Vertical wall effects on a fluid heated from below: linear theory.

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The convective motion of a fluid in a container heated from below is considered. Exact solutions of the linearized Boussinesq equations are found when the container is either a circular dish or a rectangular channel, and when the horizontal boundaries are free boundaries. Solutions which are weakly stable or weakly unstable when the Rayleigh number has a value near its critical value, are discussed in some details. The stabilizing effect of the walls on different cell forms is also considered. While the linear theory predicts the orientation of the cells in a channel in accordance with experiments, it does not predict the azimuthal variation of the motion in a circular dish. The axisymmetric cell pattern seen in experiments is thought to be determined by non-linear terms as well as wall effects.

1. Introduction.

In most investigations on the convective motion in a fluid heated from below, the fluid layer is assumed to be of infinite horizontal extent. In many respects the solutions thus obtained, have the characteristic features of the convective motion in a container of finite size. Among the well known results for the model of infinite extent, we mention that the linearized equations determine the critical Rayleigh number for which the convective motion is set up, and also determine the size of the convection cells which are formed. The form of the cells can not be predicted from the linearized equations. However, a stability analysis of the non-linear solutions gives the ranges of the Rayleigh number R above its critical value R_c, where the different cell pattern (hexagons and rolls) may occur. This was investigated in several papers, notably by Schlüter, Lortz & Busse (1965) and by Palm, Ellingsen & Gjevik (1967).

It is found from experiments, however, that the cell pattern under certain conditions strongly depend on the form of the container in which the motion takes place. Some attention was paid to such phenomena in papers by Koschmieder (1966, 1967) and by Sommerscales & Dougherty (1970). The most characteristic features in this respect are (i) in a rectangular dish, the rolls are most likely to develop with tneir axis parallel to the short side of the dish (reported in the former of the papers cited above), and (ii) in a circular dish, there is a tendency to a formation of concentric circular cells when R exceeds R_c. When R is further increased, these circular cells tend to break up and develop into other cell forms, for instance hexagons.

Theoretical investigations of the influence of the vertical walls were undertaken by Davis (1967, 1968), Segel (1969) and Joseph (1970). Both Davis and Segel considered a set of rolls in a rectangular container. While Davis determined stationary solutions of the linear and non-linear equations applying a Galerkin procedure, the work of Segel is based on the idea that an amplitude modulation of a roll solution is sufficient to form a solution satisfying the boundary conditions. Furthermore, a stability analysis of a non-linear solution turns out to fit into this scheme. Joseph's paper is mainly concerned with the discreteness of the stationary linear solutions and how this may affect the non-linear branching problem. But he also points out that when the horizontal boundaries of a circular dish are free and the motion is axisymmetric, the linear equations can be solved by separation of the variables and the boundary conditions can be satisfied. A similar separation of variables is applied by Müller (1966) for the problem of two-dimensional convection in a channel with a given temperature difference between the vertical walls.

In the present paper the possibility of solving the equations by separation of the variables is discussed further. It is found that this can be done not only for the axisymmetric solution in a circular dish, but also for solutions with an arbitrary azimuthal wavenumber. The solution for an infinite channel with arbitrary channel width is also determined in a similar way. The eigensolutions thus obtained are discussed in some details and the growth rates σ are found in terms of R. It is also pointed out that the solution of the non-linear equation can be solved by a series expansion by means of these eigensolutions.

2. Basic equations and boundary conditions.

The fluid layer under consideration is bounded by two horizontal planes a distance H apart, and some vertical walls to be specified below. With $\vec{u}$ denoting the velocity vector, $\vec{k}$ a vertical unit vector, and θ and p the deviations of temperature and pressure from those of the purely heat conducting (motionless) case, the governing equations can be written

$$\nabla^2\vec{u} + R\theta\vec{k} = \nabla p + P^{-1}(\vec{u}_t + \vec{u}\cdot\nabla\vec{u}), \tag{2.1}$$

$$\nabla^2\theta + \vec{k}\cdot\vec{u} = \theta_t + \vec{u}\cdot\nabla\theta, \tag{2.2}$$

$$\nabla\cdot\vec{u} = 0 . \tag{2.3}$$

Here the Boussinesq approximations are used, and the equations are written in dimensionless form with the scaling length, time, velocity, temperature and pressure chosen as H, $H^2\kappa^{-1}$, $H^{-1}\kappa$, ΔT and $H^{-2}\kappa\nu\rho_o$, repsectively. κ is the thermal diffusivity, ν the kinematic viscosity, ΔT the temperature difference between the lower and the upper boundary and ρ_o is a standard density. The density ρ is assumed to be a linear function of the temperature T with the coefficient of expansion α, $\alpha = -\rho_o^{-1}d\rho/dT$. R is the Rayleigh number and P the Prandtl number defined by

$$(2.4) \qquad R = \frac{\alpha g H^3 \Delta T}{\kappa \nu} , \qquad P = \frac{\nu}{\kappa} .$$

