On the non-linear stability of a liquid film flowing down an inclined plane.

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Lin (1969, 1970a) and Gjevik (1970a, 1970b) have by different methods demonstrated that steady surface waves on a thin liquid film flowing down an inclined plane can exist under certain flow conditions. In this study we discuss more thoroughly the special features of the non-linear stability analysis for a parallel flow with a free surface. The similarities and differences between the two different approaches mentioned above are emphasized and the difficulties encounter by applying expansion methods developed for the study of parallel flows between rigid planes to the present problem are clearly revealed. Moreover we study analytically the stability of the steady wave solutions obtained by Gjevik (1970a, 1970b) with respect to a certain class of perturbations. This analysis supports the experimental findings that finite-amplitude waves with a certain wavelength appear to be steady under certain flow conditions.

We also study the mechanism of energy conversion during the development of steady finite-amplitude waves on falling liquid films. The negligible effect which the Reynolds stress and the corresponding distortion of the mean velocity profile have on the finite-amplitude energy balance, is pointed out.

1. Introduction.

Since falling liquid films are widely used in different technological processes as in cooling and absorbtion systems, the flow characteristics of a liquid film have been a subject of many studies. Especially the occurrence of surface waves, which significantly affects the rate of heat or mass transfer into the bulk of fluid has received much interest. A survey of the literature on this topic is given by Levich (1962) and also by Levich and Krylov (1969). Despite all efforts the development of finite amplitude waves on falling liquid films has not been satisfactorily explained. Recent independent studies by Lin (1969, 1970a) and Gjevik (1970a, 1970b), the latter works will hereafter be referred to as (I) and (II) respectively, however, clearly demonstrate the existence of steady waves under certain flow conditions. Moreover, these studies show that the problem can be formulated as a non-linear stability problem for a parallel flow. It should, however, be noticed that the linear stability problem for a parallel flow with a free surface presents some special features which make the non-linear stability analysis different from that for parallel flows between rigid planes. Therefore the methods developed by Stuart (1960), Watson (1960) and Eckhaus (1965) cannot generally be applied to the present problem. In two works by Lin (1969) and (1970a), a modification of Stuart's (1960) amplitude expansion technique, originally proposed for the non-linear stability study of parallel flow between rigid planes, has been applied for stability studies of a parallel flow with a free surface. For this problem the method leads to rather cumbersome calculations, and the limitations of the approach are not discussed by the author. On the other hand, the method used in (I) and (II), which consists of a long wave expansion combined with a Fourier expansion, is analytically more attractive. This method provides a more general formulation of the problem and enables us to discuss more thoroughly the validity of the expansions. Such a discussion is included in the present study. The connection between the method used in (I) and (II) and the method used by Lin (1969, 1970a) was only briefly discussed in (I) and (II). We will now demonstrate that under certain specified conditions we also are lead to the same type of governing equation for the amplitude of the basic wave component as discussed by Lin. This, however, only applies to the spatially periodic waves studied in (I). This analysis shows that some of the numerical results given in Lin (1970a) are erroneous. We have also in this study discussed analytically the stability of the steady wave solutions, derived in (I) and (II). Although the analysis includes both space and time periodic perturbations it must be admitted that the analysis is still quite restricted.

Moreover in this study we discuss the mechanism of energy conversion during the development of finite-amplitude waves on falling liquid films. One of the results obtained, namely the explanation of the role of the Reynolds stress on the finite-amplitude stability is already described in Lin's work (1970a). However, since this result was derived independently and by a different method by us it is also presented here. In experiments or technological processes involving film flows it is difficult to attain a a completely clean surface and the presence of surface active agents will in some cases modify the wave motion, Levich (1962). Models taking these effects into account for film flows have been studied by Benjamin (1964) and also by Lin (1970b). Of course the method used in (I) and (II) can easily be extended to include the effect of surface active agents. This analysis is certainly necessary if more precise comparisons between theoretical and experimental results are required. For the time being we have, however, completely discarded the effect of surface active agents and focused our interest on the non-linear processes involved in surface wave motion on falling liquid films.

