ON THE MOTION OF LONG BUBBLES IN VERTICAL TUBES

by

Kjell H. Bendiksen

ABSTRACT: Potential flow theory has been applied to study the shape and speed of an infinitely long bubble in vertical tubes. In particular, the combined effects of surface tension and externally forced liquid motion are examined. An analytical formula for the bubble velocity in stagnant liquid is proposed, and shown to be in very good agreement with experimental data for all values of surface tension. The predicted changes in bubble shape have to a large extent been confirmed through comparisons with photographic evidence for a wide range of parameters.

1. INTRODUCTION

The motion of single, large gas bubbles through stagnant liquid in vertical tubes under the influence of gravity, has been studied theoretically by several workers, Dumitrescu (1943), and Davies and Taylor (1943). Goldsmith and Mason (1962) included viscous forces, and Zukoski (1966) investigated the effect of surface tension experimentally.

The present study was initiated by the results of a more extensive series of experiments, Bendiksen (1983), on the motion of long bubbles through non-stationary liquid. It was shown that for all tube inclinations, θ, the bubble propagation rate is well represented by

$$v_B = C_0 v_L + v_0 \qquad [1]$$

where v_L is the average liquid velocity at infinity. For vertical or near vertical tubes $C_0 \approx 1.20$ for $Fr \gtrsim 3.5$, quite independent of Re-number, and v_0 is close to the bubble rise velocity in stagnant liquid. The coefficient C_0 was found to increase with decreasing Re-number, approaching 1.90 at $Re \approx 100$, Nicklin et al. (1962). Thus, the bubble appears to propagate at a rate slightly less than the maximum liquid velocity at infinity plus v_0.

The following paper presents a theoretical investigation of the motion of bubbles through liquids in infinitely long vertical pipes of circular cross-section. The liquid is assumed to be at rest or obeying a parabolic velocity profile at infinity. The velocity field caused by the bubble motion is assumed to be axis-symmetric and irrotational.

A particular objective has been the combined effect of externally forced liquid motion and surface tension on bubble velocity and shape. The theoretical results have been compared with the data of Nicklin et al. (1962), and to some extent Bendiksen (1983) for laminar liquid flow, and Zukoski (1966) for stationary liquid at infinity.

2. THEORY

2.1 Potential flow equation

A cylindrical coordinate system (r,ϕ,z) following the bubble is applied, as shown in figure 1. Assuming rotational symmetry around the z-axis, and steady bubble motion, the vorticity, ω_ϕ, may be expressed as

$$\omega_\phi = -\frac{1}{r}\left[\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r}\right] \qquad [2]$$

where ψ is the stream function.

For purely inertial flow under the influence of gravity, only, dimensional analysis yields $v_0 = v_0^\star\sqrt{gR}$, and to obtain non-dimensional equations we use

$$\eta = z/R,\quad \xi = r/R,\quad u^\star = u/\sqrt{gR},\quad v^\star = v/\sqrt{gR},\quad \psi^\star = \psi/\sqrt{gR^5}$$

where the star indicates a dimensionless quantity, which will be subsequently dropped, whenever there is no possibility of confusion.

Assuming a parabolic (or zero) velocity profile at infinity, and $v_B = v_m + C(\Sigma,v_m) + v_0$ the boundary condition there becomes

$$v_L^\infty(\xi) = v_m(1-\xi^2) - v_0 - C(\Sigma,v_m) - v_m \qquad [3]$$

or

$$\psi^\infty(\xi) = -\frac{1}{4}v_m\xi^4 - \frac{1}{2}v_{0m}\xi^2 \qquad [4]$$

where

$$v_{0m} = v_0 + C(\Sigma,v_m)$$

and

$$\Sigma = \sigma/(\rho_L gR^2). \qquad [5]$$

In the applied reference system the motion is steady, and we have from Bernoullis equation (with dimensional quantities):

$$\frac{1}{2}(u^2+v^2) + \frac{p}{\rho} = H(\psi) \qquad [6]$$

With $\omega_z=\omega_r=0$, ω_ϕ may be obtained from

$$w\omega_z - v\omega_\phi - \frac{\partial u}{\partial t} = \frac{\partial H}{\partial r} \qquad [7]$$

which gives in non-dimensional notation

$$\omega_\phi = -\frac{1}{v_L^\infty(\xi)}\frac{\partial H}{\partial \xi} = -\frac{1}{2}\frac{1}{v_L^\infty(\xi)}\frac{\partial}{\partial \xi}[v_L^\infty(\xi)]^2 \qquad [8]$$

$$\omega_\phi = 2v_m\xi$$

Then [2] may be written on dimensionless form as

$$\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2} - \frac{1}{\xi}\frac{\partial\psi}{\partial\xi} = -2v_m\xi \qquad [9]$$

A solution of [9] is most easily obtained by decomposing the stream-function into its up-stream value and local deviation from this, see for instance Batchelor (1980), pp.544:

$$\psi(\xi,\eta) = -\frac{1}{2}v_{0m}\xi^2 - \frac{v_m}{4}\xi^4 + \xi F(\xi,\eta) \qquad [10]$$

Equation [9] then yields an equation for F

$$\frac{\partial^2 F}{\partial\eta^2} + \frac{\partial^2 F}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial F}{\partial\xi} - \frac{F}{\xi^2} = 0 \qquad [11]$$

A well-known solution of [11] satisfying the boundary condition $u=0$ at the tube surface and being bounded everywhere, provided $\eta<-\infty$, is obtained by the method of separation of variables, and is given by

$$F(\xi,\eta) = \sum_i k_i J_1(\beta_i\xi)e^{-\beta_i\eta} \qquad [12]$$

where β_i are the i-th zeroes of the Bessel function of the 1. kind and order. The complete stream-function is then

$$\psi(\xi,\eta) = -\frac{v_{0m}}{2}\xi^2 - \frac{v_m}{4}\xi^4 + \xi\cdot\sum_i k_i J_1(\beta_i\xi)e^{-\beta_i\eta} \qquad [13]$$

and the velocities are given by

$$\begin{aligned} u &= +\sum_i k_i\beta_i J_1(\beta_i\xi)e^{-\beta_i\eta} \\ v &= -v_{0m} - v_m\xi^2 + \sum_i k_i\beta_i J_0(\beta_i\xi)e^{-\beta_i\eta} \end{aligned} \qquad [14]$$

2.2 Boundary conditions

The conditions already satisfied in equations [13,14] are those at $\eta=+\infty$ [3,4], and at the tube surface, $\xi=1$, where $u(1,\eta)=0$ for all η.

