On steady convection in a porous medium and Bénard convection at high Rayleigh numbers

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Abstract

Utilizing Kuo's method (Kuo 1960) the dependence of the Nusselt number (N) on the Rayleigh number (R) is examined for a porous medium. This dependence is also derived by simple physical arguments. It is a good resemblance between theory and experiments. In the last part of the paper ordinary Bénard convection is studied, and it is found that asymptotically $N \sim R^{1/3}$, valid both for laminar and turbulent motion.
1. Introduction.

We shall in the present paper mainly be concerned with convection in a porous medium. The last section, however, is devoted to ordinary Bénard convection at high Rayleigh numbers. Convection in a porous medium uniformly heated from below is of considerable geophysical interest, it is thus believed that this phenomenon occurs within the mantle of the earth. It may also be mentioned that this problem has an important technical aspect, the theory having an application to the study of (preventing) convection and thereby freezing in road and railroad constructions (frozen soil, frozed heaw). It may also be worth mentioning that convection in a porous medium may be well adapted to demonstrate experimentally non-linear effects in convection such as the preferred cell pattern or hysteresis. In ordinary Bénard convection it is necessary with extreme thin fluid layers to detect these phenomenons (see Palm, Ellingsen & Gjevik (1967)). In a porous medium, however, the necessary depth of the fluid layer will be of another order of magnitude since the friction force now is much larger.

The possibility of free convection in a porous medium heated uniformly from below and the similarity to Bénard convection was pointed out by Horton & Rogers (1945) and Lapwood (1948). Wooding (1957 and following papers) has extended these studies. Elder (1958, 1967) and Schneider (1963) have performed laboratory experiments and Elder (1967) has also attacked the problem by a numerical method.

In the present paper the non-linear equations will be solved by applying an amplitude expansion proposed by Kuo (1960) for
ordinary Bénard convection. This expansion seems to converge very rapidly and gives a very good agreement with the observed data. With the terms retained in the expansion, the result is valid up to about 6 times the critical Rayleigh number.

The connection between the Nusselt number and Rayleigh number is also found by physical arguments for moderate and high Rayleigh numbers. When the Reynolds number become about unity, Darcy's law ceases to be valid and new phenomena appear.

The last part of the paper is concerned with ordinary Bénard convection at very high Rayleigh numbers. It is concluded that both in the laminar and turbulent range the Nusselt number is proportional to the Rayleigh number in the power of 1/3.

2. The equations of motion for a porous medium.

In order to examine the behaviour of steady free convection in a porous medium, we shall expand the dependent variables in series of orthogonal functions and expand the coefficients of these functions in power series of a parameter \( \eta \) which is less than unity. These expansions which were applied by Kuo (1960) in the theory of ordinary Bénard convection, turns out to converge very rapidly.

The porous medium may be thought of as composed of closely packed uniform spheres (grains), completely surrounded by a homogeneous fluid. The equations governing the motion of the
porous medium for the steady case may be written

\begin{equation}
- \nabla p - \rho_0 a T \mathbf{g}^t + \Sigma \mathbf{T}_S + \Sigma \mathbf{P}_S = 0
\end{equation}

\begin{equation}
\nabla \cdot \mathbf{V} = 0
\end{equation}

\begin{equation}
\mathbf{V} \cdot \nabla T = \kappa_m \mathbf{V}^2 T
\end{equation}

(a derivation of these equations are given in Palm & Weber (1971)).

Here \( p \) is the pressure, \( \rho_0 \) a standard density, \( a \) the coefficient of expansion, \( T \) the temperature, \( g \) the acceleration of gravity and \( \kappa_m \) the thermal diffusivity for the porous medium. \( \Sigma \mathbf{T}_S \) and \( \Sigma \mathbf{P}_S \) denote the viscous drag and pressure drag, respectively, acting on the grains per unit volume. For small Reynolds number flow the total drag is a linear function of the velocity \( \mathbf{V} \) (Darcy's law) such that

\begin{equation}
\Sigma \mathbf{T}_S + \Sigma \mathbf{P}_S = - \frac{\mu}{k} \mathbf{V}.
\end{equation}

Here \( \mu \) is the viscosity and \( k \) the permeability.