2. Derivation of the Landau equation and some further comments on the validity of the expansions used in (I) and (II).

In order to study plane waves we have in (I) and (II) introduced a Fourier expansion of the surface deflection ζ which for two-dimensional waves is a function of the distance along the plane, x, and of time, t. Thereby the partial differential equation for ζ was reduced to sets of ordinary differential equations. For the sake of the following discussion we will recapitulate the procedure for the spatially periodic waves studied in (I). We scale ζ by the mean thickness of the fluid film, h, and write

$$\zeta(x,t) = 1 + \eta(x,t) , \qquad (1a)$$

where

$$\eta(x,t) = \sum_{k=-N}^{k=N} A_k(t)e^{ikx} . \qquad (1b)$$

According to the scaling introduced in (I) the dimensional wave number of the fundamental harmonic in (1b) is α/h, where $\alpha << 1$. Moreover $A_{-k}(t) = \overset{*}{A}_k(t)$ where asterisk denotes complex conjugate and by our scaling of ζ, $A_o = 0$. We assume $|A_k|$ to be of order $\epsilon\delta_k$ with the understanding that $\epsilon << 1$ and $\delta_1 = 1$. Initially ϵ and δ_k depend on the initial conditions. However, when a steady finite-amplitude wave has developed ϵ and δ_k are determined by the flow parameters namely; the Reynolds number and the Weber number which are defined in equation (9) below. For steady waves these parameters therefore determine the rate of convergence of the expansion (1b).

We shall attempt to solve the equation for ζ correct to terms of order ϵ^2 in the expansion (1b). Hence if δ_k is assumed to be of order $\epsilon^{|k-1|}$ this leads to a simplified set of amplitude equations :

$$\frac{dA_1}{dt} = \beta_1 A_1 + qA_2\overset{*}{A}_1 + m|A_1|^2 A_1 + O(\epsilon^5), \tag{2a}$$

$$\frac{dA_2}{dt} = \beta_2 A_2 + pA_1{}^2 + O(\epsilon^4), \tag{2b}$$

where the complex valued coefficients have the same values as in the Appendix in (II).

Under certain conditions (2a) and (2b) can be combined to an equation for $|A_1|$. This equation will be of the Landau type which is referred to frequently in non-linear stability theory (Eckhaus 1965). However, as known from non-linear stability theory of parallel flows between rigid planes, conditions under which a Landau equation for the wave amplitude can be derived, is restricted (Ellingsen et al. 1970). In order to discuss the restrictions for the free surface case we write the solution of (2b) as

$$A_2 = e^{\beta_2 t}\left[p\int_0^t A_1{}^2 e^{-\beta_2 t}dt + A_2(t=o)\right] + O(\varepsilon^4). \qquad (3)$$

where $A_2(t=o)$ denotes the initial value of A_2. Let the indices r or i denote respectively the real or imaginary part of a complex number. Under two conditions the integral in (3) can be evaluated and expressed by $A_1{}^2$;

i) $\frac{dA_1}{dt} \cong \beta_1 A_1$. A condition which is valid during the initial growth of a sufficiently small disturbance $|A_1|$.

ii) $\frac{dA_1}{dt} \cong i\beta_{1i}A_1$. A condition which is valid for a steady wave propagating approximately with the velocity β_{1i}.

If we moreover assume $\beta_{2r} < 0$ then regardless of the initial value of A_2 the resulting equation for A_1, valid for $t > \beta_{2r}{}^{-1}$ and as long as terms of order ε^6 can be neglected, will be

$$\frac{d}{dt}|A_1|^2 = 2\beta_{1r}|A_1|^2 + 2P_r|A_1|^4, \qquad (4)$$

Where P_r is the real part of the socalled second Landau coefficient. The conditions i) and ii) lead to different values of P_r. However, if $|\beta_{1r}| << |\beta_{2r}|$ the difference will be negligibly small. The steady wave amplitudes obtained in (I) and (II) correspond to the stationary solution of (4) for the case ii) above. Therefore an expression for P_r for this case is already defined in (II) and need not be recapitulated. It should be noticed that in order to have steady finite-amplitude waves we found in (II) that $\beta_{1r} > 0$ and $P_r < 0$, for small values of α the latter condition implies that $\beta_{2r} < 0$. These conditions are in agreement with (4). Due to the special features of the linear stability diagram the essential requirement for the derivation of (4), $\beta_{2r} < 0$, can only be satisfied for wave numbers in the neighbourhood of the neutral curve in the linear stability diagram. In either of the cases i) or ii) a Landau type equation valid for $t > \beta_{2r}^{-1}$ can therefore be derived only in a certain range of wave numbers. On the other hand, for all values of the wave number satisfying the requirements of the long wave expansion the set of equation (2) as well as the more general set treated in (I) are valid in a certain time span as long as the discarded higher harmonics can be neglected.

