ON TIME DEPENDENT EKMAN THEORY

by

Lars Petter Røed

Abstract

The time dependent Ekman transport is discussed and shown to approach Ekman's steady state transport in the limit of large time. It is found that the transport in the Ekman layer evolves rather slowly and actually requires 12.5 pendulum days to reach 80% of its stationary value.

Introduction

The problem of wind induced velocity in the ocean making allowance for the earth's rotation was first discussed by Ekman (1905). Following Ekman's classical work a number of authors have investigated the same problem subject to various boundary conditions. A summary of results is given in the textbook by Greenspan (1968).

The present investigation on time dependent Ekman theory is mainly concerned with the time evolution of the mass transport. An investigation by Crepon (1967 a,b) discusses the initial value problem with special emphasis on the governing equation for the mass transport. To this end it is necessary to impose a condition on the stress at the bottom. This is accomplished in either of two ways. (i) The stress is put equal to zero or (ii) set proportional to the transport. In the first case the transport never reaches any stationary value. Rather it has the form of indefinite oscillations about the steady state transport obtained by Ekman (1905). In the second case a stationary value exists which is in agreement with Ekman's solution if the stress is put equal to zero subsequent to making the passage to the large time limit. This is discussed in part 1.

Gonella (1971 a) has given the solution to the Ekman problem with an impulsive wind and an arbitrary initial velocity. In order to gain a solution he invokes distribution theory. The solution presented for the transport in the case of zero initial velocity yields the same behaviour as Crepon's solution with zero bottom stress.

This is in contrast to the conclusions reached in section 2 of the present investigations where it is found that the transport does indeed tend to Ekman's steady state transport. The analysis is based on the time dependent solution attributed to Fredholm and reported by Ekman (1905). It necessitates a careful formulation of the initial value problem.In part 3 it is found that the transport in the Ekman layer requires 12.5 pendulum days to reach 80% of its steady state value. Based on time scale conclusions inferred from a related problem, which has the same steady state solution, the slow evolution of the Ekman layer transport toward its steady state value may come as a surprise. This motivates the formulation of "the related problem" in part 4 where the transients are shown to subside an order of magnitude more rapidly than in the Ekman problem.

1. Formulation of the Ekman problem

In the following the assumptions are the same as those of Ekman (1905). The model is a homogeneous ocean being infinitely deep where there are no lateral boundaries. On the surface the wind provides a stress with no spatial dependence but which may depend on time. The Cartesian coordinate system chosen has z pointing downwards and x and y tangent to the globe at latitude $\hat{\varphi}$. The y-axis coincides with the stress vector. Since the wind stress is the energy source, the induced motion is independent of the horizontal coordinates. Hence no vertical motion is generated, implying that

(1.1) $$w = 0 , \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0 ,$$

where w is the vertical component of the velocity. The continuity equation is, therefore, trivially satisfied and the exact equation of motion is

$$(1.2) \qquad \frac{\partial \vec{v}}{\partial t} + f\vec{k} \times \vec{v} = \nu \frac{\partial^2 \vec{v}}{\partial z^2} .$$

Here $\vec{v} = u\vec{i} + v\vec{j}$ is the horizontal velocity, f is the Coriolis parameter, $f = 2\Omega \sin\hat{\varphi}$, (assumed constant), ν the constant turbulent kinematic viscosity and $\vec{k}$ the unit vector along the z-axis. The solution to Eq. (1.2) is subject to the following boundary and initial conditions :

$$(1.3) \qquad \begin{aligned} -\nu \frac{\partial \vec{v}}{\partial z} &= \frac{\vec{T}(t)}{\rho} ; && z = 0 ; && t > 0 \\ \vec{v} &\to 0 ; && z \to \infty ; && t < \infty \\ \vec{v} &= 0 ; && t = 0 && \end{aligned}$$

where $\vec{T}(t)$ is the wind generated stress and ρ the density of the fluid.

