ON THE STABILITY OF UNIFORM AND NONUNIFORM FLOWS IN AN OPEN INCLINED CHANNEL

By

LEIV STORESLETTEN
Department of Mathematics
Agder College
Box 607, 4600 Kristiansand, Norway

SUMMARY

The linear stability of steady flows of an open channel of constant inclination is studied. The analysis is based upon the nonlinear equations of shallow water theory, augmented by a frictional force due to the resistance against the channel walls.

The stability properties of the uniform basic flow is thoroughly discussed by using both temporally and spatially growing modes. Attention is also devoted to the dispersion and damping of these modes.

The stability of nonuniform basic flows is studied by two different methods: The normal mode method based on spatial modes and a generalized progressing wave expansion method (called Eckhoff's method).

CONTENTS

0. LIST OF SYMBOLS

α angle of inclination

h depth of channel flow

v mean velocity of channel flow

$S = \tan\alpha$

$g^* = g\cos\alpha$, g gravity acceleration

c_f friction coefficient

x distance along the channel

t time coordinate

h_0 depth of steady flow

v_0 mean velocity of steady flow

η perturbation of depth

w perturbation of mean velocity

$h_0^* = h_0(0)$

$v_0^* = v_0(0)$

$\bar{h}_0$ normal depth , see (3.5)

$\bar{v}_0$ normal velocity , see (3.6)

h_c critical depth , see (3.5)

ζ_1 , ζ_2 perturbations defined by (5.1)

$F = \frac{v_0}{\sqrt{g^* h_0}}$ Froude number

A, B, C, D, E, E_0 2×2 matrices

k wave number , $k_r = \mathrm{Re}(k)$, $k_i = \mathrm{Im}(k)$

ω frequency , $\omega_r = \mathrm{Re}(\omega)$, $\omega_i = \mathrm{Im}(\omega)$

$i = \sqrt{-1}$

c , c^* phase velocities

u_0 , u_1 , see (6.5) and (6.7)

1. INTRODUCTION

Flow in open inclined channels has been studied by many authors, and there exists an extensive literature on the subject, see Dressler (1949), Stoker (1957), Ven Te Chow (1959) and Henderson (1969), and the references quoted there. A variety of wave motions occurs in connection with flow in rivers and open channels, such as roll waves, flood waves and tidal bores. While these phenomena are thoroughly investigated, not so much attention has been devoted to the stability problems. In his book, Whitham (1974) discusses the stability properties of the uniform steady flow, using his own approach. The basic equations have, however, a great variety of nonuniform steady solutions. One of the aims of this paper is to analyse the stability properties of these classes of solutions.

We base our analysis upon the nonlinear equations of shallow water theory, augmented by a frictional force due to the resistance against the channel walls. Following Whitham (1974), we consider very broad channels where the frictional term according to the Chezy formula is given by

$$- c_f \frac{v|v|}{h} \qquad (1.1)$$

where c_f is the friction coefficient, v the mean velocity and h the local depth of the channel. Restricted to broad channels we can assume the motion to be two-dimensional, i.e. the actual flow is well approximated by a flow with uniform velocity over each cross-section.

The Chezy formula needs some additional comments. As pointed out by Dressler (1949) this formula is strictly speaking valid only for uniform flows, and it will not be accurate unless such flows vary slowly with respect to x (position) and t (time). In the present paper the mean velocity v is, with one exception, assumed to be positive such that the velocity term $v|v| = v^2$ in (1.1). The exceptional case will be commented on separately.

The basic equations of the fluid motions are treated in section 2. In the following section we have studied the steady state solutions and their asymptotic behaviour. The stability properties of the uniform basic flow is thoroughly discussed by using both temporally growing modes (section 4) and spatially growing modes (section 6). Attention is also devoted to the dispersion and damping of these modes. The stability of non-uniform basic flows is studied by two different methods, a generalized progressing wave expansion method (section 5) and the normal mode method based on spatial modes (section 6).

Our model needs a final remark. In the basic equations (2.1) the pressure is assumed to satisfy a hydrostatic law. This approximation is physically realistic only for relatively long waves. We have, however, analysed the stability properties of the uniform basic flow for all wavelengths and it is shown that the stability condition is independent of the wavelengths. On the other hand, propagation velocities of normal modes, dispersion and damping obtained for short waves are not expected to give sufficiently accurate results.

2. EQUATIONS OF FLUID MOTION

We consider flow in a broad rectangular channel of constant inclination α and work with the depth h and mean velocity v as basic variables.

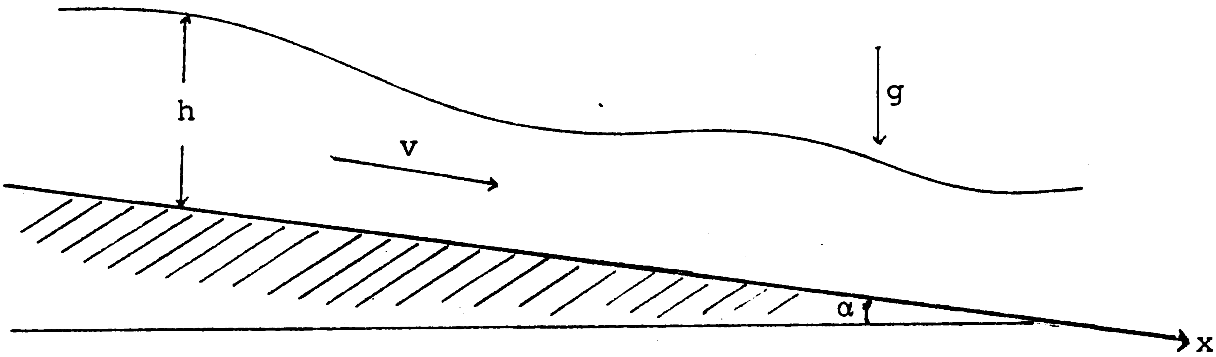

Fig. 1 Inclined channel

The basic equations governing the flow are

$$\frac{\partial h}{\partial t} + v\frac{\partial h}{\partial x} + h\frac{\partial v}{\partial x} = 0 \qquad (2.1\ a)$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + g^*\frac{\partial h}{\partial x} = g^*S - c_f\frac{v^2}{h} \qquad (2.1\ b)$$

where $S = \tan\alpha$, $g^* = g\cos\alpha$, g denotes the gravity acceleration, c_f the friction coefficient, x the distance along the channel and t denotes the time. Here (2.1 a) is the equation of continuity and (2.1 b) is the momentum equation (see Whitham (1974), pp. 80-87).

Let $h = h_0(x)$ and $v = v_0(x)$ be a steady state solution of (2.1). In order to study the stability properties of this basic flow we introduce the perturbation by

$$\begin{aligned} h &= h_0 + \eta \\ v &= v_0 + w \end{aligned} \qquad (2.2)$$

where η and w represent small disturbances superimposed on the basic flow. By substituting (2.2) into (2.1), the linearized equations for the perturbations are found to be

$$\frac{\partial \eta}{\partial t} + v_0 \frac{\partial \eta}{\partial x} + h_0 \frac{\partial w}{\partial x} + v_0' \eta + h_0' w = 0 \qquad (2.3\ a)$$

$$\frac{\partial w}{\partial t} + g^* \frac{\partial \eta}{\partial x} + v_0 \frac{\partial w}{\partial x} - c_f \frac{v_0^2}{h_0^2} \eta + [v_0' + 2c_f \frac{v_0}{h_0}] w = 0 \qquad (2.3\ b)$$

where a prime denotes differentiation with respect to x. Here we have used the fact that $v_0 h_0' + h_0 v_0' = 0$ which follows from (2.1 a).

3. ASYMPTOTIC BEHAVIOUR OF STEADY STATE SOLUTIONS

The basic equations governing the steady channel flow $h = h_0(x)$, $v = v_0(x)$ of (2.1) are

$$v_0 h_0' + h_0 v_0' = 0 \qquad (3.1\text{ a})$$

$$v_0 v_0' + g^* h_0' = g^* S - c_f \frac{v_0^2}{h_0} \qquad (3.1\text{ b})$$

We need to study the asymptotic behaviour of the solution of (3.1) corresponding to given values at $x = 0$:

$$h_0(0) = h_0^* , \quad v_0(0) = v_0^* \qquad (3.2)$$

It is suitable to keep $v_0^* > 0$ fixed and discuss the solution for different values of $h_0^* > 0$.

From (3.1) and (3.2) it follows that $h_0 v_0 = h_0^* v_0^*$, i.e.

$$v_0 = \frac{h_0^* v_0^*}{h_0} \qquad (3.3)$$

Substituting (3.3) into (3.1 b) we obtain the following equation for h_0

$$h_0' = S \frac{h_0^3 - \bar{h}_0^3}{h_0^3 - h_c^3} \qquad (3.4)$$

where the <u>normal depth</u> $\bar{h}_0$ and the <u>critical depth</u> h_c are given by

$$\bar{h}_0 = \left\{\frac{c_f}{S} \frac{(h_0^* v_0^*)^2}{g^*}\right\}^{1/3} , \quad h_c = \left\{\frac{(h_0^* v_0^*)^2}{g^*}\right\}^{1/3} \qquad (3.5)$$

From (3.4) it follows that we have to choose $h_0^* \neq h_c$ if $S \neq c_f$. In general no continuous solution corresponds to the critical depth.

If we choose $h_0^* = \bar{h}_0$, (3.1) has the unique constant solution

$$h_0 \equiv \bar{h}_0 \ , \quad v_0 \equiv \frac{h_0^* v_0^*}{\bar{h}_0} = \bar{v}_0 \tag{3.6}$$

where

$$c_f \frac{\bar{v}_0^2}{\bar{h}_0} = g^* S \ . \tag{3.7}$$

If $h_0^* \neq \bar{h}_0$ (3.1) has the unique solution given implicity by

$$h_0 + \frac{1}{6} \bar{h}_0 [\frac{S}{c_f} - 1] \{ \ln \frac{h_0^2 + h_0 \bar{h}_0 + \bar{h}_0^2}{(h_0 - \bar{h}_0)^2} + 2\sqrt{3} \text{ Arctan } \frac{2}{\sqrt{3}} (\frac{h_0}{\bar{h}_0} + \frac{1}{2}) \}$$

$$= Sx + c_0 \text{ (const.)} \tag{3.8 a}$$

$$v_0 = \frac{h_0^* v_0^*}{h_0} \tag{3.8 b}$$

The non-uniform solution given by (3.8) has different asymptotic behaviour as $S > c_f$ (steep slope), $S = c_f$ (critical slope) or $S < c_f$ (mild slope). Thus we distinguish between these three cases:

(i) $S > c_f$, i.e. $\bar{v}_0 > \sqrt{g^* \bar{h}_0}$ (See figure 2, p. 9)

(a) $h_0^* < \bar{h}_0$.