For the steady wave solutions, discussed in (I) and (II) the peculiarities of the linear stability diagram in the present problem are also crucial for the rate of convergence of the expansion (1b). The assumption that δ_k is of order $\varepsilon^{|k-1|}$, is only satisfied close to the neutral curve in the linear stability diagram and then only within certain ranges of the flow parameters. This can easily be seen from the numerical result given in (I) and (II). For example for values of the flow parameters such that β_{2r} tends to zero δ_k will not be of order $\varepsilon^{|k-1|}$.An estimate of the rate of convergence is then obtained by retaining higher order terms in (1b) as we have done in (I) and (II).

If we assume δ_k to be of order $\varepsilon^{|k-1|}$ and attempt to determine the steady wave amplitude correct to order ε^3 we are lead to the set of amplitude equations given in (I) (equation 15). These equations were solved numerically in (I) and (II). The accuracy obtained in this way corresponds to that obtained by the evaluation of the terms of order ε^6 in equation (4).

Lin's analysis (1969) lead to a similar type of equation for the amplitude of the basic harmonic as given in (4), but there is, independent of scaling procedures, some discrepancy between the values of the coefficients given by Lin and those given above. Our numerical results for the steady wave amplitudes in the neighbourhood of the line $\beta_{2r} = 0$ in the stability diagram contradict these given by Lin (1970a). According to these results the second Landau coefficient becomes zero in this range of the flow parameters, however, his analysis for steady waves seems to imply that the second Landau coefficient is derived for a similar condition as stated in i) above. On the other hand, our analysis, which correspond to the more realistic condition ii), implies that P_r becomes of order $1/\alpha$ when $\beta_{2r} \to 0$, a result which also is supported by numerical results. Also for other wave numbers there is a discrepancy between the value of the steady wave amplitudes given by Lin (1969) and and those given in (II). For example our data for the steady wave amplitudes at flow rates corresponding to Kapitza & Kapitza (1949) observations are about 20% higher than those given by Lin. Although the data are not quite comparable since the perturbation methods are different and besides that in our computation the effect of the α^2 terms as well as the effect of the non-linear correction to the volume flux are taken into account, these effects cannot fully explain this large discrepancy which we so far have found no reason for.

The long wave expansion applied in (I) and (II) can only be expected to give a reasonable approximation as long as $\alpha << 1$ and also $\alpha R << 1$, where R is the Reynolds number of the flow. In the range of wave numbers corresponding to steady waves ($\beta_{1r} > 0$, $\beta_{2r} < 0$) the requirement $\alpha R << 1$ soon becomes invalid for $R > 1$. Since the long wave expansion used in (I) and (II) also leads to extremely involved algebra when it is carried out to higher order approximations, numerical methods might be the only way to investigate the problem in these cases. For example a Taylor series expansion of the stream function for the wave motion in terms of the coordinate normal to the plane, where the coefficients are functions of x and t, combined with a Fourier expansion of these coefficients in x will reduce the problem to a set of ordinary differential equations. With a suitable truncation these equations might be solved numerically.

There is also another restriction which should be noted. Although the numerical results in (II) indicate that the scaling in (I) is valid for steady wave solutions even at an inclination angle down to 7.5°, the scaling will obviously become invalid at sufficiently small angles of inclination. In these cases a similar scaling as used by Gjevik (1970c) would be more appropriate.