Considering now the boundary conditions, we shall assume that the horizontal boundaries are free boundaries. This assumption is necessary to obtain the solutions in a tractable form when the lateral walls are taken into account. The restriction is not thought to be severe since it is known from any investigations that the solutions for different boundary conditions show many of the same features. In the paper by Palm et al. (1967) it is well demonstrated how the effect of different boundary conditions is mainly to change the ranges of R where hexagons or rolls are stable.

With $\vec{\tau}$ denoting the viscous stress vector, we therefore write the conditions

$$(2.5) \qquad \vec{k} \times \vec{\tau} = 0, \qquad w = 0, \qquad \theta = 0,$$

at the horizontal boundaries. The vertical boundaries are assumed to be rigid and perfectly conducting walls whose temperatures are kept with the same linear decrease with height as in the purely heat conducting case. Accordingly we can write

$$(2.6) \qquad \vec{u} = 0, \qquad \theta = 0,$$

at the vertical boundaries.

We shall find it convenient to rewrite (2.1) in the following way. Letting w and ζ denote the vertical components of velocity and vorticity

$$(2.7) \qquad w = \vec{k} \cdot \vec{u} , \quad \zeta = \vec{k} \cdot \nabla \times \vec{u} ,$$

we can write

(2.8) $$\nabla^4 w + R\nabla_1{}^2\theta = P^{-1}(\nabla^2 w_t + \nabla^2(\vec{u}\cdot\nabla w) - \nabla\cdot(\vec{u}\cdot\nabla\vec{u})_z),$$

(2.9) $$\nabla^2\zeta = P^{-1}(\zeta_t + \vec{u}\cdot\nabla\zeta + (\nabla\times\vec{u})\cdot\nabla w).$$

Here z is the vertical coordinate and $\nabla_1{}^2$ is the two-dimensional Laplacian $\nabla_1{}^2 = \nabla^2 - \partial^2/\partial z^2$. As our intention is to investigate how the lateral walls affect the onset of convection, only the linear equations will be considered. The linearized version of (2.1) to (2.3) together with the boundary conditions defined above constitute a self-adjoint eigenvalue problem for the time factor, as shown by Schlüter et al. (1965). The critical Rayleigh number R_c will therefore be associated with a steady solution. But we shall also be interested in the spectrum of stable and unstable solutions for R near R_c, and the equations we are going to discuss are therefore

(2.10) $$\nabla^4 w + R\nabla_1{}^2\theta = P^{-1}\nabla^2 w_t,$$

(2.11) $$\nabla^2\theta + w = \theta_t,$$

(2.12) $$\nabla^2\zeta = P^{-1}\zeta_t,$$

together with (2.3) and the boundary conditions (2.5) and (2.6).

3. Solutions for a circular dish.

In this section we consider the convective motion of a fluid in a circular dish with depth H and diameter D, such that the dimensionless radius a is

$$a = D/2H. \tag{3.1}$$

A coordinate system is chosen such that the free boundaries are located at $z = 0$ and $z = 1$, and the rigid walls at $r = a$ in terms of the cylindrical coordinates (r,ϕ,z). The velocity components (u,v,w) are given by

$$\vec{u} = \vec{i}_r u + \vec{i}_\phi v + \vec{k} w, \tag{3.2}$$

and the boundary conditions discussed above may be written

$$u_z + w_r = r^{-1} w_\phi + v_z = w = \theta = 0, \tag{3.3}$$

at $z = 0$ and $z = 1$,

$$u = v = w = \theta = 0, \quad \text{at} \quad r = a. \tag{3.4}$$

In addition to this, we require the solution to be regular at $r = 0$. Analogous to the stationary and axisymmetric solution of Joseph (1970), our solutions are assumed to be separable in the following way