It is assumed that the material is of infinite horizontal extent and bounded by two horizontal, impermeable boundaries, the distance between them being \( h \). Furthermore, let \( \Delta T \) denote the temperature difference between these horizontal boundaries. The field variables may then conveniently be made dimensionless by choosing as units of length, temperature, pressure, velocity

\begin{equation}
h, \Delta T, \frac{\mu \kappa_m}{k}, \frac{\kappa_m}{h}
\end{equation}
Equations (2.1) - (2.3) then take the form (Wooding 1957)

\begin{align*}
(2.6) \quad - \nabla p + R'Tk - \hat{v} & = 0 \\
(2.7) \quad \nabla \cdot \hat{v} & = 0 \\
(2.8) \quad \hat{v} \cdot \nabla T & = \nabla^2 T
\end{align*}

Here \( R' \) is a Rayleigh number defined by

\begin{equation}
(2.9) \quad R' = \frac{k \rho \Delta \theta}{k_m \nu}
\end{equation}

and \( \hat{k} \) is the unit vertical vector.

Comparing (2.6) - (2.8) with the corresponding equations in ordinary Bénard convection, it is noted that (2.6) corresponds to Bénard convection with infinite Prandtl number. This is due to the fact that the frictional effect is much larger in a porous medium than in a fluid. Actually, according to Rumer & Drinker (1966) the permeability \( k \) may be approximated by

\begin{equation}
(2.10) \quad k = 6.54 \cdot 10^{-4} d^2
\end{equation}

where \( d \) is the diameter of the grains, which shows that the assertion above is valid for a motion with characteristic length scale equal or larger than about \( d \).

We shall assume that the horizontal boundaries are perfect heat conductors and that the vertical velocity is zero at the boundaries. The critical \( R' \) number is then found to be \( 4\pi^2 \) (Lapwood (1948)). It will further be assumed that the motion is two-dimensional. According to Schlüter, Lortz & Busse (1965) this is the only stable mode in ordinary Bénard convection for moderate overcritical values of the Rayleigh number.
Introducing $\theta$ defined by

\begin{equation}
T = T_0 - z + \theta
\end{equation}

where $T_0$ is a standard (dimensionless) temperature, eliminating the pressure and applying the equation of continuity, we end up with

\begin{equation}
\nabla^2 w + R' \nabla^2 \tilde{w} = R' \nabla^2 (\nabla \cdot \theta)
\end{equation}

\begin{equation}
\nabla^2 \theta + w = \nabla \cdot \theta
\end{equation}

\begin{equation}
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0
\end{equation}

Here the $x$-axis is horizontal and the $z$-axis vertical. $u$ and $w$ denote the horizontal and vertical velocity, respectively, and $\nabla^2$ is the two-dimensional Laplacian.


Following Kuo (1960) we expand the solution in a power series of the parameter $\eta$ which is defined by

\begin{equation}
\eta = \frac{R'_0 - R'}{R'_0} = \frac{\Delta R'}{R'}
\end{equation}

where $R'_0$ is the critical Rayleigh number. It is noted that $\eta < 1$. The solution of (2.12) - (2.14) may then be written

\begin{equation}
w = \eta \omega^{(1)} + \eta^2 \omega^{(2)} + \ldots \eta^n \omega^{(n)} + \ldots
\end{equation}

\begin{equation}
\theta = \eta \theta^{(1)} + \eta^2 \theta^{(2)} + \ldots \eta^n \theta^{(n)} + \ldots
\end{equation}
Here \( w^{(n)} \) and \( \theta^{(n)} \) themselves are functions of \( n \), characterized by tending towards the order of unity when \( n \to 0 \). For the actual boundary conditions \( w^{(n)} \) and \( \theta^{(n)} \) may be expanded in the following series

\[
\begin{align*}
    w^{(n)} &= \sum_{p,q} w_{pq}^{(n)} \cos \rho x \sin q\pi z \\
    \theta^{(n)} &= \sum_{p,q} \theta_{pq}^{(n)} \cos \rho x \sin q\pi z
\end{align*}
\]

(3.3)

where \( w_{pq}^{(n)} \) (and \( \theta_{pq}^{(n)} \)) may be written as a power series in \( n \)

\[
\begin{align*}
    w_{pq}^{(1)} &= A_{pq}^{(0)} + A_{pq}^{(1)} n + A_{pq}^{(2)} n^2 + \cdots \\
    w_{pq}^{(2)} &= B_{pq}^{(0)} + B_{pq}^{(1)} n + B_{pq}^{(2)} n^2 + \cdots
\end{align*}
\]