The boundary condition at the bubble surface is more complicated, due to the bubble shape being unknown. If the pressure in the bubble is assumed to be constant $(\rho_G \ll \rho_L)$, p_G^0, the pressure in the liquid at the interface may be approximated as

$$p_L(z) = p_G(z) - \sigma\left[\frac{1}{R_1(z)} + \frac{1}{R_2(z)}\right] \qquad [15]$$

where R_1, R_2 are the effective local radii of curvature in and perpendicular to the rz-plane. In the applied coordinate system, $z=r=0$ is a stagnation point, and assuming that the bubble surface is also a stream-line, Bernoullis equation yields

$$\frac{p_L(z)}{\rho_L} + \frac{1}{2}(u^2+v^2) - g|z| = \frac{p_L(0)}{\rho_L} \qquad [16]$$

Substituting the liquid pressure, p_L, from [15] we get

$$u^2+v^2=2g|z| - \frac{2\sigma}{\rho_L}\left[\left(\frac{1}{R_1(0)} + \frac{1}{R_2(0)}\right)-\left(\frac{1}{R_1(z)} + \frac{1}{R_2(z)}\right)\right] \qquad [17]$$

or on non-dimensional form

$$u^2+v^2 = 2|\eta| - 2\cdot\Sigma\cdot[(\frac{1}{R_1(0)} + \frac{1}{R_2(0)})-(\frac{1}{R_1(\eta)} + \frac{1}{R_2(\eta)})]$$

where $\Sigma=\sigma/(\rho_L gR^2)$.

The radii of curvature are rather complicated functions, which will be derived from [13]. First, observe that [14] at the stagnation point $(\xi=\eta=0)$ yields

$$v_{0m} = \Sigma k_i\beta_i \qquad [18]$$

which determines v_{0m} as a function of the constants k_i, β_i. An equation for the bubble shape is then obtained from [13] and [18]:

$$\frac{1}{2}\xi\sum_i k_i\beta_i + \frac{v_m}{4}\xi^3 - \sum_i k_i J_1(\beta_i\xi)e^{-\beta_i\eta} = 0 \qquad [19]$$

on the bubble surface $(\psi=0)$.

The other boundary condition [17], using [14] and [18], yields

$$[-\sum_i k_i\beta_i J_1(\beta_i\xi)e^{-\beta_i\eta}]^2 + [\sum_i k_i\beta_i+v_m\xi^2-\sum_i k_i\beta_i J_0(\beta_i\xi)e^{-\beta_i\eta}]^2$$
$$= 2|\eta| -2\cdot\Sigma[\frac{1}{R_0} - \frac{1}{R(\eta)}] \qquad [20]$$

where $\frac{1}{R(\eta)} = \frac{1}{R_1(\eta)} + \frac{1}{R_2(\eta)}$.

Except for the surface tension terms Dumitrescu (1943) obtained an equivalent set of equations for the special case of no motion at $\eta=+\infty$ $(v_m=0)$. In order to solve equations [18-20], however, keeping three terms in the series expansion, he had to impose an additional boundary condition in $\eta=-\infty$. This replaces one of the equations, and reduces the complexity of the problem considerably, but it does lead to an overdetermined solution. Furthermore, he assumed a spherical bubble shape at origo, and this actually leads to a double solution.

In the following, equations [18-20] are therefore solved directly, making no apriori assumptions on bubble shape. Provided an explicit equation for the bubble surface on the form $\eta=\eta(\xi)$ may be found, the principal radii of curvature are easily shown to be given by

$$\frac{1}{R_1(\eta)}+\frac{1}{R_2(\eta)} = \left[-\frac{\partial^2\eta}{\partial\xi^2}\left(1+\left(\frac{\partial\eta}{\partial\xi}\right)^2\right)^{-1}+\frac{1}{\xi}\left|\frac{\partial\eta}{\partial\xi}\right|\right]\left(1+\left(\frac{\partial\eta}{\partial\xi}\right)^2\right)^{-\frac{1}{2}} \qquad [21]$$

where the second term has to be replaced by its limit as $\xi \to 0$.

2.3 Power series solution

The problem is now to determine the coefficients k_i so that ξ,η satisfies equations [19,20] simultaneously.

Retaining N terms in [19,20], there are three obvious angles of attack in solving the resulting $2N$ equations. The first is a purely numerical one; choosing different ξ_i; $i=1,N$ we get $2N$ unknowns $(k_1,k_2,\ldots,k_N,\eta_1,\eta_2,\ldots,\eta_N)$. Another approach is to utilize the orthogonality properties of the Bessel functions to get an explicit expression $\eta=\eta(\xi)$. This method was investigated, and will work, provided the velocity profile in the film at $\eta=-\infty$ may be assumed known. This, however, seems a rather severe assumption, of the same nature as that of Dumitrescu (1943). Therefore, the expansion of the Bessel and exponential functions in power series, also applied by him, was finally chosen.

A power series expansion of the Bessel and exponential functions in [19] to $O(\xi^6)$ is shown in the Appendix to yield an explicit equation for the bubble surface [A6] on the form

$$\eta = -\frac{1}{2}\{a_1\xi^2+a_2\xi^4+a_3\xi^6\}\cdot\{2B_3-B_5\xi^2\}^{-1} \qquad [A6]$$

Similarly, the boundary condition [20] reduced to

$$B_2\xi^2 + (B_{3v}-B_2B_4)\xi^4 + (\tfrac{1}{4}B_4^2 + \tfrac{1}{6}B_2B_6 - \tfrac{1}{2}B_{3v}B_5)\xi^6$$
$$+ \{2B_2v_m\xi^2 - [2B_3B_4-2B_4v_m-B_2B_5]\xi^4\}\eta + \{4B_2^2-(2B_3v_m+4B_2B_4)\}\eta^2$$
$$= \tfrac{1}{2}|\eta| - \tfrac{1}{2}\Sigma(\tfrac{1}{R_0} - \tfrac{1}{R(\eta)}) \qquad [A9]$$

where B_m, m=2,...,N+1, is given by [A2], η by [A6], the radii of curvature by [A11], and the other coefficients are defined in the appendix.

The applied method of solution consists of collecting terms of order ξ^2, ξ^4, ξ^6 and determine the coefficients B_m, m=2,3,4 form the three resulting equations. To avoid the last, and most complicated equation, Dumitrescu (1943) introduced the boundary condition in $\eta=-\infty$, as remarked in the past section.

After some algebra the equations become, from [A13]

$$B_2^3 = \tfrac{1}{8}B_{3v} + 8\cdot\Sigma\cdot B_2\cdot(b_2-b_1^3) \qquad [22a]$$

$$5B_{3v}^2 - 4B_2B_4 - 2\frac{B_2^2}{B_3}B_5 - 2B_{3v}\cdot v_m - \tfrac{1}{2}a_2B_3^{-1}$$
$$- \Sigma\{72(b_3-4b_1^2b_2+2b_1^5)+16\frac{B_5}{B_3}(b_2-b_1^3)\} = 0 \qquad [22b]$$

$$[\frac{B_3}{B_2}B_{3v}^2+2B_{3v}(B_3+v_m)+2B_2B_5]B_4 + B_3B_4^2$$
$$+[-(\frac{v_m}{2}(\frac{B_3}{B_2})^2 + \tfrac{3}{2}B_5)B_{3v}-3B_3B_5]B_{3v} + \tfrac{2}{3}B_2B_3B_6+2B_2(1-v_m)a_2 - \tfrac{1}{2}a_3 \qquad [22c]$$
$$-\Sigma\cdot[64B_3(b_4+30b_1^4b_2-10b_1^7-12b_1b_2^2-9b_1^2b_3)-36B_5(b_3-4b_1^2b_2+2b_1^5)]=0$$

These three equations yield the unknown coefficients B_{2-4}, from which the k_i's are obtained by solving the linear set [A2]:

$$\sum_i^N k_i(\frac{\beta_i}{2})^m = B_m \;;\; m=2,\ldots,N+1 \qquad [23]$$

Actually, the set [22] has to be solved iteratively, as the coefficients B_m, $m>5$ also are given by [23]. This in not a closure problem, but rather a consequence of introducing the short-hand notation B_m; the real unknowns are the coefficients k_i $i=1,3$, for which we do have three equations. Incidentally, there is a closure problem associated with the surface tension terms, as stated in the appendix. If the interface condition [20] is sought to $O(\xi^6)$, because of the presence of 1. and 2. order derivatives of the bubble surface equation [19] it has to be of $O(\xi^8)$ for the sixth order term in [20] to be accurate.