$$w = \exp(\sigma t) \sin \pi z \cos n\phi f(r), \tag{3.5}$$

$$\theta = \exp(\sigma t) \sin \pi z \cos n\phi g(r), \tag{3.6}$$

$$\zeta = \exp(\sigma t) \cos \pi z \sin n\phi k(r). \tag{3.7}$$

Introducing these expressions into (2.10) to (2.12), we obtain three equations for the determination of $f(r)$, $g(r)$ and $k(r)$. Once these equations are solved and w and ζ are known, u and v can be determined from the equation of continuity and the definition of ζ,

$$(3.8) \qquad (ru)_r + v_\phi = - rw_z,$$

$$(3.9) \qquad (rv)_r - u_\phi = r\zeta.$$

The regular solutions of (2.10) and (2.11) are

$$(3.10) \qquad f(r) = A_1 J_n(q_1 r) + A_2 J_n(q_2 r) + A_3 I_n(q_3 r) ,$$

$$(3.11) \qquad \begin{aligned} g(r) &= (q_1{}^2 + \pi^2 + \sigma)^{-1} A_1 J_n(q_1 r) \\ &+ (q_2{}^2 + \pi^2 + \sigma)^{-1} A_2 J_n(q_2 r) - (q_3{}^2 - \pi^2 - \sigma)^{-1} A_3 I_n(q_3 r). \end{aligned}$$

J_n and I_n denote Bessel functions, I_n being defined by $I_n(z) = (-i)^n J_n(iz)$, and q_1, q_2 and q_3 are chosen such that $q = q_1{}^2$, $q = q_2{}^2$ and $q = -q_3{}^2$ are the roots of the third order equation

$$(3.12) \qquad (q + \pi^2 + \sigma)(q + \pi^2 + \sigma P^{-1})(q + \pi^2) - qR = 0.$$

In the solutions we are going to discuss, q_1, q_2 and q_3 are real and positive. From (3.7) and (2.12) we find

$$(3.13) \qquad k(r) = A_4 I_n(q_4 r) , \qquad q_4{}^2 = \pi^2 + \sigma P^{-1},$$

and by writing

(3.14) $$u = \exp(\sigma t)\cos \pi z \cos n\phi h_1(r) ,$$

(3.15) $$v = \exp(\sigma t)\cos \pi z \sin n\phi h_2(r) ,$$

for the horizontal velocity components, $h_1(r)$ and $h_2(r)$ are found from (3.8) and (3.9) to be

(3.16) $$h_1(r) = \pi q_1^{-1}A_1J_n'(q_1r) + \pi q_2^{-1}A_2J_n'(q_2r) - \pi q_3^{-1}A_3I_n'(q_3r) - nq_4^{-2}A_4r^{-1}I_n(q_4r) ,$$

(3.17) $$h_2(r) = -n\pi q_1^{-2}A_1r^{-1}J_n(q_1r) - n\pi q_2^{-2}A_2r^{-1}J_n(q_2r) + n\pi q_3^{-2}A_3r^{-1}I_n(q_3r) + q_4^{-1}A_4I_n'(q_4r) .$$

The solutions obtained in this way satisfy the boundary conditions at $z = 0$ and $z = 1$. The conditions at $r = a$ constitute a homogenous set of equations for the constants A_1, A_2, A_3 and A_4, and when the determinant for this set of equations is put equal to zero, a relationship between q_1, q_2, q_3, σ and a is obtained. Since q_1, q_2 and q_3 are given functions of R and σ through (3.12), this relationship determines the growth rate σ for the various modes as a function of R and a. By putting $\sigma = 0$, we also obtain the values of R for marginal instabilities as a function of a.

By introducing

(3.18) $$Q_1 = q_1^{-2}(q_1^2 + \pi^2 + \sigma)(q_2^2 + q_3^2),$$

(3.19) $$Q_2 = q_2^{-2}(q_2^2 + \pi^2 + \sigma)(q_1^2 + q_3^2),$$

(3.20) $$Q_3 = q_3^{-2}(q_3^2 - \pi^2 - \sigma)(q_2^2 - q_1^2),$$

the characteristic equation may be written

$$q_1 Q_1 \frac{J_n'(q_1 a)}{J_n(q_1 a)} - q_2 Q_2 \frac{J_n'(q_2 a)}{J_n(q_2 a)} - q_3 Q_3 \frac{I_n'(q_3 a)}{I_n(q_3 a)}$$

(3.21)

$$- \frac{n^2}{q_4 a^2}(Q_1 - Q_2 - Q_3) \frac{I_n(q_4 a)}{I_n'(q_4 a)} = 0 \ .$$

4. Solutions for a channel.

Rectangular containers are frequently used in experiments on convection. Our method of obtaining solutions by separation of the variables is not applicable in such cases. But if the length of the rectangle is so large, compared to the width, that the effect of the distant walls may be neglected and the container can be treated as a channel, some exact solutions are easily found. In comparison to the works of Davis (1967, 1968) and Segel (1969), this model is simpler than theirs since only two of the side walls are assumed to affect the motion, but the analytic solutions obtained here will be valid for any value of the channel width.