(3.4)

The Rayleigh number is according to (3.1)

\[
R' = \frac{R'_0}{1 - \eta^2}
\]

(3.5)

This expression may be developed in a power series of \( \eta \). Instead we may apply the finite formula

\[
R' = R'_0 + \frac{R'_0}{1 - \eta^{2s}} (\eta^2 + \cdots \eta^{2s})
\]

(3.6)

or

\[
R' = R'_0 + R'_{os} (\eta^2 + \cdots \eta^{2s})
\]

(3.7)

where \( R'_{os} \) contains terms of order larger than \( 2s \). Let us assume that we are working to the order of \( 2s \). \( R'_{os} \) may then
be replaced by \( R'_o \) as usually done, or we may retain \( R'_o \) as a function of \( \eta \) which means that we are working with too large accuracy in \( R' \). This last procedure was proposed by Kuo (1960) and turns out to lead to a more rapid convergence. By this method we are working with a correct value for \( R' \), in contrast to the ordinary way of proceeding when \( R' \) is approximated by a power series to the order 2s.

The first order solution is found from

\[
(3.8) \quad V^i w^{(1)} + R'_o V^i w^{(1)} = 0
\]

which together with the boundary conditions leads to

\[
(3.9) \quad w^{(1)} = A \cos \alpha x \sin \pi z
\]

with

\[
(3.10) \quad R'_o = \frac{\left(\frac{\pi^2 + \alpha^2}{\alpha^2}\right)^2}{a^2}
\]

The minimum value of \( R'_o \), \( R_o \) say, is given by

\[
(3.11) \quad R_o = 4\pi^2, \quad a = \pi
\]

In the following calculations we shall assume that \( a \) is given by (3.11). With the notations of (3.4) we thus have, valid to the first order

\[
(3.12) \quad w^{(1)}_{11} = A_{11}^{(0)} \cos \pi x \sin \pi z
\]

\[
\theta^{(1)} = \frac{A_{11}^{(0)}}{2\pi^2} \cos \pi x \sin \pi z
\]

By a straightforward procedure we obtain the second order solution and thereby the equation for the third order solution.
From this the condition for solvability determines \( A_{11}^{(0)} \), which is found to be

\[
A_{11}^{(0)} = 4\pi \left( \frac{R_{OS}'}{R_0} \right)^{\frac{1}{2}}
\]

The condition for solvability for the fourth order equation leads to

\[
A_{11}^{(1)} = 0
\]

Correspondingly, the fifth order equation gives

\[
A_{11}^{(2)} = \frac{R_{OS}'}{R_0}^{\frac{1}{2}} \left( 1 + \frac{7}{24} \frac{R_{OS}'}{R_0} \right)
\]

The calculations are carried out up to sixth order terms. It is then found that

\[
A_{11}^{(3)} = 0
\]

and

\[
A_{11}^{(4)} = \frac{3\pi}{2} \left( \frac{R_{OS}'}{R_0} \right)^{\frac{1}{2}} \left( 1 + \frac{7}{12} \frac{R_{OS}'}{R_0} - \frac{173}{3 \times 24 \times 24} (\frac{R_{OS}'}{R_0})^2 \right)
\]

The heat transport is measured by the Nusselt number \( N \) which may be written

\[
N = 1 - \frac{\partial T}{\partial z} \bigg|_{z=0}
\]

where the bar indicates a horizontal mean. Let \( N^{(2)} \), \( N^{(4)} \) and \( N^{(6)} \) denote the second order, fourth order and sixth order approximation for the Nusselt number. Applying the results derived above we find
(3.19) \[ N(2) = 1 + 2 \left( \frac{R'_S}{R'_o} \right) \eta^2 \]

(3.20) \[ N(4) = N(2) + 2 \frac{R'_S}{R'_o} (1 - \frac{17}{24} \frac{R'_S}{R'_o}) \eta^4 \]

(3.21) \[ N(6) = N(4) + 2 \frac{R'_S}{R'_o} (1 - \frac{17}{12} \frac{R'_S}{R'_o} + \frac{191}{288} (\frac{R'_S}{R'_o})^2) \eta^6 \]

