AN INVERSE PROBLEM OF MATHEMATICAL PHYSICS
WITH APPLICATION TO THE MAGNETOTELLURIC
METHOD OF GEOPHYSICAL EXPLORATION

by

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Abstract

Redundant boundary conditions are shown to permit the
identification of an unknown, spatially varying coefficient
in the diffusion equation. The solution to this "inverse"
problem of mathematical physics is applied to the interpre-
tation of magnetotelluric observations.

1. Introduction.

The inverse problem of mathematical physics is encountered in the study of diffusion through a medium with unknown property $p(z)$. Consider the diffusion equation

$$U_{zz}(z,t) - p(z)U_t(z,t) = 0 \tag{1}$$

subject to the boundary conditions

$$U(o,t) = f(t) \tag{2}$$

$$U_z(o,t) = g(t) \tag{3}$$

$$U(\infty,t) = 0 \tag{4}$$

The boundary conditions (2) and (4) are sufficient for the solution of the direct problem. That is, given $p(z)$, $U(z,t)$ can be found from the system (1), (2) and (4). The additional requirement, (3), while redundant in the solution of the direct problem, provides the additional information necessary for the synthesis of the unknown property $p(z)$ in the inverse problem. In the case of magnetotelluric method of geophysical exploration the unknown function $p(z)$ describes the electrical conductivity profile of the earth. A similar need for inversion of boundary data occurs in the interpretation of continental heat flow and temperature data to determine the earth's thermal conductivity (Minster, 1970).

2.   The magnetotelluric method.

Natural fluctuations in the magnetic field of the earth induce currents in the earth known as telluric currents. The magnetotelluric method employs observations of the fields generated by the combined system of ionospheric and telluric currents to determine the electrical characteristics of the earth.

Tikhonov (1950) first recognized that the electrical properties of the earth could be identified through observations of the tangential electric and magnetic fields at the earth's surface. His work, which treated an essentially homogeneous medium, was followed by Cagniard's (1953) in which the conducting medium was allowed to be horizontally stratified. Cagniard showed that variations with frequency of the tangential components of the electric and magnetic fields at the surface could be diagnostic of the medium as a whole. An important assumption of Cagniard's work was that the incident electromagnetic excitation was represented by a plane wave. Following a large number of investigations, both theoretical and experimental, Tikhonov (1965) has shown that the electrical characteristics of a layered medium are uniquely determined as a function of depth by the complete spectrum of tangential electrical and magnetic fields at the surface of the medium.

3. Maxwell's equations in a one-dimensionally inhomogeneous medium.

Consider a medium of arbitrarily varying conductivity in the z-direction above a homogeneously conducting basement. This medium will be represented by the conductivity profile

$$\sigma = \begin{cases} \sigma(z) & 0 \le z \le H \\ \\ \sigma_B & z > H \end{cases} \tag{5}$$

The electric and magnetic fields in this inhomogeneously conducting medium are described by the relations

$$\nabla \times \overline{E} = - \mu \frac{\partial \overline{H}}{\partial t} \tag{6}$$

$$\nabla \times \overline{H} = \sigma \overline{E} + \varepsilon \frac{\partial \overline{E}}{\partial t} \tag{7}$$

Taking the curl of (6)

$$\nabla(\nabla \cdot \overline{E}) - \nabla^2 \overline{E} = - \mu_0 \frac{\partial}{\partial t}(\sigma \overline{E} + \varepsilon \frac{\partial \overline{E}}{\partial t}) \tag{8}$$

where $\mu = \mu_0$ for all z. If displacement currents are small (7) requires that

$$\nabla \cdot (\sigma \overline{E}) = 0$$

and consequently

$$\sigma(z)\{\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\} + E_z \frac{\partial \sigma}{\partial z} = 0 \tag{9}$$

For the horizontally stratified medium treated in this study the currents induced by a time varying external magnetic field

necessarily flow parallel to the layers of equal conductivity (Price, 1950). Thus, in this example

$$E_z = 0$$

and from (9)

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0$$

or

$$\nabla \cdot \bar{E} = 0 \tag{10}$$

Neglecting displacement currents (8) and (10) are combined to yield

$$\nabla^2 \bar{E} - \mu_0 \sigma \frac{\partial \bar{E}}{\partial t} = 0 \tag{11}$$