The channel is taken to be of depth H and width B. A in cartesian coordinate system (x,y,z) is applied to the channel such that the horizontal boundaries coincide with the planes $z = 0$ and $z = 1$, and the vertical walls coincide with the planes $x = \pm b$. With the scaling defined in section 2, b is then given by

(4.1) $\quad b = B/2H$.

(u,v,w) are the velocity components in this coordinate system, and the boundary conditions (2.5) and (2.6) take the form

(4.2) $\quad v_z + w_y = u_z + w_x = w = \theta = 0$, at $z = 0$ and $z = 1$,

(4.3) $\quad u = v = w = \theta = 0$, at $x = \pm b$.

The motion is assumed to be periodic in the y-direction (along the channel) and we seek a solution of the form

(4.4) $\quad w = \exp(\sigma t)\sin\pi z \, \cos\kappa y \, f(x)$,

(4.5) $\quad \theta = \exp(\sigma t)\sin\pi z \, \cos\kappa y \, g(x)$,

(4.6) $\quad \zeta = \exp(\sigma t)\cos\pi z \, \sin\kappa y \, k(x)$.

The functions $f(x)$, $g(x)$ and $k(x)$ are determined from (2.10) to (2.12), and the horizontal velocity components are determined from the equation of continuity and the definition of ζ. There are two cases to be treated separately, the symmetric case in which $f(-x) = f(x)$, and the antisymmetric one with $f(-x) = -f(x)$. The solutions are quite analogous to those for the circular dish discussed in the last section and it should be sufficient just to write down the result of the calculations. The characteristic equations are found to be as follows :

(i) For the symmetric case :

$$(q_1{}^2 - \kappa^2)^{\frac{1}{2}}Q_1\tan(q_1{}^2 - \kappa^2)^{\frac{1}{2}}b - (q_2{}^2 - \kappa^2)^{\frac{1}{2}}Q_2\tan(q_2{}^2-\kappa^2)^{\frac{1}{2}}b$$

(4.7) $$+ (q_3{}^2 + \kappa^2)^{\frac{1}{2}}Q_2\tanh(q_3{}^2+\kappa^2)^{\frac{1}{2}}b + \kappa^2(q_4{}^2 + \kappa^2)^{-\frac{1}{2}}$$

$$\cdot (Q_1 - Q_2 - Q_3)\tanh(q_4{}^2+\kappa^2)^{\frac{1}{2}}b = 0.$$

(ii) For the anti-symmetric case :

$$(4.8)\quad \begin{aligned} &(q_1{}^2 - \kappa^2)^{\frac{1}{2}} Q_1 \cot(q_1{}^2 - \kappa^2)^{\frac{1}{2}} b - (q_2{}^2 - \kappa^2)^{\frac{1}{2}} Q_2 \cot(q_2{}^2 - \kappa^2)^{\frac{1}{2}} b \\ &- (q_3{}^2 + \kappa^2)^{\frac{1}{2}} Q_3 \coth(q_3{}^2 + \kappa^2)^{\frac{1}{2}} b - \kappa^2 (q_4{}^2 + \kappa^2)^{-\frac{1}{2}} \cdot \\ &\cdot (Q_1 - Q_2 - Q_3) \coth(q_4{}^2 + \kappa^2)^{\frac{1}{2}} b = 0 \; . \end{aligned}$$

q_1, q_2, q_3 and q_4 have the same meaning as above, such that $q = q_1{}^2$, $q = q_2{}^2$ and $q = -q_3{}^2$ are solutions of (3.12). q_4 is defined in (3.13) and Q_1, Q_2 and Q_3 are given in (3.18) to (3.20).