4. Discussion of the solution.

In Fig. 1 are shown \( N(2) \), \( N(4) \) and \( N(6) \) as functions of \( R'/R'_o \) obtained from (3.19) - (3.21), choosing \( s = 1, 2, 3 \) respectively (solid lines). In the figure is also displayed a curve illustrating the general trend of the various experimental data. It turns out that for moderate values of \( R'/R'_o \), larger than about 2-3, say, this curve is close to a straight line. We are thus in the position of being able to compare our approximate results with the right answer. It is seen that \( N(2) \) is no good approximation at all whereas \( N(6) \) is a very good approximation for \( R'/R'_o \) less than about 6. For larger values of \( R'/R'_o \) it obviously is necessary to take into account higher order approximations. For the sake of comparison we have also drawn the curves for \( N(2) \), \( N(4) \) and \( N(6) \) which would have been obtained by replacing \( R'_S \) with \( R'_o \) in the formulas above. It is noted that these set of values converge much poorer than the first set. In the figure is also shown the values obtained by Elder (1967) for \( R'/R'_o \leq 2.5 \) by using a suitable finite difference method.
5. **Comparison with experiments.**

We shall first by simple physical arguments derive that $N \sim R'$, compare Fig. 1. For values of $R'$ larger than 2-3, the mean temperature profile in the central region of the convection cell is spatially constant. Thus $\vartheta$ is here of order unity. Assuming that in this region the friction force and the buoyancy force are of the same order of magnitude, we have from (2.6)

\[(5.1) \quad w \sim R' \vartheta\]

Multiplying this with $\vartheta$ and integrating horizontally over the cell, we obtain

\[(5.2) \quad \bar{w}\vartheta \sim R' \bar{\vartheta}^2 \sim R'\]

where the bar denotes horizontal integration. From the definition of the Nusselt number

\[(5.3) \quad N \sim \bar{w}\vartheta\]

which gives

\[(5.4) \quad N \sim R'\]

It is important to note that this result is only true for moderate values of $R'$.

For larger values of $R'$ horizontal as well as vertical boundary layers develop. The central region of the cell is now characterized by constant temperature, as shown by Pillow (1952) for the case of ordinary Bénard convection. From the vorticity
equation, obtained from (2.6), it then follows that the vorticity, and accordingly also the velocity, vanish in the central region. Thus the cell can be divided into an isothermal, motionless core surrounded by boundary layers.

We now must introduce

\[(5.5) \quad \delta^2 \sim \delta\]

in (5.2) where \(\delta\) is the (dimensionless) thickness of the boundary layer. Since

\[(5.6) \quad N = \frac{1}{\delta}\]

we obtain

\[(5.7) \quad N^2 \sim R^1\]

or

\[(5.8) \quad N \sim R^{1/2} .\]

Thus asymptotically (5.8) is the correct relation.

Equation (5.4) has been derived by Elder (1967). We find that his derivation is not free from objections since he uses boundary layer considerations which actually should lead to (5.8). Elder claims that \(w \sim \delta^{-1}\) which is only true very close to the horizontal boundaries. It is easily seen that in the derivation above for the asymptotic case, \(w \sim \delta^{-2}\), which is necessary in order to obtain the correct convective heat flux.

Formulas (5.4) and (5.8) are only valid as long as our basic equations (2.12) - (2.14) are correct. When the diameter of the grains is not small, (2.12) - (2.14) cease to be valid for moderate
values of the Rayleigh number, and we must expect another relation between the Nusselt number and the Rayleigh number. This is seen in Fig. 2 where experimental data by Schneider (1963) and Elder (1967) show for moderate Rayleigh numbers a marked deviation from the straight line. It is noted that this deviation starts for smaller Rayleigh number the larger the grain diameter is. It has been suggested by Wooding (1958) and Elder (1967) that the deviation is due to the boundary layers becoming of the same order of magnitude as the grain diameter. There is, however, also another effect which may render the basic equations invalid, viz. the occurrence of quadratic terms in Darcy's law. This effect is expected to be sensible for Reynolds numbers about unity.

