SHEAR LAG IN BEAMS OF ORTHOTROPIC MATERIAL

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Abstract

FEM parametric study of the stress distribution in the orthotropic flanges of beams with various boundary conditions and cross sections is performed. A 2D plane stress model is used. An empirical formula is established for the shear lag coefficient $\lambda$ which is used for computing the effective flange width $B_0 = \lambda B$ for stress calculations. Flange stiffeners along the length of the beam can be accounted for by modifying the E/G ratio. The formula has a good accuracy and to the authors knowledge is the only one that gives reliable results for high E/G ratios as is the case in fibre reinforced composites.
NOTATIONS

E - Young's modulus of flange in x direction
G - Shear modulus of flange
L - (Half) length of beam, Fig. 2
\( \kappa L \) - Distance between the zero and maximum moment sections.
Values of \( \kappa \) are given in Table 1.
B - Half flange width, Fig. 3
\( B_e \) - Effective half Flange width, defined by eqns (2a)
H - (Half) Height of web, Fig. 3
\( T_w \) - (Half) Thickness of web, Fig. 3
\( T_f \) - (Half) Thickness of flange, Fig. 3
\( \lambda \) - Shear lag coefficient, defined by eqn (2b)
\( \sigma_x \) - Normal stress in beam cross section
\( \sigma_{xo} \) - Normal stress at web-flange intersection
\( M_y \) - Bending moment in beam
\( I_y \) - Moment of inertia of beam cross section
\( \psi \) - Coefficient to account for flange stiffeners, defined by eqn (8)
\( x,y,z \) - Cartesian coordinate system, Fig. 2, Fig. 3b
S1 ÷ S4 - Beam support cases, Fig. 2
F1 ÷ F4 - Beam loading cases, Fig. 2
1. INTRODUCTION

According to the elementary beam theory (an assumption that plane sections remain plane after deformation) the normal stress $\sigma_x$ at a point of the cross section with $y, z$ coordinates, due to bending, is:

$$\sigma_x = \frac{M_y}{I_y} z$$

(1)

and implies a constant stress in $y$ direction. In the case of a box, T, I, etc. cross section, due to the action of in-plane shear strain in the flanges the longitudinal displacements in the parts of the flange remote from the web (i.e. in $y$ direction) lag behind those near the web. This results in a distribution of the normal stress as shown on Fig. 1. This phenomenon is called shear lag. If the width of the flange is big, eqn (1) will significantly underestimate the stress at the web-flange intersection. However, it is still possible to obtain from eqn (1) a correct value for the maximum stress by using an effective width of the flange $B_e$:

$$B_e = \frac{1}{\sigma_{x_0}} \int_0^B \sigma_x(y) \, dy$$

(2a)

Eqn (2a) is of little practical use since $\sigma_x(y)$ is unknown. An alternative formulation is:

$$B_e = \lambda \cdot B$$

(2b)

where $\lambda$ is the shear lag coefficient. The purpose of this paper is to provide a simple way for its calculation.

One of the earliest investigations into the shear lag problem is due to von Kármán (1924). It is well documented by Timoshenko (1970) and criticized by Vendhan & Bhattacharyya (1987). A series solution form of Airy’s stress function for the plane stress field in the flange with infinite width is chosen to satisfy the boundary conditions. The unknown coefficients are
Figure 1

Shear lag in the cross section of a beam with a wide flange
determined by minimizing the strain energy (due only to $\sigma_x$) in the beam.

The same form of a stress function is used by Miller (1937) and the unknown coefficients are determined from the solution of a third order partial differential equation. The flange width is finite.

Reissner (1941) assumes a parabolic variation of the bending stress across the width of the flange. A second order differential equation is obtained for the spanwise variation of the vertex curvature of the stress parabolas.


The shear lag effect in box girders of varying depth has been addressed by Chang & Yun (1988). The prestress influence on shear lag effect in continuous box girder bridges - by Chang (1992).

Apparently the analytical research could not provide a general, reliable and suitable for design purposes solution of the shear lag problem and a specialized computer program for the analysis of box girder bridges, based on the finite element method, has been developed by Hinton and Hewitt (1975).

Moffatt & Dowling (1972, 1975) have performed a comprehensive parametric FEM study on the shear lag in steel box girder. Their results were used as a basis for the formulation of the British shear lag rules, Moffatt & Dowling (1978).

