2}$ . We clearly see that the regularity of the function  $\theta_{B^h}$  is equal to the regularity of the function  $h$ . Indeed, we have that  $|\theta_{B^h}(t, \omega) - \theta_{B^h}(s, \omega)| = |h(t) - h(s)| \lesssim |t - s|^\beta$ .

### A note on the construction of iterated integrals of mBm's.

There is currently no straight forward way to construct iterated integrals of mBm's in higher dimensions. However, in this note we will present an idea on how one may go forth to construct these integrals. If we let the mBm  $B_t^h$  be given by a simple function of fractional Brownian motions, in the sense that, if  $\{t_i\}_{i=1}^n$  is a dissection of  $[0, T]$ , let  $\tilde{B}(t, H)$  be a fBm, then

$$B^{(n)}(t, h(t)) := \sum_{i=1}^n 1_{[t_i, t_{i+1})}(t) \tilde{B}(t, h(t_i)).$$

One would then investigate if  $B^{(n)}(t, h(t))$  converges to  $B_t^h$  in  $L^2$  as  $n \rightarrow \infty$  for any dissection  $\{t_i\}_{i=1}^n$  such that the length of all intervals  $[t_i, t_{i+1})$  converges to 0 as  $n \rightarrow \infty$ . We see that each term  $\tilde{B}(t, h(t_i))$  is a fBm on the interval  $[t_i, t_{i+1})$ . If it converges, we could try to construct the iterated integral as

$$\int_0^T B_{0,r}^h dB_r^h := \lim_{n \rightarrow \infty} \sum_{i=1}^n \int_0^T 1_{[t_i, t_{i+1})}(t) \tilde{B}(r, h(t_i)) - \tilde{B}(0, h(0)) d\tilde{B}(r, h(t_i)).$$

As we want  $h$  to vary around  $\frac{1}{2}$ , we would encounter to different types of integrals depending on the Hürst parameter  $h(t_i)$  of the fBm is below or above  $\frac{1}{2}$ . A way to deal with this, would be to introduce stopping times in the following way.

**Definition 6.13.** Let  $h \in \mathcal{C}^\beta$  for  $\beta > \sup_{t \in [0, T]} h(t)$ , and  $h(0) > \frac{1}{2}$ . Define the stopping times  $\{\nu_n\}_{n=1}^N$  and  $\{\tau_n\}_{n=1}^N$  such that ,

$$\begin{aligned} \tau_1 &:= \inf \{t > 0 : h(t) > \frac{1}{2}\} \\ \nu_1 &:= \inf \{t > 0 : h(t) \leq \frac{1}{2}\}, \end{aligned}$$

and then iteratively,

$$\begin{aligned} \tau_n &= \inf \{t > \tau_{n-1} : h(t) > \frac{1}{2}\} \wedge T \\ \nu_n &= \inf \{t > \nu_{n-1} : h(t) \leq \frac{1}{2}\} \wedge T. \end{aligned}$$

And define  $\tau_{N+1} := T$ . We can see that  $\nu_N, \tau_N \rightarrow T$  as  $N \rightarrow \infty$ . Then the partitions

$$\begin{aligned} \mathcal{P}_N^{h > \frac{1}{2}} &= \left\{ [\tau_1, \nu_1], [\tau_2, \nu_2], [\tau_3, \nu_3], \dots, [\tau_N, \nu_N] \right\} \\ \mathcal{P}_N^{h \leq \frac{1}{2}} &= \left\{ [\nu_1, \tau_2], [\nu_2, \tau_3], [\nu_3, \tau_4], \dots, [\nu_{N-1}, \tau_N], [\nu_N, \tau_{N+1}] \right\}, \end{aligned}$$

are well defined partitions of  $[0, T]$ .

We assumed  $h(0) > \frac{1}{2}$ , this is just to keep the right order on the intervals in the partition. If  $h(0) < \frac{1}{2}$ , we just need to re-arrange the order of the partitions, and the problem is solved.

From the proposition we see that  $N$  can be a random number, depending on the function  $h$ , and how many times it “crosses”  $\frac{1}{2}$ . We say random number, as we may choose  $h$  to be a stochastic process. Now we have three partitions on  $[0, T]$  to deal with. One partition  $D_n = \{[t_i, t_{i+1}] : i = 1, 2, \dots, n\}$ , and the two partitions  $\mathcal{P}_N^{h \leq \frac{1}{2}}$  and  $\mathcal{P}_N^{h > \frac{1}{2}}$ . Combining these, we see that,

$$\begin{aligned} &\sum_{i=1}^n \int_0^T 1_{[t_i, t_{i+1})}(t) \tilde{B}(r, h(t_i)) - \tilde{B}(0, h(0)) d\tilde{B}(r, h(t_i)) \\ &= \sum_{k=1}^N \sum_{i=1}^n \int_0^T 1_{[\tau_k, \nu_k] \cap [t_i, t_{i+1})}(t) \tilde{B}(r, h(t_i)) - \tilde{B}(0, h(0)) d\tilde{B}(r, h(t_i)) \end{aligned}$$

$$+ \sum_{k=1}^N \sum_{i=1}^n \int_0^T 1_{[\nu_k, \tau_{k+1}] \cap [t_i, t_{i+1})}(t) \tilde{B}(r, h(t_i)) - \tilde{B}(0, h(0)) d\tilde{B}(r, h(t_i)).$$

Now, the integrals in the first double sum are integrals where the hurst parameter  $h(t_i) > \frac{1}{2}$  on the whole interval  $[t_i, t_{i+1})$ , and hence Young's integral theory will be used to construct the integrals. In the second double sum  $h(t_i) \leq \frac{1}{2}$ , and fortunately, as the process on the interval  $[\nu_k, \tau_{k+1}] \cap [t_i, t_{i+1})$  is a fBm with  $h(t_i)$  as regularity parameter, we are able to construct the iterated integrals from the covariance function of  $B(t, h(t_i))$  on the respective interval. Therefore both integrals exist, and the “discrete” mBm integral then exist. Now, if one is able to show that this converges in  $L^2$  to the the mBm integral, one would have constructed the iterated integrals for mBm's. We believe that this is possible, And hope to write another article later on the possibility.

In this thesis we are mostly concerned with rough path theory, and we want to construct an Itô formula for fBm's and mBm's. In the case of mBm's, we really don't need the iterated integral of the process, we will only deal with the symmetric part of the geometric choice of iterated integral. The symmetric part of the iterated integral of a geometric path, as we have seen before, is given by the increment of the first order process to the second power. Therefore, as long as we have a proper construction of the mBm process itself, the Itô formula will hold.

## 7 Itô formula for reduced multifractional rough paths when $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ .

As we have discussed in previous sections, the geometric choice for the iterated integrals, is the one which obeys the usual rules of simple calculus. Therefore we will give a definition of what we call a reduced rough path, and soon see the link to geometric rough paths. We begin by introducing the concept on paths in the space  $\mathcal{C}^\alpha$ ,  $\alpha \in (\frac{1}{3}, \frac{1}{2}]$  and show the Itô formula of Friz and Hairer [16] applied to fractional Brownian motion. Next, we develop an Itô formula for mBm's with  $h : [0, T] \rightarrow [\frac{1}{3} + \varepsilon, 1 - \varepsilon]$ ,  $h \in \mathcal{C}^1$  for some small  $\varepsilon > 0$ . First we will define the space of reduced rough paths. A reduced rough path is a rough path where we ignore the anti symmetric part of the iterated integrals. Formally, a definition is given as follows.

**Definition 7.1.** We define the tuple  $(X^{(1)}, S^{(2)})$  to be a reduced rough path if  $S^{(2)}$  takes values in  $\text{Sym}(\mathbb{R}^{d \times d})$  and the following conditions holds: a) A reduced Chen relation

$$S_{s,t}^{(2)} - S_{s,u}^{(2)} - S_{u,t}^{(2)} = \text{Sym} \left( X_{s,u}^{(1)} \otimes X_{u,t}^{(1)} \right)$$

and b) The usual regularity conditions holds,  $|X_{s,t}^{(1)}| \lesssim |t - s|^\alpha$  and  $|S_{s,t}^{(2)}| \lesssim |t - s|^{2\alpha}$  for  $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ . We formally write that  $(X^{(1)}, S^{(2)}) \in \mathcal{C}_r^\alpha$ .

Naturally, the simplest choice of reduced iterated integral, i.e  $S_{s,t}^{(2)} := \frac{1}{2} X_{s,t}^{(1)} \otimes X_{s,t}^{(1)} = \text{Sym}(X_{s,t}^{(2)})$  where  $X_{s,t}^{(2)}$  is geometric, yields a reduced rough path. If we perturbate this choice by a  $2\alpha$  regular path  $f \in \text{Sym}(\mathbb{R}^{d \times d})$ , the perturbed path will again yield a reduced rough path.