To examine this problem more closely, we shall derive an expression for the Reynolds number. Applying the boundary conditions, the energy equation for a cell takes the form

\[(5.9) \quad \langle v^2 \rangle = R' \langle \omega \theta \rangle \]

where \(< >\) denotes vertical integration and the bar horizontal integration. It is easily derived that

\[(5.10) \quad \langle \omega \theta \rangle = N - 1 \]

which leads to

\[(5.11) \quad \langle v^2 \rangle = R'(N-1) \]

Let \((\langle v^2 \rangle)^{\frac{1}{2}}\) be the characteristic (dimensionless) velocity. We then may write the Reynolds number

\[(5.12) \quad Re = \frac{(\langle v^2 \rangle)^{\frac{1}{2}} d}{Pr} \]
where $Pr$ is the Prandtl number

\begin{equation}
Pr = \frac{\nu}{\kappa m}
\end{equation}

and $d$ is the (dimensionless) grain diameter. From (5.11) we obtain

\begin{equation}
Re = R_{1}^{1/2}(N-1)^{1/2}Pr^{-1}d
\end{equation}

Schneider (1963) has measured the Nusselt number as a function of the Rayleigh number for several granular materials, and from his experimental curves (Fig. 3a, p. 250) we may estimate the values of $N$ and $R'$ where the deviations from the straight line start. We have only been able to do this for three of his experimental series which lead to

1. Steel spheres in turpentine
   \begin{align*}
   R' &= 60, \quad N = 1.5, \quad Pr = 1.42, \quad d = \frac{15}{40}
   \end{align*}

2. Glass spheres in turpentine
   \begin{align*}
   R' &= 180, \quad N = 4, \quad Pr = 4.15, \quad d = \frac{1}{4}
   \end{align*}

3. Glass spheres in water
   \begin{align*}
   R' &= 300, \quad N = 7, \quad Pr = 5.35, \quad d = \frac{7}{40}
   \end{align*}

(5.14) then gives

1. $Re = 1.47$
2. $Re = 1.40$
3. $Re = 1.38$

In spite of some inaccuracy in the reading of $N$ and $R'$, the Reynolds number in all three cases is very close to unity, indicating
that the non-linear terms in Darcy's law become important. We also note that the Reynolds number is remarkable constant in the three cases.

Multiplying (5.6) with the grain diameter, we obtain

\[ \frac{d}{\delta} = N d \]  

(5.15)

Since \( R' \) is proportional to \( N \), it is noted by comparing (5.14) and (5.15) that \( d \) and \( N \) enter almost identically in the two formulas. Only the Prandtl number dependence is essentially different, small Prandtl numbers favouring the effect due to the appearance of non-linear terms in Darcy's law.

Introducing the actual values in (5.15), we find for the three cases considered

\[ (1) \frac{d}{\delta} = 0.6, \quad (2) \frac{d}{\delta} = 1.0, \quad (3) \frac{d}{\delta} = 1.2 \]

which shows that also \( \frac{d}{\delta} \) is close to unity. Our conclusion is therefore that at least for the cases considered, both effects may be important.

According to Elder (1967) experiments give that the deviating curves approach \( N \propto R^{1/4} \) with increasing Rayleigh numbers, and \( N \propto R^{1/2} \) for very large Rayleigh numbers. It is of interest to note that if Darcy's law is replaced by a quadratic law, we may write (5.11) (Rumer & Drinker (1966))

\[ \text{ow} Re \sim R'0 \]

(5.16)

where \( c \) is dependent on the porosity, but in most cases of interest is close to \( 10^{-1} \). Instead of (5.4) we then obtain

\[ N \sim R^{1/2} \]

(5.17)
Furthermore, at Rayleigh numbers for which vertical boundary layers have developed, (5.17) is replaced by

\[(5.18) \quad N \sim R^{1/4}\]

which is in agreement with the experiments referred to above.

The observed Rayleigh number dependence may also be explained by the fact that the boundary layers have become much less than the grain diameter and the porous medium therefore behaves like a fluid (Schneider (1963), Elder (1967)). Introducing the ordinary Rayleigh number \( R \), both Schneider and Elder show that the various experimental data fit the curve

\[(5.19) \quad N \sim R^{1/4}\]

nearly independent of \( Pr \) and \( d \), apparently favouring the boundary layer explanation.