The solution to Eq. (1.2) subject to (1.3) is attributed to Fredholm and reported by Ekman (1905) and may formally be written:

$$(1.4) \qquad \varphi = \frac{2\pi i}{\rho D f} \int_0^\tau T(t-\xi) \frac{e^{-2\pi i \xi}}{\sqrt{\xi}} e^{-\frac{\pi}{4\xi}\left(\frac{z}{D}\right)^2} d\xi ; \quad \vec{T} = T(t)\vec{j}$$

as shown by Fjeldstad (1930). Here $\varphi = u + iv$ is the complex velocity, $\tau = \frac{ft}{2\pi}$ is a dimensionless time and $D = \pi\sqrt{\frac{2\nu}{f}}$ is the Ekman depth. The steady state solution presented by Ekman (1905) which is obtained by putting $\frac{\partial}{\partial t} = 0$ in Eq. (1.2) is

$$(1.5) \qquad \varphi_{st} = \frac{\pi T_0}{\rho D f} (1+i) e^{-\pi(1+i)\frac{z}{D}} .$$

Defining

$$(1.6) \qquad \vec{M}_T = \int_0^\infty \vec{v}\, dz = M_x^{(T)}\vec{i} + M_y^{(T)}\vec{j} ,$$

with the complex representation $M_T = M_x^{(T)} + i M_y^{(T)}$ for the total transport, the governing equation for M_T, obtained by integrating Eq. (1.2) with respect to z, may be written

$$(1.7) \qquad \frac{\partial M_T}{\partial t} + i f M_T = \frac{i T_0}{\rho} + \nu \left.\frac{\partial \varphi}{\partial z}\right|_{z\to\infty} .$$

Here use has been made of the first condition in Eq. (1.3). Putting $\nu \left.\frac{\partial \varphi}{\partial z}\right|_{z\to\infty} = 0$ as did Crepon (1967a) and noting that M_T is initially zero, the solution to Eq. (1.7) becomes

$$(1.8) \qquad M_T = \frac{T_0}{\rho f}(1 - e^{-2\pi i \tau}) .$$

The constant factor in the above expression is the solution to the concomitant steady state equation ($\frac{\partial}{\partial t} = 0$) (Ekman (1905)), viz.

$$(1.9) \qquad M_{st} = \frac{T_0}{\rho f} .$$

Thus M_T given by Eq. (1.8) does not tend to M_{st} in the limit of large time. Rather M_T has an oscillatory behaviour about its steady state value. As pointed out by Crepon this inconsistency may be resolved by allowing for a dissipative mechanism. The commonplace devise is to let the stress at the bottom be assumed proportional to the integrated velocity, viz.

$$(1.10) \qquad - \nu \left.\frac{\partial \varphi}{\partial z}\right|_{z\to\infty} = R\, f\, M_T \ ,$$

the factor of proportionality being R. Here R is put equal to zero subsequent to making the passage to the limit $\tau \to \infty$. Upon introducing Eq. (1.10) in Eq. (1.7) the solution becomes

$$M_T = \frac{iT_0}{\rho f} \quad \frac{1-e^{-2\pi(R+i)\tau}}{R+i} \quad ,$$

which has the large time value

$$M_T = \frac{T_0}{\rho f}\,\frac{i}{R+i}$$

Thus M_T tends to M_{st} as $R \to 0$.

Hence in order to reproduce Ekman's steady state solution it is essential that the time is allowed to approach infinity before R is put equal to zero. Interpreting M_T as the limit of the transport in the layer 0,z when z tends to infinity, this is tentamount to letting the time approach infinity before z. For an initial value problem in an unbounded medium this is the usual devise by which the large time behaviour of the solutions for large values of the spatial coordinates are obtained. The introduction of an artificial bottom stress resembles the artificial Rayleigh friction introduced in the problem of fluid flow over a corrugated bed. Being an artifice, however, it does not give any information about the pertinent time scales for the solution to become practically stationary.

2. The transport as a function of depth and time.

In order to display the time dependency it is convenient to define the complex representation of the transport, M, in a layer of thickness z, viz.

$$(2.1) \qquad M(z,\tau) = \int_0^z \varphi(\xi,\tau)\delta\xi.$$

Inserting the expression for φ, Eq. (1.4) in the above equation and interchanging the order of integration one obtains

$$(2.2) \qquad M(z,\tau) = \frac{2\pi i T_0}{\rho f} \int_0^\tau e^{-2\pi i \xi} \operatorname{erf}\left(\frac{z}{D}\sqrt{\frac{\pi}{4\xi}}\right)\delta\xi \quad .$$

Here erf is the error function defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad .$$

The remaining integral can be expressed in closed form (Abramowitz and Stegun (1964) p. 304), viz.