From stochastic calculus of regular Brownian motion, we have the relation between the second iterated integral from Stratonovich and Itô calculus as follows

$$B_{s,t}^{(2)Itô} = B_{s,t}^{(2)Strat} + \frac{1}{2} (t - s) I^{d \times d}.$$

Where  $I^{d \times d}$  is a  $d \times d$  identity matrix. As we know, the Stratonovich iterated integral is the geometric choice of iterated integrals of brownian motion, and we obtain the Itô integral from perturbation of the Itô integral by a function  $\frac{1}{2}(t - s)I^{d \times d}$ . We are interested in extending this kind of relation to fBm's, and one could think the relation

$$\bar{B}_{s,t}^{(2)Itô} = \bar{B}_{s,t}^{(2)Strat} + \frac{1}{2} (t^{2H} - s^{2H}) I^{d \times d}$$

should hold, when  $\bar{B}_{s,t}^{(2)}$  is the iterated integral of a the fBm  $B_{s,t}^{(1)}$ . In a rough path setting, we can clearly see that the choice

$$\left( B_{s,t}^{(1)}, \frac{1}{2} B_{s,t}^{(1)} \otimes B_{s,t}^{(1)} + (t^{2H} - s^{2H}) I^{d \times d} \right) = \left( B_{s,t}^{(1)}, \text{Sym}(\bar{B}_{s,t}^{(2)Strat}) + (t^{2H} - s^{2H}) I^{d \times d} \right) \in \mathcal{C}_r^\alpha,$$

and we will from now on denote  $S_{s,t}^{(2)Strat} = \text{Sym}(\bar{B}_{s,t}^{(2)Strat}) = \frac{1}{2}B_{s,t}^{(1)} \otimes B_{s,t}^{(1)}$  and in the same way for Itô, and the relation  $S_{s,t}^{(2)Itô} = S_{s,t}^{(2)Strat} + \frac{1}{2}(t^{2H} - s^{2H})I^{d \times d}$ . We now present an Itô formula to describe the behavior of a sufficiently smooth function evaluated in a fBm, by using the relation between Stratonovich and Itô Calculus. A more general result for reduced rough paths was first presented in [16].

**Lemma 7.2.** *Let  $F : \mathbb{R}^d \rightarrow \mathbb{R}^n$  be  $\mathcal{C}_b^3$  and let  $\mathbf{B}_{s,t}^{Itô} = (B_{s,t}^{(1)}, S_{s,t}^{(2)Itô}) \in \mathcal{C}_r^\alpha$ ,  $\alpha \in (\frac{1}{3}, \frac{1}{2}]$ , Then*

$$F(B_t^{(1)}) - F(B_0^{(1)}) = \int_0^t DF(B_s^{(1)})d\mathbf{B}_s^{Itô} + H \int_0^t D^2F(B_s)s^{2H-1}ds$$

Where the rough integral is given by

$$\int_0^t DF(B_s^{(1)})d\mathbf{B}_s^{Itô} := \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} DF(B_s^{(1)})B_{s,t}^{(1)} + D^2F(B_s^{(1)})S_{s,t}^{(2)Itô},$$

here,  $\mathcal{P}$  is a partition of  $[0, t]$ , and the second integral is well defined as a Young integral.

*Proof.* Following the proof from chp. 5 in [16], we start by considering a taylor approximation of  $F$ ,

$$\begin{aligned} F(B_t^{(1)}) - F(B_0^{(1)}) &= \sum_{[u,v] \in \mathcal{P}} F(B_v^{(1)}) - F(B_u^{(1)}) \\ &= \sum_{[u,v] \in \mathcal{P}} DF(B_u^{(1)})B_{u,v}^{(1)} + \frac{1}{2}D^2F(B_u^{(1)})B_{u,v}^{(1)} \otimes B_{u,v}^{(1)} + o(|u - s|) \end{aligned}$$

Now, using the fact that  $S_{s,t}^{(2)Strat} := \frac{1}{2}B_{u,v}^{(1)} \otimes B_{u,v}^{(1)} = S_{s,t}^{(2)Itô} + (t^{2H} - s^{2H})I^{d \times d}$  we see that last equality above

$$= \sum_{[u,v] \in \mathcal{P}} DF(B_u^{(1)})B_{u,v}^{(1)} + \frac{1}{2}D^2F(B_u^{(1)})\left(S_{u,v}^{(2)Itô} + (v^{2H} - u^{2H})I^{d \times d}\right) + o(|u - s|)$$

Splitting the second derivative part in two, we recognize the Rough integral, and a sum of equal to  $\sum_{[u,v] \in \mathcal{P}} D^2F(B_u^{(1)})(v^{2H} - u^{2H})I^{d \times d}$ . Let  $\mathcal{P}$  be a partition of  $[0, t]$ . We will prove that

$$\sum_{[u,v] \subset \mathcal{P}} \frac{1}{2}D^2F(B_u)(v^{2H} - u^{2H}) \rightarrow H \int_0^t D^2F(B_s)s^{2H-1}ds.$$

Let us write

$$\lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \subset \mathcal{P}} \frac{1}{2}D^2F(B_u)(v^{2H} - u^{2H}) = \lim_{|\mathcal{P}| \rightarrow 0} \int_0^T \sum_{[u,v] \subset \mathcal{P}} D^2F(B_u)1_{[u,v]}(s)Hs^{2H-1}ds$$

We define the measure  $\lambda$  such that  $d\lambda(s) = Hs^{2H-1}ds$ . We can rewrite the integral with respect

to  $\lambda$

$$\lim_{|\mathcal{P}| \rightarrow 0} \int_0^T \sum_{[u,v] \subset \mathcal{P}} D^2 F(B_u) 1_{[u,v]}(s) d\lambda(s)$$

We recognize the integrand as a simple function  $f^{|\mathcal{P}|}$  in this way

$$f^{|\mathcal{P}|}(s) := \sum_{[u,v] \subset \mathcal{P}} D^2 F(B_u) 1_{[u,v]}(s) \rightarrow f(s) := D^2 F(B_s) 1_{[0,t]}(s)$$

We will continue with a simple dominated convergence argument. Since  $F \in \mathcal{C}_b^4$ , we know  $|f^{|\mathcal{P}|}(s)| \leq D$ . Therefore define  $g(s) = D 1_{[0,t]}(s)$  s.t  $g \in L^1([0, T], \lambda)$  and  $|f^{|\mathcal{P}|}(s)| \leq g(s)$ , and therefore by the DCT,

$$\lim_{|\mathcal{P}| \rightarrow 0} \int_0^T \sum_{[u,v] \subset \mathcal{P}} D^2 F(B_u) 1_{[u,v]}(s) H s^{2H-1} ds = H \int_0^t D^2 F(B_s) s^{2H-1} ds.$$

And our proof is done.  $\square$

We now move on to look at multifractional Brownian motions, and how we can construct the Itô formula for such processes. As we have discussed earlier, we do not have a canonical construction of the iterated integral with respect to an mBm. However, as we have seen in the formula for fBm's above, the second iterated integral will be appear in the second derivative of the function. The second derivative of this function is a symmetric operator, and hence, one only uses the symmetric part of the iterated integral. We are, in a sense, stating that the Stratonovich iterated integral is the geometric choice of the iterated integral, and we know that the symmetric part of a geometric iterated integral is the increments of the first-order process squared. Therefore, we do not need to have a canonical construction of the iterated integral, or a lift to a geometric rough path to find an Itô formula, we only need the concept of reduced rough paths. As we have seen, all rough paths induces a reduced rough path, but the converse is in general not true. We will first define the mBm in a reduced rough path sense.