2.4 Analytical expression for the bubble velocity

Due to the surface tension terms equations [22] has to be solved numerically. Based on the physical features of the problem, however, a very simple analytical expression for the bubble velocity may be obtained as follows. The basic idea is to reformulate [22a,b] in terms of the mean radius of curvature at the nose (ρ), and replace [22c] by a known relation

$$\rho = \rho(\Sigma) \qquad [24]$$

Then, from [A11] we get, with $\rho=R_i(0)$, $i=1,2$

$$\rho = \frac{1}{2b_1} = \frac{2B_3}{a_1} = \frac{2B_2}{B_{3v}} \qquad [25]$$

Rearranging [22a] yields:

$$2B_2 = B_{3v}^{1/3}[1+64\cdot\Sigma\cdot\frac{B_2}{B_{3v}}(b_2-b_1^3)]^{1/3} \qquad [26]$$

Equation [25] then yields

$$B_{3v} = \left(\frac{1}{\rho}\right)^{3/2}(1+\Sigma\cdot f(\rho))$$

and [27]

$$B_2 = \frac{1}{2}\,\frac{1}{\rho}(1+\Sigma\cdot f(\rho))$$

where $f(\rho)$ depends on N:

$$f(\rho) = 64\frac{B_2}{B_{3v}}(b_2 - b_1^3) \qquad [28]$$

For $N=1$, using [23] the computation is straightforward and yields

$$f_1(\rho) = \frac{4}{3}\cdot\rho^{-2}$$

and for $N=2$ it becomes

$$f_2(\rho) = 4\left\{-5\rho^{-2} + \frac{1}{3(\beta_2-\beta_1)}\left[(\beta_2 - \frac{4}{\rho})(\frac{\beta_1}{2})^2(6 - \frac{\beta_1}{2}\rho)\right.\right.$$
$$\left.\left. + (\frac{4}{\rho} - \beta_1)(\frac{\beta_2}{2})^2(6 - \frac{\beta_2}{2}\rho)\right]\right\} \qquad [29]$$

For $N<2$ the closed expressions for the bubble velocity then follow from [23] and [18]. If $N>3$ additional equations [22b,c...] for the coefficients $B_4, B_5\ldots$ are required, and these soon become prohibitively complicated. With $N=1$, however, the calculation again is staightforward and yields

$$v_0(\Sigma) = \beta^{-\frac{1}{2}}(1 + \frac{4}{3}\Sigma\rho^{-2})^{\frac{1}{2}} \qquad [30]$$

where from [25]

$$\rho = 4/\beta_1 = 1.0439$$

For $N=2$ equations [23] with B_m from [27] reduces, after some more algebra, to a surprisingly simple expression

$$v_0(\Sigma) = 2\frac{\beta_1+\beta_2}{\beta_1\,\beta_2}\rho^{-\frac{1}{2}}\{1 - \frac{4}{\beta_1+\beta_2}\rho^{-1}\}\cdot\{1 + \Sigma\cdot f_2(\rho)\}^{\frac{1}{2}} \qquad [31]$$

It is immediately obvious from [30] that keeping the first term, only, although giving a rather accurate bubble velocity for $\Sigma=0$, leads to an incorrect dependence on surface tension. If the radii of curvature [24] are known, for instance by the method of the past section, the velocities [30-31] should agree with those of the last section for $N=1,2$. A comparison with experiment to be presented in the next chapter shows very good agreement with [31] for all values of Σ.

2.5 Numerical method

Equations [22],[23] and [18] have been solved numerically for different values of surface tension parameter, Σ, and liquid velocity in $\eta=\infty$, v_m. Keeping in mind that the coefficients $b_i=b_i(B_j)$; $j=2,8$, an iterative procedure is required for the solution of B_{2-4} from [22] with B_{5-8} from [23]. The outer loop yields B_3 from [22b]. For a given B_3 the inner iteration then consists of solving [22a,c] for B_2 and B_4, simultaneously. For $\Sigma<<1$ the B_2-calculation may be decoupled from that of B_4, which is a very significant simplification. It is also numerically advantageous to start the B_2 calculation with $B_2=\frac{1}{2}B_{3v}^{1/3}+\Delta(\rho)$, where $\Delta<<B_3$. If the resulting changes in B_{5-8} are greater than, say 1%, the outer iteration is repeated with the same B_3-values, but with $B_n=(B_n^{old}+B_n^{new})/2$; $n=5,\ldots 8$. For $\Sigma<<1$ this proved unnecessary, if the initial values were properly selected, for example from the analytical expressions [30] or [31]. For $\Sigma \gtrsim .20$ with $N=3$ the method fails, as might be expected, due to the large deformation of the bubble requiring increasingly higher order terms in [19],[20]. However, as will be shown, in this region the analytical formulas still yield surprisingly good results.

With our choice of coordinate system, $v_0>0$, and $a_1/4B_3>0$ require $B_2,B_3>0$. With this assumption we always found a unique solution, if it existed at all. As is most easily seen from [22a,b], for $\Sigma=v_m=0$, there is a "mirror" solution $B_2,B_3<0$, corresponding to the boundary condition being given at $\eta=-\infty$ instead of $\eta=+\infty$.

3. RESULTS

3.1 Bubble motion

The influence of surface tension on bubble velocity is shown in figure 2 for $v_m=0$. The theoretical values are presented for N=1-3, and the measurements are from Zukoski (1966) and Bendiksen (1983). For $\Sigma=0$ $v_0=.511$, .487 and .495, respectively, and the latter compares very well with that of Dumitrescu (1943) of .496 based on asymptotic matching, Keeping the first term of the series [19,20], only, yields a somewhat too high value of .511, in contrast to that reported by Davies and Taylor (1949) of .464. The latter, however, was obtained for $\xi=1/2$, and this, obviously, is inconsistent with the assumption $\xi<<1$, permitting the higher order terms to be neglected. For $\xi \to 0$, the result of Davies and Taylor would be in accordance with ours.