$$(2.3) \qquad M(z,\tau) = \frac{T_0}{\rho f}\left[1 - e^{-\pi(1+i)\frac{z}{D}} - \operatorname{erf}\left(\frac{z}{D}\sqrt{\frac{\pi}{4\tau}}\right)e^{-2\pi i \tau} - H_2(z,\tau)\right]$$

where

$$(2.4) \qquad H_2(z,\tau) = \tfrac{1}{2}\left[e^{\pi(1+i)\frac{z}{D}} \operatorname{erf c}\left(\sqrt{2\pi i \tau} + \frac{z}{D}\sqrt{\frac{\pi}{4}}\right) - e^{-\pi(1+i)\frac{z}{D}} \operatorname{erf c}\left(\sqrt{2\pi i \tau} - \frac{z}{D}\sqrt{\frac{\pi}{4\tau}}\right)\right]$$

is expressed in terms of the complementary error function

$$\operatorname{erf c}(x) = 1 - \operatorname{erf}(x).$$

Expression (2.3) for the transport enables one, through the defining equation (2.1), to express the velocity in closed form as well, viz.

$$\varphi = \varphi_{st} - \frac{\pi T_0}{\rho D f}(1+i)H_1(z,\tau) \quad , \tag{2.5}$$

where H_1 is the sum of the two terms constituting H_2, Eq. (2.4). Now following the procedure outlined in the last part of the preceding section the time is at this stage allowed to approach infinity. Since H_2 and H_1 both tend to zero for large time and since

$$\text{erf}\left(\frac{z}{D}\sqrt{\frac{\pi}{4\tau}}\right) = 0 \quad ; \qquad \tau \to \infty \quad , \tag{2.6}$$

one obtains

$$\lim_{\tau\to\infty} M = \frac{T_0}{\rho f}\left(1 - e^{-\pi(1+i)\frac{z}{D}}\right) . \tag{2.7}$$

Hence

$$\lim_{z\to\infty}(\lim_{\tau\to\infty} M) = M_{st} . \tag{2.8}$$

Since H_1 and H_2 also tend to zero if z is allowed to approach infinity, the inadmissibility of interchanging the two limit processes may be brought to the fore upon taking the asymptotics of the term

$$\text{erf}\left(\frac{z}{D}\sqrt{\frac{\pi}{4\tau}}\right)e^{-2\pi i\tau}$$

in Eq. (2.3). If the space variable is allowed to approach infinity before time τ, this term tends to $e^{-2\pi i\tau}$. On the other hand, if the two limits are interchanged, the term vanishes. This is the crucial point since for an initial value problem the time must approach infinity before the space variable z.

From Eq. (2.5) the expression obtained for the stress is

$$-\nu\frac{\partial\varphi}{\partial z} = \frac{iT_0}{\rho}\left[e^{-\pi(1+i)\frac{z}{D}} + H_2(z,\tau)\right] . \tag{2.9}$$

Thus, altheug the assumption made by Crepon (1967 a) and Gonella (1971 a), i.e. that there be no bottom stress, leads to a false steady state, the assumption is supported by the fact that the stress does indeed tend to zero for all times in the limit of large values of z, even if τ is allowed to approach infinity before z.

The fact that the time dependent transport does indeed tend to Ekman's steady state transport, Eq. (1.9), may also be established by making use of the following argument. The ocean is so far thought of as being infinitely deep. However, the phrase "infinitely deep" means that the depth, h, of the ocean is very large compared with some other characteristic length parameter, here the Ekman depth, D. Hence the solution for an infinitely deep ocean should follow from the problem with a finite depth scaled on D in the limit $h/D \to \infty$. Indeed changing the second condition in Eq. (1.3) to $\vec{v} = 0$ at $z = h$, the solution to Eq. (1.2) in the domain $0 \leq z \leq h$ may be expressed as [c.f. Fjeldstad (1930)]

$$(2.10) \qquad \varphi = \frac{2\pi i T_0}{\rho D f}\left(\frac{D}{h}\right) \int_0^{\tau} e^{-2\pi i \xi}\, \theta_2\left[\frac{z}{2h}\middle|\frac{\xi}{\pi}\left(\frac{D}{h}\right)^2\right] d\xi \ .$$

Here θ_2, the second theta function, is defined by [Roberts and Kaufmann 1966]

$$(2.11) \qquad \theta_2(\nu|x) = \frac{1}{\sqrt{\pi x}} \sum_{n=-\infty}^{n=\infty} (-1)^n e^{-\frac{(\nu+n)^2}{x}} \ .$$