5. Solutions of the characteristic equations.

When the container in which the motion takes place is of large horizontal extent, the critical Rayleigh number will exceed that for an unbounded fluid layer by a small amount. In the case of free horizontal boundaries this critical value is $27\pi^4/4$, and by putting $R = 27\pi^4/4$ and $\sigma = 0$ in (3.12), the solutions are $\pi^2/2$, $\pi^2/2$ and $-4\pi^2$. The numbers q_1, q_2 and q_3 are in this case $q_1 = q_2 = \pi/\sqrt{2}$ and $q_3 = 2\pi$. We shall be interested in the solutions of the characteristic equations in the case of such large containers, considering only those solutions which have a slow growth rate when R is given a value near R_c.

To discuss the characteristic equations, we first need the solutions of (3.12). q_1, q_2 and q_3 introduced above are given by

$$(5.1) \qquad q_1{}^2 = 2\alpha^{\frac{1}{2}}\cos(\gamma + \tfrac{\pi}{3}) - \pi^2 - \tfrac{\sigma}{3}(1 + P^{-1}) \ ,$$

$$(5.2) \qquad q_2{}^2 = 2\alpha^{\frac{1}{2}}\cos(\gamma - \tfrac{\pi}{3}) - \pi^2 - \tfrac{\sigma}{3}(1 + P^{-1}) \ ,$$

$$(5.3) \qquad q_3{}^2 = 2\alpha^{\frac{1}{2}}\cos\gamma + \pi^2 + \tfrac{\sigma}{3}(1 + P^{-1}),$$

where

$$(5.4) \qquad \cos 3\gamma = \alpha^{-\frac{3}{2}}\beta \ ,$$

and

$$(5.5) \qquad \alpha = \tfrac{1}{3}R + \tfrac{1}{9}\sigma^2((1+P^{-1})^2 - 3P^{-1}) \ ,$$

$$(5.6) \qquad \beta = \tfrac{1}{2}\pi^2 R + \tfrac{1}{9}\sigma(1+P^{-1})(R + \tfrac{2}{9}\sigma^2(1+ \tfrac{1}{2}P^{-1})(1-2P^{-1})).$$

Introducing η by

$$(5.7) \qquad R = \frac{27\pi^4}{4}(1+\eta)$$

and seeking the approximate solutions for η and σ small, we obtain

$$(5.8) \qquad q_1 = \pi/\sqrt{2}(1 - \varepsilon) + O(\eta,\sigma) \ ,$$

$$(5.9) \qquad q_2 = \pi/\sqrt{2}(1 + \varepsilon) + O(\eta,\sigma) \ ,$$

$$(5.10) \qquad q_3 = 2\pi + O(\eta,\sigma) \ ,$$

where ε is given by

$$(5.11) \qquad \varepsilon^2 = \frac{3\eta}{4} - \frac{\sigma}{4\pi^2}(1 + P^{-1}) \ .$$

With the use of these approximations the characteristic equations can be discussed in some details. The effect of finite P compared to P infinite is to lower σ by a factor $(1+P^{-1})$, and P^{-1} is therefore set equal to zero in the following.

a) The circular dish.

The characteristic equation (3.22) takes the asymptotic form

$$(5.12) \qquad \frac{J_n'(q_1 a)}{J_n(q_1 a)} - \frac{J_n'(q_2 a)}{J_n(q_2 a)} + \frac{5\varepsilon}{9}\left(\frac{J_n'(q_1 a)}{J_n(q_1 a)} + \frac{J_n'(q_2 a)}{J_n(q_2 a)} - \frac{2\sqrt{2}}{5}\right) = O(\eta,\sigma) .$$

By using the asymptotic expansions for the Bessel functions, we find the first approximation to give

$$(5.13) \qquad q_2 a - q_1 a = k\pi + O(\varepsilon).$$

Since $q_2 - q_1 = \sqrt{2}\,\pi\,\varepsilon + O(\varepsilon^2)$ and accordingly $\varepsilon = O(a^{-1})$, the solutions of (5.12) may be written

$$(5.14) \qquad \varepsilon = \frac{k}{\sqrt{2}\,a}\left(1 + \frac{\lambda}{2a}\right) + O(a^{-3})$$

where k is an integer and λ is determined by the second approximation in the asymptotic expansions. The result is

$$(5.15) \qquad \lambda = \frac{2}{9\pi}\left(1 + (-1)^{n+k}\left(\sin\sqrt{2}\pi a - \frac{5}{\sqrt{2}}\cos\sqrt{2}\pi a\right)\right).$$

