Derivation of the equations for thermal convection in a porous material.

Enok Palm and Jan Erik Weber

In this note the equations valid for thermal convection are derived, assuming that the (particle) Reynolds number is small. The note is an extension of the hydrodynamical considerations leading to Darcy's law.

We shall consider a porous material which, for simplicity, is assumed to consist of uniform,spherical grains, the space between the grains being filled with a fluid. The present considerations are an extension of the hydrodynamical considerations leading to Darcy's law as given by Rumer and Drinker (1966).

Applying the Boussinesq approximation the equations governing the motion of the fluid may be written

$$\rho_o\left(\frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}\right) = -\nabla p + \rho\vec{g} + \mu\nabla^2\vec{v} \qquad (1)$$

$$\nabla\cdot\vec{v} = 0 \qquad (2)$$

$$\frac{\partial\theta}{\partial t} + \vec{v}\cdot\nabla\theta = \kappa\nabla^2\theta \qquad (3)$$

$$\rho = \rho_o(1-\alpha(\theta-\theta_o)) \qquad (4)$$

Here (1) and (2) are the equations of motion and continuity, respectively. (3) denotes the equation for conduction of heat and (4) the equation of state. Furthermore, ρ is the density, ρ_o a standard density, $\vec{v}$ the velocity, t the time, p the pressure, g the acceleration of gravity, μ the viscosity, θ the temperature, κ the thermal diffusivity, α the coefficient of expansion and θ_o a standard temperature.

Consider the porous material contained within a cube of length L' where L' is much larger than the diameter d of the grains and much smaller than the characteristic length scale of the "macroscopic" motion (which for moderate values of the Rayleigh number equals the distance between the two boundaries), Integrating (1) over this cube, the first term may be written

$$\rho_o \int_\tau \frac{\partial \vec{v}}{\partial t}\, d\tau = \rho_o n \, \overline{\frac{\partial \vec{v}}{\partial t}}\, \tau \qquad (5)$$

where a bar denotes the mean, and n is the volume porosity (the ratio of the fluid volume to the volume of the porous material).

Applying (2) and the Gaussian theorem, the second term in (1) leads to

$$\int_\tau \vec{v}\cdot\nabla\vec{v}\,d\tau = \int_\sigma \vec{v}\vec{v}\cdot d\vec{\sigma} = n(\overline{\vec{v}}\overline{\vec{v}}\cdot\ \vec{\sigma})_\sigma + \int_\sigma \vec{v}'\vec{v}'\cdot d\vec{\sigma} \qquad (6)$$

where σ denotes the surface of the cube and prime denotes deviation from the mean. It is assumed that, owing to the homogeneity of the porous material, the mean defined by an area integration is equal to the mean defined by volume integration. Strictly speaking, n is here the area porosity which, however, for spheres is equal to the volume porosity. It seems reasonable that the last term in (6) is small, and this will therefore be cancelled. Utilizing that the scale of motion is assumed large compared to L',(6) may then be written

$$\int \vec{v}\cdot\nabla\vec{v}\,d\tau = n\ \overline{\vec{v}}\cdot\nabla\overline{\vec{v}}\tau \qquad (7)$$

Applying (4) and disregarding an insignificant constant, right hand side of (1) may be written

$$-n\tau\rho_o\alpha\overline{\theta}\vec{g} + n\tau\mu\nabla^2\overline{\vec{v}} - n\tau\nabla\overline{p} - \Sigma\vec{f}_s - \Sigma\vec{p}_s \tag{8}$$

Here

$$\Sigma\vec{f}_s = \int_i d\vec{\sigma}\cdot\nabla\vec{v} = \int_i \frac{\partial\vec{v}}{\partial n}\, d\sigma$$

and

$$\Sigma\vec{p}_s = -\int_i p\,d\vec{\sigma}$$

where i denotes the total surface of the spherical grains within the cube. Thus $\Sigma\vec{f}_s$ are the total viscous drag and $\Sigma\vec{p}_s$ the total pressure drag acting on the grains.

The terms $+\Sigma\vec{f}_s + \Sigma\vec{p}_s$ give rise to Darcy's law. In the case of one single sphere the viscous and pressure drag may be written (Stoke's law)

$$\vec{p} + \vec{f} = \lambda\mu d\overline{\vec{v}} \tag{9}$$

where λ equals 3π. Assuming that the form (9) of the drag also is true in the presence of a number of uniform spheres, we get

$$\Sigma\vec{p}_s + \Sigma\vec{f}_s = N(\vec{f}+\vec{p}) \tag{10}$$

where N is the number of grains within the cube. Since

$$N = \frac{(1-n)\tau}{\frac{1}{6}\pi d^3} \tag{11}$$

(10) takes the form

$$\Sigma\vec{p}_s + \Sigma\vec{f}_s = \frac{\lambda(1-n)\mu\tau}{\frac{1}{6}\pi d^2}\,\bar{\vec{v}} \qquad (12)$$