Upon introducing Eq. (2.11) into Eq. (2.10) the latter may be recast in the form

$$(2.12) \qquad \varphi = \frac{2\pi i T_0}{\rho D f} \int_0^\tau \frac{e^{-2\pi i \xi}}{\sqrt{\xi}} \left[e^{-\frac{\pi}{4\xi}\left(\frac{z}{D}\right)^2} + f(z,\xi) \right] d\xi$$

where

$$f(z,\xi) = \sum_{n=1}^{\infty} (-1)^n \left[e^{-\frac{\pi}{4\xi}\left(\frac{h}{D}\right)^2\left(2n+\frac{z}{h}\right)^2} + e^{-\frac{\pi}{4\xi}\left(\frac{h}{D}\right)^2\left(2n-\frac{z}{h}\right)^2} \right].$$

If Eq. (2.12) were integrated with respect to z from zero to h, the result would be an expression for the total transport. However, all the integrands in such an integration is seen to be of the same form as the right hand side of Eq. (1.4). Following the procedure of the first part of this section all terms are, therefore, easily integrated. The first term of Eq. (2.12) when integrated from 0 to h gives the right hand side of Eq. (2.3) with z/D substituted by h/D. Hence this term alone will produce Ekman's steady state transport in the limit of $\tau \to \infty$ neglecting terms of order $e^{-h/D}$. The second term in Eq. (2.12) is upon integration seen to be of order $e^{-h/D}$ and may consequently be neglected. Hence, however large the ratio h/D, the transport will in the limit of large time tend to Ekman's steady state transport.

3. The Ekman layer transport.

Upon integrating Eq. (1.5) with respect to z it is readily seen that the bulk of the transport takes place in the upper layer of the ocean. The transport in the Ekman layer, $0 \leq z \leq D$, is

$$(3.1) \qquad \int_0^D \varphi_{st} dz = \frac{T_0}{\rho f}\left(1+e^{-\pi}\right) \simeq 1.04\ M_{st}$$

A relevant time scale characterizing the transients is therefore the time it takes for the time dependent transport in this layer to reach the value given by Eq. (3.1).

The transport in the Ekman layer at an arbitrary time is obtained from Eq. (2.3) by putting $z = D$, viz.

$$(3.2) \qquad M_D = M_{st}\left[1+e^{-\pi} - \mathrm{erf}\left(\sqrt{\frac{\pi}{4\tau}}\right)e^{-2\pi i\tau} - H_2(D,\tau)\right].$$

Making use of the asymptotic formula

$$(3.3) \qquad \mathrm{erfc}\,(x) = \frac{e^{-x^2}}{x\sqrt{\pi}} \quad ; \qquad x \to \infty$$

there results

$$(3.4) \qquad M_D = M_{st}\left[1+e^{-\pi} - \tau^{-\frac{1}{2}}\, e^{-2\pi i\tau} + O(\tau^{-3/2})\right].$$

Thus the steady state is approached rather slowly, oscillating about the steady state with an amplitude which decreases as $\tau^{-\frac{1}{2}}$.

Based on Eq. (2.2) the exact time evolution of the transport in layers of different thickness has been plotted in fig. 1, 2 and 3 for $z = 0.2\ D$, $z = 0.5\ D$ and $z = D$, where the y-component is drawn as a function of the x-component. (Hodograph-plane). The figures exhibit the oscillatory behaviour as inferred from the asymptotic expression Eq. (3.4), and, moreover, they display how the oscillations will be more important the greater the depth. For example, the Ekman layer transport (fig. 3) requires 12.5 pendulum days to reach approximately 80% of its steady state value.

This runs contra to expectations, since it may be inferred from a problem which has exactly the same steady state solution that the transport will reach its steady state value on a time scale equal to the time scale which characterizes the decay of the transient terms in the velocity, i.e. on a scale of a couple of pendulum days only. This problem is dubbed "the related problem" for reasons which the next section will justify.

4. The related problem.

The related problem bears a strong resemblance to the Ekman problem. Instead of generating the motion by a wind sweaping over the surface and providing for a wind stress, the motion is generated by a moving plate at the surface. The governing equation is again (1.2) subject to the conditions (1.3) except for the first condition which has to be replaced by $\vec{v} = \vec{U}$ at the surface, $z = 0$. The solution to this problem may be written [c.f.Howard (1969)]

$$(4.1) \qquad \varphi_r = W\left[e^{-\pi(1+i)\frac{z}{D}} + H_2(z,\tau)\right]$$

where φ_r is the complex representation of the horizontal velocity. H_2 is given by Eq. (2.4), and W are the complex representation of the surface velocity $\vec{U}$. Correspondingly the integrated velocity becomes

$$(4.2) \qquad M_r = \frac{WD}{\pi(1+i)}\left[1 - e^{-\pi(1+i)\frac{z}{D}} - \operatorname{erf} c\,(\sqrt{2\pi i\tau}) + H_1(z,\tau)\right]$$

where H_1 is as before, the sum of the two terms constituting H_2.