(3.4) and (3.8) then imply that $h_0' > 0$, $h_0' \to 0$, $h_0 \to \bar{h}_0$ and $v_0 \to \bar{v}_0$ as $x \to \infty$, where $\bar{h}_0$ and $\bar{v}_0$ are determined by (3.7).

(b) $\bar{h}_0 < h_0^* < h_c$.

In this case $h_0' < 0$, $h_0' \to 0$, $h_0' \to \bar{h}_0$ and $v_0 \to \bar{v}_0$ as $x \to \infty$.

(c) $h_0^* > h_c$.

Then $h_0' > S$, $h_0' \to S$, $h_0 \to \infty$ and $v_0 \to 0$ as $x \to \infty$. It is easily seen that h_0 tends asymptotically to a straight line with slope S (in the x, h_0-plane) as $x \to \infty$. This means that the flow surface tends to a horizontal level.

(ii) $S = c_f$, i.e. $\bar{v}_0 = \sqrt{g^* \bar{h}_0}$ (see figure 3, p. 9)

In this case the solution is given by

$$h_0 = Sx + h_0^* \quad , \qquad v_0 = \frac{v_0^* h^*}{Sx+h^*}$$

which means that the flow surface is horizontal.

(iii) $S < c_f$, i.e. $\bar{v}_0 < \sqrt{g^* \bar{h}_0}$ (see figure 4, p. 9)

(a) $h_0^* < h_c$.

In this case $h_0' > 0$ and $h_0' \to +\infty$ on a finite x-interval, which indicates that the surface profile will be vertical in crossing the critical depth line. The continuous solution breaks down at this point and a hydraulic jump may occur. It should be noted that near the critical depth line the flow may become so curvilinear or rapidly varied that our basic equations are not valid. Using the Chezy formula, we have assumed the flow to vary slowly with respect to x and t .

(b) $h_c < h_0^* < \bar{h}_0$.

Then $h_0' < 0$ and $h_0' \to -\infty$ on a finite x-interval, which again indicates that the surface profile will be vertical in crossing the critical depth line. The same arguments are valid as in case (a) above.

(c) $h_0^* > \bar{h}_0$.

In this case $0 < h_0' < S$, $h_0' \to S$, $h_0 \to \infty$ and $v_0 \to 0$ as $x \to \infty$ which means that h_0 tends asymptotically to a straight line with slope S , i.e. the flow profile tends to a horizontal level.

*

The conclusions on the asymptotic behaviour of the solution (3.8) may be more easily seen by considering x as a function of h_0. The graphs are sketched in figure 5 and 6 (p. 10).

The depth h_0 sketched as a function of the position x in three different cases:

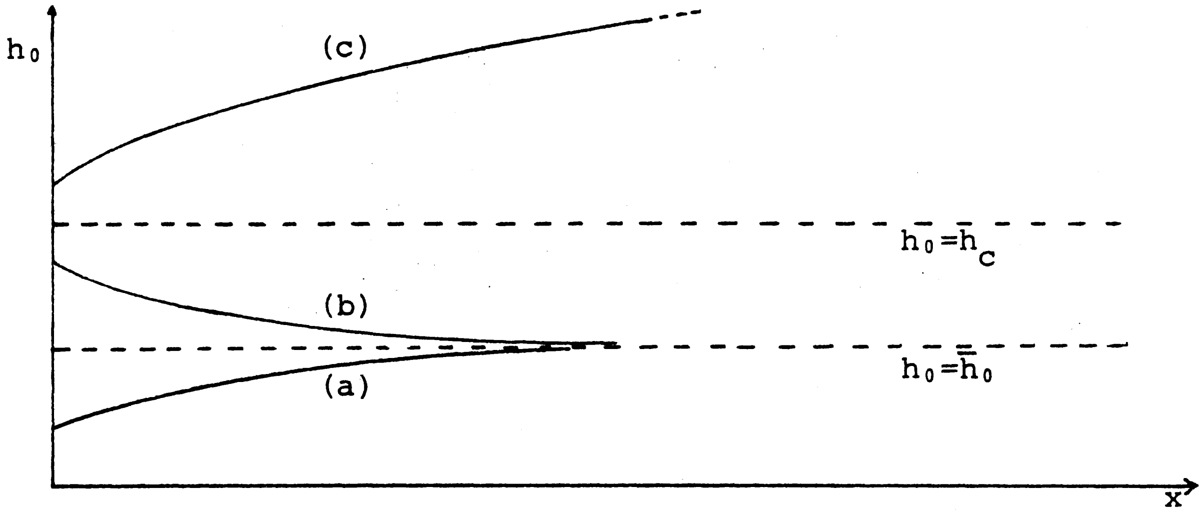

Figure 2. Steep slope: $S > c_f$, i.e. $\bar{v}_0 > \sqrt{g^*\bar{h}_0}$

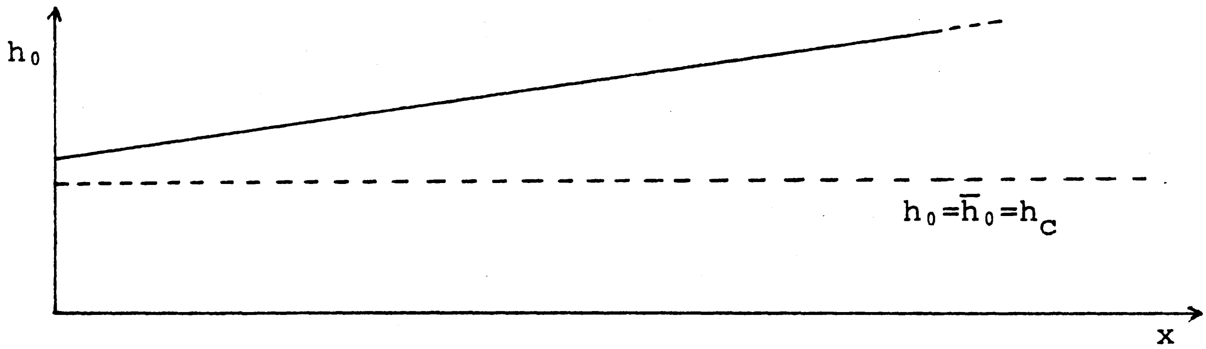

Figure 3. Critical slope: $S = c_f$, i.e. $\bar{v}_0 = \sqrt{g^*\bar{h}_0}$

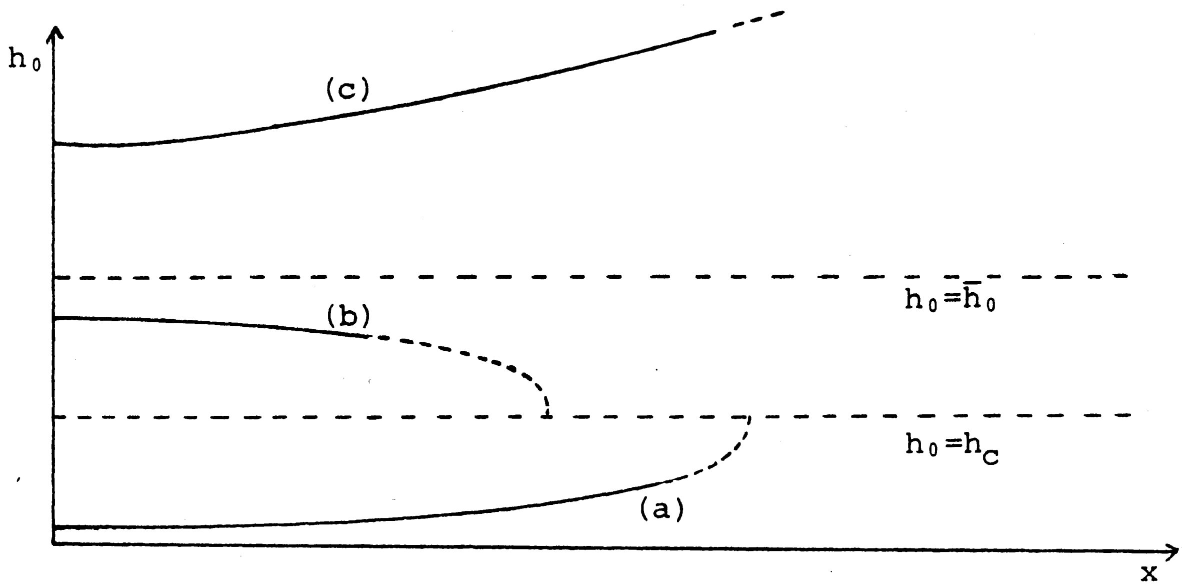

Figure 4. Mild slope: $S < c_f$, i.e. $\bar{v}_0 < \sqrt{g^*\bar{h}_0}$

The position x sketched as a function of the depth h_0 .