**Lemma 7.3.** *Let  $B^h$  be a  $d$ -dimensional mBm on  $[0, T]$ . Let  $0 < \varepsilon$  be small such that,  $h : [0, T] \rightarrow [\frac{1}{3} + \varepsilon, 1 - \varepsilon] \subset (\frac{1}{3}, 1)$ , and be such that  $h \in \mathcal{C}^1$ . Then,  $B_{s,t}^h = B_t^{h(t)} - B_s^{h(s)}$  can be lifted canonically to a reduced rough path by choosing  $\mathbf{B}_{s,t}^h = (B_{s,t}^h, \frac{1}{2} B_{s,t}^h \otimes B_{s,t}^h) \in \mathcal{C}_r^{\frac{1}{3}+}([0, T], \mathbb{R}^d)$ . We call this a reduced multifractional rough path.*

*Proof.* We know there exist a canonical lift of  $B_{s,t}^h$  to a reduced rough path by choosing the iterated integral to be  $\frac{1}{2} B_{s,t}^h \otimes B_{s,t}^h$ . Indeed, we see that

$$\begin{aligned} & \frac{1}{2} B_{s,t}^h \otimes B_{s,t}^h - \frac{1}{2} B_{s,u}^h \otimes B_{s,u}^h - \frac{1}{2} B_{u,t}^h \otimes B_{u,t}^h \\ &= \frac{1}{2} \left( B_{s,u}^h + B_{u,t}^h \right) \otimes \left( B_{s,u}^h + B_{u,t}^h \right) - \frac{1}{2} B_{s,u}^h \otimes B_{s,u}^h - \frac{1}{2} B_{u,t}^h \otimes B_{u,t}^h \end{aligned}$$



$$= \frac{1}{2} \left( B_{s,u}^h \otimes B_{u,t}^h + B_{u,t}^h \otimes B_{s,u}^h \right) = \text{Sym} \left( B_{s,u}^h \otimes B_{u,t}^h \right).$$

Choosing  $\varepsilon$  small enough, we see that the path  $\mathbf{B}_{s,t}^h := (B_{s,t}^h, \frac{1}{2} B_{s,t}^h \otimes B_{s,t}^h) \in \mathcal{C}_r^{\frac{1}{3}+}([0, T], \mathbb{R}^d)$ , since it satisfies the reduced Chen's relation and the analytical conditions. The analytical conditions follows trivially from lemma 6.10.  $\square$

Remember that mBm's are just generalizations of fBm's, and hence, in the case of  $h(t) = H$  where  $H$  is a constant in  $(0, 1)$ , we obtain a fBm.

We assume the relation between the iterated Itô integral and iterated Stratonovich integral are as follows, let  $\varepsilon > 0$  and  $h : [0, T] \rightarrow [\frac{1}{3} + \varepsilon, 1 - \varepsilon] \subset (\frac{1}{3}, 1)$ , where  $h \in \mathcal{C}_b^1$ , then

$$B_{s,t}^{h, \text{Itô}} = B_{s,t}^{h, \text{Strat}} + (t^{2h(t)} - s^{2h(s)}) I^{d \otimes d},$$

The stratonovich integral is the geometric choice of iterated integral, and we therefore see that,

$$\text{Sym} \left( B_{s,t}^{h, \text{Strat}} \right) = \frac{1}{2} B_{s,t}^h \otimes B_{s,t}^h.$$

We will follow up with an Itô formula for reduced mBm's.

**Lemma 7.4.** *Let  $h : [0, T] \rightarrow [\frac{1}{3} + \varepsilon, 1 - \varepsilon] \subset (\frac{1}{3}, 1)$ , for some small  $\varepsilon > 0$ , be a  $\mathcal{C}_b^1$  function, and let*

$$\mathbf{B}_{s,t}^{h, \text{Itô}} = \left( B_{s,t}^h, S_{s,t}^{h, \text{Itô}} \right) = \left( B_{s,t}^h, \text{Sym}(B_{s,t}^{h, \text{Strat}}) + (t^{2h(t)} - s^{2h(s)}) \right) \in \mathcal{C}_r^{\frac{1}{3}+}([0, T], \mathbb{R}^d),$$

and  $F : \mathbb{R}^d \rightarrow \mathbb{R}^m$  be a  $\mathcal{C}_b^3$  function. Then

$$F(B_t^h) - F(B_0^h) = \int_0^t DF(B_s^h) d\mathbf{B}_s^{h, \text{Itô}} + \frac{1}{2} \int_0^t D^2 F(B_s^h) 2s^{2h(s)-1} (s \ln(s) h'(s) + h(s)) ds.$$

Where the rough integral is given by

$$\int_0^t DF(B_s^h) d\mathbf{B}_s^{h, \text{Itô}} := \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} DF(B_s^h) B_{s,t}^h + D^2 F(B_s^h) S_{s,t}^{h, \text{Itô}},$$

and the second integral is well defined as a Young integral.

*Proof.* The proof is very similar to the proof of lemma 7.2, but the difference is the convergence of the young integral. We will take a closer look at this part. We obtain a sum of the form

$$\sum_{[u,v] \in \mathcal{P}} D^2 F(B_u) \left( v^{2h(v)} - u^{2h(u)} \right)$$

We know that since  $h \in \mathcal{C}_b^1$ , the existence is assured from Young theory. Further, we have  $\frac{d(u^{2h(u)})}{du} =$

$2u^{2h(u)-1} (u \ln(u)h'(u) + h(u))$ , and therefore since  $F \in \mathcal{C}_b^3$  we have that  $|D^2F(B_u)| \leq K$ , and we know that  $|2u^{2h(u)-1} (u \ln(u)h'(u) + h(u))| \leq D$  as a consequence of the continuity of  $\frac{d(u^{2h(u)})}{du}$  on  $[0, T]$  for  $h : [0, T] \rightarrow [\frac{1}{3} + \varepsilon, 1 - \varepsilon]$ . Since the lebesgue measure of  $[0, T]$  is finite, and

$$\begin{aligned} f^{|\mathcal{P}|}(s) &= \sum_{[u,v] \in \mathcal{P}} D^2F(B_u) 2u^{2h(u)-1} (u \ln(u)h'(u) + h(u)) 1_{[u,v]}(s) \\ &\rightarrow D^2F(B_s) 2s^{2h(s)-1} (s \ln(s)h'(s) + h(s)) 1_{[0,t]}(s) =: f(s) \end{aligned}$$

pointwise. We use the bounded convergence theorem, to find that

$$\begin{aligned} &\lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} D^2F(B_u) (v^{2h(v)} - u^{2h(u)}) \\ &= \lim_{|\mathcal{P}| \rightarrow 0} \int_0^t \sum_{[u,v] \in \mathcal{P}} D^2F(B_u) 2u^{2h(u)-1} (u \ln(u)h'(u) + h(u)) 1_{[u,v]}(s) ds \\ &= \int_0^t D^2F(B_s) 2s^{2h(s)-1} (s \ln(s)h'(s) + h(s)) ds. \end{aligned}$$

Which concludes the proof.  $\square$

*Remark 7.5.* The Itô formulas obtained in this section for mBm's with  $h(t)$  restricted to  $[\frac{1}{3} + \varepsilon, 1 - \varepsilon]$  for each  $t \in [0, T]$  are based on reduced mBm's. However, as the second derivative of  $F$ ,  $D^2F : \mathbb{R}^d \rightarrow \mathcal{L}(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^m))$  is symmetric, the Itô formula will hold for any geometric multifractional Brownian rough path. We have assumed that  $h \in \mathcal{C}^1$ . However, the result will hold for  $h \in \mathcal{C}^\beta$  as long as  $\beta + \frac{1}{3} + \varepsilon > 1$ , such that the Young theory may be applied to the second integral. This imply, that we may let  $h$  be a reflected fractional Brownian motion, or a similar process which is stochastic.

*Remark 7.6.* Interestingly, the Itô formula above corresponds very well to the one proved by C. Bender [2] in the case of fractional Brownian motion, i.e for a constant  $a$ ,  $h(t) = a \in (\frac{1}{3}, \frac{1}{2}]$ . Bender proved this by using white noise theory. The formula stated above for fBm was first proved by Hairer and Friz in [16] and uses (of course) Rough path theory, which is a path-wise approach, in contrast to the white noise approach. The white noise approach in finance have received critics for admitting arbitrage opportunities under Wick-Itô products (see [24]), while The authors of this thesis has yet to investigate for arbitrage opportunities in a multifractional “Black-Scholes” model in a rough path setting, the rough path based Itô formula for mBm's will hopefully become useful. An Itô formula has previously been constructed by Lebovits et.al. for mBm's using white noise theory [11], and the Itô formula presented here corresponds very well with this formula. The only difference is actually the choice of integral, i.e Skorohod vs rough integral. As far as the authors of this thesis know, the Ito formula presented for mBm's in a rough path setting is new.

The next section will deal with fractional Brownian rough paths in a lower regularity setting. We have yet to generalize this to Multifractional Brownian motions, but we will look at this for future work.