The analytical expression [31] is applied for $\Sigma>.20$ with radii of curvatures at the nose from the theoretical calculation. For $\Sigma \lesssim .2$ the predictions agree to within 5% for $N>2$, whereas keeping the first term, only, is clearly insufficient to reproduce the observed dependence on surface tension. Although higher order terms in [19,20] ($N>3$) are needed for $\Sigma \gtrsim .2$, the analytical expression [31] with ρ extrapolated from figure 5, [32], is surprisingly accurate.

A convenient way of representing the effect of a liquid velocity profile at infinity is through the distribution slip-ratio, C_0, defined by [1].

It is seen from figure 3 that for $\Sigma=0$, $C_0 \approx 2.29$ and the bubble actually moves faster than the center liquid at infinity, plus v_0. This is in accordance with earlier results for purely viscous flow with $v_0=0$, where the acknowledged value is $C_0 \approx 2.27$ for $\Sigma=0$, Taylor (1962). For $v_L<<\Sigma$, C_0 decreases, but for $v_L<<.01$ the accuracy also decreases sharply, due to numerical difficulties.

The predicted values of C_0 are in reasonable agreement with the experimentally observed ones of from 1.80 to 1.95 of Nicklin et al. (1962). Furthermore, the predicted fall in C_0 with Σ indicates that the slight, but significant difference between the experimentally observed value of $C_0 \approx 1.20$ in turbulent flow and the corresponding maximum to average liquid velocity ratio of 1.22 for all inclinations and $D=.025$m, Bendiksen (1983), may be due to surface tension effects.

3.2 Bubble shape

The predicted radii of curvature [A11], presented in figure 4, show only a slight change in bubble shape for low values of surface tension. Both decrease with Σ near origo (N=3), which implies a more pointed nose; but R_1 decreases far more with ξ than R_2, the latter being practically constant. For $\Sigma>>.10$ the power series approximation in [A11] breaks down, and both radii of curvature near origo start to increase for rather low values of ξ. This implies a flattening of the bubble cap and there may ultimately form a "break point" near the tube wall, where R_1 becomes very small.

This rather complicated influence of surface tension on bubble shape provides an explanation of the apparent contradicting experimental evidence of Dumitrescu (1943) that R_1 decreases with increasing $\Sigma(\lesssim .30)$ near origo, and of Zukoski (1965) that the bubble shape is approximately constant up to $\Sigma \simeq .10$, with the radii of curvature near origo increasing with Σ from then on. The data of Dumitrescu also confirm the predicted trend in the dependence of R_1 on $\xi(\lesssim .40)$ of figure 4.

Note that the bubble shape becomes increasingly non-spherical for $\xi \gtrsim .20$, invalidating the assumption of Dumitrescu of spherical shape up to $\xi=.60$.

The principal radius of curvature at origo with N terms in [19,20], ρ, is shown in figure 5 for $N=1,\ldots,3$. It is believed that the predicted fall in ρ with Σ for $N=3$ is slightly too large. The extrapolated values to be applied in the analytical velocity formula [31] are very well represented, for $\Sigma < 1$, by

$$\rho(\Sigma) = \rho_0(1-c_1 e^{-c_2/\Sigma}) \qquad [32]$$

where $\rho_0=.795$, $c_1=.5173$, and $c_2=.0661$. Obviously, at least three terms are required to yield a correct shape, although the analytical expression for the bubble velocity with $\rho(\Sigma)$ from [32] is surprisingly accurate (figure 2). For large Σ increasingly higher order terms are required, as well as the incorporation of the exact bubble surface equation [A3] in the calculation of R_1, R_2, to represent the break-point near the wall.

The substantial increase in complexity, however, makes it an extremely doubtful procedure, particularly in view of the methodical limitations, and the only marginal potential improvements with respect to the velocity formula [31].

Finally, a last important effect is the predicted increase in curvature near origo caused by liquid motion at infinity, figure 6. This has been qualitatively confirmed by photographic evidence, figure 7, from a series of experiments, Bendiksen (1983). Actually this effect was observed for other inclinations as well, even in horizontal tubes.

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APPENDIX: Coefficients of the power series expansion

Consider the following power series expansion of the Bessel and exponential functions to the sixth order in ξ and fourth order in η:

$$J_0(\beta_i\xi)=1 - \frac{1}{1}\left(\frac{\beta_i}{2}\right)^2\xi^2 + \frac{1}{(2!)^2}\left(\frac{\beta_i}{2}\right)^4\xi^4 - \frac{1}{(3!)^2}\left(\frac{\beta_i}{2}\right)^6\xi^6+O(\xi^8)\ldots$$

$$J_1(\beta_i\xi)=\left(\frac{\beta_i}{2}\right)\xi - \frac{1}{1!2!}\left(\frac{\beta_i}{2}\right)^3\xi^3 + \frac{1}{2!3!}\left(\frac{\beta_i}{2}\right)^5\xi^5 - O(\xi^7) \qquad [A1]$$

$$e^{-\beta_i\eta}=1 -2\left(\frac{\beta_i}{2}\right)\eta + \frac{2^2}{2!}\left(\frac{\beta_i}{2}\right)^2\eta^2 - \frac{2^3}{3!}\left(\frac{\beta_i}{2}\right)^3\eta^3 + \frac{2^4}{4!}\left(\frac{\beta_i}{2}\right)^4\eta^4+O(\eta^5)\ldots$$

Define the constants B_m by

$$B_m = \sum_{i=1}^{N} k_i\left(\frac{\beta_i}{2}\right)^m \;; \quad m=2,\ldots,N+1 \qquad [A2]$$

The bubble surface equation [19] is then to the order $O(\xi^6)$ and $O(\eta^3)$ for $\xi \neq 0$:

$$\xi\cdot\left\{-\frac{1}{2}\sum_i k_i\beta_i - \frac{v_m}{4}\xi^2 + \frac{1}{\xi}\sum_i k_i J_1(\beta_i\xi)e^{-\beta_i\eta}\right\} = 0$$

Applying the expansions [A1] yields

$$\left\{-\left(\frac{B_3}{1!2!} + \frac{v_m}{4}\right)\xi^2 + \frac{B_5}{2!3!}\xi^4 - \frac{B_7}{3!4!}\xi^6 + \ldots\right\}$$
$$- \frac{2}{1!}\left\{B_2 - \frac{B_4}{1!2!}\xi^2 + \frac{B_6}{2!3!}\xi^4\right\}\eta$$
$$- \frac{2^2}{2!}\left\{B_3 - \frac{B_5}{1!2!}\xi^2 + \frac{B_7}{2!3!}\xi^4\right\}\eta^2 - \frac{2^3}{3!}\left\{B_4 - \frac{B_6}{1!2!}\xi^2\right\}\eta^3 = 0$$

From Dumitrescu (1943), Davies and Taylor (1949), and others, we know that for small Σ, at least, the bubble shape is nearly spherical at its nose, and η is at least of order 2 in ξ. Thus to $O(\xi^6)$ we have

$$- \frac{8}{6}B_4\eta^3 + [2B_3-B_5\xi^2]\eta^2 - 2\left[B_2 - \frac{B_4}{2}\xi^2 + \frac{B_6}{12}\xi^4\right]\eta$$
$$- \frac{1}{2}\left[B_3 + \frac{v_m}{2}\right]\xi^2 + \frac{B_5}{12}\xi^4 - \frac{B_7}{144}\xi^6 = 0 \qquad [A3]$$