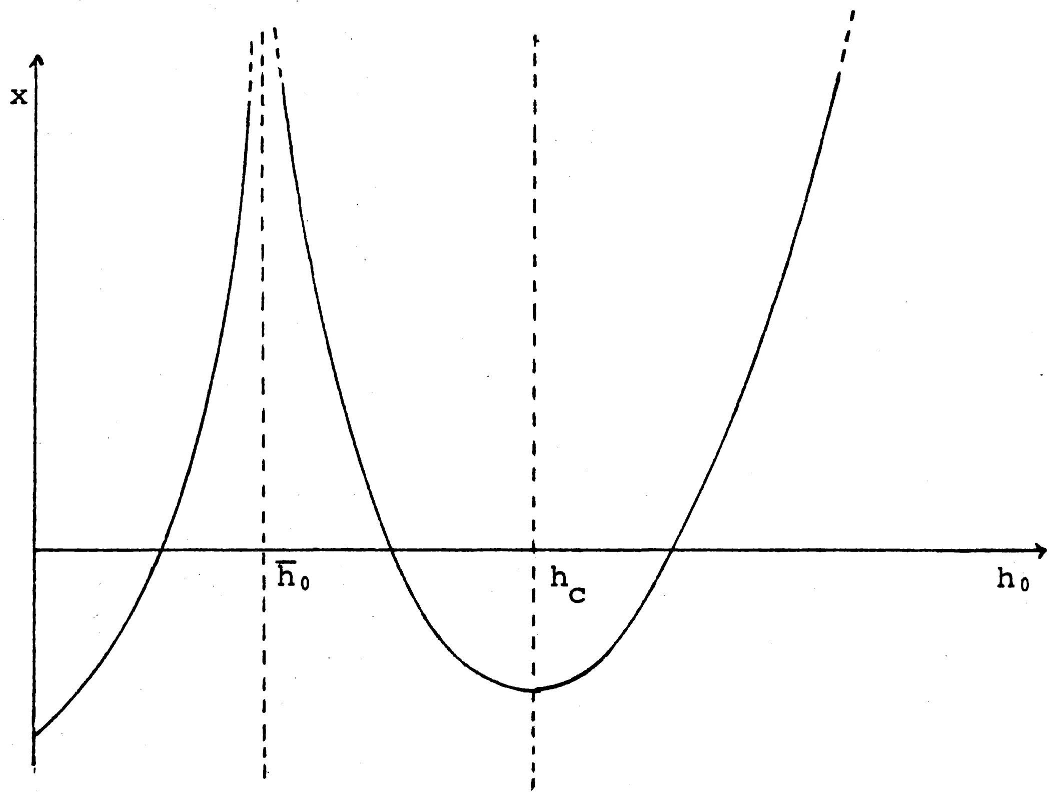

Figure 5. Steep slope

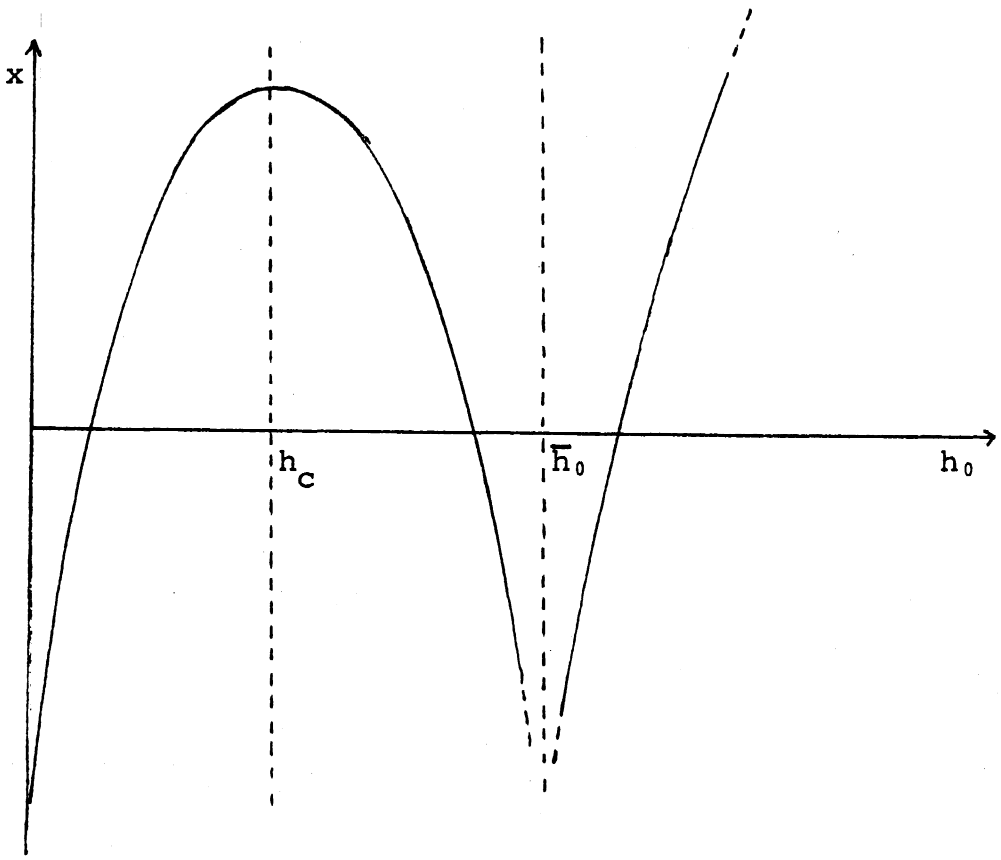

Figure 6. Mild slope

4. STABILITY ANALYSIS OF THE UNIFORM BASIC FLOW BY THE NORMAL MODE METHOD. TEMPORALLY GROWING MODES.

In this section we will thoroughly analyse the stability properties of the uniform basic flow given by (3.6). This case is easily studied by the normal mode method. Assuming the channel to be infinite in the x-direction, there is no appropriate boundary conditions to satisfy. We suppose that the perturbations η and w of (2.2) have the form:

$$\eta = \eta_0 e^{i(kx-\omega t)} \quad , \quad w = w_0 e^{i(kx-\omega t)} \tag{4.1}$$

where the real parts of η and w represent the physical quantities. In (4.1) the wave number k is taken as real whereas we seek complex frequencies ω to determine the temporally growing modes. If there exists modes where $Im(\omega) > 0$, the uniform basic flow is judged unstable. By this method the basic flow is disturbed slightly at $t = 0$ by periodic wave trains in space.

By introducing (3.6) and (4.1) into (2.3) we get the algebraic equations:

$$\begin{aligned} & i(k\overline{v}_0-\omega)\eta_0 + i\,k\,\overline{h}_0 w_0 = 0 \\ & i(kg^* + ic_f \frac{\overline{v}_0^{\,2}}{\overline{h}_0^{\,2}})\eta_0 + i(k\overline{v}_0-\omega-i2c_f \frac{\overline{v}_0}{\overline{h}_0})w_0 = 0 \end{aligned} \tag{4.2}$$

which give the dispersion relation

$$(\omega-k\overline{v}_0)^2 - k^2 g^* \overline{h}_0 = -\,i\,c_f \frac{\overline{v}_0}{\overline{h}_0}(2\omega-3k\overline{v}_0) \tag{4.3}$$

The real and imaginary parts of (4.3) give the equations

$$\omega_i^2 + 2c_f \frac{\overline{v}_0}{\overline{h}_0}\,\omega_i - k^2\{(c-\overline{v}_0)^2 - g^*\overline{h}_0\} = 0 \tag{4.4}$$

and

$$\omega_i = - c_f \frac{\bar{v}_0}{\bar{h}_0} \frac{c-\frac{3}{2}\bar{v}_0}{c-\bar{v}_0} \tag{4.5}$$

where $\omega_r = \mathrm{Re}(\omega)$, $\omega_i = \mathrm{Im}(\omega)$ and the phase velocity $c = \frac{\omega_r}{k}$. Substitution of (4.5) into (4.4) leads to the following equation for c :

$$c_f \frac{\bar{v}_0^2}{\bar{h}_0^2} \frac{(c-\bar{v}_0)^2-\frac{1}{4}\bar{v}_0^2}{(c-\bar{v}_0)^2} + k^2\{(c-\bar{v}_0)^2-g^*\bar{h}_0\} = 0 \tag{4.6}$$

(i) <u>The stability discussion</u>

The perturbations (4.1) are stable provided $\omega_i \leq 0$ for both of the roots ω of (4.3). This condition is satisfied (see (4.5)) when

$$\frac{c-\frac{3}{2}\bar{v}_0}{c-\bar{v}_0} \geq 0 \tag{4.7}$$

for both of the phase velocities c , which are the real roots of (4.6). In order to find the stability criterion we have to investigate these roots. Equation (4.6) implies that

$$(c-\bar{v}_0)^2 = \frac{1}{2}\left[g^*\bar{h}_0-\left(\frac{c_f\bar{v}_0}{k\ \bar{h}_0}\right)^2 \pm \sqrt{\{g^*\bar{h}_0-\left(\frac{c_f\bar{v}_0}{k\ \bar{h}_0}\right)^2\}^2 + \left(\frac{c_f\bar{v}_0^2}{k\ \bar{h}_0}\right)^2}\ \right] \tag{4.8}$$

It is now easily seen that (4.6) has two real roots , $c_+ > \bar{v}_0$ and $c_- < \bar{v}_0$, corresponding to waves travelling in opposite directions relative to the basic flow. Let ω_i^+ and ω_i^- denote the damping factors (roots of (4.4)) associated with these waves.

It follows immediately that c_- satisfies the condition (4.7), i.e. $\omega_i^- < 0$ and thus the c_--wave is always stable. The c_+-wave, however, is stable if and only if

$$c_+ \geq \frac{3}{2}\bar{v}_0 \tag{4.9}$$

Since $\omega_i^- < 0$ it also follows from (4.4) that $\omega_i^+ \leq 0$ (stability) if and only if

$$(c_+ - \bar{v}_0)^2 \leq g^* \bar{h}_0$$

or, equivalently

$$\bar{v}_0 - \sqrt{g^* \bar{h}_0} \leq c_+ \leq \bar{v}_0 + \sqrt{g^* \bar{h}_0} \; . \tag{4.10}$$

In order to avoid a contradiction we have to conclude: If $\frac{3}{2}\,\bar{v}_0 \leq \bar{v}_0 + \sqrt{g^* \bar{h}_0}$, it follows that $\frac{3}{2}\,\bar{v}_0 \leq c_+ \leq \bar{v}_0 + \sqrt{g^* \bar{h}_0}$. On the other hand, if $\bar{v}_0 + \sqrt{g^* \bar{h}_0} \leq \frac{3}{2}\,\bar{v}_0$ it follows that $\bar{v}_0 + \sqrt{g^* \bar{h}_0} \leq c_+ \leq \frac{3}{2}\,\bar{v}_0$. Consequently, condition (4.7) is satisfied if and only if

$$\frac{3}{2}\,\bar{v}_0 \leq \bar{v}_0 + \sqrt{g^* \bar{h}_0}$$

or, equivalently

$$\bar{v}_0 \leq 2\sqrt{g^* \bar{h}_0} \; . \tag{4.11}$$

Using (3.7), this stability condition may also be written

$$S \leq 4\, c_f \; . \tag{4.12}$$

Thus the condition (4.12), being independent of the wave number, is necessary and sufficient for the perturbations (4.1) to be stable. If the basic flow (3.6) does not satisfy (4.12), exponential instabilities arise. On the other hand, it is shown by Dressler (1949) that $S > 4\, c_f$ is a necessary condition for the formation of roll waves.

Concerning the uniform basic flow we have established the following result

(4,I) In order for the basic flow (3.6) to be stable it is necessary that

$$S \leq 4\, c_f \; .$$

This result is consistent with that obtained by Whitman (1974).