At this point the plane wave assumption, fundamental to Cagniard's analysis is made. Suppose that a plane wave

$$\bar{E} = \{E_x, 0, 0\} \tag{12}$$

$$\bar{H} = \{0, H_y, 0\} \tag{13}$$

is vertically incident upon the semi-infinite half space described by the properties of (5). Then E_x satisfies the scalar form of (11) in both the inhomogeneous region, $0 \leq z \leq H$, and the conducting basement. Thus

$$\frac{d^2 E_x}{dx^2} - i\omega\mu_0\sigma(z)E_x = 0 \qquad 0 \leq z \leq H \tag{14}$$

$$\frac{d^2 E_x}{dx^2} - i\omega\mu_0\sigma_B E_x = 0 \qquad z > H \tag{15}$$

where $e^{i\omega t}$ time dependence has been assumed.

The general solution to (15) which satisfies the condition that the fields remain finite for all z is

$$E_x(z) = A\ e^{-(1+i)z\sqrt{\frac{\omega\mu_0\sigma_B}{2}}}$$

At z = H

$$E_x'(H) = -(1+i)\sqrt{\frac{\omega\mu_0\sigma_B}{2}}\ E_x(H)$$

and consequently

$$E_x'(H) + \frac{(1+i)}{\delta_B}\ E_x = 0 \tag{16}$$

where

$$\delta_B = \sqrt{\frac{2}{\omega\mu_0\sigma_B}}$$

Normalizing the electric field to its value at the surface, we substitute

$$u_z = \frac{E_x(z)}{E_x(o)}$$

into (14) and (16) to obtain

$$u_{zz}(z) + i\omega\mu_0\sigma(z)u(z) = 0 \tag{17}$$

$$u(o) = 1 \tag{18a}$$

$$u'(H) + \frac{(1+i)}{\delta_B}\ u(H) = 0 \tag{18b}$$

Solution of equations (17) and (18) will describe the diffusion of a plane electromagnetic wave into an inhomogeneously conducting, horizontally stratified medium.

4. **A Green's function and associated integral form for the diffusion equation.**

The diffusion of electromagnetic energy into an inhomogeneously conducting medium has been shown to be governed by the equation

$$u_{zz}(z) + i\omega\mu_0\sigma(z)u(z) = 0 \tag{17}$$

subject to the boundary conditions

$$u(o) = 1 \tag{18a}$$

$$u'(H) + \frac{(1+i)}{\delta_B} u(H) = 0 \tag{18b}$$

Let F be a twice differentiable function satisfying the boundary conditions

$$F(o) = 1$$
$$F'(H) + \frac{(1+i)}{\delta_B} F(H) = 0 \tag{19}$$

If F is chosen so that

$$F'' = 0 \tag{20}$$

the substitution of

$$U = u - F \tag{21}$$

in (17) and (18) will result in the inhomogeneous equation

$$U'' - i\omega\mu_0\sigma(z)U(z) = i\omega\mu_0\sigma(z)F(z) \tag{22}$$

with homogeneous boundary conditions

$$U(o) = 1$$
$$U'(H) + \frac{(1+i)}{\delta_B} U(H) = 0 \tag{23}$$

Rewriting (22)

$$U''(z) = i\omega\mu_0\sigma(z)\left[F(z) + U(z)\right]$$

and substitution of (21) yields

$$U''(z) = i\omega\mu_0\sigma(z)u(z) \tag{24}$$

A function $F(z)$ which satisfies (19) and (20) is

$$F(z) = \frac{\delta_B + (1+i)(H-z)}{\delta_B + (1+i)H} \tag{25}$$

In employing the Green's function to cast (24) into integral form we must look for a solution to the homogeneous equation

$$G''(x;z) = 0 \tag{26}$$

subject to the following conditions

$$G(o) = 0 \tag{27}$$

$$G'(H) + \frac{(1+i)}{\delta_B} G(H) = 0 \tag{28}$$

and

$$G(x;z^+) = G(x;z^-) \tag{29}$$

$$G'(x;z^+) - G'(x;z^-) = -1 \tag{30}$$

With these conditions satisfied,

$$U(z) = i\omega\mu_0 \int_o^H \sigma(x)u(x)G(x;z)dx \tag{31}$$

One such solution to (25) is

$$G(x;z) = \frac{\delta_B + (1+i)(H-z)}{\delta_B + (1+i)H} \qquad 0 \le x < z$$

$$\tag{32}$$

$$-\frac{(1+i)z}{\delta_B + (1+i)H} \; x+z \qquad z < x \le H$$

and the solution to (26) becomes

$$U(z) = -i\omega\mu_0 \{ \frac{\delta_B + (1+i)(H-z)}{\delta_B + (1+i)H} \int_0^H x\sigma(x)u(x)dx$$