Choosing, $W = U_0(1+i)$, i.e. the velocity at the surface to be 45° to the right of the y-axis, the stress in the steady state exerted by the fluid on the plane, $z = 0$, is $-i\,\frac{U_0 Df}{\pi}$. Consequently the stress on the fluid is along the positive y-axis and 45° to the left of the surface velocity. Hence with $U_0 = \frac{\pi T_0}{\rho Df}$ the steady state solution becomes identical with the corresponding solution obtained by Ekman (1905). This justifies the phrase "related problem".

Since the steady state solutions are identical one might be tempted to infer from the related problem the time evolution of the Ekman layer transport. The different ways the motions are generated will, however, lead to different evolutions in time. For instance, the work done on the fluid by the surface, $z = 0$, in the related problem is proportional to $-(\partial\varphi/\partial z)$ which is greater than (or as $\tau \to \infty$ equal to) the constant work provided by the wind stress in the Ekman problem. The transfer of motion to the interior will, therefore, take place at a greater speed, and hence a shorter time is needed to reach the steady state solution. One may reach the same conclusion by the following argument. In the Ekman problem the surface water has to be accelerated by a constant stress and the gradient necessary to transfer motion to the interior will develop slowly. In the related problem the surface water is already given its velocity and hence the necessary gradient is established initially.

In fact, the transient terms of the transport in the related problem may be shown to subside an order of magnitude more rapidly than in the Ekman problem, Eq. (3.4).

Making use of the formula (3.3) there results

$$M_{rD} = M_{st}\left[1+e^{-\pi} - \frac{1}{8}\,\tau^{-\frac{3}{2}}(1+i)e^{-2\pi i\tau} + O(\tau^{-\frac{5}{2}})\right].$$

This difference in the asymptotic behaviour of the solutions of the Ekman and the related problem demonstrates that although the two problems have the same steady state solutions, care need to be exercised in drawing any inferences from the related problem with respect to the time evolution of the Ekman layer transport.

5. Summary and final remarks.

The time dependent solutions to both the Ekman problem and the related problem have been studied for the following purposes.

(1) To show that the initial value problem indeed tends to Ekman's steady state solution.

(2) To point out that the transport evolves rather slowly towards it steady state value compared to the transport in the related problem.

(3) To estimate how long time it actually takes before the stationary Ekman layer transport is established.

It may be worth while to make some remarks regarding part (3). As has been shown Ekman's steady state solution does not constitute a "first approximation" to a problem with a non-steady windfield which varies on a time scale of order 1 - 10 pendulum days. The time dependent solutions will still have rapid variations at least as far as the transport is concerned, and allowance has to be made of this fact. On the other hand, the solution expressed by Eq.(1.8) may be a good approximation within one half pendulum day.

Figure captions

The hodograph diagrams fig. 1-3 display the time evolution of the transport in layers of three different thicknesses. The unit for the transport is M_{st}, Eq. (1.9). The points on the curve markes the time in units of $\frac{1}{2}$ pendulum days.

Figure 1 $z = 0.2$ D.

Figure 2 $z = 0.5$ D.

Figure 3 $z =$ D.

REFERENCES :

Abramowitz, M, & I.A. Stegun (1964) "Handbook of Mathematical Functions", Nat.Bureau of Stand.Appl.Math.Ser.55.

Crepon, M (1967 a) "Hydrodynamique marine en régime impulsionel, 1ère partie". Cah. Océanogr., 19 (8) pp.627-655.

Crepon, M. (1967 b) "Hydrodynamique marine en Régime impulsionel, 2ème partie". Cah.Océanogr., 19 (10) pp.647-880.

Ekman, V.W. (1905) "On the influence of earth's rotation on ocean currents". Ark.Mat.Ast.Fys. 2, 11.

Fjeldstad, J.E. (1930) "Ein Problem aus der Windstromtheorie", Z.A.M.M. 10, pp. 121-137.

Gonella, J. (1971 a) "A local study of inertial oscillations in the upper layers of the ocean". Deep-sea Res., 18, pp. 776-788.

Gonella, J. (1971 b) "The drift currents from observations made on the Bouée-Laboratoire". Cah.Océanogr., 23 (1) pp. 1-15.