3. Stability of the steady wave solution.

Since the equations (2a) and (2b) in a certain range of the flow parameters can be reduced to a Landau type equation for $|A_1|$, it follows immediately from previous results (Eckhaus 1960) that the steady solution of equation (4), for $\beta_{1r} > 0$ and $P_r < 0$, is stable for perturbations in $|A_1|$. The stability of the steady wave solution to perturbations which have a more general x-dependence is obviously a difficult problem. We will, however, study some types of perturbations which are to be expected under experimental conditions. Consider now the case where the steady waves are generated by a wave generator (idealized). This situation can be modelled by the analysis in (II) which we briefly will recapitulate. The surface deflection is described by (1a) where

$$\eta(x,t) = \sum_{k=-N}^{k=N} \tilde{A}_k(x)e^{ik(x+ct)} \tag{5}$$

In (5) c denotes the dimensionless frequency and $\tilde{A}_k = \overset{*}{\tilde{A}}_{-k}(x)$. For weakly non-linear waves the $\tilde{A}_k$ are slowly varying functions of x. If (5) is introduced in the equation for the surface deflection, we obtain a set of equations for $\tilde{A}_1$ and $\tilde{A}_2$ while $\tilde{A}_0$ is determined by the rate of volume flux. (For details see II)*) With much the same analysis as that leading to (4) we find, approximately:

$$\frac{d}{dx}|A_1|^2 = \beta_{1r}|\tilde{A}_1|^2 + P_r|\tilde{A}_1|^4 + \tfrac{1}{2}r_r|\tilde{A}_1|^2\tilde{A}_0 \, , \tag{6}$$

where

$$\tilde{A}_0 = \tfrac{1}{2} Q_0 - \tfrac{1}{3} - 2|\tilde{A}_1|^2 \tag{7}$$

Equation (7) expresses that the mean volume flux (with respect to time) is constant and equal to Q_0. The coefficient r_r in (6) will be proportional to α and need not be recapitulated here. For $\beta_{1r} > 0$ and $P_r < 0$, by a proper choice of h, $\tilde{A}_0$ can be set equal zero at values of x where the steady wave has developed fully. Therefore the steady wave solution of (6) corresponds to

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*) Note that for the temporally periodic waves,(5),the scaling length h for ζ must be interpreted as the mean film thickness with respect to time at a certain position x along the plane.

that of equation (4). Suppose that the wave generator is vibrating with an amplitude corresponding to a steady wave amplitude and that a small perturbation in $|\tilde{A}_1|$ is introduced such that the mean volume flux is kept constant. According to (7) this latter condition requires a perturbation of the mean layer thickness. For sufficiently small values of $\alpha, |P_r| >> |r_r|$, and it follows from (6) that in this equation the perturbation in $\tilde{A}_0$ can be neglected compared to the perturbation in $|\tilde{A}_1|$. Therefore we conclude, with the same arguments as for the spatially periodic perturbation referred to above, that the steady wave solution is stable for a perturbation in $|\tilde{A}_1|$. The stability analysis for the steady wave train does not directly apply to cases with spatially varying wave trains downstream from the wave generator. Nevertheless, for a finite amplitude wave train with spatial variation it is reasonable to expect that this motion will have a similar although weaker, stabilizing effect on a perturbation of $|\tilde{A}_1|$ as the steady wave motion is found to have.

Perturbations with a slightly different frequency than that of a steady wave motion will, according to the linear stability analysis, be unstable. Therefore this type of perturbation might grow and overshadow the basic steady motion. However, we will argue that the basic steady motion will also have a stabilizing effect on this type of perturbation. Consider, for example, two spatially periodic perturbations, PP, with wave numbers $\alpha' = \alpha + \delta$ and $\alpha'' = \alpha - \delta$ respectively, where α is the wave number of the steady finite-amplitude wave and the parameter δ is much less than α. Then by non-linear interaction between the PP and the steady wave, wave components, S, with wave numbers close to that of the higher harmonics for the steady wave as well as a wave component, Q, with wave number δ will be generated. When the wave number of the steady wave is chosen so that $\beta_{1r} > 0, \beta_{2r} < 0$ then according to linear stability analysis the S components are stable while the Q component will be unstable. If we assume Q to be small initially, it might be neglected for a certain time span. The weak coupling between the Q and the PP components suggests that this time span might be large compared to the time it takes for the steady wave pattern to travel a distance equal to its own wavelength. Some estimates which we have done support this suggestion. As long as the Q component can be neglected the stability analysis for the PP component will be very similar to that already given for the same type of perturbations to a steady wave motion in parallel flows between rigid planes. (Eckhaus 1965, Chp. 8). Therefore the result obtained by Eckhaus that the PP components will be damped also applies here, but then only for a certain limited time span.