From the brief survey the following conclusions can be made:

1. For a given problem (eg.: simple beam, uniform load, box cross section) there are several analytical solutions for the shear lag effect, due to the different initial assumptions. To simplify the results and to make them useful to the design engineer more assumptions and approximations are introduced and,
Figure 2

(a) Full beam
(b) Beam to be analyzed when symmetry is taken into account
(c) Load cases applied on beam model (b):

F1 - concentrated, F2 - uniform, F3 - linear load
usually, there is no information about the error involved. Explicit results for beams with a variety of support and loading conditions are rarely presented.

2. More confidence is placed on FEM analysis and results rather than analytical solutions (the British shear lag rules are based on FEM results).

3. For beams whose flanges are made of laminated fiber reinforced composites the E/G ratio has a significant influence on the shear lag. Design codes [Moffatt & Dowling (1978), Thein Wah (1960)] deal only with beams made of isotropic material (steel). Even a code for fibre composite and sandwich constructions [Det Norske Veritas (1991)] makes no provisions for varying E/G.

The aim of this paper is to give a simple empirical formula for the computation of shear lag coefficient $\lambda$ and the effective width of flanges made of orthotropic material. Several support and loading conditions are considered. Numerous FEM analyses provide the data from which the empirical formula is derived.

2. THE FEM MODEL

Several beams, shown on Fig. 2a, are considered. Taking into account symmetry, the beam models to be analyzed are shown on Fig. 2b. This simplifies the FEM data/results management because only the support conditions on both ends of the beam has to be changed and it is always the right end of the beam, point M, where the maximum moment occurs.

Only beams with a symmetrical cross section - box, T, I and U, loading in the plane of symmetry, shown on Fig. 3, will be considered. Due to symmetry a half or a quarter of the cross section, which is shown by the broken line rectangulares, are modeled. This results in identical FEM meshes with only different boundary conditions to account for the missing parts.

The shear lag is analyzed in details only at the cross section with the
Figure 3

Cross sections used in the shear lag analysis

(a), (b), (c), (d) - T, U, I and box section, resp.

Dotted rectangular - the part for FEM analysis (due to symmetry).
maximum bending moment (point M, Fig. 2b).

For the FEM model the straightforward choice is to use 3D shell or solid elements. However, since a large number of FE analyses are required, a special attention is paid to reduce the computer effort. The problem is modeled by 2D plane stress elements, Tenchev (1994).

For a simple beam with a T cross section subjected to uniformly distributed load the 2D plane stress FEM model is shown on Fig. 4. The bending stiffness of the flange is ignored. (The same assumption or its equivalent that $\sigma_x$ is constant through the thickness of the flange is made in all analytical solutions, too. However, if required, the flange bending stiffness can be accounted for in the 2D model, too, Tenchev (1994)) A constraint is imposed that corresponding nodes on lines $B_w-B_w'$ and $B_f-B_f'$ (which represent the web-flange intersection) have equal x displacements.

In the case of I and box cross section the x displacements of all nodes on line $C-C'$ (the neutral line) are set to zero. In the case of U and box cross section the y displacements of all nodes on line $A-A'$ are set to zero, too.

In the FEM model a concentrated load is represented as a constant distributed load along the height of the web.

In the real 3D model, for a given cross section the deflection $z$ of a point at the web-flange intersection is different from the deflection of a point at the end of the flange. This is not accounted for by the 2D model. FEM 3D shell analyses show that: (a) in the case of pin supports (simple beam, etc., when the beam cross section at the support is not restricted to deform out of plane) the stresses at the maximum moment cross section are identical to those computed by the 2D model; (b) in the case of built-in supports (i.e. the cross section is forced to remain plane) the differences are negligible.

The FEM program is developed by the author and extensively tested in the course of several years. Quadratic (8-nodes), isoparametric elements are used.
Figure 4

The 2D plane stress FEM model for a half of a simple beam (case S1 on Figure 2b) with T cross section.

Lines $B_fB_f'$ and $B_wB_w'$ have equal displacements along x.
The constraint is modeled by stiff bar elements connecting the corresponding nodes on lines $B_{w}B_{w}'$ and $B_{r}B_{r}'$ in x direction. Their stiffness is calculated and assembled after the assembling of the global stiffness matrix in such a way that the coefficients on the main diagonal where the bar stiffness is added will keep about half of their most significant digits.