A natural next step would be to represent η by a power series function of ξ^2, possibly spherical near origo, as assumed by Dumitrescu (1943). With surface tension effects present, however, the bubble shape near origo will be quite deformed, and the above approximation becomes too crude. Instead we solve for η from [A3], considering it a second order equation in η

$$\eta = \{2(B_2 - \frac{B_4}{2}\xi^2 + \frac{B_6}{12}\xi^4) \pm 2\{(B_2 - \frac{B_4}{2}\xi^2 + \frac{B_6}{12}\xi^4)^2 + (2B_3 - B_5\xi^2) \cdot$$

$$\cdot [\frac{1}{2}(B_3 + \frac{v_m}{2})\xi^2 - \frac{B_5}{12}\xi^4 + \frac{B_7}{144}\xi^6 - \frac{8}{6}B_4\eta^3]\}^{\frac{1}{2}}\} \cdot \{2[2B_3 - B_5\xi^2]\}^{-1}$$

With the actual choice of coordinate system, the bubble surface is completely below origo, and the negative sign must be retained. Expanding the square root in powers of ξ^2 we get to $O(\xi^6)$:

$$\eta = \{(B_2 - \frac{B_4}{2}\xi^2 + \frac{B_6}{12}\xi^4) - B_2 - B_2[\frac{a}{2}\xi^2 - \frac{1}{2}(b - \frac{1}{4}a^2)\xi^4$$

$$+ \frac{1}{6}(3c - \frac{3}{2}ab + \frac{3a^3}{8})\xi^6]\} / \{2B_3 - B_5\xi^2\} \qquad [A4]$$

where

$$a = (B_3^2 + \frac{1}{2}B_3 v_m - B_2 B_4) B_2^{-2}$$

$$b = (\frac{1}{4}B_4^2 - \frac{2}{3}B_3 B_5 - \frac{1}{4}B_5 v_m + \frac{1}{6}B_2 B_6) B_2^{-2} \qquad [A5]$$

$$c = \frac{1}{12}(B_5^2 - B_4 B_6 + \frac{1}{6}B_3 B_7 + \frac{8}{3}B_4\eta^3) B_2^{-2}$$

Rearranging [A4] in increasing order of ξ we finally get

$$\eta = \{-a_1\xi^2 - a_2\xi^4 - a_3\xi^6\} \cdot \{2(2B_3 - B_5\xi^2)\}^{-1} \qquad [A6]$$

where $B_{3v} = B_3 + \frac{v_m}{2}$

$$a_1 = \frac{B_3}{B_2} B_{3v}$$

$$a_2 = B_2(b - \tfrac{1}{4}a^2 - \tfrac{1}{6}\frac{B_6}{B_2}) \qquad [A7]$$

$$a_3 = \tfrac{1}{3}B_2(3c - \tfrac{3}{2}ab + \tfrac{3}{8}a^3)$$

Equation [A6] is an explicit one for η, i.e. if the coefficients B_m are known, the bubble shape near origo may be obtained. With N=3 in [19,20] three of the coefficients B_m, m=2,4 may be obtained from [20] as follows.

Firstly, the expansion [A1] is substituted in [20]; yielding to $O(\xi^6)$

$$\begin{aligned}
&\{4[B_2^2\xi^2-B_2B_4\xi^4 + \tfrac{1}{4}B_4^2\xi^6] + \tfrac{2}{3}B_2B_6\xi^6-16B_2B_3\xi^2\eta+8[B_3B_4+B_2B_5]\xi^4\eta \\
&+16B_2B_4\xi^2\eta^2+16B_3^2\xi^2\eta^2\} + \{4[B_{3v}^2\xi^4 - \tfrac{1}{2}B_5B_{3v}\xi^6] \\
&-16[-B_{3v}\xi^2 + \tfrac{1}{4}B_5\xi^4]\cdot[(B_2-B_4\xi^2)\eta-B_3\eta^2] \\
&+16[B_2^2-2B_2B_4\xi^2]\eta^2\} = 2|\eta| - 2\cdot\Sigma[\frac{2}{R_0} - \frac{2}{R(\eta)}]
\end{aligned} \qquad [A8]$$

Rearranging this equation, we get

$$\begin{aligned}
&B_2^2\xi^2 + [B_{3v}^2-B_2B_4]\xi^4 + [\tfrac{1}{4}B_4^2 + \tfrac{1}{6}B_2B_6 - \tfrac{1}{2}B_{3v}B_5]\xi^6 \\
&+ \{2B_2v_m\xi^2-[2B_3B_4+2B_4v_m-B_2B_5]\xi^4\}\eta + \{-[2B_3v_m+4B_2B_4]\xi^2+4B_2^2\}\eta^2 \\
&= \tfrac{1}{2}|\eta| - \tfrac{1}{2}\Sigma(\frac{1}{R_0} - \frac{1}{R(\eta)})
\end{aligned} \qquad [A9]$$

To proceed expressions for the radii of curvature are needed. Approximating the surface equation, [A3] with a power series in ξ^2, $\eta=-\Sigma_{i=1}^N|b_i|\xi^{2i}$ in [21], differentiating with respect to ξ, yields

$$\frac{\partial\eta}{\partial\xi} = -\sum_{i=1}^{N+1} 2ib_i\xi^{2i-1}$$

and [A10]

$$\frac{\partial^2\eta}{\partial\xi^2} = -\sum_{i=1}^{N+1} 2i(2i-1)b_i\xi^{2(i-1)}$$

where

$$b_1 = a_1/4B_3$$

$$b_2 = (-2B_3b_1^2+B_4b_1 - \frac{1}{12}B_5)/2B_2$$

$$b_3 = [-4B_3b_1b_2+B_5b_1^2+B_4b_2 - \frac{1}{6}B_6b_2 + \frac{1}{144}B_7 - \frac{4}{3}B_4a_1^3]/2B_2$$

$$b_4 = [-2B_3(b_2^2+2b_1b_3)+2B_5b_1b_2+B_4b_3 - \frac{1}{6}B_6b_2 - 4B_4b_1^2b_2 - (\frac{1}{4!5!} - \frac{2}{3!4!})B_8 - \frac{1}{6}B_7b_1^2 + \frac{2}{3}B_6b_1^3]/2B_2$$

Thus

$$\frac{2}{R_0} = 2b_1 + 2b_1 = 4b_1$$

Obviously, including all surface tension terms of order six, requires the bubble surface equation [A6] to be of order $O(\xi^8)$, and this was done in order to obtain b_4. Then, from [21] after some algebra, we get

$$\frac{1}{R_1(\eta)} = 2b_1 + 12[b_2-b_1^3]\xi^2 + 30[b_3-4b_1^2b_2+2b_1^5]\xi^4 + 28[b_4-12b_1b_2^2-9b_1^2b_3+30b_1^4b_2-10b_1^7]\xi^6$$

and [A11]

$$\frac{1}{R_2(\eta)} = 2b_1 + 4[b_2-b_1^3]\xi^2 + 6[b_3-4b_1^2b_2+2b_1^5]\xi^4 + 4[b_4-12b_1b_2^2-9b_1^2b_3+30b_1^4b_2-10b_1^7]\xi^6$$

It is now easily ascertained that for a spherical bubble surface the expressions in the brackets are zero to $O(\xi^6)$; thus the curvatures are constant and equal for all ξ, as they should.