## 8 Itô formula for reduced fractional rough paths when $\alpha \in (\frac{1}{4}, \frac{1}{3}]$ .

In this subsection we will show how we may perturbate the second iterated integral in the Itô vs. Stratonovich sense, and how this affects the third iterated integral. We will then develop a similar formula for fractional Brownian rough paths, as the one developed in the previous section. As in the previous section, we have the relation between the stratonovich and Itô integral as follows,

$$B_{s,t}^{(2),Strat} = B_{s,t}^{(2),Itô} + (t^{2H} - s^{2H}) I^{d \otimes d}.$$

If we define a function  $f : [0, T] \rightarrow \mathbb{R}^{d \times d}$  by  $f(t) = t^{2H} I^{d \times d}$ , where  $I^{d \times d}$  is the identity operator in  $\mathbb{R}^{d \times d}$ , as we did in the previous section, we see that this is the function perturbing the Stratonovich integral to obtain the itô integral. When we are considering lower regularities, we need to check how this perturbation affects the third iterated integral. To give some intuition, let us first have a look at typical stochastic calculus with a regular brownian motion, to see how the third iterated Itô integral relates to the Stratonovich integral. For simplicity, let  $B_t$  be a one dimensional Brownian motion on a probability space  $(\Omega, P, \mathcal{F})$ . We know from Itô calculus that

$$\int_s^t B_{s,r} dB_r = \frac{1}{2} B_{s,t}^2 - \frac{1}{2} (t - s).$$

The Stratonovich calculus will follow the regular rules of calculus, and hence  $\int_s^t B_{s,r} \circ dB_r = \frac{1}{2} B_{s,t}^2$ , where  $\circ$  denotes the Stratonovich integration. Therefore, we have the relation

$$\int_s^t B_{s,r} \circ dB_r = \int_s^t B_{s,r} dB_r + \frac{1}{2} (t - s).$$

We may also calculate the third iterated Itô integral. It is straight forward to check that,

$$\int_s^t \int_s^{t_1} B_{s,r} dB_r dB_{t_1} = \frac{1}{6} B_{s,t}^3 - \int_s^t B_{s,t_1} dt_1 - \int_s^t (t_1 - s) dB_{t_1}.$$

As usual will the third iterated Stratonovich integral be given by  $\int_s^t \int_s^{t_1} B_{s,r} \circ dB_r \circ dB_{t_1} = \frac{1}{6} B_{s,t}^3$ . Let us define a new variable  $\tilde{Y}_{s,t} = \int_s^t B_{s,t_1} dt_1 + \int_s^t (t_1 - s) dB_{t_1}$ . Then the following relation occur between the third iterated Stratonovich and Itô integral,

$$\int_s^t \int_s^{t_1} B_{s,r} \circ dB_r \circ dB_{t_1} = \int_s^t \int_s^{t_1} B_{s,r} dB_r dB_{t_1} + \tilde{Y}_{s,t}.$$

We clearly see that the perturbation of the second iterated integral affects the third iterated integrals with a function  $\tilde{Y}_{s,t}$  consisting of two cross integrals. We will see that this also must be the case for fBm's. The next lemma will give a way to construct the third integral in the case of a perturbation of the second integral.

**Lemma 8.1.** Let  $\mathbf{B} = (B^{(1)}, B^{(2)}, B^{(3)}) \in \mathcal{C}^\alpha([0, T], \mathbb{R}^d)$  and let  $f \in \mathcal{C}^{2\alpha}$ , with  $f \in \text{Sym}(\mathbb{R}^d \otimes \mathbb{R}^d)$ . Assume the integrals  $\int_s^t B_{s,r}^{(1)} df_r$  and  $\int_s^t f_{s,r} dB_r^{(1)}$  are well defined for all  $s, t \in [0, T]$ , and assume they are such that  $|\int_s^t X_{s,r} df_r| \lesssim |t - s|^{3\alpha}$  and  $|\int_s^t f_{s,r} dB_r^{(1)}| \lesssim |t - s|^{3\alpha}$ . Now, define the perturbed iterated integrals as follows

$$\begin{aligned} B_{s,t}^{(1)} &= B_{s,t}^{(1)} \\ B_{s,t}^{(2),f} &= B_{s,t}^{(2)} + f_{s,t} \\ B_{s,t}^{(3),f} &= B_{s,t}^{(3)} + \int_s^t f_{s,r} dB_r + \int_s^t B_{s,r} df_r \end{aligned}$$

Then  $(B^{(1)}, B^{(2),f}, B^{(3),f})$  is a rough path in  $\mathcal{C}^\alpha([0, T], V)$

*Proof.* It is straight forward to check that  $B_{s,t}^{(2),f}$  satisfies the second order Chen's relation, the trick is to prove it for the third iterated integral. We have seen that the third iterated integral need to satisfy

$$B_{s,t}^{(3)} - B_{s,u}^{(3)} - B_{u,t}^{(3)} = B_{s,u}^{(1)} B_{u,t}^{(2)} + B_{s,u}^{(2)} B_{u,t}^{(1)}.$$

Let us first look at the left hand side of the above equation, and insert  $B^{(3),f}$ ,

$$\begin{aligned} B_{s,t}^{(3),f} - B_{s,u}^{(3),f} - B_{u,t}^{(3),f} &= B_{s,t}^{(3)} - B_{s,u}^{(3)} - B_{u,t}^{(3)} \\ &+ \int_s^t f_{s,r} dB_r + \int_s^t B_{s,r} df_r - \left( \int_s^u f_{s,r} dB_r + \int_s^u B_{s,r} df_r \right) - \left( \int_u^t f_{u,r} dB_r + \int_u^t B_{u,r} df_r \right). \end{aligned}$$

We know  $B^{(3)}$  satisfy the relation, and hence, our objective is to look at the three additive perturbing terms. We see that,

$$\begin{aligned} \int_s^t f_{s,r} dB_r + \int_s^t B_{s,r} df_r - \left( \int_s^u f_{s,r} dB_r + \int_s^u B_{s,r} df_r \right) - \left( \int_u^t f_{u,r} dB_r + \int_u^t B_{u,r} df_r \right) \\ = f_{s,u} B_{u,t}^{(1)} + B_{s,u}^{(1)} f_{u,t}. \end{aligned}$$

Inserting this in the equation two above, and using the fact that the path  $B^{(3)}$  is following the usual Chen's relation, we find that

$$\begin{aligned} B_{s,t}^{(3),f} - B_{s,u}^{(3),f} - B_{u,t}^{(3),f} &= B_{s,t}^{(3)} - B_{s,u}^{(3)} - B_{u,t}^{(3)} + f_{s,u} B_{u,t}^{(1)} + B_{s,u}^{(1)} f_{u,t} \\ &= B_{s,u}^{(2)} B_{u,t}^{(1)} + B_{s,u}^{(1)} B_{u,t}^{(2)} + f_{s,u} B_{u,t}^{(1)} + B_{s,u}^{(1)} f_{u,t} \\ &= B_{s,u}^{(2),f} B_{u,t}^{(1)} + B_{s,u}^{(1)} B_{u,t}^{(2),f}, \end{aligned}$$

and then we see that the Chen's relation holds.  $\square$

We see by the previous lemma that we need to perturbate the third integral by two integrals,

similar to the case of the third iterated integral of a regular Brownian motion.

The next definition will show how we construct the space of reduced rough paths in a lower regularity setting.

**Definition 8.2.** We call the triple  $\mathbf{B} = (B^{(1)}, S^{(2)}, S^{(3)})$  a reduced rough path, formally  $\mathbf{B} \in \mathcal{C}_r^\alpha([0, T], V)$ , if  $B^{(1)}$  takes values in  $\mathbb{R}^d$ ,  $S^{(2)}$  takes values in  $\text{Sym}(\mathbb{R}^d \otimes \mathbb{R}^d)$ , and  $S^{(3)}$  takes values in  $\text{Sym}((\mathbb{R}^d)^{\otimes 3})$ , and the following hold.

1. The reduced Chen relation

$$\begin{aligned} S_{s,t}^{(2)} - S_{s,u}^{(2)} - S_{u,t}^{(2)} &= \text{Sym} \left( B_{s,u}^{(1)} \otimes B_{u,t}^{(1)} \right) \\ S_{s,t}^{(3)} - S_{s,u}^{(3)} - S_{u,t}^{(3)} &= \text{Sym} \left( B_{s,u}^{(1)} \otimes B_{u,t}^{(2)} \right) + \text{Sym} \left( B_{s,u}^{(1)} \otimes B_{u,t}^{(1)} \right). \end{aligned}$$

2. The usual analytical conditions,  $\left| B_{s,t}^{(1)} \right| \lesssim |t - s|^\alpha$ ,  $\left| B_{s,t}^{(2)} \right| \lesssim |t - s|^{2\alpha}$  and  $\left| B_{s,t}^{(3)} \right| \lesssim |t - s|^{3\alpha}$  holds.

Now, one can see that all paths  $\mathbf{B} \in \mathcal{C}_g^\alpha$  induces a reduced rough path, by ignoring its anti-symmetric parts of the second and third iterated integrals. The lift of a rough path to a reduced rough path is essentially trivial by setting  $S_{s,t}^{(2)} := \frac{1}{2} B_{s,t}^{(1)} \otimes B_{s,t}^{(1)}$  and  $S_{s,t}^{(3)} := \frac{1}{6} B_{s,t}^{(1)} \otimes B_{s,t}^{(1)} \otimes B_{s,t}^{(1)}$ . We will now give two results, showing that the integrals  $\int_s^t f_{s,r} dB_r$  and  $\int_s^t B_{s,r} df_r$  from lemma 8.1 exists for a particular choice of  $f$ , namely  $f(t) = t^{2H} I^{d \times d}$  where  $H$  is the Hürst parameter of the fBm. First, we will prove a nice inequality.