The FEM results for $\sigma_{x}$ in the middle section (line $A'-C'$ on Fig. 4) of a simple beam (case S1 on Fig. 2) are shown on Fig. 5. The dimensions used are: $L=200\text{mm}, B=100\text{mm}, H=25\text{mm}, T_{w}=10\text{mm}, T_{r}=1\text{mm}$; isotropic material $E/G=2.5$; uniform distributed load $q=10\text{N/mm}$. The shear lag is evident. The maximum stress at the web-flange intersection is $-90\text{N/mm}^2$ while at the end of the flange it is $-43.4\text{N/mm}^2$. The $\sigma_{x}$ stress distribution in the flange is quadratic (correlation coefficient $= 0.9999626$) and Reissner's (1941) assumption is quite accurate.

The FEM shear lag coefficient is computed according to eqns (2):

$$\lambda_{\text{FEM}} = \frac{1}{B\sigma_{x_{0}}} \int_{0}^{B} \sigma_{x}(y) \, dy$$

3. PARAMETRIC STUDY

For all combinations of beam boundary conditions (Fig. 2) and cross sections (Fig. 3) FEM parametric studies are performed to show the sensitivity of the shear lag coefficient $\lambda$. On Fig. 6+10 the curves are cubic spline interpolation of the FEM results. The latter are shown on some of the graphs as rotated squares.

(a) $B/L$ and $E/G$

FEA show that the shear lag coefficient $\lambda$ depends strongly on the ratios
Figure 5

\( \sigma_x \) distribution along line A'-C' on Figure 4
B/L and E/G, Fig. 6. This is recognized by all researchers but many of them have considered only isotropic materials and the ratio E/G is fixed, depending only on the Poisson's ratio.

(b) Web dimensions, flange thickness

FEA show that the web dimensions (height, thickness) has negligible influence on the shear lag coefficient $\lambda$, Fig. 7. The flange thickness has negligible influence, too.

The results shown on Fig. 6 and Fig. 7 are for a simple beam, uniform distributed load, T section. For the other cases of boundary conditions and cross sections similar results were obtained.

Reissner (1941) included a ratio $m$ defined by the moment of inertia of the web(s) and flange(s) in his analysis and final results. However, it is reported in Roark & Young (1975) that for the practical range of $m$ the variation in the shear lag is small enough to be disregarded.

Horie et al (1984) concluded that the shear lag may be regarded as independent on these dimensions for uniform loading but for a concentrated load their effect is not negligible.

In the present study local effects of concentrated forces will not be accounted for and a concentrated force in the FEM model is represented by a uniform load along the height of the web.

(c) Loading and Supports

FEA show that there is a moderate (to strong) dependence of the shear lag coefficient $\lambda$ on the loading, Fig. 8, and a strong dependence on the support conditions, Fig. 9. This is recognized by all researchers, too.

(d) Type of cross section

The FEA show that there is a negligible dependence of the shear lag coefficient $\lambda$ on the type of cross section - box, T, I or U. The same
Figure 6

Strong dependence of the shear lag coefficient $\lambda$ on:

(a) $B/L$ (flange_width / beam_length) ratio

(b) $E/G$ (Young / Shear modulus) ratio for the flange
conclusion has been already reported by Winter (1943), Roark & Young (1975) and Song & Scordelis (1990).

For the parametric study of those cross sections one and the same mesh is used but with different boundary conditions to account for the symmetry(-ies) in the cross section. (For box and I cross section the height of the web is reduced twice in order to have one and the same ratio height/span and the same accuracy of the assumption of linear $\sigma_x$ distribution along the height.)

(e) Variation of $E_y$, $v_{xy}$ of flange and the material properties of the web

FEA show that those parameters have negligible influence on the shear lag.

(d) Variation of $\lambda$ along the length of the beam

The variation of the shear lag coefficient $\lambda$ along the length of the beam is shown on Fig. 10a (simple beam) and Fig. 10b (cantilever beam); uniform load, T cross section. The origin of $\bar{x}$ axis is at the maximum bending moment (point M on Fig. 2). On all other Figures showing graphs $\lambda$, the shear lag coefficient is computed for the cross section at $x=0$.

In the case of cantilever beam it can be seen that at some distance from the built-in end the shear lag coefficient is greater than one. This means that the maximum $\sigma_x$ is not at the web-flange intersection but at the end of the flange. This 'anomaly' or 'negative shear lag' has been already reported by Foutch & Chang (1982) and Kristek & Studnicka (1991).