From [A11] the surface tension term in [A9] becomes

$$-\frac{1}{2}\Sigma[\frac{1}{R_0} - \frac{1}{R(\eta)}] = \frac{1}{2}\Sigma\{16(b_2-b_1^3)\xi^2+36(b_3-4b_1^2b_2+2b_1^5)\xi^4 +32(b_4-12b_1b_2^2-9b_1^2b_3+30b_1^4b_2-10b_1^7)\xi^6\} \qquad [A12]$$

Inserting [A6] for η and [A12] in [A9] finally yields

$$\{4B_3B_2^2 - \frac{1}{2}a_1 - 8\Sigma(b_2-b_1^3)4B_3\}\xi^2$$
$$+\{5B_3B_{3v}^2-4B_2B_3B_4-2B_2^2B_5-2B_3B_{3v}v_m - \frac{1}{2}a_2-\Sigma\cdot[18(b_3-4b_1^2b_2+2b_1^5)4B_3$$
$$- 16B_5(b_2-b_1^3)]\}\xi^4$$
$$+\{[\frac{B_3}{B_2}B_{3v}^2+2B_{3v}B_3(B_3+v_m)+2B_2B_5]B_4 + B_3\cdot B_4^2 \qquad [A13]$$
$$+[(-\frac{3}{2}B_5 - \frac{v_m}{2}\frac{B_3^2}{B_2^2})B_{3v} - 3B_3B_5]B_{3v} + \frac{2}{3}B_2B_3B_6 + 2B_2(1-v_m)a_2 - \frac{1}{2}a_3$$
$$-\Sigma[64B_3(b_4-12b_1b_2^2-9b_1^2b_3+30b_1^4b_2-10b_1^7)-36B_5\cdot(b_3-4b_1^2b_2+2b_1^5)]\}\xi^6 = 0$$

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FIGURE CAPTIONS

Figure 1. The applied coordinate system.

Figure 2. The influence of surface tension (Σ) on bubble velocity in stagnant liquid (Theoretical: —— 3 terms, -•- 2 terms [31], --- 1 term [30]. Experimental: Zukoski (1966); o Air/Water, Δ Alcohol/Water. Bendiksen (1983); Air/Water).

Figure 3. The dependence of C_0 [1] on liquid velocity at infinity (v_m) with surface tension as parameter (oΣ=0., $\nabla\Sigma$=.0026, $\Delta\Sigma$=.013).

Figure 4. Predicted non-dimensional radii of curvature (R_1,R_2) vs ξ in stagnant liquid (v_m=0) with surface tension as parameter (—— R_1, --- R_2).

Figure 5. Predicted radius of curvature at origo, ρ, vs surface tension for v_m=0 (—— N=3, -•- N=2, --- N=1).

Figure 6. Predicted effects of liquid motion at infinity and surface tension on bubble shape at origo (ρ) (Δ v_m=0, o v_m=.05, ∇ v_m=.10).

Figure 7. Measured effect of liquid motion at infinity on bubble shape (ρ) for Σ=.042: (a) v_m=0, (b) v_m=.07, (c) v_m=.94, (d) v_m=1.00, (e) v_m=1.40, (f) v_m=1.93).

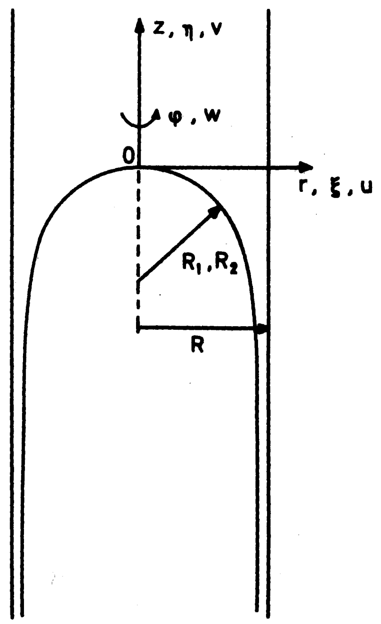

Figure 1.

