A note on the stability of linear flow

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Abstract

It is shown that a finite disturbance independent of the streamwise coordinate may lead to instability of linear flow, even though the basic velocity does not possess any inflexion point.

It was shown already by Rayleigh [1] that a necessary condition for instability of linear flow in an inviscid, non-stratified fluid is that the velocity profile possesses an inflexion point. The criterion was sharpened by Fjørtoft [2] and Høiland [3] who showed that, in addition, the numerical value of the vorticity must be maximum at the inflexion point. Rayleigh and Høiland base their analysis on a normal mode expansion in time whereas Fjørtoft avoids this assumption. The normal modes do not usually form a complete set in these problems. Fjørtofts proof is therefore far more general.

These results are obtained by considering infinitesimal two-dimensional perturbations. We show in this note, however, that three-dimensional disturbances may lead to another kind of instability, independent of the existence of an inflexion point, which possibly may be responsible for the break-down of the laminar motion. This type of instability was originally suggested by the late Professor Høiland in his lecture notes. He did not, however, draw the full conclusions of his idea.

The fluid is assumed to be inviscid, incompressible and non-stratified, bounded by two horizontal parallel planes. The basic velocity $U(z)$ is directed along the horizontal x-axis and dependent on the vertical z-coordinate only. We consider a disturbance independent of the x-coordinate. The disturbance is, however, not strictly two-dimensional since it contains a velocity component along the x-axis. With this assumption the momentum equation for the x-component of the total velocity reduces to

$$\frac{Du}{dt} = 0 \ . \qquad (1)$$

Similarly the x-component of the vorticity ξ is given by

$$\frac{D\xi}{dt} = 0 . \tag{2}$$

Let v and w denote the velocities in the y- and z-direction, respectively. Since $\partial u/\partial x$ is zero, we may write

$$v = \frac{\partial\psi}{\partial z} , \qquad w = -\frac{\partial\psi}{\partial y} ,$$

where ψ is a streamfunction. Linearizing for the moment (1) and (2), we have

$$\frac{\partial u}{\partial t} + wU' = 0 , \tag{3}$$

$$\frac{\partial}{\partial t} \nabla_1{}^2\psi = 0 , \tag{4}$$

where $\nabla_1{}^2$ is the two-dimensional Laplacian. It follows from (4) that $\nabla_1{}^2\psi$, and therefore also w, is independent of time. (3) may therefore be integrated to give

$$u = u(o) - wU't , \tag{5}$$

showing that u increases linearly with time. We therefore deduce that the basic flow $U(z)$ is unstable for this special kind of infinitesimal disturbance.

The equations (1) and (2) may, however, also be solved for finite perturbations. (2) may be written

$$\frac{\partial}{\partial t}(\nabla_1{}^2\psi) + \frac{\partial\psi}{\partial z}\frac{\partial}{\partial y}(\nabla_1{}^2\psi) - \frac{\partial\psi}{\partial y}\frac{\partial}{\partial z}(\nabla_1{}^2\psi) = 0 . \tag{6}$$

A well known class of solutions are those satisfying

$$\nabla_1{}^2\psi = f(\psi) , \qquad \frac{\partial\psi}{\partial t} = 0 , \tag{7}$$

where f is an arbitrary function. Choosing f as a linear function, one solution is

$$\psi = A \sin k_1 y \sin k_2 z \ . \tag{8}$$

Here A, k_1 and k_2 are constants. k_2 must be chosen properly to satisfy the boundary conditions. (8) represents a set of closed, steady streamlines. From (1) it follows that u is conserved for the projection of the motion into these streamlines. Let us assume that initially u is equal to the basic velocity $U(z)$. A particle in its orbit in the yz-plane will therefore always have a u-velocity equal to the value of the basic flow at the initial position of that particle. This value may, however, be very different from the local value of the basic flow. The difference is largest for streamlines with large vertical extent. The u-velocity thus experience a complete redistribution, the variation of u with y and t becoming just as dominant as the variation with z. Though the motion at a fixed point is periodic, the period is different for different streamlines. The entire motion is therefore aperiodic. This last effect must lead to large gradients in u, i.e. large vorticity concentrations.

We notice that this distortion of the basic profile is independent of the initial amplitude of the disturbance. The time for development of the motion will, however, increase when the amplitude decreases. Since the asymptotic motion is very different from the basic flow, we conclude that this is unstable. It is possible, of course, that the developed motion is unstable. Owing to the large vorticity concentrations this seems indeed very likely so that the motion discussed above is valid only for a short span of time. Also this possibility means instability of the basic flow.

In equation (1) and (2) it is assumed that the disturbance is independent of x. The equations are also valid if the basic flow ia oblique to the x-axis. For sufficiently small angles the way of reasoning will be the same as in the case discussed above. For larger angles, however, the streamline field will contain a noticeable component of the basic flow, and the streamlines are usually not closed. The local changes in u will therefore be much smaller, and an initial small disturbance will not lead to a large distortion of the mean flow, even for a large span of time. However, for sufficiently large disturbances the concentration of vorticity will be significant and may lead to local breakdowns of the basic velocity.

(1) and (2) may be applied to find exact solutions of the problem. Such a solution is

$$\psi = A \sin k_1 y \sin k_2 z + B \cos \kappa z \,, \tag{9}$$

where

$$\kappa^2 = k_1{}^2 + k_2{}^2$$

and A and B are arbitrary constants. u is then found from the linear equation (2). (9) represents a basic velocity proportional to $\sin \kappa z$, superposed a finite disturbance. Steady solutions of (1) and (2) for $f(\psi) = \exp(-2\psi)$ has been discussed by Stuart [4].

(8) also describes a finite disturbance in a channel flow, i.e. a flow bounded by vertical as well as horizontal planes. By a similar reasoning we obtain the result that an inviscid channel flow is always unstable for perturbations independent of the streamwise coordinate.

The same result is also true for the flow in a circular pipe. This is seen by considering the solution

$$\psi = A J_n(kr) \cos n\theta \tag{10}$$

($n \neq 0$) where J_n denotes a Bessel function and r and θ are polar coordinates.

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References

1. Lord Rayleigh, Proc.Lond.Math.Soc. 11, 57 (1880)

2. E. Høiland, Geofys. Publ. 18, No.9 (1953)

3. R. Fjørtoft, Geofys. Publ. 17, No.5 (1950)

4. J. T. Stuart, J. Fluid Mech. 29, 417 (1967)