The aim of this paper is to produce an easy-to-use formula for the calculation of the shear lag coefficient $\lambda$. From the presented parametric study it is concluded that the formula must take into account:

(a) The ratios $B/L$ and $E/G$.

(b) The loading and the support conditions.

Since it is basically for design purposes it will consider only the cross section with the maximum bending moment.
Figure 7

Negligible dependence of the shear lag coefficient $\lambda$ on the web dimensions:

(a) Height,  (b) Thickness
4. EMPIRICAL FORMULA FOR THE SHEAR LAG COEFFICIENT $\lambda$

Empirical formula for calculating the shear lag coefficient $\lambda$ at the section with the maximum bending moment will be established for several beam support conditions and loading cases, Fig. 2, valid for the cross sections on Fig. 3, as a function of the ratios B/L and E/G.

On Fig. 11 FEM results for the shear lag coefficient (a simple beam, uniform load) are approximated by a geometric least square fit. It can be clearly seen that the approximation is quite accurate for both cases - the B/L and E/G dependence. This finding is crucial for the success of the empirical approach. The following expression for the empirical evaluation of shear lag coefficient $\lambda$ is proposed:

$$\lambda_{\text{Emp}} = \frac{p}{C_1} \left( \frac{B}{\kappa L} \right)^q \left( \frac{E}{G} \right)^r$$

(4)

where:

$$C_1 = 1 + s e^{iX}, \quad X = \frac{B}{\kappa L} \left( 1 - \frac{1}{2} \frac{E}{G} \right)$$

(4b)

$$C_2 = 1 + t e^{iY}, \quad Y = \left( \frac{E}{G} \right) \left( \frac{B}{\kappa L} \right)^{-1}$$

(4c)

$C_1$, $C_2$ - coefficients to enhance the approximation at the extremities of B/L and E/G.

e = 2.7183 is the base of the natural logarithm.

$p, q, r, s, u, t, v$ - coefficients to be determined from a least square fit.

$\kappa L$ - distance between the zero moment and the maximum moment cross section. Values of $\kappa$ are given in Table 1.

To acquire sufficient experimental data (i.e. FEM results) the ratio B/L has been varied on 25 steps from 0.05 to 2.0 and E/G - on 16 steps from 1 to 30, simultaneously. This gives 400 FEM results for each loading and support
Figure 8

Moderate to strong dependence of the shear lag coefficient \( \lambda \) on loading

F1-concentrated; F2-uniform; F3-linear load, (Fig.2).

(a) simple beam, (b) cantilever beam.
conditions, an amount of data which is considered adequate to reveal the dependency of $\lambda$ on $B/L$ and $E/G$.

Because of the $C_1$ and $C_2$ coefficients iterative least square fit is used, but without the constraint in eqn (4a). The values of the unknown coefficients $(p, q, \ldots, v)$ are then manually adjusted to enhance the results when eqn (4a) is taken into account and to have some identical coefficient when the support conditions are not changed.

The results are presented in Table 1. The support conditions S1 to S4 and loading F1 to F4 are given on Fig. 2. The shear lag estimation is valid for the cross section with the maximum moment and for types of cross section given on Fig. 3.

If $B/L > 2$ one may use $B/L = 2$ either for the FEA or the empirical calculation without introducing any error. When $B/L \geq 2$ and any value of $E/G$ $\sigma_x = 0$ at the part of the flange remotest from the web(s).

Comments:

For the simple beam, Case S1, the type of the distributed load (F2 uniform, F3 - linear, F4 - uniform, acting on a part of the beam) has little influence on the shear lag, and all (but one) coefficients $p, q, \ldots, v$ of eqns (4), have identical values for the different load cases. Those values may be used for other types of distributed loads or several concentrated loads along the length of the beam.

In the case of the statically indeterminate beams, cases S3 and S4, the distance between the zero moment section and the maximum moment section (point M on Fig. 2b) is about one quarter (see coeff. $\kappa$) of the span between adjacent supports. The type of the moment diagram at that part seems to have little influence on the shear lag and the coefficients $p, q, \ldots, v$ are chosen to have identical values.
Figure 9

Strong dependence of the shear lag coefficient $\lambda$ on support conditions S1 to S4 (Fig. 2), uniform load (F2).