A closer examination of the phase velocities as functions of the wave number k shows:

$$c_+ \to \tfrac{3}{2}\,\bar{v}_0 \quad \text{and} \quad \frac{dc_+}{dk} \to 0$$

$$c_- \to \tfrac{1}{2}\,\bar{v}_0 \quad \text{and} \quad \frac{dc_-}{dk} \to 0 \qquad \text{as} \quad k \to 0 \; ,$$

and

$$c_+ \to \bar{v}_0 + \sqrt{g^*\bar{h}_0} \quad \text{and} \quad \frac{dc_+}{dk} \to 0$$

$$c_- \to \bar{v}_0 - \sqrt{g^*\bar{h}_0} \quad \text{and} \quad \frac{dc_-}{dk} \to 0 \qquad \text{as} \quad k \to \infty \; .$$

Furthermore, both of the functions are monotonic.
Here we have used the relation

$$\frac{dc}{dk} = - \frac{\{(c-\bar{v}_0)^2 - g^*\bar{h}_0\}(c-\bar{v}_0)k}{\{2(c-\bar{v}_0)^2 - g^*h_0\}k^2 + c_f\left(\frac{\bar{v}}{\bar{h}_0}\right)^2} \tag{4.13}$$

which is easily derived from (4.6).

In the figures 7, 8, 9 and 10 (p. 17 & 18) c_+ and c_- are sketched as functions of k in the mild slope case $\bar{v}_0 < \sqrt{g^*\bar{h}_0}$ and the steep slope cases $\sqrt{g^*\bar{h}_0} < \bar{v}_0 < 2\sqrt{g^*\bar{h}_0}$, $\bar{v}_0 = 2\sqrt{g^*\bar{h}_0}$ (non-dispersive waves) and $\bar{v}_0 > \sqrt{g^*\bar{h}_0}$.

(ii) Some special cases

It is also of interest to discuss the dispersion relation, waves modes and damping in three special cases: Non-dispersive waves, long wave approximation and short wave approximation.

(a) Non-dispersive waves ($v_0 = 2\sqrt{g^*\bar{h}_0}$, see figure 9):

From (4.6) it follows that the waves are non-dispersive if and only if

$$(c-\bar{v}_0)^2 - g^*\bar{h}_0 = 0$$

which implies

$$c_+ = \bar{v}_0 + \sqrt{g^*\bar{h}_0} = \frac{3}{2}\bar{v}_0 , \quad \omega_i^+ = 0 \tag{4.14}$$

$$c_- = \bar{v}_0 - \sqrt{g^*\bar{h}_0} = \frac{1}{2}\bar{v}_0 , \quad \omega_i^- = -2 c_f \frac{\bar{v}_0}{\bar{h}_0} . \tag{4.15}$$

The wave associated with c_+ is undamped in this case, whereas the c_--wave is damped. As expected, both are stable.

(b) Long wave approximation:

As pointed out, the basic equations (2.1) are derived for shallow water theory, and are therefore especially appropriate for long waves. In the extremely long wave region $k \to 0$, (4.6) requires that

$$(c-\bar{v}_0)^2 \to \frac{1}{4}\bar{v}_0^2$$

which implies

$$c_+ \to \frac{3}{2}\bar{v}_0 , \quad \omega_i^+ \to 0 , \tag{4.16}$$

and

$$c_- \to \frac{1}{2}\bar{v}_0 , \quad \omega_i^- \to -2 c_f \frac{\bar{v}_0}{\bar{h}_0} . \tag{4.17}$$

As expected, the c_--wave is always stable, whereas the c_+-wave is stable if and only if $\bar{v}_0 \leq 2\sqrt{g^*\bar{h}_0}$.

Furthermore

$$\omega_i^+ \to 0- \quad \text{if} \quad \bar{v}_0 < 2\sqrt{g^*\bar{h}_0} \quad \text{and} \quad k \to 0 ,$$

$$\omega_i^+ \to 0+ \quad \text{if} \quad \bar{v}_0 > 2\sqrt{g^*\bar{h}_0} \quad \text{and} \quad k \to 0 .$$

(c) Short wave approximation:

We will also examine our model for short waves in order to compare our results to those obtained by the method used in section 5. In the extreme short wave region $k \rightarrow \infty$, (4.6) requires that

$$(c-\bar{v}_0)^2 \rightarrow g^*\bar{h}_0$$

which implies

$$c_+ \rightarrow \bar{v}_0 + \sqrt{g^*\bar{h}_0}\ , \qquad \omega_i^+ \rightarrow - c_f \frac{\bar{v}_0}{\bar{h}_0}\left(1 - \frac{\bar{v}_0}{2\sqrt{g^*\bar{h}_0}}\right) \tag{4.18}$$

$$c_- \rightarrow \bar{v}_0 - \sqrt{g^*h_0}\ , \qquad \omega_i^- \rightarrow - c_f \frac{\bar{v}_0}{\bar{h}_0}\left(1 + \frac{\bar{v}_0}{2\sqrt{g^*\bar{h}_0}}\right). \tag{4.19}$$

The conclusions on stability are the same as for long waves, but the c_+-wave is in general more strongly damped or amplified. For long waves the damping is nearly independent of gravity g, for short waves, however, gravity may play a dominant role in damping.

As pointed out in the introduction, our model is a long wave approximation and propagation velocities, dispersion and damping obtained for short waves are not expected to give sufficiently accurate results.

The phase velocities c_+ and c_- sketched as functions of the wave number k in four different cases:

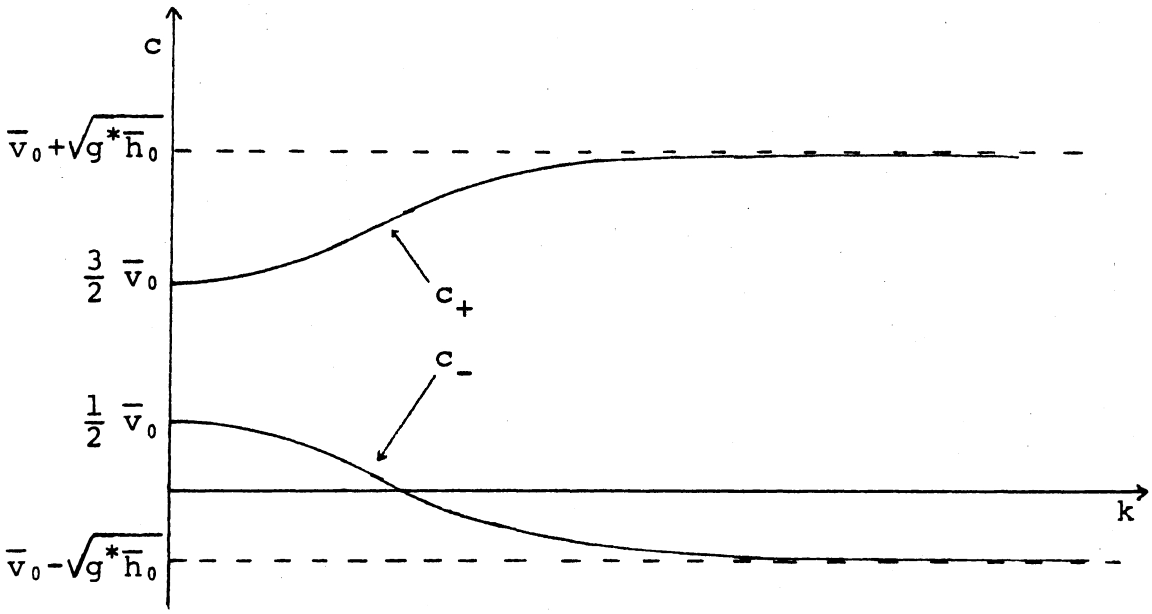

Figure 7: $\bar{v}_0 < \sqrt{g^*\bar{h}_0}$

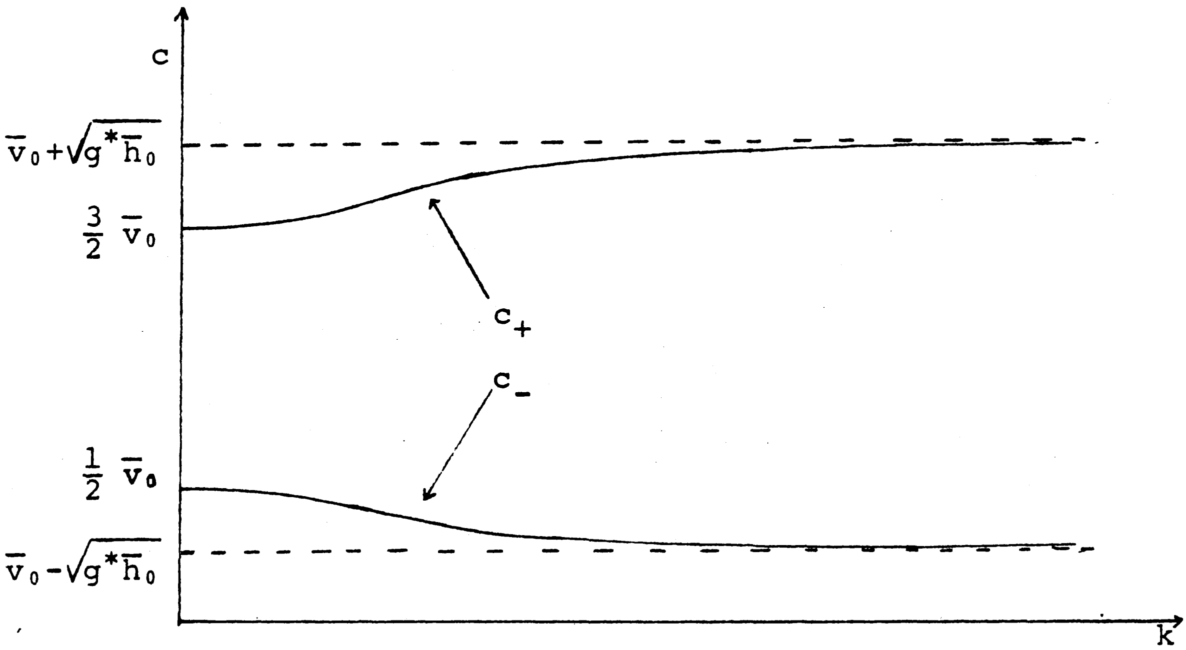

Figure 8: $\sqrt{g^*\bar{h}_0} < \bar{v}_0 < 2\sqrt{g^*\bar{h}_0}$

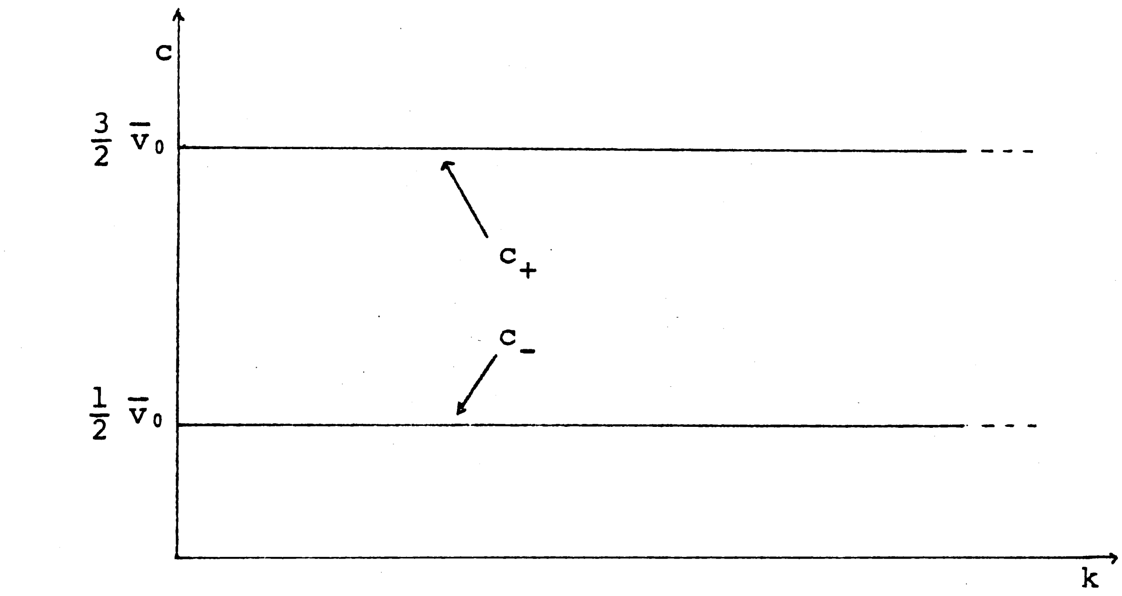

Figure 9: $\bar{v}_0 = 2\sqrt{g^*\bar{h}_0}$ (non-dispersive waves)

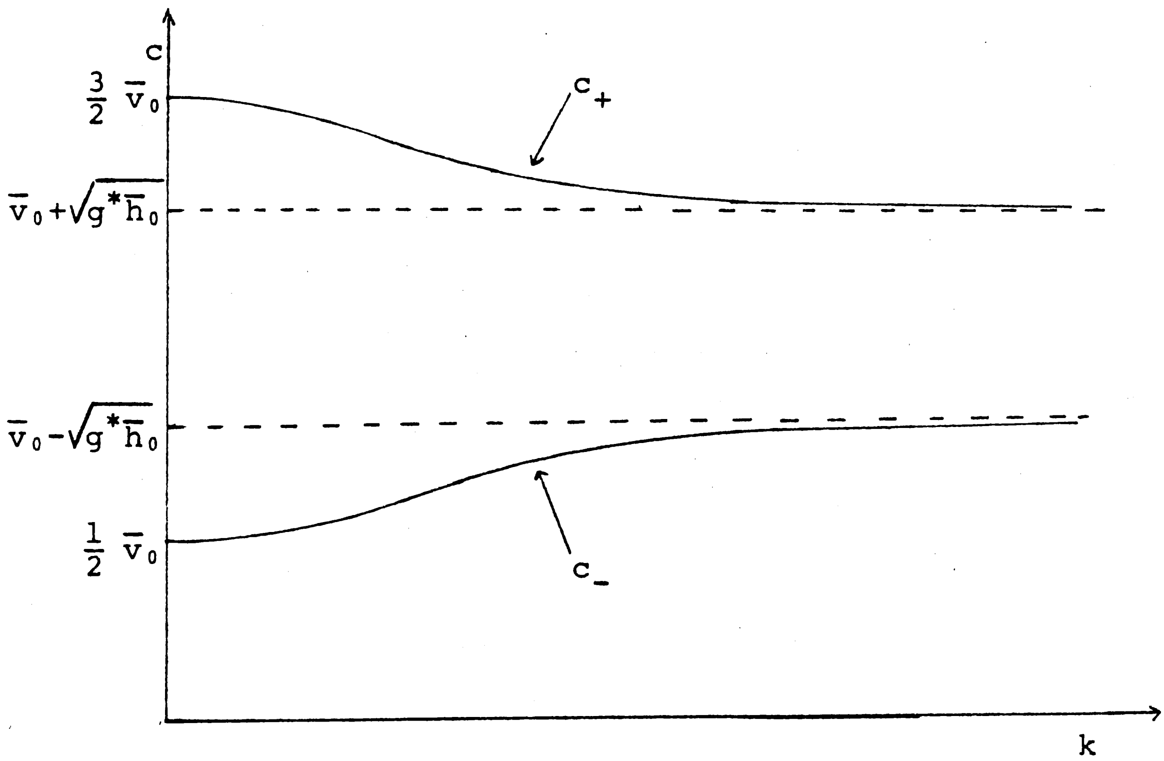

Figure 10: $\bar{v}_0 > 2\sqrt{g^*\bar{h}_0}$

5. STABILITY ANALYSIS OF STEADY FLOWS BY ECKHOFF'S METHOD

We are now going to investigate the stability properties of the steady flows given by (3.6) and (3.8). Our analysis is based on Eckhoff's method (see Eckhoff 1981), which is especially developed for stability investigations of short wave disturbances. Thus it may seem suspect to apply this method to a model which is a long wave approximation. However, the discussion in the previous section concerning uniform basic flow shows that the stability condition (4.12) is independent of the wavelengths. This fact justifies the use of Eckhoff's method to our problem.

When applying this method we have to transform the hyperbolic system (2.3) into a symmetric form. We introduce new variables ζ_1 and ζ_2 by

$$\eta = \frac{1}{2} h_0(\zeta_1 + \zeta_2)$$
$$w = \frac{1}{2} \sqrt{g^* h_0}(\zeta_1 - \zeta_2) \tag{5.1}$$

which transform (2.3) into the symmetric hyperbolic system

$$\frac{\partial \underline{w}}{\partial t} + A \frac{\partial \underline{w}}{\partial x} + B \underline{w} = 0 \tag{5.2}$$

where the vector variable $\underline{w}$ and the coefficient matrices A and B are

$$\underline{w} = \begin{Bmatrix} \zeta_1 \\ \zeta_2 \end{Bmatrix} , \quad A = \begin{Bmatrix} v_0 + \sqrt{g^* h_0} & 0 \\ 0 & v_0 - \sqrt{g^* h_0} \end{Bmatrix} \tag{5.3}$$

$$B = \frac{1}{4} \frac{h_0'}{h_0} \sqrt{g^* h_0} \begin{Bmatrix} -F+5 & F-1 \\ F+1 & -F-5 \end{Bmatrix} + \frac{1}{2} c_f \frac{v_0}{h_0} \begin{Bmatrix} -F+2 & -F-2 \\ F-2 & F+2 \end{Bmatrix} \tag{5.4}$$

Here $F = \dfrac{v_0}{\sqrt{g^* h_0}}$ is the local Froude number.

The characteristic equation associated with (5.2) is

$$\det\{\xi A-\lambda I\} = [(v_0+\sqrt{g^*h_0})\xi-\lambda][(v_0-\sqrt{g^*h_0})\xi-\lambda] = 0 \tag{5.5}$$

Thus the characteristic roots are seen to be

$$\Omega_1 = (v_0+\sqrt{g^*h_0})\xi \ , \qquad \Omega_2 = (v_0-\sqrt{g^*h_0})\xi \tag{5.6}$$

and the associated eigenvectors may be chosen

$$\underline{r}_1 = (1,0) \quad , \quad \underline{r}_2 = (0,1) \ . \tag{5.7}$$

These roots correspond to waves travelling with speeds $v_0 + \sqrt{g^*h_0}$ and $v_0 - \sqrt{g^*h_0}$ respectively. The ray equation and the stability equation (see Eckhoff (1981), eq. (5.2) and (6.15)) corresponding to Ω_1 are found to be

$$\frac{dx}{dt} = v_0 + \sqrt{g^*h_0} \tag{5.8}$$

$$\frac{dP}{dt} = -\{\frac{h_0'}{h_0}\sqrt{g^*h_0}\left(\frac{v_0}{4\sqrt{g^*h_0}}+1\right)+c_f\,\frac{v_0}{h_0}\left(-\frac{v_0}{2\sqrt{g^*h_0}}+1\right)\}P \tag{5.9}$$

and the equations corresponding to Ω_2 are

$$\frac{dx}{dt} = v_0 - \sqrt{g^*h_0} \tag{5.10}$$

$$\frac{dP}{dt} = -\{\frac{h_0'}{h_0}\sqrt{g^*h_0}\left(\frac{v_0}{4\sqrt{g^*h_0}}-1\right) + c_f\,\frac{v_0}{h_0}\left(\frac{v_0}{2\sqrt{g^*h_0}}+1\right)\}P \tag{5.11}$$

These equations are the basis for our stability analysis.

(i) <u>Waves associated with the Ω_1-root</u>

Let us first discuss the stability properties of the waves corresponding to the characteristic root Ω_1. The stability equation (5.9) is valid along the ray determined by (5.8). Substituting the solution $x = x(t)$ of (5.8) (satisfying $x = x_0$ at $t = 0$) into h_0 , v_0 and h_0' of (5.9), we obtain a linear ordinary differential equation

$$\frac{dP}{dt} = a(t)P \ . \tag{5.13}$$

It follows from the theory of Eckhoff (1981) that the basic flows given by (3.6) or (3.8) cannot be stable unless the trivial solution $P = 0$ of (5.13) is stable.

The basic flow corresponding to the constant solution (3.6) is easily handled. In this case (5.13) is autonomous, and the constant coefficient is

$$\bar{a} = - c_f \frac{\bar{v}_0}{\bar{h}_0}\left(-\frac{\bar{v}_0}{2\sqrt{g^*\bar{h}_0}} + 1\right) . \tag{5.14}$$

The trivial solution $P = 0$ is thus stable if and only if $\bar{a} \leq 0$, i.e.

$$\bar{v}_0 \leq 2\sqrt{g^*\bar{h}_0} \tag{5.15}$$

or, equivalently

$$S \leq 4c_f . \tag{5.16}$$

Consequently (5.16) is a necessary condition for the uniform basic flow (3.6) to be stable, which agrees with the result found in the previous section.