**Lemma 8.3.** *Let  $0 \leq s \leq t \in [0, T]$ , and  $0 \leq a \leq 1$ . then the following inequality holds*

$$(t^a - s^a) \leq C(a) (t - s)^a$$

For some constant  $C(a) \in \mathbb{R}$ .

*Proof.* If  $a = 1$  or  $0$ , then the statement is obvious, therefore fix  $a \in (0, 1)$ , and  $C := C(a)$ . We reformulate the inequality and find that

$$(t^a - s^a) \leq C (t - s)^a \iff \frac{(1 - (\frac{s}{t})^a)}{(1 - \frac{s}{t})^a} \leq C.$$

Define a function  $f : [0, 1] \rightarrow \mathbb{R}$  by  $f(x) = \frac{(1-x^a)}{(1-x)^a}$ . Now, if we can prove  $f$  is continuous, then it is bounded. We need to check what happens with  $f(x)$  when  $x \uparrow 1$ , as this is the possible singularity. We can easily check that  $\lim_{x \uparrow 1} f(x) = 0$ . For all other values of  $x$  the function is well defined and continuous on the compact interval  $[0, 1]$ , hence, bounded. This concludes the proof.  $\square$

We will use this inequality to prove the existence of the perturbing integrals.

**Theorem 8.4.** *Let  $f(t) = t^{2H} I^{d \times d}$ , and  $B_{s,t}$  be a fBm with  $H \in (\frac{1}{4}, \frac{1}{3}]$ . Let the partition integral over a partition  $\mathcal{P} \subset [s, t]$  be given by  $\int_{\mathcal{P}} f_{s,r} dB_r := \sum_{[u,v] \in \mathcal{P}} f_{s,u} B_{u,v}$ . Then the integral*

$$\int_s^t f_{s,r} dB_r := \lim_{|\mathcal{P}| \rightarrow 0} \int_{\mathcal{P}} f_{s,r} dB_r$$

are well defined, and

$$\left| \int_s^t f_{s,r} dB_r \right| \lesssim |t - s|^{3H-}.$$

*Proof.* We will start to prove existence of the integral  $\int_s^t f_{s,r} dB_r$ . We will use a rough path methodology similar to the one we used for proving the iterated integral for fBm's. We define the integral over a partition  $\mathcal{P} \subset [0, 1]$  by

$$\int_{\mathcal{P}} f_{0,r} dB_r := \sum_{[u,v] \in \mathcal{P}} f_{0,u} B_{u,v}.$$

We then look at the  $L^2(\Omega)$  norm of the integral, and notice that

$$E \left[ \left( \int_{\mathcal{P}} f_{0,r} dB_r \right)^2 \right] = \sum_{[u',v'], [u,v] \in \mathcal{P}} f_{0,u} f_{0,u'} E [B_{u,v} B_{u',v'}].$$

We know that the covariance function  $R \begin{pmatrix} u & v \\ u' & v' \end{pmatrix}$  is defined by  $R \begin{pmatrix} u & v \\ u' & v' \end{pmatrix} := E [B_{u,v} B_{u',v'}]$ .

As we have seen in the previous sections, we know that  $\|R\|_{\rho; [s,t]^2} \leq M |t - s|^{\frac{1}{\rho}}$  when  $\rho = \frac{1}{2H}$  for a fBm. We then define a new four variable function  $\bar{R} : [0, T]^4 \rightarrow \mathbb{R}^d$  such that  $\bar{R} \begin{pmatrix} u & v \\ u' & v' \end{pmatrix} := f_{u,v} f_{u',v'} = (v^{2H} - u^{2H}) (v'^{2H} - u'^{2H})$ . Now, we see that

$$\begin{aligned} \|\bar{R}\|_{\bar{\rho}; [s,t]^2} &= \left( \sup_{\mathcal{P} \subset [s,t]} \sum_{[u,v], [u',v'] \in \mathcal{P}} |f_{u,v} f_{u',v'}|^{\bar{\rho}} \right)^{\frac{1}{\bar{\rho}}} \\ &\leq \left( \sup_{\mathcal{P} \subset [s,t]} \sum_{[u,v], [u',v'] \in \mathcal{P}} |v - u|^{2H\bar{\rho}} |v' - u'|^{2H\bar{\rho}} \right)^{\frac{1}{\bar{\rho}}} \leq \left( \sup_{\mathcal{P} \subset [s,t]} \left( \sum_{[u,v] \in \mathcal{P}} |v - u|^{2H\bar{\rho}} \right)^2 \right)^{\frac{1}{\bar{\rho}}}. \end{aligned}$$

Let  $\bar{\rho} = \frac{1}{2H}$ , then we find that

$$\|\bar{R}\|_{\bar{\rho}; [s,t]^2} \leq |t - s|^{\frac{2}{\bar{\rho}}}.$$

Knowing that  $\|\bar{R}\|_{\bar{\rho}; [s,t]^2}, \|R\|_{\rho; [s,t]^2} < \infty$ , and  $\theta = \frac{1}{\rho} + \frac{1}{\bar{\rho}} > 1$ , we use Young's maximal inequality,

by Towghi [25], and find that

$$\sup_{\mathcal{P} \subset [0,1]} \left| \sum_{[u',v'],[u,v] \in \mathcal{P}} \bar{R} \begin{pmatrix} 0 & u \\ 0 & u' \end{pmatrix} R \begin{pmatrix} u & v \\ u' & v' \end{pmatrix} \right| \leq C(\theta) \|\bar{R}\|_{\bar{\rho};[0,1]^2} \|R\|_{\rho;[0,1]^2}.$$

Therefore we have that

$$\sup_{\mathcal{P} \subset [0,1]} E \left[ \left( \int_{\mathcal{P}} f_{0,r} dB_r \right)^2 \right] \leq C(\theta) \|\bar{R}\|_{\bar{\rho};[0,1]^2} \|R\|_{\rho;[0,1]^2}.$$

Now, the rest of the proof is essentially equal to that of proposition 6.6. Indeed, just look at two partitions, and check that

$$\lim_{\varepsilon \rightarrow 0} \sup_{\substack{\mathcal{P}, \mathcal{P}' \subset [0,1] \\ |\mathcal{P}| \vee |\mathcal{P}'| < \varepsilon}} E \left[ \left( \int_{\mathcal{P}} f_{0,r} dB_r - \int_{\mathcal{P}'} f_{0,r} dB_r \right)^2 \right] = 0,$$

by the same argument as in proposition 6.6. We conclude that the integral is invariant under different partitions. Therefore, we may define

$$\int_0^1 f_{0,r} dB_r := \lim_{|\mathcal{P}| \rightarrow 0} \int_{\mathcal{P}} f_{0,r} dB_r.$$

Then we have that,

$$E \left[ \left( \int_0^1 f_{0,r} dB_r \right)^2 \right] \leq C(\theta) \|\bar{R}\|_{\bar{\rho};[0,1]^2} \|R\|_{\rho;[0,1]^2}.$$

We have looked at the interval  $[0, 1]$ , but we can extend this to any interval  $[s, t]$  by reparametrization of  $\{B_t; 0 \leq t \leq 1\} \sim \{B_{s+(t-s)\Theta}; 0 \leq \Theta \leq 1\}$  as the variation norm are invariant under reparametrization. We therefore get the bounds,

$$E \left[ \left( \int_s^t f_{s,r} dB_r \right)^2 \right] \leq C(\theta) \|\bar{R}\|_{\bar{\rho};[s,t]^2} \|R\|_{\rho;[s,t]^2} \leq M(\theta) |t-s|^{\frac{2}{\bar{\rho}}} |t-s|^{\frac{1}{\rho}}$$

where  $\rho = \bar{\rho} = \frac{1}{2H}$ , and hence

$$E \left[ \left( \int_s^t f_{s,r} dB_r \right)^2 \right] \leq M(\theta) |t-s|^{6H}.$$

By  $L^p$ - $L^2$  equivalence, and by Kolmogorov's continuity theorem, we know there exist a continuous version of  $\int_s^t f_{s,r} dB_r$  such that  $\int_s^t f_{s,r} dB_r \in \mathcal{C}^{3H-}$ .  $\square$



Next we will show that the “opposite” cross-integral exist and is of sufficient regularity.