(a) $E/G=1.0$; (b) $E/G=25.0$
Table 1.
Values of the coefficients in eqns (4). The code for Beam and Load is according to Fig. 2.

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<th>q</th>
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<th>s</th>
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Figure 10

Shear lag coefficient $\lambda$ along the length of the beam.

$\ddot{x} = 0$ at point M (the maximum bending moment) on Figure 2.

(a) Simple beam  (b) Cantilever beam
For each boundary condition (Fig. 2) 400 FEM analyses are performed as explained in the previous section.

The error of $\lambda_{\text{Emp},i}$ is:

$$\Delta \lambda_i = \frac{\lambda_{\text{Emp},i} - \lambda_{\text{FEM},i}}{\lambda_{\text{FEM},i}} \times 100\% \quad (i=1,400) \quad (5)$$

The mean square error, $\Delta \lambda_{\text{MSq}}$, may be considered as an overall measure of the accuracy:

$$\Delta \lambda_{\text{MSq}} = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (\Delta \lambda_i)^2} \quad (N=400) \quad (6)$$

The effective flange width according to eqn (2b) is:

$$B_e = \lambda_{\text{Emp}} B \quad (7)$$

and it is used to compute the moment of inertia $I_y$ of the cross section and to compute the constant stress $\sigma_x$ in the flange using the elementary beam theory. This stress is compared to the FEM stress at the web-flange intersection. Stress errors $\Delta \sigma_i$ and $\Delta \sigma_{\text{MSq}}$ are computed similarly to eqns (5) and (6).

For each case the maximum, minimum (i.e. maximum negative) and the mean square errors are given in Table 2.

In the case of the statically indeterminate beams, S3 and S4, there is a relatively large negative error in the shear lag coefficient, which means that the shear lag is overestimated which result in a big (about 20%) positive error of the maximum stress, i.e. the stress is overestimated. Since the result is on the safer side, this is not a very serious drawback. Those errors increase as the ratios $B/L$ and $E/G$ increase and the maximum errors given in Table 2 are when $B/L=1.0$ ($B/kL=4.0$) and $E/G=30$. The error distribution for case S3-F2 is shown on Fig. 12. The error of the stress prediction is not only due to the error of $\lambda$. Due to the shear lag the real moment diagram and the
Figure 11
FEM results for the shear lag coefficient $\lambda$ and a geometric least square interpolation.
value of the maximum moment are different from the one given by beam theory.

Having in mind that the empirical formula, eqns (4) and Table 1, is to be used for design calculations the errors are considered acceptable.

Table 2

Errors [%]: maximum, minimum (maximum negative) and mean square for the shear lag coefficient $\lambda$ and the maximum stress in the flange.

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<td>24.42</td>
<td>-0.09</td>
<td>7.12</td>
</tr>
<tr>
<td>S4</td>
<td>F1</td>
<td>10.25</td>
<td>-13.35</td>
<td>5.10</td>
<td>21.15</td>
<td>-3.07</td>
<td>4.41</td>
</tr>
<tr>
<td></td>
<td>F2</td>
<td>5.87</td>
<td>-22.69</td>
<td>5.01</td>
<td>22.67</td>
<td>-1.93</td>
<td>5.41</td>
</tr>
<tr>
<td></td>
<td>F3</td>
<td>5.13</td>
<td>-26.90</td>
<td>5.85</td>
<td>25.20</td>
<td>-0.13</td>
<td>7.16</td>
</tr>
</tbody>
</table>
Figure 12

Typical error distribution [%] of the predicted $\sigma_x(\lambda)$ for the statically indeterminate beams (Case S3 and S4)
6. COMPARISON WITH OTHER FORMULAS

To check the performance of the proposed empirical formula the following test problems are considered:

- Test problem S1-F2: simple beam, \(2L=200\text{mm}\), uniform load.
- Test problem S2-F2: cantilever beam, \(L=200\text{mm}\), uniform load.

In both cases it is a box cross section (Fig. 3d), with dimensions \(2H=25\text{mm}\), \(T_f=1\text{mm}\), \(T_w=10\text{mm}\). The \(B/L\) ratio is varied from 0.05 to 1.0 on 20 steps.

On Fig. 13† 16 'FEM' graphs are based on the FEM computation of \(\lambda\), eqn (3) and 'Emp.' graphs - on the empirical formula, eqns (4). A 'FEM' result for \(\lambda\) is considered to be the true one and a target value for all other analytical (but with several approximations) or empirical results.