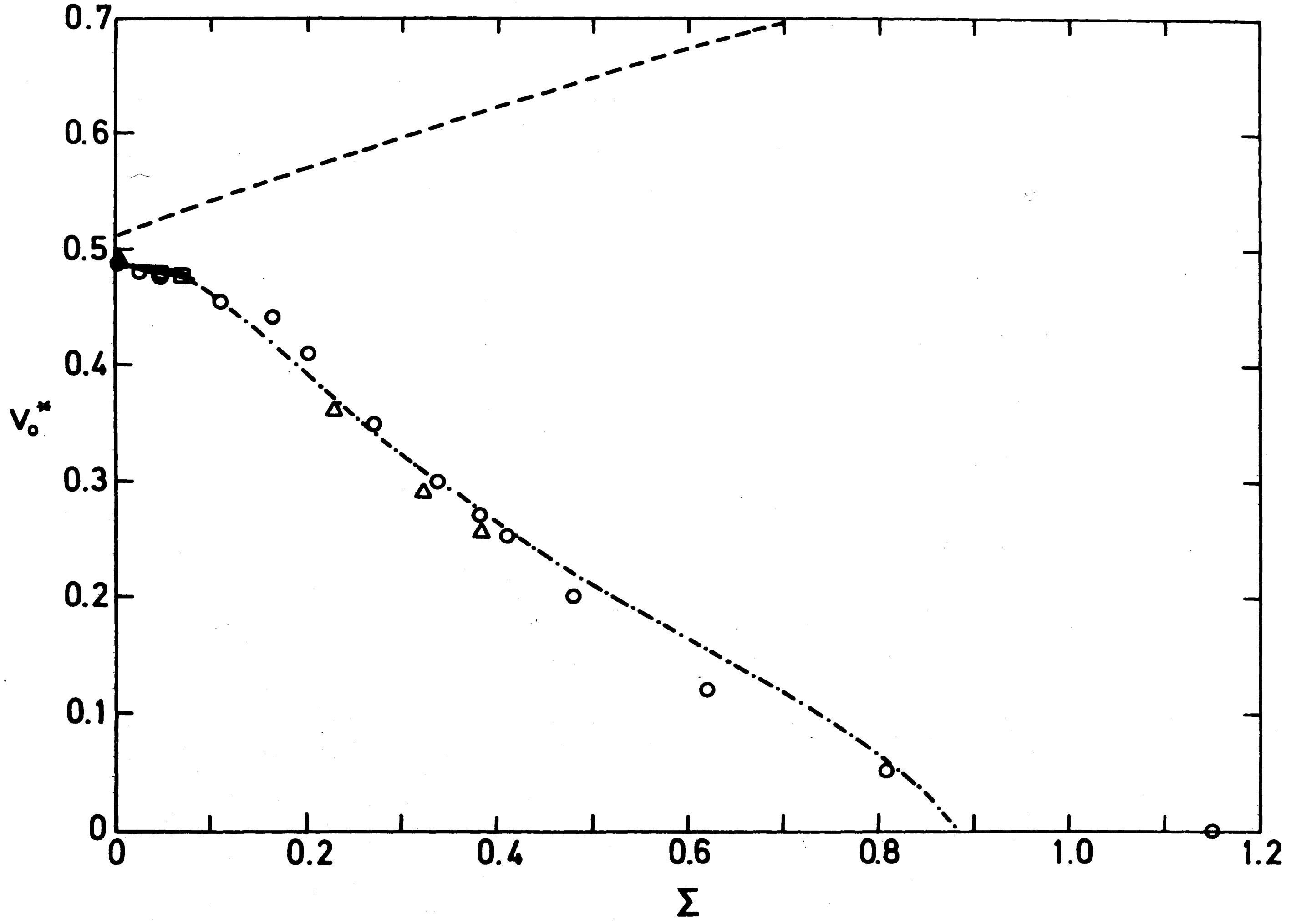

Figure 2.

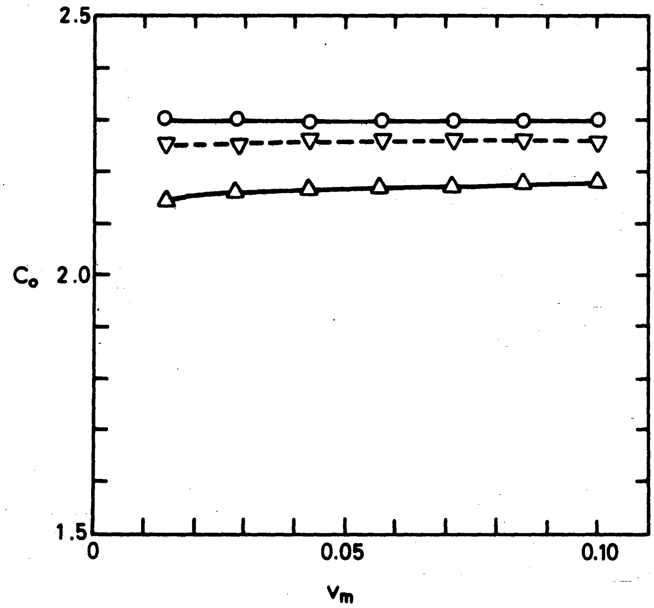

Figure 3.

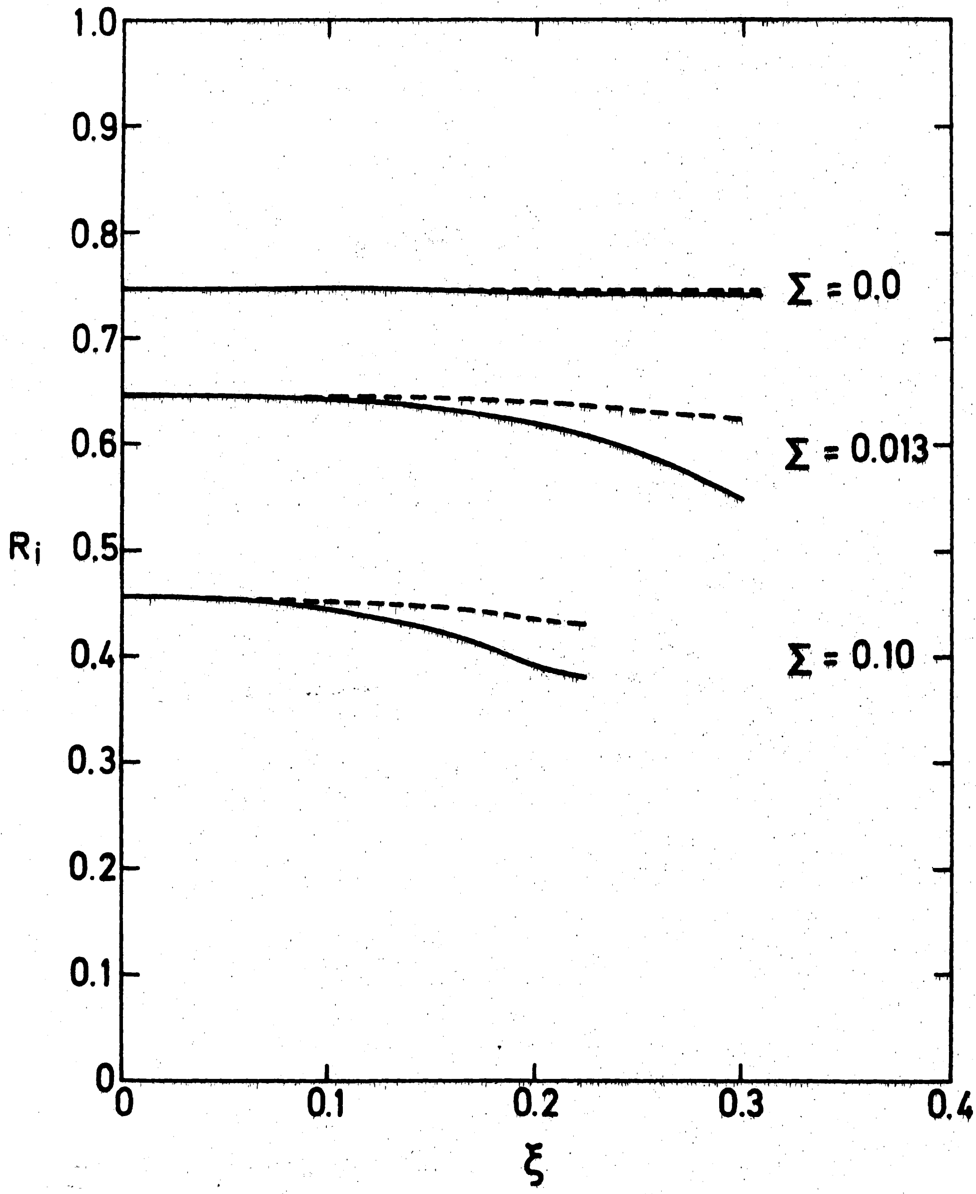
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
R_i
$\Sigma = 0.0$
$\Sigma = 0.013$
$\Sigma = 0.10$
0
0.1
0.2
0.3
0.4
ξ

Figure 4.