**Lemma 8.5.** *Let  $f : [0, T] \rightarrow \mathbb{R}^d$ , and  $f(t) = t^{2H}$ . Let  $B_t$  be a fractional brownian motion with  $H \in (0, 1)$ , Define*

$$\int_s^t B_{s,r} df_r := \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \in \mathcal{P}} B_u f_{u,v} - B_s f_{s,t}.$$

*Then the integral*

$$\int_s^t B_{s,r} df_r = 2H \int_s^t B_{s,r} r^{2H-1} dr$$

*is well defined, and*

$$| \int_s^t B_{s,r} df_r | \lesssim |t - s|^{3H}.$$

*Proof.* We look at

$$| \int_s^t B_{s,r} df_r | \leq \sup_{u,v \in [s,t]} |B_{u,v}| \int_s^t 2H r^{2H-1} dr \lesssim |t - s|^{3H}.$$

Where we have used that  $(t^{2H} - s^{2H}) \leq C(t - s)^{2H}$ , and  $H \in (\frac{1}{4}, \frac{1}{3}]$ . This implies that the integral exists, and is of sufficient regularity. Next, we see that if we consider the Lebesgue Stieltjes integral, and know that  $f$  is of finite variation since  $f$  is differentiable, and hence,

$$\int_s^t B_{s,r} df_r = 2H \int_s^t B_{s,r} r^{2H-1} dr.$$

Which concludes the proof.  $\square$

We now want to use the integrals we just showed to define a reduced rough path. To simplify notation, we will define the function  $Y : [0, T]^{\otimes 2} \rightarrow \mathbb{R}^{d \otimes 3}$  by  $Y_{s,t} := \text{Sym}(\int_s^t f_{s,r} dB_r + \int_s^t B_{s,r} df_r)$ , where  $f(t) = t^{2H}$ . We will use  $Y_{s,t}$  in the rest of this section.

**Proposition 8.6.** *The path  $(B^{(1)}, B^{(2),f}, B^{(3),f}) \in \mathcal{C}^{H-}([0, T], V)$ ,  $f(t) = t^{2H}$ , induces a reduced rough path  $(B^{(1)}, S^{(2),f}, S^{(3),f}) \in \mathcal{C}_r^{H-}([0, T], V)$  by considering the map*

$$\begin{aligned} & (B^{(1)}, B^{(2),f}, B^{(3),f}) \\ & \longmapsto \left( B^{(1)}, \text{Sym} \left( B_{s,t}^{(2)} \right) + f_{s,t}, \text{Sym} (B_{s,t}^{(3),f}) + Y_{s,t} \right). \end{aligned}$$

*Proof.* As we know the symmetry operator is linear, and therefore the reduced Chen’s relation is easy to check. the regularity of the objects remains unchanged by the symmetry operator.  $\square$

Next, we will give a proposition relating the reduced Brownian rough path the iterated integrals given by exponents of  $B_{s,t}^{(1)}$ . As we know, the relation we have stated between the Itô integral and

Stratonovich is given by

$$B_{s,t}^{(2),Itô} = B_{s,t}^{(2),Strat} + (t^{2H} - s^{2H}) I^{d \times d}.$$

We know  $\text{Sym} \left( B_{s,t}^{(2),Strat} \right) = B_{s,t}^{(1)} \otimes B_{s,t}^{(1)}$ , therefore, the next proposition will become useful for the proof of the Itô formula.

**Proposition 8.7.** *Let  $B^{(1)} \in \mathcal{C}^\alpha$ ,  $\alpha > \frac{1}{4}$ . Consider the geometric choice  $\bar{S}_{s,t}^{(2)} = \frac{1}{2} B_{s,t}^{(1)} \otimes B_{s,t}^{(1)}$  and  $\bar{S}_{s,t}^{(3)} = \frac{1}{3!} B_{s,t}^{(1)} \otimes B_{s,t}^{(1)} \otimes B_{s,t}^{(1)}$ . Let  $f \in \mathcal{C}^{2\alpha}$ , then the reduced path induced by perturbing the geometric choice of  $S^{(2)}$  and  $S^{(3)}$  is again a reduced rough path,*

$$\left( B_{s,t}^{(1)}, S_{s,t}^{(2)}, S_{s,t}^{(3)} \right) = \left( B_{s,t}^{(1)}, \bar{S}_{s,t}^{(2)} + f_{s,t}, \bar{S}_{s,t}^{(3)} + Y_{s,t} \right) \in \mathcal{C}_r^\alpha \left( [0, T], \mathbb{R}^d \right)$$

*Proof.* This follows directly from proposition 8.6. □

Just as with the second iterated integral, we need a relation between the third iterated Itô integral and Stratonovich integral. We will define

$$S_{s,t}^{(3),Strat} = S_{s,t}^{(3),Itô} + \frac{1}{6} Y_{s,t}.$$

Where  $Y_{s,t}$  is the function described above. In the next theorem we present an Itô formula for reduced fractional rough paths with  $\alpha \in (\frac{1}{4}, \frac{1}{3}]$ .

**Theorem 8.8.** *Let  $F : V \rightarrow W$  be in  $C_b^4$  and let  $Y_{s,t}$  be given by*

$$Y_{s,t} = \text{Sym} \left( \int_s^t f_{s,r} dB_r + \int_s^t B_{s,r} df_r \right) \lesssim |t - s|^{3H}.$$

Let

$$\begin{aligned} \mathbf{B}^{Itô} &= \left( B_{s,t}^{(1)}, S_{s,t}^{(2),Itô}, S_{s,t}^{(3),Itô} \right) = \\ &\left( B_{s,t}, S_{s,t}^{(2),Strat} + \frac{1}{2} (t^{2H} - s^{2H}) I^{d \otimes d}, S_{s,t}^{(3),Strat} + \frac{1}{6} Y_{s,t} \right) \in \mathcal{C}_r^{H-}([0, T], V), \end{aligned}$$

with  $H \in [\frac{1}{4}, \frac{1}{3})$ . Then

$$F(B_t) - F(B_0) = \int_0^t DF(B_s) d\mathbf{B}_s^{Itô} + H \int_0^t D^2 F(B_s) s^{2H-1} ds + \int_0^t D^3 F(B_s) dY_s$$

for  $0 \leq t \leq T$ . Where the integral  $\int_0^t DF(B_s) d\mathbf{B}_s^{Itô}$  is understood as the rough integral in the following way; Let  $\mathcal{P}$  be a partition on  $[0, t]$ ,

$$\int_0^t DF(B_s) d\mathbf{B}_s^{Itô} := \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \subset \mathcal{P}} DF(B_u) B_{u,v}^{(1)} + D^2 F(B_u) S_{u,v}^{(2),Itô} + D^3 F(B_u) S_{u,v}^{(3),Itô}.$$

The second integral  $H \int_0^t D^2 F(B_s) s^{2H-1} ds$  is well defined as a Young integral, and the third integral

$$\int_0^t D^3 F(B_s) dY_s := \lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \subset \mathcal{P}} D^3 F(B_u) Y_{u,v}$$

Which exists, as all other terms in the equation, is finite.

*Proof.* When  $\mathbf{B}$  is a perturbed geometric path, we know that it can be reduced, by considering the symmetric choice. hence looking at the taylor expansion of  $F(B)$  we see that

$$\begin{aligned} F(B_t) - F(B_0) &= \sum_{[u,v] \subset \mathcal{P}} F(B_v) - F(B_u) \\ &= \sum_{[u,v] \subset \mathcal{P}} DF(B_u) B_{u,v} + \frac{1}{2} D^2 F(B_u) B_{u,v}^{\otimes 2} + \frac{1}{6} D^3 F(B_u) B_{u,v}^{\otimes 3} + o(|v - u|^{4H}) \\ &= \sum_{[u,v] \subset \mathcal{P}} DF(B_u) B_{u,v} + D^2 F(B_u) S_{u,v}^{(2),Strat} + D^3 F(B_u) S_{u,v}^{(3),Strat} + o(|\mathcal{P}|). \end{aligned}$$

Now, knowing that  $S_{s,t}^{(2),Strat} = S_{s,t}^{(2),It\acute{o}} + \frac{1}{2} (t^{2H} - s^{2H}) I^{d \otimes d}$  and  $S_{s,t}^{(3),Strat} = S_{s,t}^{(3),It\acute{o}} + \frac{1}{6} Y_{s,t}$  we substitute in the above equations and find

$$\begin{aligned} &= \sum_{[u,v] \subset \mathcal{P}} DF(B_u) B_{u,v} + D^2 F(B_u) \left( S_{u,v}^{(2),It\acute{o}} + \frac{1}{2} (u^{2H} - v^{2H}) I^{d \otimes d} \right) \\ &\quad + D^3 F(B_u) \left( S_{u,v}^{(3),It\acute{o}} + \frac{1}{6} Y_{u,v} \right) + o(|v - u|^{4H}). \end{aligned}$$

We rearrange the terms, and get the expression

$$\begin{aligned} &= \sum_{[u,v] \subset \mathcal{P}} DF(B_u) B_{u,v} + D^2 F(B_u) S_{u,v}^{(2),It\acute{o}} + D^3 F(B_u) S_{u,v}^{(3),It\acute{o}} \\ &\quad + \sum_{[u,v] \subset \mathcal{P}} \frac{1}{2} D^2 F(B_u) (v^{2H} - u^{2H}) + \frac{1}{6} D^3 F(B_u) Y_{u,v} + o(|v - u|^{4H}). \end{aligned}$$

The first line in the last equality we recognize as the rough path integral  $\int_0^t DF(B_s) d\mathbf{B}_s$ . We need to prove the last two objects in the last sum. Let  $\mathcal{P}$  be a partition of  $[0, t]$ . We know from lemma 7.2 that,

$$\sum_{[u,v] \subset \mathcal{P}} \frac{1}{2} D^2 F(B_u) (v^{2H} - u^{2H}) \rightarrow H \int_0^t D^2 F(B_s) s^{2H-1} ds.$$

The last sum is defined to be such that

$$\lim_{|\mathcal{P}| \rightarrow 0} \sum_{[u,v] \subset \mathcal{P}} D^3 F(B_u) Y_{u,v} = \int_0^t D^3 F(B_s) dY_s.$$

Although one would expect Young theory to apply to this integral as well, since  $|Y_{s,t}| \lesssim |t-s|^{3\alpha}$  and  $|D^3 F(B)_{s,t}| \lesssim |t-s|^\alpha$ , the Although we do not have an explicit expression for the last integral, we know it exists, as all other terms in the formula exists, and is finite.