On Fig. 13 graph 'Br.R.', (i.e. British shear lag Rules) is based on Moffatt & Dowling's (1978) tabulated results and graph S&S is based on Song & Scordelis (1990b). Test problem S1-F2 is analyzed but only for \(E/G=2.5\) (isotropic material with Poisson's ratio 0.25) since they do not make provisions for varying \(E/G\) ratio. Song & Scordelis's (1990b) formula is said to be valid for \(B/L<0.5\) but it can be seen that an extension to 0.75 is acceptable. All results show good agreement with FEM. The empirical formula, eqns (4), gives the best approximation in the range \(0.2 < B/L < 1.0\).

On Fig. 14 graph 'Reis' is based on Reissner (1941), eqns 46 and 50 in his paper. Graph 'Br.R.' - as explained previously. Test problem S2-F2 is analyzed with \(E/G=2.5\) (Fig. 14a) and \(E/G=25\) (Fig. 14b). Reissner's formula severely underestimates the shear lag while Moffatt's results for the isotropic case overestimate it. It can be seen that the empirical formula has again a very good agreement with the FEM results.

On Fig. 15 the formula given by Horie et al (1984) is compared with FEM and the empirical formula. Test problem S1-F2, \(E/G=2.5\) (Fig. 15a) and \(E/G=25\) (Fig. 15b) is analyzed. For the case of \(E/G=2.5\) the 'Emp.' graph is not
Figure 13

Comparison of shear lag coefficient $\lambda$

Simple beam, uniform load (Case S1-F2); $E/G=2.5$

**FEM** - Finite Element results
**Emp.** - Empirical formula, Eqns (4)
**S&S** - Song-Scordelis (1990b)
**Br.R.** - British shear lag rules, Moffatt & Dowling (1978)
plotted since its agreement with the FEM results has been already shown on Fig. 13. Horie’s formula uses coefficients which are complicated functions of the dimensions of the cross section (parameters $B/H$ and $T_f/T_w$) and for small values of $B/H$ ($B/H<1.5$, i.e. $B/L<0.19$ on Fig. 15b) the formula fails because a square root of a negative number must be calculated. When it works it gives good results for isotropic materials.

On Fig. 16 results based on Kristek & Evans (1985) are plotted for test problem S1-F2. Their formula estimates the true stress when shear lag is taken into account rather than the effective width. For the purpose of comparison the corresponding shear lag coefficient $\lambda$ is calculated from this stress. Their formula has a drawback that $\lambda$ is strongly dependent on the height $H$ of the web, while the results on Fig. 7a show the opposite. Graphs ‘K&E’ is for $2H=25$mm, ‘K&E_H/2’- for $2H=12.5$mm, ‘K&E_H/4’- for $2H=6.25$mm and ‘K&E_H*2’ for $2H=50.0$mm.

Kristek & Evans (1985) have solved an example: simple beam, span $2L=9144$mm, uniform load $q=1$ kN/mm, box cross section with dimension (according to Fig. 3d) $2H=1829$mm, $2B=3632.6$mm, $T_f=25.4$mm, $T_w=12.7$mm. Isotropic material (steel).

A comparison of some results is made in Table 3.

Table 3.

Steel box girder (without stiffeners), simply supported, under uniform load

<table>
<thead>
<tr>
<th>Kristek &amp; Evans (1985)</th>
<th>Moffatt &amp; Dowling (1975)</th>
<th>FEM (Fig. 4)</th>
<th>Empirical (Eqs. 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_x$</td>
<td>-69.0 MPa</td>
<td>-68.9 MPa</td>
<td>-70.6 MPa</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>0.807</td>
<td>0.810</td>
<td>0.798</td>
</tr>
</tbody>
</table>
Figure 14
Comparison of shear lag coefficient $\lambda$

Cantilever beam, uniform load (Case S2-F2)

**FEM** - Finite Element results

**Emp.** - Empirical formula, Eqns (4)

**Reis** - Reissner (1941)

**Br.R.** - British shear lag rules, Moffatt & Dowling (1978)

(a) $E/G = 2.5$  (b) $E/G = 25.0$
There is a good agreement in the results. The error of the empirical formula, eqns (4), is $\Delta \lambda=4.6\%$ and $\Delta \sigma_x=3.2\%$ and it is within the error range shown in Table 2.