By means of (5.14) and (5.11) we therefore obtain

$$(5.16) \qquad \sigma = 3\pi^2\left(\eta - \frac{2k^2}{3a^2}\left(1 + \frac{\lambda}{a}\right)\right) + O(a^{-4}).$$

This equation defines the growth rates for the modes belonging to that part of the spectrum for which σ is small. By choosing $\sigma = 0$ it also gives the values of R where the modes become unstable, the least of these values is R_c as a function of the given dish radius a. R_c is obtained for $k = 1$ and is found to be

$$(5.17) \qquad R_c = \frac{27\pi^4}{4}\left(1 + \frac{2}{3a^2}\left(1 + \frac{\lambda}{a}\right)\right) + O(a^{-4}).$$

From (5.16) and (5.17) it is seen that the azimuthal wave number n plays a minor role in the determination of σ and R_c. In the first approximation (neglecting terms of order a^{-3}) the conclusion is that for a given k, all the modes $n = 1,2,\cdots$ become unstable at the same value of R, and they also have the same value of σ for R near R_c. When the terms of order a^{-3} are considered, it is found that the difference between the σ-values for two different values of n is proportional to $(\sin\sqrt{2}\pi a - 5/\sqrt{2}\cos\sqrt{2}\pi a)$. Since this factor is either positive or negative, depending on the value of a, the conclusion seems to be that the azimuthal wave number n (and thus the form) of the solution which first becomes unstable may be different for different values of a.

b) The infinite channel.

The asymptotic forms of the characteristic equations (4.7) and (4.8) are found to be

$$(q_1{}^2-\kappa^2)^{\frac{1}{2}}\tan(q_1{}^2-\kappa^2)^{\frac{1}{2}}b - (q_2{}^2-\kappa^2)^{\frac{1}{2}}\tan(q_2{}^2-\kappa^2)^{\frac{1}{2}}b = O(\eta,\sigma) ,$$

for the symmetric case and

$$(5.19) \qquad (q_1{}^2-\kappa^2)^{\frac{1}{2}}\cot(q_1{}^2-\kappa^2)^{\frac{1}{2}}b - (q_2{}^2-\kappa^2)^{\frac{1}{2}}\cot(q_2{}^2-\kappa^2)^{\frac{1}{2}}b = O(\eta,\sigma),$$

for the antisymmetric case. If $\pi^2/2 - \kappa^2$ is positive and not small of order ε, both (5.18) and (5.19) give

$$(5.20) \qquad (q_2{}^2-\kappa^2)^{\frac{1}{2}}b - (q_1{}^2-\kappa^2)^{\frac{1}{2}}b = k\pi + O(\varepsilon),$$

where k is an integer. By means of (5.8), (5.9) and (5.11) we derive

$$(5.21) \qquad \sigma = 3\pi^2\left(\eta - \frac{4k^2}{3\pi^2b^2}\left(\frac{\pi^2}{2} - \kappa^2\right)\right) + O(b^{-3}).$$

An increase in κ clearly has a destabilizing effect, and the value of κ^2 for the solution which first becomes unstable will differ from $\pi^2/2$ by an amount of order ε. We therefore write

$$(5.22) \qquad \kappa^2 = \frac{\pi^2}{2}(1 + 2\delta\varepsilon),$$

and obtain the solution

$$(5.23) \qquad \sigma = 3\pi^2\left(\eta - \frac{4\lambda^4}{3\pi^4b^4}\right) + O(b^{-5}).$$

Here λ is any solution of

$$(5.24) \qquad (1+\delta)^{\frac{1}{2}}\tanh(1+\delta)^{\frac{1}{2}}\lambda + (1-\delta)^{\frac{1}{2}}\tan(1-\delta)^{\frac{1}{2}}\lambda = 0$$

for the symmetric case and

$$(5.25) \qquad (1+\delta)^{\frac{1}{2}}\coth(1+\delta)^{\frac{1}{2}}\lambda - (1-\delta)^{\frac{1}{2}}\cot(1-\delta)^{\frac{1}{2}}\lambda = 0$$

for the antisymmetric case.