Under experimental conditions where the steady waves are initiated artificially it is reasonable to expect that the most dominant two-dimensional perturbations occurring are contained among the types of perturbations investigated above. Our analysis can, however, only indicate what experiments show, namely that finite-amplitude waves with a certain wavelength will appear steady under certain flow conditions.

Finally we add some remarks on Kapitza's (1948) analysis of the present non-linear stability problem. It follows from (7) that among the steady wave motions having the same mean volume flux, the wave with maximum amplitude of the fundamental harmonic will correspond to the lowest value of the mean thickness of the fluid layer. Kapitza (1948), stated without proof that only this wave motion would correspond to a stable motion and consequently used this assumption to determine the flow uniquely. Our results obtained in (I) and (II), that the steady flow depends strongly on the initial or boundary condition imposed, together with the results of the stability analysis above indicate, however, that Kapitza's assumption is inappropriate and that only the wave motion having approximately the largest amplitude for a given mean volume flux can be determined by the method suggested by Kapitza.

4. The energy conversion during the development of steady finite-amplitude waves on falling liquid films.

We shall consider two-dimensional motion down an inclined plane. The basic parabolic flow we denote U, while the velocity components of the wave motion, along the plane and normal to the plane respectively, we denote u and w. The velocity components of the total motion are then:

$$\tilde{u} = U + u \qquad \tilde{w} = w \tag{8}$$

Otherwise we shall adopt the same notation and scaling procedure as introduced in (I) and (II). The motion will then be characterized by a Reynolds number and a Weber number which can be written:

$$R = \frac{gh^3 \sin\theta}{2\nu^2}, \qquad W = \frac{T}{\rho g h^2 \sin\theta}, \tag{9}$$

where θ is the inclination angle of the plane, g is the acceleration of gravity, ν is the kinematic viscosity, ρ is the density of the fluid and T denotes the surface tension. The scaling length h is a characteristic thickness of the fluid layer and is defined differently for spatially periodic wave motion and for temporally periodic wave motion (See § 2 and 3 above).

Consider the case where the wave motion is spatially periodic. The temporal rate of change of the kinetic energy, $\tilde{E}$, per wavelength and per unit span of the plane, is caused by the following effects. The work done by the component of gravity along the inclined plane which we denote $\tilde{N}$, the rate of energy dissipation which we denote $\tilde{M}$, the work done against the surface tension which we denote $\tilde{L}$ and finally the work done to deform the free surface in the gravity field which we denote $\tilde{P}$. If all these different contributions per wavelength and per unit span of the flow are scaled according to (I) and (II) by $\rho R^3\nu^3/h^2$ we obtain, by using the equation of motion and the corresponding boundary conditions given in (I):

$$\frac{d\tilde{E}}{dt} = \tilde{N} - \tilde{M} + \tilde{L} + \tilde{P} , \qquad (10)$$

where the right hand side is defined in Appendix A. Obviously for the parabolic flow, U, the energy balance in (10) is $\tilde{L} = \tilde{P} = 0$ and $\tilde{N} = \tilde{M}$. For steady finite-amplitude waves ζ is a periodic function of $x \mp ct$, where c is the dimensionless wave velocity and consequently, in this case, a similar equilibrium energy balance exists. The different constraints on the spatially periodic and the temporally periodic wave motion are clearly illustrated by energy considerations. Consider an infinitesimal amplitude surface perturbation of the form