However, if \( R \) is introduced in (5.18), we obtain a formula similar to (5.19), having a relatively weak dependence of \( Pr \) and \( d \). Which of these two explanations are the correct one, seems difficult to decide. Perhaps the effect expressed by (5.18) is the most important one for moderate Rayleigh numbers whereas the other effect takes over for larger values of the Rayleigh number.

The next section will be devoted to ordinary Bénard convection for high Rayleigh numbers.
6. Bénard convection for large Rayleigh numbers.

a. The periodic case.

It is assumed that the motion is periodic, two-dimensional and that the Rayleigh number is large. The horizontal boundaries are perfect conductors and may be either free (no shear stresses) or rigid (no slip). The equations (2.6) - (2.8) will now be replaced by

\begin{align}
(6.1) & \quad \text{Pr}^{-1} \nabla \cdot \nabla \mathbf{v} = - \frac{1}{\rho_0} \nabla p + RTk + \mathbf{v}^2 \mathbf{v} \\
(6.2) & \quad \nabla \cdot \mathbf{v} = 0 \\
(6.3) & \quad \nabla \cdot \mathbf{v} T = \mathbf{v}^2 T
\end{align}

where \( T = T_o - z + \theta \) as before. \( R \) is the ordinary Rayleigh number defined by

\begin{align}
(6.4) & \quad R = \frac{ga \Delta \theta \beta^3}{\kappa \nu}
\end{align}

This problem has been discussed by Pillow (1952), and Robinson (1967) by asymptotic considerations, and by Fromm (1965) and Veronis (1966) by computational methods. The cell may be divided in a central region, and vertical and horizontal boundary layers. The central region is here characterized by uniform temperature and constant vorticity (Pillow (1952)).

We shall apply the vorticity equation at the point \( B' \) at the vertical cell boundary (see Fig. 3). It is expected, and also confirmed by computations (Fromm (1965)), that the streamlines in and just outside the boundary layer at \( B' \) are almost parallel to the cell boundary \( BC \). The vorticity equation may therefore to a
very good approximation be written

(6.5) \[ w_{xxx} = R\theta_x \]

By integration

(6.6) \[ w_{xx} = R(\theta - \theta_0) \]

since \( w_{xx} \) outside the boundary layer is of smaller order. \( \theta_0 \) denotes the temperature in the central region. Since \( \theta - \theta_0 \) is of order unity, equation (6.6) leads to

(6.7) \[ w_{xx} \sim R \]

By repeated integration, applying the boundary condition \( w_x = 0 \) at the cell boundary,

(6.8) \[ w^0_x \sim R\delta \]

where \( w^0_x \) denotes the vertical velocity just outside the boundary layer. \( w^0_x \) is of the same order as \( w \), and \( w \) is determined by the temperature equation (6.3). This gives

(6.9) \[ \bar{w}\theta = \left(\frac{\partial \bar{\theta}}{\partial z}\right)_{z=0} \]

where the bar denotes horizontal integration, and the left hand side is to be computed where \( \frac{\partial \theta}{\partial z} = 0 \). We then obtain

(6.10) \[ w \sim \frac{1}{\delta^2} \]

which, combined with (5.6) and (6.8) gives

(6.11) \[ N \sim R^3 \]

It is noted that (6.11) is true for free as well as rigid horizontal boundaries.
For rigid horizontal boundaries this is in contrast to the result obtained by Pillow (1952) who found that \( N \sim R^{1/4} \). This result may be obtained by assuming that in (6.6)

\[
(6.12) \quad w_{xx} \sim \frac{w}{\delta^2}
\]

This seems, however, to be opposed by the boundary condition \( w_x = 0 \).

The relation (6.11) was also obtained by Robinson (1967) by a somewhat other line of arguments. His reasoning has been questioned by Wesseling (1969), whose conclusions clearly are in favour of Pillow's procedure and results. It seems to us that some of Wesseling's objections are relevant. It is thus not convincing, as claimed by Robinson, that the frictional force along AB in the rigid case is balanced by a second order pressure.