□

The connection here seen between rough path theory and stochastic analysis is very interesting. We want to conduct further research to gain knowledge about how to define a proper integral with respect to the integral process  $Y$  such that the Itô Lemma for reduced fBm's is explicit and gives a translation from Stratonovich calculus to Itô calculus for fractional brownian motion with low regularity. We have not extended this result to multifractional Brownian motions, as we have to little knowledge of the explicit form of the process  $Y$ .

## Part III

# Financial applications of Rough Path theory, and the Heston model.

Recent research done by Jim Gatheral, Peter Friz, and others, suggest that volatility as a stochastic process, is driven by a fractional brownian motion with Hürst exponent somewhere in the interval  $(0.1, 0.2)$ , see [9] ,[5]. This is highly interesting findings, which directly translates to negative autocorrelation in the volatility process. To model the phenomena, we will have use for a theory to handle noise of low regularity. In the paper “volatility is rough” by J. Gatheral et.al. [9] the authors look at an exponential Ornstein-Uhlenbeck model driven by a fractional brownian motion, i.e if  $v_t$  is the solution to

$$dv_t = k(\mu - v_t)dt + dB_t^H,$$

then take the exponential of  $v_t$ ,

$$V_t = \exp(v_t).$$

The existence of such a model is essentially trivial, as the brownian motion is additive, but as we have seen in part I of this thesis, the existence of such solutions in the case where the differential equation is given by

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t^H,$$

can be more tricky to prove. Given some regularity conditions on  $\sigma$ (i.e  $\sigma \in \mathcal{C}_b^4$ ) we have shown the existence of such equation when  $H \in (\frac{1}{4}, \frac{1}{3}]$ . Still if we consider a general square root model, where  $\sigma(X_t) = \sqrt{X_t}$ , We do not have the desired regularity on  $\sigma$  to show existence of a solution by established theory. However, we will show that we are able to prove existence of a solution to a square root process, based on a type of “Wong-Zakai” approximation of smooth paths, and choosing the geometric lift of a fractional brownian motion. Or even better, of a multifractional Brownian motion. When considering financial applications of such noise, it is important to be aware of the lack of market completeness, from the use of fractional brownian motion. The process itself gets, in some sense, predictable and arbitrage will arise. We will discuss some recent empirical findings relating to fBm’s and mBm’s and suggest some ideas for future research into how one can remove arbitrage in a multifractional Black-Scholes universe.

## 9 A discussion of the use of multifractional Brownian motions in Finance

We discussed in section 6.2 the construction of multifractional Brownian motions when the function  $h : [0, T] \rightarrow [a, b] \subset (0, 1)$ , and used the Mandelbrot Van Ness representation. In applications, there

have been a lot of discussion on whether or not the market (be it prices, volatility, etc.) behaves like a fractional Brownian motion, see [3] and [13]. In the article by Morales et.al. [18] the authors investigate financial time series of prices and observe that the Hürst parameter change over time. They show that the Hürst parameter in various blue-chip companies fluctuates around  $H = \frac{1}{2}$  and up/down to approximately  $\sim 0.3, 0.7$  over time. This could shed some light on the debate of existence of long memory in prices, as it seem to be depending on the time frame of observation. This is very interesting from a rough path point of view. As we have shown, we can use rough path theory to give meaning to SDE's etc. driven by multifractional brownian motions of low regularity. As the mBm capture fluctuations of the Hürst parameter in time, we believe that this could appropriately model these financial time series. The difficulty will still be to construct a proper function  $h$ , and making a “multifractional Black-Scholes” market arbitrage free. As shown by t. Björk et .al. [24] and L. C. G. Rogers [19], fractional brownian motion admits for arbitrage, both in the regular probability, and white noise sense. However, if one restricts admissible trading strategies, there is possible to show no arbitrage with fBm's (at least when  $H > \frac{1}{2}$ ), see C. Bender et. al [4]. Although there has not been a lot of research into arbitrage opportunities in the case of price/volatility processes driven by multifractional Brownian motions, it seem to be reasonable to believe that there exist arbitrage opportunities in such models as well. At least if  $h : [0, T] \rightarrow [\varepsilon, 1 - \varepsilon]$  for small  $\varepsilon > 0$ , is a deterministic function.

### 9.1 A Wong-Zakai type approximations of a rough square root process driven by a mBm.

The square root process (SRP) has many applications in finance. Introduced by Cox-Ingersoll-Ross in the infamous paper [12], where the process was used to model interest rates, it quickly became a standard model in financial business. In 1993 Steven L. Heston used SRP to model volatility and proved a closed form solutions to options with stochastic volatility [8]. However, in recent years, especially the Heston model has gained critics from both practitioners and for example J. Gatheral (see [9]) for its lack of accuracy to the observed volatility surface. Still the model has some very interesting mathematical properties, as the square root has a derivative which is singular in 0. In stochastic analysis we can still show existence, as proved by Yamada and Watanabe, see [22] page 291.

We will in this section use a simple Wong-Zakai type of approximation of a SRP given by

$$dY_t = -\alpha Y_t dt + \sqrt{Y_t} d\mathbf{B}_t^h,$$

where  $\mathbf{B}^h$  is a one-dimensional multifractional brownian rough path with  $h : [0, T] \rightarrow [a, b] \subset (0, 1)$  for  $0 < a < b < \beta \leq 1$  such that  $h \in \mathcal{C}^\beta$ , to show the existence of a solution. When we consider one-dimensional geometric processes, the iterated integrals is essentially given by powers of the

increment of the first order process. That is, the  $n$ 'th iterated integral in one dimension of a mBm  $B_{s,t}^h$  is given by  $B_{s,t}^{(n),h} = \frac{1}{n!} (B_{s,t}^h)^n$ . If we now define the  $N$ 'th order lift by

$$S_N(B_{s,t}^h) = \left(1, B_{s,t}^h, B_{s,t}^{(2),h}, \dots, B_{s,t}^{(N-1),h}\right) = \left(1, B_{s,t}^h, \frac{1}{2} (B_{s,t}^h)^2, \dots, \frac{1}{(N-1)!} (B_{s,t}^h)^{N-1}\right).$$

This rough path satisfies a higher order Chen's relation by considering the truncated algebra multiplication in the space  $T^{(N)}(\mathbb{R})$  (see section 3), such that  $S_N(B_{s,u}^h) \otimes S_N(B_{u,t}^h) = S_N(B_{s,t}^h)$ , and the regularity conditions holds. When considering rough integration, and differential equations in previous sections, we have seen that we need 4 elements in the rough path, i.e  $S_4(X) = (1, X^{(1)}, X^{(2)}, X^{(3)})$  when we were dealing with paths with regularity in  $\frac{1}{4} < \alpha \leq \frac{1}{3}$ . One would believe that we need more iterated integrals, the lower regularity we have on the process we integrate. Actually, we need  $N = \inf \{n \in \mathbb{N} : \frac{1}{a} < n\}$  iterated integrals, where  $a = \inf \{h(t) : 0 \leq t \leq T\}$  for a mBm  $B_{s,t}^h$  with  $h : [0, T] \rightarrow [a, b] \subset (0, 1)$  for  $0 < a < b < \beta \leq 1$  such that  $h \in \mathcal{C}^\beta$ . We may define a space of such one dimensional geometric multifractional rough paths of arbitrary regularity, by letting

$$S_N(B^h) \in \mathcal{C}_g^a([0, T], \mathbb{R}) \iff \sum_{i=0}^{N-1} \|B^{(i),h}\|_{ia} < \infty, \text{ and } \text{Chen's relation}.$$

Where,  $a = \inf \{h(t) : 0 \leq t \leq T\}$  and  $B^{(0),h} := 1$ . Further we could define a metric for two elements  $S_N(B^h)$  and  $S_N(\tilde{B}^h)$  of  $\mathcal{C}_g^a$  by

$$d_a(S_N(\tilde{B}^h); S_N(B^h)) := \sum_{i=0}^{N-1} \|\tilde{B}^{(i),h} - B^{(i),h}\|_{ia}$$

We will use this rough path to show existence of square root processes driven by a mBm with arbitrary regularity.