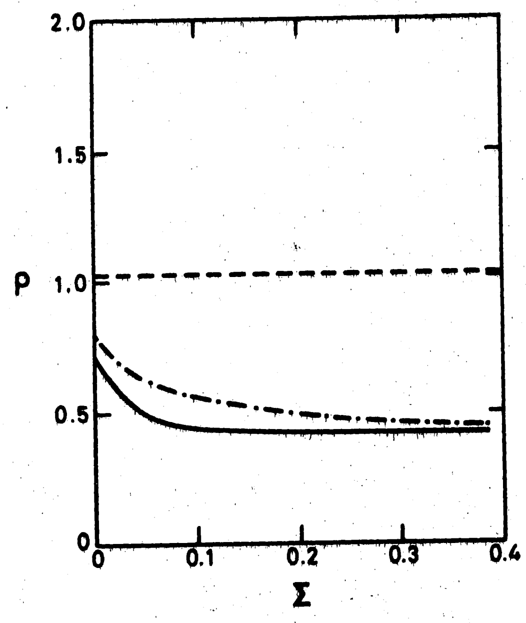

Figure 5.

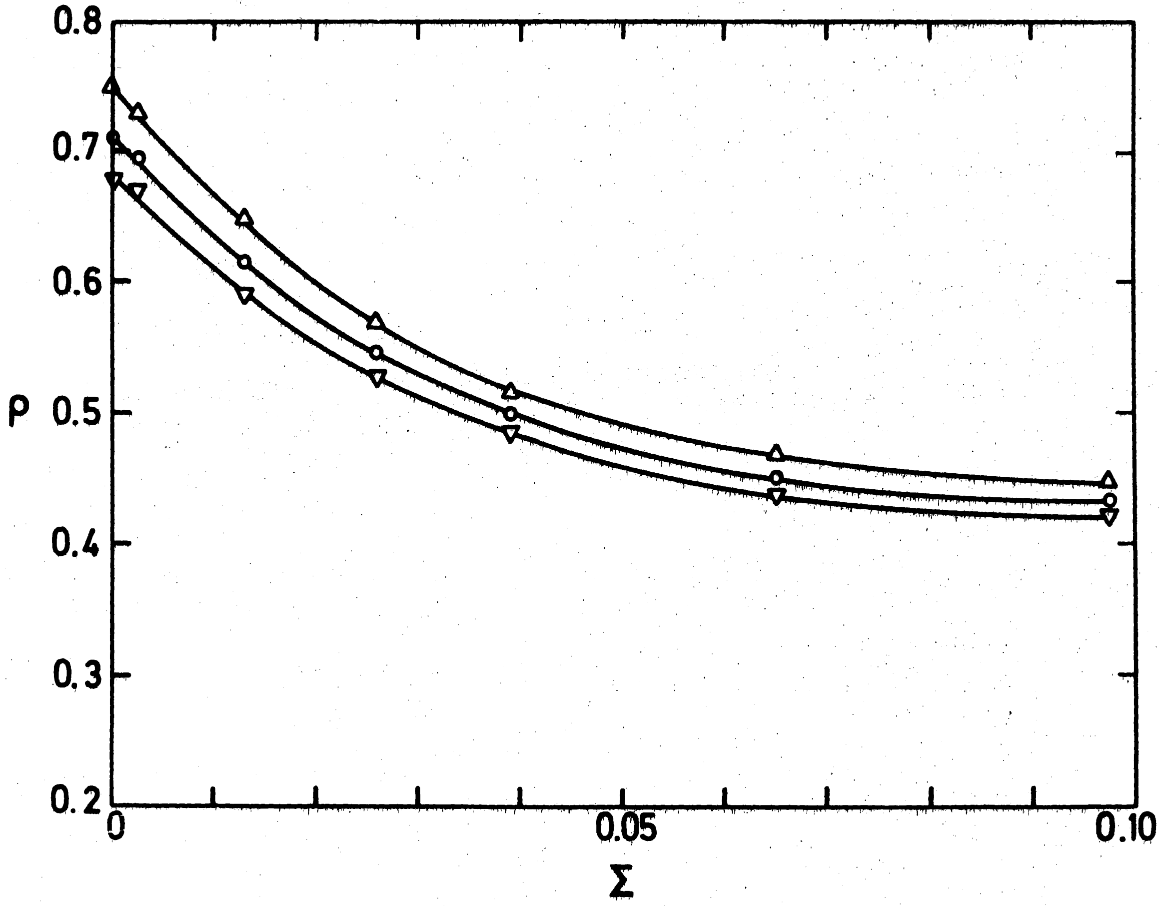

Figure 6.

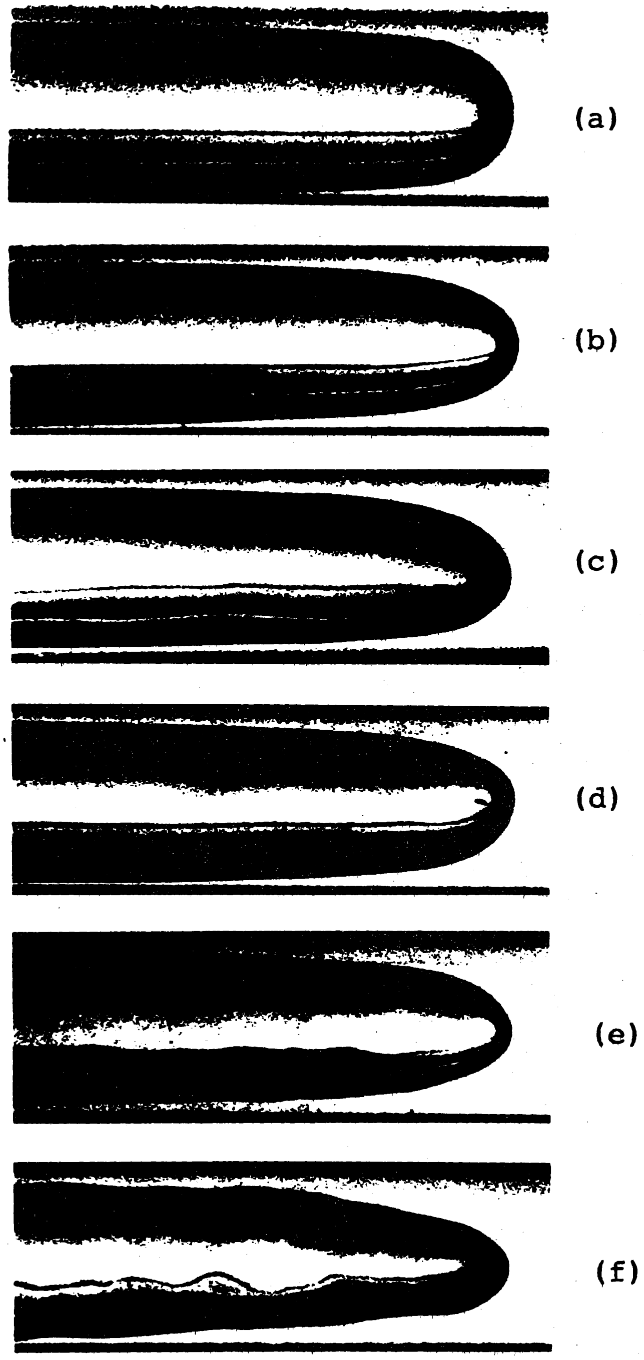

Figure 7.