To illustrate the use of the empirical formula, eqns (4) and Table 1, the steps of calculation $\lambda_{\text{Emp}}$ from Table 3 will be shown explicitly:

Case: S1-F1 (Table 1): $\kappa=1$, $p=0.55$, $q=-0.89$, $r=-0.43$, $s=5$, $t=-9$, $u=0.3$, $v=-1$.  

\[
\begin{align*}
E/G &= 2.6 \text{ (Steel, Poisson's ratio 0.3)} \\
B/\kappa L &= \frac{2B}{2L} = 3632.6 / 9144 = 0.397 \\
X &= \frac{B}{L} \left(1 - \frac{1}{2} \frac{E}{G}\right) = 0.397 \times \sqrt{0.5 \times 2.6} = 0.453 \\
C_1 &= 1 + s \ e^x = 1 + 5 \times e^{9 \times 0.53} = 1.09 \\
Y &= \frac{E/G}{B/L} = 2.6 / 0.397 = 6.55 \\
C_2 &= 1 + u \ e^y = 1 + 0.3 \times e^{-1 \times 6.55} = 1.00 \\
\lambda_{\text{Emp}} &= \frac{p}{C_1 C_2} \left(\frac{B}{L}\right)^q \left(\frac{E}{G}\right)^r = \frac{0.55}{1.09} \times 0.397^{-0.89} \times 2.6^{-0.43} = 0.761
\end{align*}
\]

7. EXTENDED APPLICATIONS

7.1. Flange with longitudinal stiffeners

Stiffeners contribute to the axial stiffness of the flange and have negligible effect on the shear stiffness. In this case the shear lag is more pronounced (Fig. 6, increasing E/G ratio).

Due to the stiffeners the axial stiffness of the flange will be increased by a factor $\psi$:

\[
\psi = \frac{AE + A_s E_s}{AE} \tag{8}
\]

where:

$A_s$ - Stiffeners total cross-sectional area at each flange
Figure 15

Comparison of shear lag coefficient $\lambda$

Simple beam, uniform load (Case S1-F2)

FEM - Finite Element results

Emp. - Empirical formula, Eqns (4)


(a) $E/G = 2.5$  (b) $E/G = 25.0$
\( E_s \) - Young's modulus of stiffeners  
\( A, E \) - Cross-sectional area and Young's modulus of the flange

If the shear stiffness is not to be changed then the shear modulus has to decreased by the same amount. In eqns (4) instead of \((E/G)\) the modified ratio \((\psi E/G)\) must be used.

Kristek & Evans (1985) have solved an example: simple beam, span 2L=9144mm, uniform load \( q=1 \) kN/mm, box cross section with dimension (according to Fig. 3d) 2H=1829mm, 2B=3632.6mm, \( T_f=12.7 \) mm, \( T_w=12.7 \) mm. Isotropic material (steel). Total cross-sectional area of the stiffeners on each flange \( A_s=46456.6 \) mm².

For this problem \( \psi = 2 \) and comparison of some results is made in Table 4.

Table 4

| Steel box girder (with stiffeners), simply supported, under uniform load |
|------------------------|------------------------|------------------------|------------------------|
| Kristek & Evans (1985) | Moffatt & Dowling (1975) | FEM (Fig. 4) | Empirical (Eqs. 4) |
| \( \sigma_x \) | -83 MPa | -82 MPa | -84.4 MPa | — |
| \( \lambda \) | — | 0.67 | 0.65 | 0.62 |

For the FEM model the total area of the stiffeners \( A_s \) is assumed to be distributed uniformly along the flange width and the new flange thickness is \( t_f+A_s/2B \). Young's modulus for the flange and the web is \( E = 2 \times 10^6 \) N/mm², Poisson's ratio - \( \nu=0.3 \), the web is isotropic \( G = \frac{E}{2(1+\nu)} \) and the shear modulus for the flange is \( G = \frac{1}{\psi} \frac{E}{2(1+\nu)} = 0.385 \) N/mm².

There is again a very good agreement of the results.
Figure 16

Spurious dependence of $\lambda$ on the height of the web in Kristek & Evans (1985)

$K&E, K&E_H/2, K&E_H/4, K&E_H*2$ - Web height $2H = 25; 12.5; 6.25; 50$ mm, resp.