The asymptotic heat transport law may also be obtained by considering the energy integral. Corresponding to (5.9) the equations (6.1) - (6.3) lead to

\[
(6.13) \quad \langle \eta^2 \rangle = R \langle w \theta \rangle
\]

where \( \eta \) denotes the vorticity. From (5.10)

\[
(6.14) \quad R \langle w \theta \rangle \sim RN
\]

The contribution to the left hand side in (6.13) from the constant vorticity region is of order \( \delta^{-1} \). The contribution from the vertical boundary layers are obviously very small and may be cancelled. This is also true for the horizontal boundary layers in the case of free boundaries. Utilizing (5.6) we again obtain the \( \frac{1}{3} \)-power
law (6.11).

The vorticity in the horizontal boundary layer is approximately equal to \( \frac{\partial u}{\partial z} \). From the heat equation it follows that \( u \sim \delta^{-2} \). One should therefore expect that in the case of rigid boundaries, the vorticity in the boundary layer is of order \( \delta^{-3} \), and this would apparently lead to a \( \frac{1}{n} \)-power law. It is, however, doubtful that the vorticity is of order \( \delta^{-3} \) in the whole horizontal boundary layer. In fact this seems to be true only in a very small region, due to the tilting of the streamlines and isodines (see Fromm (1965) p. 1764, fig. 8). From these numerical computations it is apparent that the contribution from the horizontal boundary layer does not exceed the contribution from the central region. Accordingly, energy considerations also in this case lead to a \( \frac{1}{3} \)-power law.

It may be worth noting that in the case of free boundaries the streamlines must be tilted, though this is not easily seen from Fromm's computations. This is necessary, however, in order to produce a pressure torque which may compensate the buoyancy torque.

At last we repeat that in the discussion above it has been assumed that the mean temperature is constant in the central region. Various numerical computations show that \( \frac{\partial T}{\partial z} \) is not strictly zero, being indeed slightly positive. This seems to be due to the mushroom form of some of the isotherms, which is necessary for continuity reasons. This effect is, however, so small that it does not influence the results above.
b. The turbulent case.

In the turbulent case the convective heat transport is not limited to the vertical boundary layers and therefore, instead of (6.10)

\[(6.15) \quad w \sim \frac{1}{\delta}\]

We assume that

\[(6.16) \quad u \sim v \sim w \quad \text{and} \quad \frac{2}{\partial x} \sim \frac{2}{\partial y} \sim \frac{2}{\partial z} \sim \frac{1}{\delta}\]

From (6.1), neglecting the "pressure" \(\frac{1}{\rho_0} \frac{\partial p}{\partial z} + Rz\) (or, equivalently, using the vorticity equation)

\[(6.17) \quad P^{-1} \nabla \cdot \nabla w + \nabla^2 w \sim R\]

Introducing (6.15) and (6.16) we obtain

\[(6.18) \quad (1 + cP^{-1}) \frac{1}{\delta^3} \sim R\]

where \(c\) is a pure number of order unity. Finally, by (5.6)

\[(6.19) \quad N \sim (1 + cP^{-1})^{-\frac{1}{3}} \frac{1}{\delta^{\frac{1}{3}}} R^{\frac{1}{3}}\]

We note that also in the turbulent case we obtain a \(\frac{1}{3}\)-power law, in contrast to (6.11), the Prandtl number now enters in the formula. Due to the power of \(-\frac{1}{3}\), the dependence of the Prandtl number is, however, weak.

A \(\frac{1}{3}\)-power law for the turbulent case was also obtained by Howard (1965) by requiring that the heat transport is independent of height when the Rayleigh number tends towards infinity.
REFERENCES


Schneider, K.J. 1963 11th Int. Cong. of Refrigeration, Paper 11-4, Munich.


Figure legends

Figure 1  Values of $N_{s=3}$ vs. $R'/R'_o$ for the second, fourth and sixth order solution.

--- --- general trend of various experimental data for $R'/R'_o$ larger than 2-3.

OOO numerical values obtained by Elder (1967).

Figure 2  Sixth order Nusselt number, $N^{(6)} (s=3)$, vs. the Rayleigh number $R'$ (solid line) compared with experimental data from Schneider (1963):

Open circle, 4 mm glass spheres in water
Closed triangle, 10 mm glass spheres in turpentine
and Elder (1967):

Open triangle, 8 mm glass spheres
Closed circle, 18 mm glass spheres
Closed square, small glass spheres

Figure 3  Sketch of a single convection cell.
Figure 1.