$$\zeta = 1 + A(t)e^{i(x+ct)} + \text{complex conjugate}, \qquad (11)$$

where $A(t)$ is an exponential growing or decaying function of time and c is determined by linear theory. Since $\tilde{L}$ and $\tilde{P}$ are negative for a temporally growing wave, i.e. energy is converted to both surface energy and to potential energy, the gravity component along the inclined plane is the only energy source for an unstable wave component. Since $\tilde{N}$ depends on A an eventual finite-amplitude balance for this wave component can only be established by a proper adjustment of $\tilde{N}$, i.e. by an adjustment of the rate of volume flow in the x-direction. A temporally periodic surface perturbation with infinitesimal amplitude is also given by (11) where A now is an exponential function of x. For weakly stable or unstable waves $A(x)$ is a slowly varying function. If we follow one individual wave the energy equation (10) will also apply (approximately) in this case. Since the mean volume flux is kept constant, $\tilde{N}$ will also be independent of x. This is achieved by an adjustment of the mean layer thickness. Therefore for the temporally periodic wave an eventual finite-amplitude balance in (10) can only be established by a non-linear modification of the energy dissipation.

The special features of non-linear stability analysis of a parallel flow with a free surface can be demonstrated clearly by energy considerations. Let us examine the energy conversion between the wave motion and the basic parabolic flow. The temporal rate of change of kinetic energy, E, in the perturbation motion is caused by: The work done by the Reynolds shear stresses which we denote

K, the rate of energy dissipation in the perturbation motion which we denote M. (This expression is defined similar to $\tilde{M}$ in equation (10), and the work done by the pressure and the viscous stresses set up by the perturbation motion. This latter contribution can as well be interpreted as work done at the deformed free surface, namely i) work done against the surface tension which we denote L, ii) work done against the hydrostatic pressure deviation in the steady flow, which we denote P, iii) the work done against the component of the viscous stress deviation in the steady flow normal to the free surface, which we denote S_n, and finally iv) the work done against the component of viscous stress deviation in the steady flow tangential to the free surface, which we denote S_t. It is pertinent to mention that the contributions ii)-iv) arise from the assumed analytical continuation of the basic parabolic velocity profile and the corresponding pressure distribution at the deformed surface. If we assume spatially periodic wave motion and evaluate all these different contributions per wavelength and per unit span of the flow, we find

$$\frac{dE}{dt} = K - M + L + P + S_n + S_t, \tag{12}$$

where the right hand side of (12) is defined in Appendix B and the scaling is as for the terms in equation (10). For parallel flows between rigid planes L, P, S_n and S_t vanish and the only energy source (in the mean) is through the action of the Reynolds stress, $\overline{uw}$. With a deformable surface, however, the situation becomes more involved. Let us now estimate the relative importance of the different terms on the right hand side of (12). We first observe that for waves of the type (11) L and P are negative for temporally growing waves and positive for temporally decaying waves. If we assume a wave motion with amplitude ε then it follows from Appendix B and Appendix C that S_n is of order $\frac{\alpha\varepsilon^3}{R}$, S_t is of order $\frac{\varepsilon^2}{\alpha R}$, and $K = \alpha R\varepsilon^2$.*) It is therefore obvious that the main energy source for the perturbation motion is S_t, while the effect of the Reynolds stress is negligible for the surface mode treated in (I) and (II). It is pertinent to mention here that by using the results in Appendix C we easily show that for the disturbance (11) the Reynolds stress is stabilizing if

$$\alpha^2 < (\tfrac{31}{56} R - \cot\theta)/W. \qquad (13)$$

The requirement (13) can be satisfied even for disturbances which are unstable according to linear stability theory.

For a steady finite-amplitude wave, obviously $L = P = 0$. Therefore for the steady weakly non-linear waves studied in (I) and (II), the equilibrium energy balance is mainly established by the non-linear modification of the terms S_t and M. It follows from the results in (I) and (II) that this modification is attained mainly by the generation of higher harmonics through non-linear interactions while the mean distortion of the basic parabolic velocity profile, given in equation (4C), Appendix C, is a negligible effect. This situation is quite different for high Reynolds number parallel flows between rigid planes where the distortion of the mean velocity profile has an important effect on the finite-amplitude equilibrium energy balance. For plane Couette flow for example this is discussed in the work by Ellingsen et al. (1970).

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*) $\alpha^2 W$ and $\cot\theta$ is assumed to be of order unity.

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Appendix A.