The values of λ found from (5.25) are seen to be larger than those found from (5.24). We therefore conclude that for these solutions the symmetric modes are more unstable than the corresponding antisymmetric modes. From (5.24) it is found that the least value of λ is obtained for $\delta = -0.69$. This value, $\lambda = 2.153$, determines the critical Rayleigh number,

$$(5.26) \qquad R_c = \frac{27\pi^4}{4}(1 + 0.294\, b^{-4}),$$

and it also defines the wave number (along the channel) of the mode which first becomes unstable. This wave number is given by

$$(5.27) \qquad \kappa^2 = \frac{\pi^2}{2}(1 - 0.648\, b^{-2})$$

and the channel walls therefore tend to increase the wave length of the most unstable mode.

6. Discussion.

The results of the last section for the motion in a channel are in agreement with those of Davis (1967, 1968) and Segel (1969) as far as they can be compared. Since in those papers, only motions in containers of finite extensions are considered, a comparison with the results for a channel can be made only for the qualitative features of the solutions.

For a given κ, the solutions discussed in section 4 are modified rectangular cells, the sides parallel to the channel walls being of length $\pi\kappa^{-1}$. (5.21) then expresses that the effect of the channel walls are to decrease the growth rate and to increase the Rayleigh number for marginal instability, the correction terms being proportional to $(H/B)^2$ whe H is the depth and B the width of the channel. But the effect of the walls will decrease as the cells become more stretched in the x-direction, and when the length of cells across the channel become equal to B, the correction terms will be proportional to $(H/B)^4$. While the critical Rayleigh number is raised by a term of order $(H/B)^4$, the corresponding wave number is lowered by a term of order $(H/B)^2$, as seen from (5.26) and (5.27). The order of magnitude of these corrections agree partly with those of Segel (1969), (leaving aside the corrections due to the length of the dish) when his results are carefully interpreted. In his paper, a solution satisfying the boundary conditions is constructed by an amplitude modulation of a set of roll solutions with a given wavenumber $\pi\alpha$. However, the solution obtained in this way (given by (3.1) in his paper) turns out to be a set of rectangular cells with wave numbers $\pi\alpha'$ and πS^{-1} $(S = B/H)$ along the longer and shorter side of the rectangel, and slowly varying amplitudes. The difference $\alpha - \alpha'$ is found to be $\alpha(H/B)^2$, this reduction leaves the overall wave number unchanged compared to that for an infinite layer, and Segel's result is therefore that the size of the cells are affected by the walls through terms of order $(H/B)^4$. If our solution is interpreted as a set of rectangular cells, we find by means of (5.27) that the overall wave number is $\pi/\sqrt{2}\,(1 - 0.296(H/B)^2)$, giving an increase in the cell size for the most unstable mode of order $(H/B)^2$.

It should be pointed out that the solutions found in section 4 are valid for any value of B while they are discussed in detail only for B/H >> 1. A solution for a rectangular dish with sides B and L, L >> B should possibly be found by means of a multiple-scale analysis like that used by Segel. Some difficulty would certainly arise in such analysis since the modulation necessary to satisfy the boundary conditions must depend on x, the direction in which B is measured.

For the motion in a circular dish it is found that the growth rate is lowered and the Rayleigh number for marginal instability is raised by an amount proportional to $(H/D)^2$, where H is the depth of the fluid layer and D is the diameter of the dish. It also appears that the azimuthal wave number has virtually no effect on the stability of the motion when H/D is small. In this respect the linear theory does not explain the experimental evidence that concentric circular rolls are the preferred cell forms in a circular dish for values of R just above R_c. In the papers by Koschmieder (1966, 1967) and Sommerscales & Dougherty (1970) this is found to be the case, they also find that by further increase of R this symmetry will break down and the motion tends to develop into other cell forms like hexagons or approximate rolls. In the experiments of these papers the ratio H/D ranges from 2.1 to 3.4 per cent, and the asymptotic solutions found in section 5 above should be adequate as far as the linear theory is considered. It is therefore clear that the form of the cell pattern to be realized in a circular dish must be determined from non-linear terms as well as from the wall effects. This is in contrast to the case of a rectangular dish where the linear solutions predict the preferred orientation of the rolls.