The class of basic flows corresponding to (3.8) is more difficult to handle since we are not able to find the solutions of (5.8) and (3.8) explicitly. It is, however, sufficient for us to know the asymptotic behaviour of these solutions as $x \to \infty$. The two different cases $h_0 \to \bar{h}_0$ and $h_0 \to \infty$ are both accessible to stability analysis:

(α) <u>Cases where $h_0(x) \to \bar{h}_0$</u> :

These cases include the solutions of (3.8) when $S > c_f$ (steep slope) and $0 < h_0^* < h_c$.

Since $h_0(x) \to \bar{h}_0$, $h_0'(x) \to 0$, $v_0(x) \to \bar{v}_0$ as $x \to \infty$, and $x \to \infty$ as $t \to \infty$, it follows that

$$\lim_{t\to\infty} a(t) = \bar{a}$$

where $\bar{a}$ is given by (5.14). From standard theory of stability we conclude that if the trivial solution $P = 0$ of (5.13) is to be stable it is necessary that $\bar{a} \leq 0$, i.e. (5.16) has to be satisfied

(β) <u>Cases where $h_0(x) \to \infty$</u> :

These cases include the solutions of (3.8) when

$S > c_f$ and $h_0^* > h_c$, or

$S = c_f$, or

$S < c_f$ and $h_0^* > \bar{h}_0$.

Since $h_0(x) \to \infty$, $h_0'(x) \to S$, $v_0(x) \to 0$ as $x \to \infty$, and $x \to \infty$ as $t \to \infty$, it follows that $a(t) \to 0$ as $t \to \infty$. To handle this case we eliminate t as independent variable in (5.9) by using (5.8) and get the equation

$$\frac{dP}{dx} = b(x)P \tag{5.17}$$

where

$$b(x) = -\frac{1}{v_0+\sqrt{g^*h_0}}\left\{\frac{h_0'}{h_0}\sqrt{g^*h_0}\left(\frac{v_0}{4\sqrt{g^*h_0}}+1\right)+c_f\,\frac{v_0}{h_0}\left(-\frac{v_0}{2\sqrt{g^*h_0}}+1\right)\right\} \tag{5.18}$$

The trivial solution of (5.17) has exactly the same stability properties as the trivial solution of (5.13). Since also $b(x) \to 0$ as $x \to \infty$, we introduce $z = \ln x$ as a new independent variable (see Eckhoff & Storesletten (1978), p. 405). Again the transformed equation maintains the stability properties at the origin. Asymptotically as $z \to \infty$ this transformed equation tends to an autonomous equation with the constant coefficient

$$\bar{b} = \lim_{x\to\infty} x\,b(x) = -1\ . \tag{5.19}$$

Here we have used the fact the $\frac{h_0}{x} \to S$ as $x \to \infty$, which follows from (3.8 a). Thus we conclude from (5.19) that the trivial solution $P = 0$ of (5.13) is stable in this case.

Furthermore, the leading term in the asymptotic expansion of the variables ζ_1 and ζ_2 tends exponentially to zero as $x \to \infty$. Since h_0 grows linearly with x we can conclude that also the leading term of the variables η and w (see 5.1)) tends to zero as $x \to \infty$.

There is an additional point which needs a comment. In this paper we have supposed the undisturbed mean velocity v_0 to be positive. Since $h_0 \to \infty$ in our case, it follows that $v_0 \to 0$ and thus the frictional term in (5.17) may change sign in spite of small disturbances. However, this fact does not alter our conclusions since the frictional term in the transformed equation of (5.17) vanishes when $x \to \infty$.

Consequently no instabilities are detected by our method in cases where $h_0(x) \to +\infty$.

(ii) Waves associated with the Ω_2-root

An analogous discussion of the stability properties of the waves corresponding to the characteristic root Ω_2, i.e. the equations (5.10-11), does not detect any instabilities. The discussion is therefore omitted here.

(iii) Conclusions

In view of the above discussion and the theory in Eckhoff (1981), we have established the following results:

(5,I) In order for the basic flow (3.6) to be stable it is necessary that

$$S \leq 4c_f .$$

(5,II) In order for the basic flow (3.8), when

$$S > c_f \quad \text{and} \quad 0 < h_0^* < h_c ,$$

to be stable it is necessary that

$$S \leq 4c_f .$$

The waves associated with the roots Ω_1 and Ω_2 correspond to the c_+- and c_--wave, respectively, in the short wave approximation studied in the previous section. Where comparable the stability results agree. In particular, no instabilities have been detected in the slowest travelling wave.

6. SPATIALLY GROWING WAVE MODES ON UNIFORM AND NONUNIFORM BASIC FLOWS

In section 4 we analysed the stability properties of the uniform basic flow. This study was concerned with temporally growing or decaying wave solutions which were assumed to be periodic in the distance along the inclined channel. Such solutions do not provide any information about the spatial development of a steady wave train downstream (or upstream) from a wave generator. In experiments where a wave generator is introduced into a basic flow to produce controlled wave disturbances a better model may be spatially growing modes.

(i) The uniform basic flow

The perturbations (4.1) in section 4 were treated as temporal modes where the wave number k was taken as real and the complex frequencies ω were determined from the dispersion relation.

For spatial modes we analyse the perturbations

$$\eta = \eta_0 e^{i(kx-\omega t)} \quad , \quad w = w_0 e^{i(kx-\omega t)} \tag{6.1}$$

where we take a real value of the frequency and seek complex eigenvalues $k = k_r + ik_i$ for the wave number. We judge the flow to be stable against spatial modes (6.1) if for all real ω

$k_i \geq 0$ for waves travelling in the positive x-direction

and

$k_i \leq 0$ for waves travelling in the negative x-direction.

The dispersion relation (4.3) derived in section 4 may be written

$$(\bar{v}_0^2 - g^*\bar{h}_0)k^2 - (2\omega\bar{v}_0 + 3ic_f \frac{\bar{v}_0^2}{\bar{h}_0})k + \omega^2 + 2i\omega c_f \frac{\bar{v}_0}{\bar{h}_0} = 0 \tag{6.2}$$

The real and imaginary parts of (6.2) give the equations

$$(\bar{v}_0^2-g^*\bar{h}_0)k_i^2 - 3c_f\,\frac{\bar{v}_0^2}{\bar{h}_0}k_i - k_r^2\{(c^*-\bar{v}_0)^2-g^*\bar{h}_0\} = 0 \qquad (6.3)$$

and

$$k_i = \frac{c_f}{\bar{h}_0}\,\frac{c^*-\frac{3}{2}\bar{v}_0}{c^*-u_0} \qquad (6.4)$$

where $c^* = \frac{\omega}{k_r}$ is the phase velocity and

$$u_0 = \frac{\bar{v}_0^2-g^*h_0}{\bar{v}_0} = \bar{v}_0 - \frac{g^*\bar{h}_0}{\bar{v}_0} \; . \qquad (6.5)$$

The relation (6.4) introduced into (6.3) leads to the equation

$$\left(\frac{c_f}{\bar{h}_0}\right)^2(2\bar{v}_0^2+g^*h_0)\frac{(c^*-\frac{3}{2}\bar{v}_0)(c^*-u_1)}{(c^*-u_0)^2} + \left(\frac{\omega}{c^*}\right)^2\{(c^*-\bar{v}_0)^2-g^*h_0\} = 0 \qquad (6.6)$$

where

$$u_1 = \frac{3}{2}\,\frac{\bar{v}_0^2-g^*\bar{h}_0}{2\bar{v}_0^2+g^*h_0}\,\bar{v}_0 = \frac{3\bar{v}_0^2}{4\bar{v}_0^2+2g^*\bar{h}_0}\,u_0 \; . \qquad (6.7)$$

The equations (6.3) - (6.6) are a little more complicated than the analogous equations for temporal modes discussed in section 4. It is suitable to start our stability discussion by a brief examination of the phase velocities as functions of the frequency ω .

The equation (6.6) has in general $(\bar{v}_0 \neq \sqrt{g^*\bar{h}_0})$ two real roots $c_+^* > \bar{v}_0$ and $c_-^* < \bar{v}_0$, corresponding to waves travelling in opposite directions relative to the basic flow. Let k_i^+ and k_i^- denote the damping factors (roots of (6.3)) associated with these waves. If $\omega \to 0$ than $c_+^* \to \frac{3}{2}\bar{v}_0$ and $c_-^* \to u_1$, and if $\omega \to \infty$ than $c_+^* \to \bar{v}_0 + \sqrt{g^*\bar{h}_0}$ and $c_-^* \to \bar{v}_0 - \sqrt{g^*\bar{h}_0}$.

Since

$$(\bar{v}_0-\sqrt{g^*\bar{h}_0}) - u_1 = \frac{\frac{1}{2}\bar{v}_0-\sqrt{g^*\bar{h}_0}}{2\bar{v}_0^2+g^*h_0}(\bar{v}_0-\sqrt{g^*\bar{h}_0})^2 \qquad (6.8)$$

it follows that

$$u_1 \geq \bar{v}_0 - \sqrt{g^*\bar{h}_0} \quad \text{and} \quad \tfrac{3}{2}\bar{v}_0 \leq \bar{v}_0 + \sqrt{g^*\bar{h}_0} \quad \text{when} \quad \bar{v}_0 \leq 2\sqrt{g^*\bar{h}_0}$$

and

$$u_1 < \bar{v}_0 - \sqrt{g^*\bar{h}_0} \quad \text{and} \quad \tfrac{3}{2}\bar{v}_0 > \bar{v}_0 + \sqrt{g^*\bar{h}_0} \quad \text{when} \quad \bar{v}_0 > 2\sqrt{g^*\bar{h}_0} \ .$$

In the critical case $\bar{v}_0 = \sqrt{g^*\bar{h}_0}$ there is only one mode since the c_-^*-wave degenerates, see some special cases below (p. 28).

In the figures 11, 12, 13 and 14 (p. 30 and 31) c_+^* and c_-^* are sketched as functions of ω in the mild slope case $\bar{v}_0 < \sqrt{g^*\bar{h}_0}$ and the steep slope cases $\sqrt{g^*\bar{h}_0} < \bar{v}_0 < 2\sqrt{g^*\bar{h}_0}$, $\bar{v}_0 = 2\sqrt{g^*\bar{h}_0}$ (non-dispersive waves) and $\bar{v}_0 > 2\sqrt{g^*\bar{h}_0}$.

From (6.4) and the above discussion it follows immediately that $c_+^* \geq \frac{3}{2}\bar{v}_0$, i.e. $k_i^+ \geq 0$, is satisfied if and only if $\bar{v}_0 \leq 2\sqrt{g^*\bar{h}_0}$. This condition is necessary and sufficient for the c_+^*-wave to be stable since it always travels in the positive x - direction.