Greenspan, H.P. (1968) "The theory of rotating fluids. Cambridge Monographs on Mechanics and Appl. Math., Cambridge University Press.

Howard, H.P. (1969) "Rotating flow". Lectures at The Royal Inst. of Techn., Stockholm, Springtime 1969,pp.43-46.

Roberts, G.E., & H.F. Kaufmann (1966) "Table of Laplace transforms", W.B. Saunders Company, Philadelphia and London.

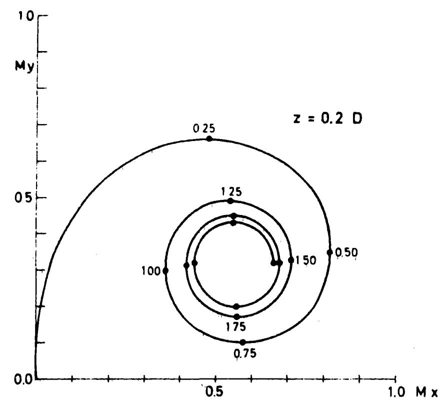

Fig. 1

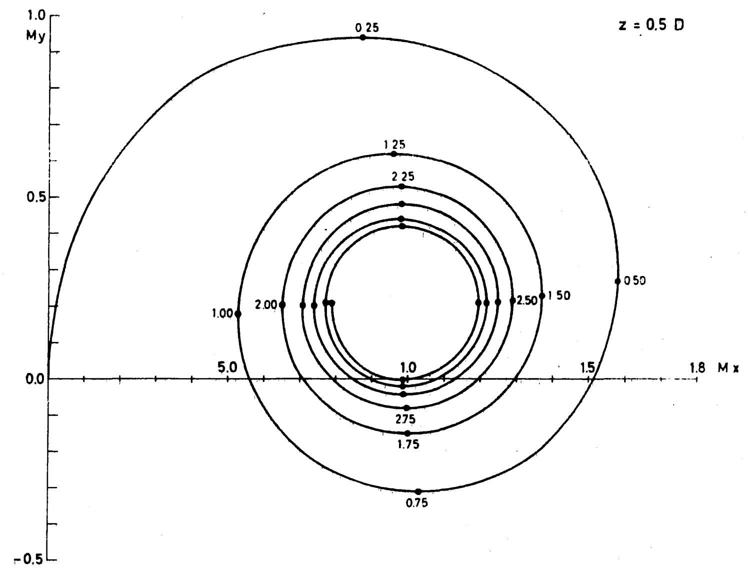

Fig. 2

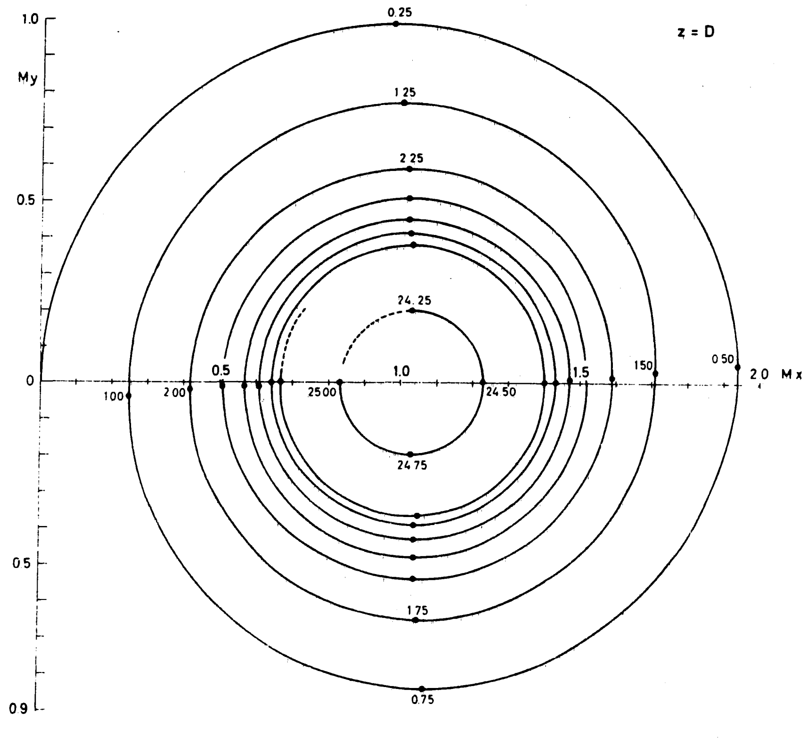

Fig. 3