If z denotes the coordinate normal to the inclined plane and a bar denotes integration over one wavelength in x-direction, we have:

$$\tilde{E} = \overline{\int_0^{\zeta} \tfrac{1}{2}(\tilde{u}^2 + \alpha^2 \tilde{w})dz}$$

$$\tilde{N} = \frac{2}{\alpha R} \overline{\int_0^{\zeta} \tilde{u}dz}$$

$$\tilde{M} = \frac{1}{\alpha R} \overline{\int_0^{\zeta} \{(\frac{\partial \tilde{u}}{\partial z})^2 + 2\alpha^2 \left[(\frac{\partial \tilde{u}}{\partial x})^2 + \frac{\partial \tilde{u}}{\partial x}\frac{\partial \tilde{w}}{\partial x} + (\frac{\partial \tilde{w}}{\partial z})^2\right] + \alpha^4 (\frac{\partial \tilde{w}}{\partial x})^2\} dz}$$

$$\tilde{L} = \frac{2\alpha^2 W}{R} \overline{\frac{\frac{\partial \zeta}{\partial t} \frac{\partial^2 \zeta}{\partial x^2}}{(1+\alpha^2(\frac{\partial \zeta}{\partial x})^2)^{\frac{3}{2}}}}$$

$$\tilde{P} = -\frac{2\cot\theta}{R} \overline{\int_0^{\zeta} w dz} = -\frac{\cot\theta}{R} \overline{\frac{\partial \zeta^2}{\partial t}} .$$

Appendix B.

With the same notation as in Appendix A, we have

$$K = -\overline{\int_0^{\zeta} \frac{dU}{dz} uw \, dz}$$

$$L = \frac{2\alpha^2 W}{R} \frac{\overline{\phi \frac{\partial^2 \zeta}{\partial x^2}}}{(1+\alpha^2(\frac{\partial \zeta}{\partial x})^2)^{\frac{3}{2}}}$$

$$P = \frac{2\cot\theta}{R} \overline{\phi(1-\zeta)}$$

$$S_n = \frac{2\alpha}{R} \frac{\overline{\phi \frac{dU}{dz} \frac{\partial\zeta}{\partial x}}}{1+\alpha^2(\frac{\partial\zeta}{\partial x})^2} \qquad \text{at} \quad z = \zeta,$$

$$S_t = -\frac{1}{\alpha R}\,(u+\alpha^2 w\frac{\partial\zeta}{\partial x})\,\frac{\frac{dU}{dz}(1-\alpha^2(\frac{\partial\zeta}{\partial x})^2)}{1+\alpha^2(\frac{\partial\zeta}{\partial x})^2} \qquad \text{at} \quad z = \zeta,$$

where

$$\phi = \frac{\partial\zeta}{\partial t} + U\,\frac{\partial\zeta}{\partial x}$$

and

$$U = 2z - z^2.$$

Appendix C.

The Reynolds stress $\overline{uw}$ in the case of spatially periodic motion is easily obtained, correct to terms of order α, from the results in (I), namely:

$$\overline{uw} = -\tfrac{1}{3}\left[\alpha^3 W\overline{(\frac{\partial^2\zeta}{\partial x^2})^2} + \alpha\cot\theta\overline{(\frac{\partial\zeta}{\partial x})^2}\right]z^4$$

$$+\ \tfrac{1}{3}\alpha R\overline{\zeta^2(\frac{\partial\zeta}{\partial x})^2}z^5 - \tfrac{1}{10}\,\alpha R\zeta\overline{(\frac{\partial\zeta}{\partial x})^2}z^6 \qquad (1C)$$

where z is the coordinate normal to the plane and a bar denotes the mean with respect to x.

For steady two-dimensional motion, the distortion of the mean velocity profile is given by

$$\frac{d^2\overline{u}}{dz^2} = \alpha R\,\frac{d}{dz}\,\overline{uw} \qquad (2C)$$

The boundary conditions for $\overline{u}$ are

$$\overline{u} = \frac{d\overline{u}}{dz} = 0 \qquad \text{for } z = 0. \qquad (3C)$$

The latter condition follows immediately from the fact that the mean layer thickness is used as the scaling length in the Reynolds number. From (2C) and (3C), we find

$$\overline{u} = \alpha R\int_0^z \overline{uw}dz \qquad (4C)$$

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