$$+ \int_z^H (z-x)\sigma(x)u(x)dx \} \tag{33}$$

Substituting (22) in (34)

$$u(z) = \frac{\delta_B + (1+i)(H-z)}{\delta_B + (1+i)H} - i\omega\mu_0 \{ \frac{\delta_B + (1+i)(H-z)}{\delta_B + (1+i)H}$$

$$\tag{34}$$

$$\times \int_0^H x\sigma(x)u(x)dx + \int_z^H (z-x)\sigma(x)u(x)dx \}$$

The method of successive substitutions can be applied to (34)
to yield a series solution for the normalized electric field
within an inhomogeneously conducting medium (Dmitriev, 1970).

5. The inverse problem.

Having obtained a solution for the electric field within an inhomogeneously conducting medium we return to the inverse or identification problem. Application of Ampere's law to a semi-infinite medium indicates that

$$H_y(o;\omega) = \int_0^\infty \sigma(x)E_x(z;\omega)dz \qquad (35)$$

where the parameter ω has been introduced to indicate the particular frequence for which $E_x(z;\omega)$ is a solution to (17). Recalling that

$$u(z;\omega) = \frac{E_x(z;\omega)}{E_x(o;\omega)}$$

(35) is written

$$Y(\omega) = \frac{H_y(o;\omega)}{E_x(o;\omega)} = \int_0^\infty \sigma(z)u(z;\omega)dz \qquad (36)$$

If $\sigma(z)$ is approximated by a piecewise constant profile

$$\sigma(z) = \sigma_i \qquad x_{i-1} \le z < x_i \qquad (37)$$

for $i = 1,2,\cdots,N$, (36) becomes

$$Y(\omega) = \sigma_1 \int_0^{x_1} u(z;\omega)dz + \sigma_2 \int_{x_1}^{x_2} u(z;\omega)dz + \cdots$$
$$+ \cdots \sigma_N \int_{x_{N-1}}^\infty u(z;\omega)dz \qquad (38)$$

Equation (38) relates the surface impedance of a stratified medium to the conductivity and normalized electric field within

the medium. If observations of $Y(\omega)$ are made at the N

frequnncies $\omega_1, \omega_2, \cdots, \omega_N$, (38) can be written in matrix

form

$$\bar{Y} = \bar{\bar{A}} \, \bar{\sigma} \qquad (39)$$

where

$$\bar{Y} = \begin{bmatrix} Y(\omega_1) \\ \circ \\ \bullet \\ \circ \\ \bullet \\ \circ \\ \bullet \\ \circ \\ \bullet \\ Y(\omega_N) \end{bmatrix} \qquad \bar{\bar{A}} = \begin{bmatrix} \int_0^{X_1} u(z;\omega_1)dz \cdots \int_{X_{N-1}}^{\infty} u(z;\omega_1)dz \\ \circ \\ \circ \\ \circ \\ \circ \\ \circ \\ \int_0^{X_1} u(z;\omega_N)dz \end{bmatrix} \qquad \bar{\sigma} = \begin{bmatrix} \sigma_1 \\ \bullet \\ \circ \\ \bullet \\ \circ \\ \bullet \\ \circ \\ \bullet \\ \sigma_N \end{bmatrix}$$

With a profile $\bar{\sigma}$ assumed initially, $\bar{\bar{A}}$ is determined from

the solution of (34). Thus, if an initial profile is assumed,

(34) can be solved for $u(z)$ at each frequency for which boundary

data is observed and the results integrated to determine $\bar{\bar{A}}$.

Equation (39) can then be solved for an improved approximation

to $\bar{\sigma}$.

It remains to show that the resulting sequence of approxima-

tions, $\bar{\sigma}_i$, will converge to the true profile $\bar{\sigma}$. Tikhonov's

demonstration of uniqueness indicates that there is only one solu-

tion to (39) so that convergence, if it occurs, will necessarily

be to $\bar{\sigma}$.

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