**Theorem 9.1.** *Let  $B_{s,t}^h$  be a one-dimensional multifractional Brownian motion with  $h : [0, T] \rightarrow [a, b]$ , for  $0 < a < b < \beta \leq 1$  such that  $h \in \mathcal{C}^\beta$ . Let  $S_N(B^h) \in \mathcal{C}_g^a(\mathbb{R})$  denote the geometric lift to a multifractional rough path as described above, with  $N = \inf \{n \in \mathbb{N} : \frac{1}{a} < n\}$ . Then there exists a solution  $Y_t$  to the differential equation given by*

$$dY_t = -\alpha Y_t dt + \sqrt{Y_t} d\mathbf{B}_t^h, Y_0 = y.$$

Where  $\mathbf{B}_{s,t}^h = S_N(B_{s,t}^h)$ .

*Proof.* From proposition 2.3 we know that if  $S_N(B^h) \in \mathcal{C}_g^a$ , where  $B_{s,t}^h$  is constructed according to section 6.2, there exist a smooth path  $S_N(B^{h,\varepsilon}) \in \mathcal{C}_g^1$  such that

$$d_a(S_N(B^{h,\varepsilon}); S_N(B^h)) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Indeed, due to the geometric nature of  $S_N(B^h)$ , and the fact that  $S_N(B^{h,\varepsilon})$  behaves by the rules of regular calculus, we may use the interpolation lemma 2.3 (extended to more integrals). Define  $\mathbf{B}_{s,t}^{h,\varepsilon} = S_N(B_{s,t}^{h,\varepsilon})$  and  $\mathbf{B}_{s,t}^h = S_N(B_{s,t}^h)$ . We start to study the smooth version of the square root process,

$$\dot{Y}_t^\varepsilon = -\alpha Y_t^\varepsilon dt + \sqrt{Y_t^\varepsilon} \dot{\mathbf{B}}_t^{h,\varepsilon} \quad Y_0^\varepsilon = y$$

We can solve this by looking at the derivative of  $X_t := \sqrt{Y_t^\varepsilon}$ , i.e

$$\dot{X}_t = \frac{\dot{Y}_t^\varepsilon}{2X_t} = -\frac{1}{2}\alpha\sqrt{Y_t^\varepsilon} + \frac{1}{2}\dot{\mathbf{B}}_t^{h,\varepsilon}$$

Using the integrating factor  $\exp(\frac{1}{2}\alpha t)$ , we find that

$$\sqrt{Y_t^\varepsilon} = \sqrt{y} \exp\left(-\frac{1}{2}\alpha t\right) + \frac{1}{2} \int_0^t \exp\left(-\frac{1}{2}\alpha(t-s)\right) \dot{\mathbf{B}}_s^{h,\varepsilon} ds$$

and hence

$$Y_t^\varepsilon = \left( \sqrt{y} \exp\left(-\frac{1}{2}\alpha t\right) + \frac{1}{2} \int_0^t \exp\left(-\frac{1}{2}\alpha(t-s)\right) \dot{\mathbf{B}}_s^{h,\varepsilon} ds \right)^2.$$

We know that the integral containing the approximation of the fBm converges to the fBm integral, as desired

$$\lim_{\varepsilon \rightarrow 0} \int_0^t \exp\left(-\frac{1}{2}\alpha(t-s)\right) \dot{\mathbf{B}}_s^{h,\varepsilon} ds = \int_0^t \exp\left(-\frac{1}{2}\alpha(t-s)\right) d\mathbf{B}_s^h.$$

Indeed, we know that the exponential function in  $s$  is smooth, and hence, Young's integral theory assures us that both integrals exists, and are well defined. Actually, we could as well write  $\int_0^t \exp(-\frac{1}{2}\alpha(t-s)) d\mathbf{B}_s^h = \int_0^t \exp(-\frac{1}{2}\alpha(t-s)) dB_s^h$  and  $\int_0^t \exp(-\frac{1}{2}\alpha(t-s)) d\mathbf{B}_s^{h,\varepsilon} = \int_0^t \exp(-\frac{1}{2}\alpha(t-s)) dB_s^{h,\varepsilon}$  as these are not a rough integrals in the sense of theorem 3.7. We therefore see that by boundedness of the exponential function on  $[0, T]$  we see that,

$$\left| \int_0^t \exp\left(-\frac{1}{2}\alpha(t-s)\right) \dot{\mathbf{B}}_s^{h,\varepsilon} ds - \int_0^t \exp\left(-\frac{1}{2}\alpha(t-s)\right) d\mathbf{B}_s^h \right| \leq K d_\alpha(\mathbf{B}^h, \mathbf{B}^{h,\varepsilon}) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Therefore, we define a solution  $Y_t := \lim_{\varepsilon \rightarrow 0} Y_t^\varepsilon$ , and see that this solution will solve the differential equation we started with, i.e

$$\begin{aligned} dY_t &= d \left( \left( \sqrt{y} \exp\left(-\frac{1}{2}\alpha t\right) + \frac{1}{2} \int_0^t \exp\left(-\frac{1}{2}\alpha(t-s)\right) d\mathbf{B}_s^h \right)^2 \right) \\ &= 2 \left( \sqrt{y} \exp\left(-\frac{1}{2}\alpha t\right) + \frac{1}{2} \int_0^t \exp\left(-\frac{1}{2}\alpha(t-s)\right) d\mathbf{B}_s^h \right) \\ &\quad \times \left( -\frac{1}{2}\alpha \exp\left(-\frac{1}{2}\alpha t\right) \sqrt{y} - \alpha \frac{1}{4} \exp\left(-\frac{1}{2}\alpha t\right) \int_0^t \exp\left(\frac{1}{2}\alpha s\right) d\mathbf{B}_t^h + \frac{1}{2} d\mathbf{B}_t^h \right) \end{aligned}$$



$$\begin{aligned}
&= \left( \sqrt{y} \exp\left(-\frac{1}{2}\alpha t\right) + \frac{1}{2} \int_0^t \exp\left(-\frac{1}{2}\alpha(t-s)\right) d\mathbf{B}_s^h \right) \\
&\times \left( -\alpha \sqrt{y} \exp\left(-\frac{1}{2}\alpha t\right) - \alpha \frac{1}{2} \exp\left(-\frac{1}{2}\alpha t\right) \int_0^t \exp\left(\frac{1}{2}\alpha s\right) d\mathbf{B}_t^h \right) \\
&+ \left( \sqrt{y} \exp\left(-\frac{1}{2}\alpha t\right) + \frac{1}{2} \int_0^t \exp\left(-\frac{1}{2}\alpha(t-s)\right) d\mathbf{B}_s^h \right) d\mathbf{B}_t^h \\
&= -\alpha Y_t + \sqrt{Y_t} d\mathbf{B}_t^h
\end{aligned}$$

and we are done.  $\square$

The result here is, of course, just a generalization of an SRP driven by a fractional Brownian motion, we can just choose  $h$ , such that for fixed  $a \in (0, 1)$ , then  $h(t) = a$ , for all  $t \in [0, T]$ . We can use this model for the regularity that J. Gatheral et.al [9] claims the volatility inherits. A square root process, has of course other properties than an exponential O-U model, but may in some occasions seem fitting to the volatility surface. One may also use this model for interest rates, where one could easily believe that the market should trend as well, but maybe with a hurst parameter more frequently around one half, to signify that there, with more volumes, should be less trends.

We used in the construction of the solution a function  $h$  of  $\beta$  regularity such that  $\beta > \sup \{h(t) : 0 \leq t \leq T\}$ . This suggest that we may model the function  $h$  by a stochastic process itself.

## 10 Conclusion

We have in this thesis studied the theory of rough paths, and shown applications of the theory both to stochastic processes, such as fractional and multifractional Brownian motion, and further to finance. The authors find the subject very promising. The theory of multifractional Brownian motions is, in our opinion, an important tool in financial applications, as we think the markets could be modeled more accurately with a Hürst parameter (either deterministic or stochastic) depending on time. However, it is important to remember that one would need to observe the Hürst function to apply the mBm to financial markets. As this is done by observing price data, one could get significant errors when estimating the function. Therefore, it would be suitable to conduct studies on how much more accurate one are able to model stock prices compared to a usual Brownian motion, and study the error relative to the estimation error of the Hürst function. For future research we also want to study how multifractional Brownian motions behave when  $h$  is stochastic, and how it behaves under no-arbitrage restrictions.

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