FEM - Finite Element results

(a) $E/G = 2.5$  
(b) $E/G = 25.0$
7.2. Load combination

In the case of load combinations, using the principal of superposition, the shear lag coefficient is:

\[ \lambda^{I+II} = \frac{1}{\sigma_x + \sigma_x^{II}} \int_{0}^{B} \left( \sigma_x(y) + \sigma_x^{II}(y) \right) dy \] (9)

or:

\[ \lambda^{I+II} = \frac{\lambda^I \sigma_x^I(\lambda^I) + \lambda^{II} \sigma_x^{II}(\lambda^{II})}{\sigma_x^I(\lambda^I) + \sigma_x^{II}(\lambda^{II})} \] (9a)

Superscripts I and II refer to the two load cases when solved separately and \( \sigma_x(\lambda) \) is the constant stress in the flange when effective flange width and beam theory is used. Eqn (9a) is to be used when the section with maximum moment remains the same, i.e. \( \sigma_x^I \) and \( \sigma_x^{II} \) have equal signs. When the stresses have opposite signs the denominator may approach zero and unrealistic values to be predicted.

For a simple beam 2L=400mm; box section: 2B=200mm, 2H=25mm, \( T_f=1 \) mm, \( T_w=10 \) mm; Load case I - uniform load \( q = 10 \) N/mm (max \( M_y = 200 \) kNm), Load case II - concentrated load at midspan \( P=1 \) kN (max \( M_y = 100 \) kNm) the empirical formula, eqns (4), gives \( \lambda^I = 0.667 \) and \( \lambda^{II} = 0.517 \). From beam theory using flange effective width - \( \sigma_x^I(\lambda^I) = 36.9 \) MPa and \( \sigma_x^{II}(\lambda^{II}) = 21.4 \) MPa. From eqn (9a) \( \lambda^{I+II} = 0.61 \). The FEM result for this load combination is \( \lambda^{I+II}_{FEM} = 0.60 \). The agreement is very good. However, with different \( B/L \) and \( E/G \) ratios the error of eqn (9a) may be greater than the one shown in Table 2, due to accumulation.

7.3. Complex cross sections

In the case of complex cross sections (multiple webs, box or U with overhangs, etc.) Moffatt & Dowling (1978) has suggested division of the cross section into basic units and estimating the effective width of each one. If the unit represents an overhang its effective width is to be reduced by 15%.
Figure 17

(a) U cross section with overhangs

(b) A half section for the FEA. Lines B_f-B'_f etc. defined on Fig. 4

(c) The U section divided into basic units
For a simple beam, uniform load, U cross section with overhangs (Fig. 17a) FEM analyses are carried out with the model on Fig. 4. The following modification of the boundary conditions are introduced:

a) On line $B_t-B'_t$ (Fig. 4 and Fig. 17b) there are no prescribed zero displacements along y (it is not a symmetry line for the cross section).

b) On line $A-A'$ there are prescribed zero displacements along y (it is symmetry line for the cross section).

c) Line $B_w-B'_w$ has prescribed equal $x$ displacements with the line $U-U'$, which position is determined by the $B_t/B_2$ ratio.

The case $B_t/B_2 =1$, $E/G=2.5$ is analyzed for varying $B/L$ and the results are shown on Fig. 18. Graph 1 is for the shear lag at the flange between the webs. Graph 2 is for the shear lag at the overhang. The shear lag coefficient at the overhang is about 15% smaller then the one at the flange between the webs, as already observed by Moffatt & Dowling (1978).

The U cross section with overhangs may be divided into two basic units with $T$ cross section, as shown on Fig. 17b. Graph 3 is for the shear lag in the basic unit (when $B_t/B_2 =1$ the two basic units are identical). It can be seen that using the basic units the shear lag will be underestimated for the overhang and overestimated for the flange between the webs. When the whole section is concerned the two errors tend to cancel out.

Based on these observations it is proposed to compute the shear lag coefficient for a complex cross section as:

$$
\lambda = \frac{\sum \lambda_i B_i}{\sum B_i}
$$

(10)

where $\lambda_i$ and $B_i$ are the shear lag coefficient and the flange width of each basic unit, respectively.

The accuracy of eqn (10) is compared with FEM results for several test problems. The errors, as defined by eqn (5), are given in Table 5. The $(B_1+B_2)/L$ ratio has been varied from 0.1 to 1.0 on 20 steps.