The c_-^*-wave, however, travels in the negative or positive x - direction according as $\bar{v}_0 < \sqrt{g^*\bar{h}_0}$ or $\bar{v}_0 > \sqrt{g^*\bar{h}_0}$. Stability requires $k_i^- \leq 0$ or $k_i^- \geq 0$, respectively. It is easily seen that $u_0 < \bar{v}_0 - \sqrt{g^*\bar{h}_0} < c_-^*$ if $\bar{v}_0 < \sqrt{g^*\bar{h}_0}$. On the other hand

$$c_-^* \leq u_1 < u_0 \quad \text{if} \quad \sqrt{g^*\bar{h}_0} < \bar{v}_0 < 2\sqrt{g^*\bar{h}_0}$$

and

$$c_-^* < v_0 - \sqrt{g^*h_0} < u_0 \quad \text{if} \quad \bar{v}_0 > 2\sqrt{g^*\bar{h}_0} \ .$$

Consequently, $k_i^- < 0$ when $\bar{v}_0 < \sqrt{g^*\bar{h}_0}$ and $k_i^- > 0$ when $\bar{v} > \sqrt{g^*\bar{h}_0}$, which implies that the c_-^*-wave is always stable.

In view of the above discussion we have now established the following result:

(6,I) For the uniform basic flow (3.6) to be stable against spatially growing modes (6.1) it is necessary and sufficient that

$$S \leq 4c_f .$$

This result agrees with that obtained for temporally growing modes in section 4. In particular, no instabilities arise in the slowest travelling wave.

Some special cases:

The phase velocities and damping factors in the following four special cases are found to be

(a) The critical case $\bar{v}_0 = \sqrt{g^* \bar{h}_0}$:

In this case the c_-^*-wave degenerates and there is only one mode, which is stable.

$$c^* = c_+^* = \frac{9\alpha^2+4\omega^2}{6\alpha^2+2\omega^2}\,\bar{v}_0 , \quad k_i = k_i^+ = \frac{c_f}{\bar{h}_0}\,\frac{\omega^2}{9\alpha^2+4\omega^2} \tag{6.9}$$

where $\alpha = c_f \dfrac{\bar{v}_0}{\bar{h}_0}$.

(b) Non-dispersive waves ($\bar{v}_0 = 2\sqrt{g^* \bar{h}_0}$, see figure 13):

$$c_+^* = \frac{3}{2}\,\bar{v}_0 , \qquad k_i^+ = 0 \tag{6.10}$$

$$c_-^* = \frac{1}{2}\,\bar{v}_0 , \qquad k_i^- = 4\frac{c_f}{\bar{h}_0} \tag{6.11}$$

(c) Long wave approximation $(\overline{v}_0 \neq \sqrt{g^*\overline{h}_0})$:

If $\omega \rightarrow 0$ it follows that

$$c_+^* \rightarrow \frac{3}{2}\,\overline{v}_0 \,, \qquad k_i^+ \rightarrow 0 \tag{6.12}$$

$$c_-^* \rightarrow \frac{3}{2}\,\frac{\overline{v}_0{}^2 - g^*\overline{h}_0}{2\overline{v}_0{}^2 + g^* h_0}\,\overline{v}_0 \,, \qquad k_i^- \rightarrow 3\frac{c_f}{\overline{h}_0}\,\frac{\overline{v}_0{}^2}{\overline{v}_0{}^2 - g^*\overline{h}_0} \tag{6.13}$$

Near the critical case the c_-^*-wave is strongly damped. Concerning propagating velocity, dispersion and damping of the c_-^*-wave, there is an essential difference for long waves between spatial and temporal modes. On account of the negligible damping or amplification of the c_+^*-wave, there is no corresponding difference for this mode.

(d) Short wave approximation $(\overline{v}_0 \neq \sqrt{g^*\overline{h}_0})$:

If $\omega \rightarrow \infty$ it follows that

$$c_+^* \rightarrow \overline{v}_0 + \sqrt{g^*\overline{h}_0} \,, \quad k_i^+ \rightarrow c_f\,\frac{\overline{v}_0}{\overline{h}_0}\,\frac{1-\overline{v}_0/2\sqrt{g^*\overline{h}_0}}{\overline{v}_0+\sqrt{g^*\overline{h}_0}} \tag{6.14}$$

$$c_-^* \rightarrow \overline{v}_0 - \sqrt{g^*\overline{h}_0} \,, \quad k_i^- \rightarrow c_f\,\frac{\overline{v}_0}{\overline{h}_0}\,\frac{1+\overline{v}_0/2\sqrt{g^*\overline{h}_0}}{\overline{v}_0-\sqrt{g^*\overline{h}_0}} \tag{6.15}$$

The phase velocities c_+^* and c_-^* sketched as functions of frequency ω in four different cases:

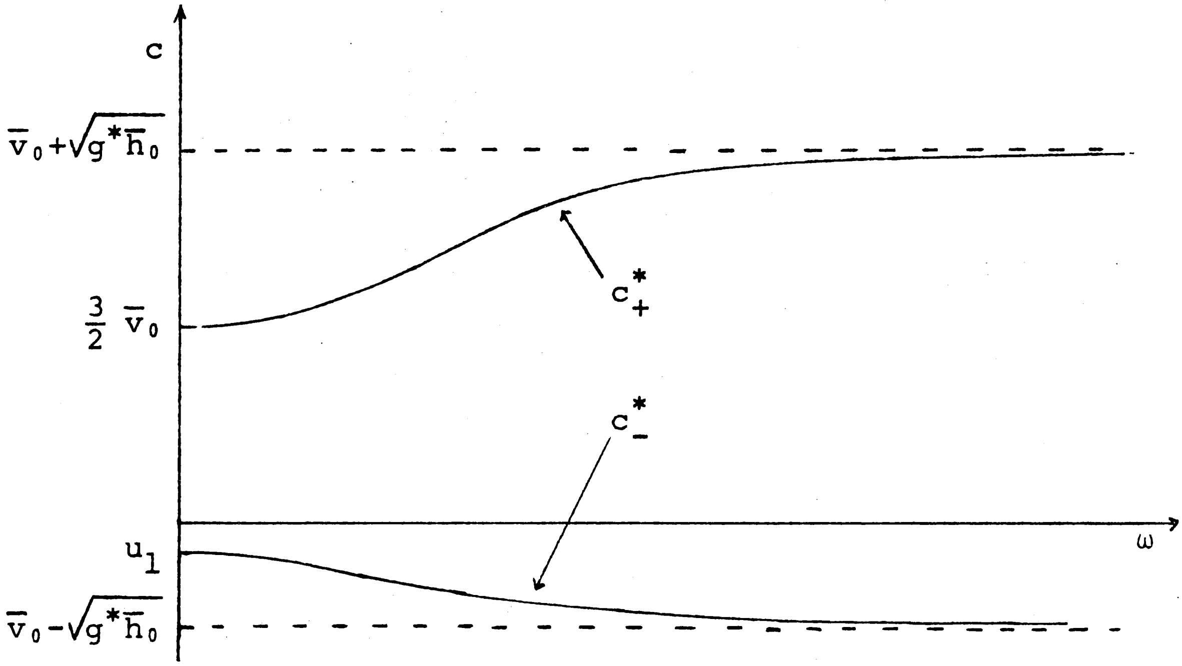

Figure 11. $\bar{v}_0 < \sqrt{g^*\bar{h}_0}$

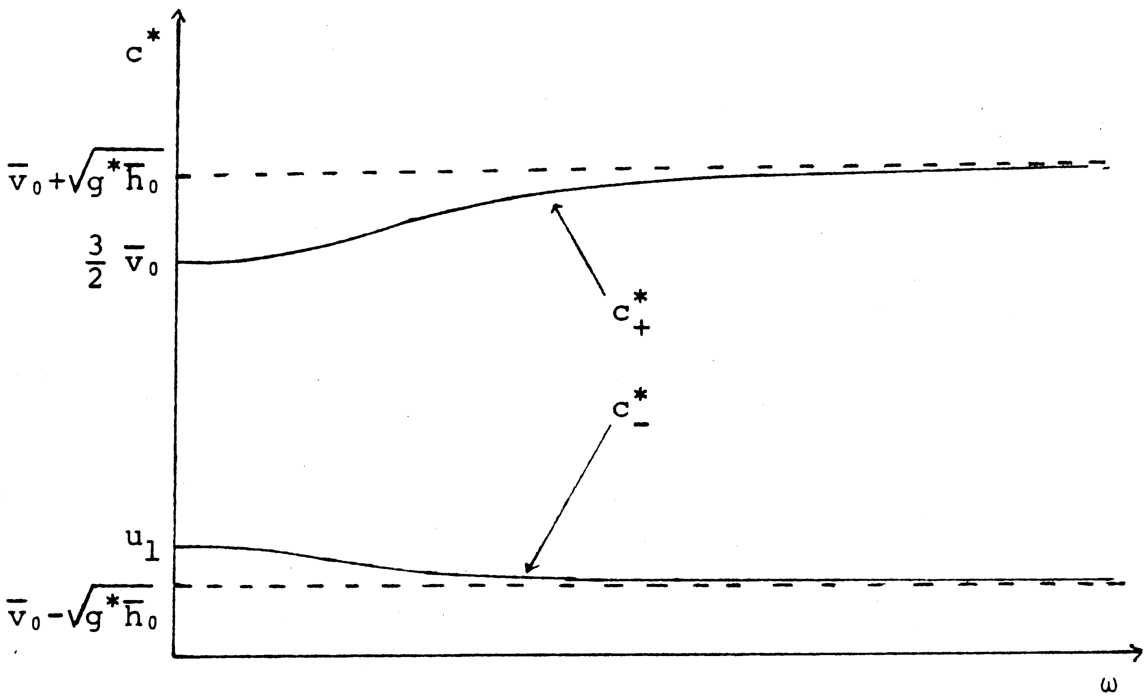

Figure 12. $\sqrt{g^*\bar{h}_0} < \bar{v}_0 < 2\sqrt{g^*\bar{h}_0}$

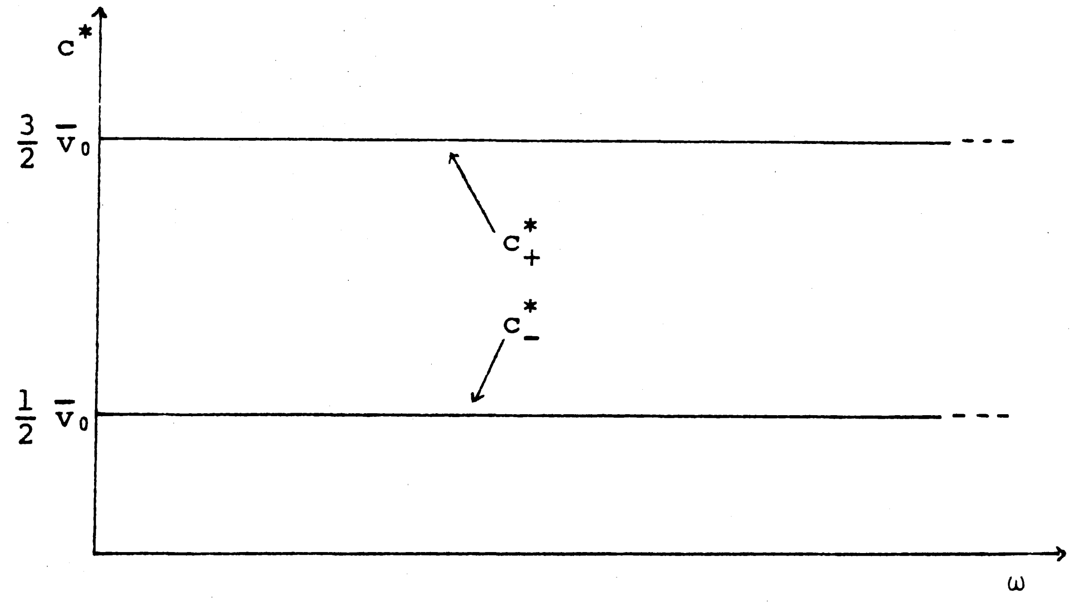

Figure 13. $\bar{v}_0 = 2\sqrt{g^*\bar{h}_0}$ (non-dispersive waves)

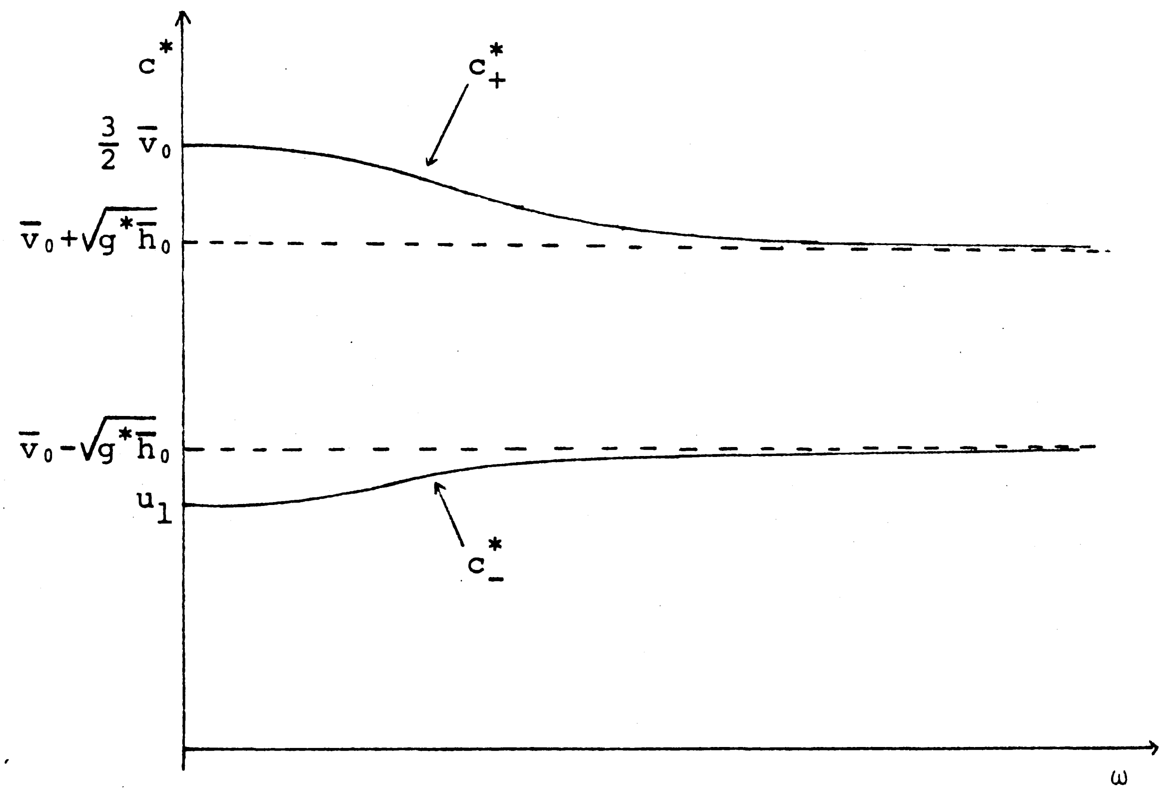

Figure 14. $\bar{v}_0 > \sqrt{g^*\bar{h}_0}$

(ii) Nonuniform basic flows

In order to study spatially growing wave solutions on nonuniform basic flows we introduce the time periodic perturbations

$$\eta = \bar{\eta}(x)e^{-i\omega t}, \qquad w = \bar{w}(x)e^{-\omega t} \tag{6.16}$$

The nonuniform basic flows may be stable only if small perturbations given by (6.16) die away or persist as perturbations of similar magnitude as $x \to \infty$ (or as $x \to -\infty$ if perturbations propagate in the negative x-direction).

The expressions (6.16) substituted into (2.3) give the equations

$$\begin{aligned} v_0 \frac{d\bar{\eta}}{dx} + h_0 \frac{d\bar{w}}{dx} &= (i\omega - v_0')\bar{\eta} - h_0' \bar{w} \\ g^* \frac{d\bar{\eta}}{dx} + v_0 \frac{d\bar{w}}{dx} &= c_f \frac{v_0^2}{h_0^2} \bar{\eta} + [i\omega - v_0' - 2c_f \frac{v_0}{h_0}]\bar{w} \end{aligned} \tag{6.17}$$

which may be written as

$$C \frac{d\underline{s}}{dx} = D\underline{s} \tag{6.18}$$

where

$$\underline{s} = \begin{Bmatrix} \bar{\eta} \\ \bar{w} \end{Bmatrix}$$, and C and D are 2×2 matrices.

Multiplying (6.18) by the inverse of C we get the system

$$\frac{d\underline{s}}{dx} = E\underline{s} \tag{6.19}$$

where

$$E = \frac{1}{v_0^2 - g^* h_0} \begin{Bmatrix} i\omega v_0 - v_0 v_0' - c_f \frac{v_0^2}{h_0} & -v_0 h_0' - i\omega h_0 + h_0 v_0' + 2c_f v_0 \\ -i\omega g^* + g^* v_0' + c_f \frac{v_0^3}{h_0^2} & g^* h_0' + i\omega v_0 - v_0 v_0' - 2c_f \frac{v_0^2}{h_0} \end{Bmatrix} \tag{6.20}$$

It is now possible to study the stability properties of the nonuniform basic flow (3.8) in the steep slope case $S > c_f$ when $0 < h_0^* < h_c$, i.e. in cases where $h_0(x) \to \bar{h}_0$ as $x \to \infty$. ($S > c_f$ implies that all perturbations propagate in the positive x-direction.) Stability of the basic flow requires that the trivial solution of (6.19) is stable (as $x \to \infty$).

According to standard theory of stability it is necessary for the trivial solution of (6.19) to be stable that both of the eigenvalues of $E_0 = \lim_{x\to\infty} E(x)$ have nonpositive real parts. Here

$$E_0 = \frac{1}{\bar{v}_0^2 - g^*\bar{h}_0} \begin{Bmatrix} i\omega\bar{v}_0 - c_f \dfrac{\bar{v}_0}{\bar{h}_0} & -i\omega\bar{h} + 2c_f\bar{v}_0 \\ -i\omega g^* + c_f \dfrac{\bar{v}_0^3}{\bar{h}_0^2} & i\omega\bar{v}_0 - 2c_f \dfrac{\bar{v}_0^2}{\bar{h}_0} \end{Bmatrix} \tag{6.21}$$

The eigenvalues of E_0 are roots of the equation

$$\det(E_0 - \lambda I) = (\bar{v}_0^2 - g^*\bar{h}_0)\lambda^2 + (-2i\omega\bar{v}_0 + 3c_f \frac{\bar{v}_0^2}{\bar{h}_0})\lambda - (\omega^2 + 2i\omega c_f \frac{\bar{v}_0}{\bar{h}_0}) = 0 \tag{6.22}$$

It is easily seen that (6.22) is transformed into (6.2) by substituting ik for λ, where i is the imaginary unit. Consequently

$$\lambda = ik = -k_i + ik_r \tag{6.23}$$

where k_i and k_r are given by (6.3) and (6.4). $Re(\lambda) = -k_i \leq 0$ if and only if $k_i \geq 0$. Since $S > c_f$, i.e. $\bar{v}_0 > \sqrt{g^*\bar{h}_0}$, it follows from the discussion in the above subsection (i) that:

(6,II) For the nonuniform basic flow (3.8), when

$$S > c_f \quad \text{and} \quad 0 < h_0^* < h_c ,$$

to be stable against spatially growing modes it is necessary that

$$S \leq 4c_f .$$

This stability result agrees with that obtained for nonuniform basic flows in section 5. In that section we also investigated the stability properties of nonuniform flows where $h_0(x) \to \infty$ as $x \to \infty$. No instabilities were detected in these basic flows. An analogous discussion by the method used in the present section gives no information since both of the eigenvalues of $E_0 \neq 0$ are zero.

Acknowledgement

This paper was carried out during my sabbatical leave of 1981-82, supported by the Norwegian Research Council for Science and the Humanities. Thanks to Professor E. Palm and his collegues in Applied Mathematics I was able to spend the year at the University of Oslo. I am grateful for their kindness and for stimulating and helpful discussions.

REFERENCES

Dressler, R.F. 1949 Mathematical solution of the problem of roll waves in inclined open channels. Comm. Pure Appl. Math. 2, 149-194.

Eckhoff, K.S. & L. Storesletten 1978 A note on the stability of steady inviscid helical gas flows. J. Fluid Mech. 89, 401-411.

Eckhoff, K.S. 1981 On stability for symmetric hyperbolic systems,I. J. Differential Equations 40, 94-115.

Henderson, F.M. 1969 Open channel flow. MacMillan.

Stoker, J.J. 1957 Water waves. Interscience Publ.

Ven Te Chow 1959 Open-channel hydraulics. McGraw-Hill.

Whitham, G.B. 1974 Linear and nonlinear waves. John Wiley & Sons.