As a demonstration of how the vertical walls affect the velocity field, let us consider the expressions for the vertical velocity. It is found that

$$w = A\exp(\sigma t)\sin \pi z \cos n\phi \Big(q_1{}^2 Q_1 \frac{J_n(q_1 r)}{J_n(q_1 a)} - q_2{}^2 Q_2 \frac{J_n(q_2 r)}{J_n(q_2 a)} + q_3{}^2 Q_3 \frac{I_n(q_3 r)}{I_n(q_3 a)}\Big) , \quad (6.1)$$

for the circular dish and

$$w = A\exp(\sigma t)\sin \pi z \cos \kappa y \Big(q_1{}^2 Q_1 \frac{\cos(q_1{}^2-\kappa^2)^{\frac{1}{2}}x}{\cos(q_1{}^2-\kappa^2)^{\frac{1}{2}}b} - q_2{}^2 Q_2 \frac{\cos(q_2{}^2-\kappa^2)^{\frac{1}{2}}x}{\cos(q_2{}^2-\kappa^2)^{\frac{1}{2}}b} + q_3{}^2 Q_3 \frac{\cosh(q_3{}^2+\kappa^2)^{\frac{1}{2}}x}{\cosh(q_3{}^2+\kappa^2)^{\frac{1}{2}}b}\Big) , \quad (6.2)$$

for the symmetric motion in a channel. The terms proportional to $q_3{}^2 Q_3$ are approximately

$$\exp\{-q_3(a-r)\} \quad \text{and} \quad \exp\{-(q_3{}^2+\kappa^2)^{\frac{1}{2}}(b-x)\},$$

respectively, and represent boundary layer solutions. The boundary layer thicknesses are of order H and are nearly independent of the size of the container. When the horizontal velocity components are considered, they are found to contain other boundary layer terms like $\exp\{-(\pi^2+\sigma P^{-1})^{\frac{1}{2}}(a-r)\}$ and $\exp\{-(\pi^2+\kappa^2+\sigma P^{-1})^{\frac{1}{2}}(b-x)\}$. The boundary layer thicknesses and the amplitudes are of the same order as those considered above. But these boundary layers have another origin since they are due to the vertical vorticity component which has to be imposed upon the fluid to get the boundary conditions satisfied.

The interpretation of the terms in the parantheses in (6.1) and (6.2) will be different for the different modes. From (5.12) and (5.18) it follows

(6.3) $$J_n(q_2 a) = (-1)^k J_n(q_1 a) + O(\varepsilon) ,$$

(6.4) $$\cos(q_2^2-\kappa^2)^{\frac{1}{2}}b = (-1)^k \cos(q_1^2-\kappa^2)^{\frac{1}{2}}b + O(\varepsilon) .$$

When k is an odd integer the dominating terms in (6.1) and (6.2) are proportional to the amplitude modulation terms $J_n(q_1 r) + J_n(q_2 r)$ for the circular dish and $\cos(q_1^2+\kappa^2)^{\frac{1}{2}}x + \cos(q_2^2+\kappa^2)^{\frac{1}{2}}x$ for the channel. The rest of the solutions tend to zero when the extent of the containers is increased. When k is an even integer, however, none of the terms in (6.1) and (6.2) are dominating. The boundary layer terms now have the same amplitudes as the amplitude modulation terms, and can not be neglected, however large the containers are.

7. Conclusion.

The main result of the present paper is the derivation of a three parameter family of solutions satisfying the given boundary conditions for any given Rayleigh number. There is a two parameter family for each z-dependence of the form $\sin \pi z$, $\sin 2\pi z$,..., only the first of these is considered here. The solutions having a small growth rate when the Rayleigh number is near its critical value are discussed in some details above. In the case of a circular dish the spectrum is discrete, the eigenvalue σ are dependent on two integers, n and k, n being the azimuthal wave number. In the solutions for a channel there are also two parameters, κ and k, where κ is a wave number measured along the channel and k is an integer. Since the channel is considered to be of infinite length, there is no restriction on κ, and the spectrum is therefore discrete in k but continuous in κ.

Due to the self-adjointness of the four dimensional, second order operator defined by the left hand sides of (2.1) and (2.2) the eigensolutions are orthogonal to each other. If $(\vec{u}', \theta')$ and $(\vec{u}'', \theta'')$ are two different eigensolutions, the orthogonality condition can be written

$$(7.1) \qquad \int_V (P^{-1}\vec{u}' \cdot \vec{u}'' + R\theta'\theta'')dV = 0 ,$$

V being the fluid volume.

The completeness of the set of eigensolutions obtained in this way is not so obvious, but assuming this to be the case, we are able to obtain also the non-linear solutions satisfying the boundary conditions by an expansion in a series of the eigensolutions for the linearized equations. For a given Rayleigh number not far above its critical value, the eigensolutions discussed above are by far the most important in such series expansion, since the rest of the spectrum will be more rapidly damped out.

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