or,

$$\Sigma\vec{p}_s + \Sigma\vec{f}_s = \frac{\tau n^2\mu}{k}\,\bar{\vec{v}} \qquad (13)$$

where k is the permeability and here given by

$$k = \frac{\frac{1}{6}\pi n^2 d^2}{\lambda(1-n)} \qquad (14)$$

$\Sigma\vec{f}_s$ is proportional to the velocity gradients which, obviously, are much larger in the case of closely packed spheres than for one single sphere. λ is therefore for a porous material much larger than the value 3π valid for one sphere. For a characteristic value of n ($n = 0.37$) Rumer and Drinker (1966) find from a set of experimental dates that

$$k = 6.54\cdot 10^{-4} d^2 \qquad (15)$$

The equations for conduction of heat for the fluid and the grains are, respectively,

$$\rho_o c_o\left(\frac{\partial\theta_o}{\partial t} + \vec{v}\cdot\nabla\theta_o\right) = k_o\nabla^2\theta_o \qquad (16)$$

$$\rho_1 c_1 \frac{\partial\theta_1}{\partial t} = k_1\nabla^2\theta_1 \qquad (17)$$

where subscript o refers to the fluid and subscript 1 to the grains. c is the specific heat and k the thermal conductivity

Integrating these equations over the volume considered, adding them, applying that the heat flux is the same on both sides of the fluid-grain boundary, we get

$$n\rho_o c_o\left(\frac{\partial\overline{\theta}_o}{\partial t} + \overline{\vec{v}}\cdot\nabla\overline{\theta}_o\right) + (1-n)\rho_1 c_1\frac{\partial\overline{\theta}_1}{\partial t} + n\rho_o c_o\overline{\vec{v}'\cdot\nabla\theta_o'} =$$

$$= nk_o\nabla^2\overline{\theta}_o + (1-n)k_1\nabla^2\overline{\theta}_1 \qquad (18)$$

As above, neglecting the term due to deviations from the mean and, furthermore, assuming that the temperature in the fluid and the grains are approximately equal, we obtain

$$(n\rho_o c_o + (1-n)\rho_1 c_1)\frac{\partial\overline{\theta}}{\partial t} + n\rho_o c_o\overline{\vec{v}}\cdot\nabla\overline{\theta} = (nk_o + (1-n)k_1)\nabla^2\overline{\theta} \qquad (19)$$

dropping unnecessary subscripts. This is the same form of the heat equation as derived by Katto and Masuoka (1967).

If the motion is steady, (19) reduces to

$$\overline{\vec{v}}\cdot\nabla\,\overline{\theta} = \kappa_m\nabla^2\overline{\theta} \qquad (20)$$

where κ_m is the thermal diffusivity for the porous material, given by

$$\kappa_m = \frac{nk_o + (1-n)k_1}{n\rho_o c_o} \qquad (21)$$

A formula of the form (20) is usually assumed to be valid in porous convection. The value (21) for κ_m, however, is to be considered as a rough estimate only, valid when k_o and k_1 are nearly equal. On the other side, when k_o and k_1 are of different order of magnitude, (21) highly exaggerates the κ_m value.

The equations (1) - (4) may for small (particle) Reynolds number now be written

$$\rho_o\left(\frac{\partial\vec{\bar{v}}}{\partial t} + \vec{\bar{v}}\cdot\nabla\vec{\bar{v}}\right) = -\nabla\bar{p} - \rho_o\alpha\bar{\theta}\vec{g} + \mu\nabla^2\vec{\bar{v}} - \frac{\mu n}{k}\vec{\bar{v}} \tag{22}$$

$$\nabla\cdot\vec{\bar{v}} = 0 \tag{23}$$

$$\left(1 + \frac{(1-n)\rho_1 c_1}{n\rho_o c_o}\right)\frac{\partial\bar{\theta}}{\partial t} + \vec{\bar{v}}\cdot\nabla\bar{\theta} = \kappa_m\nabla^2\bar{\theta} \tag{24}$$

It should be noted that in the formulas above $\vec{\bar{v}}$ is the velocity, and not the specific discharge $\vec{\bar{q}}(\vec{\bar{q}} = n\vec{\bar{v}})$. For larger Reynolds number, larger than one, say, the pressure terms $\Sigma\vec{p}_s$ become important, and Darcy's law must be replaced by a resistance law proportional to the square of the velocity.

It is easily seen that the convective term and viscous term in (22) are much less the resistance term $(-\frac{\mu n}{k}\vec{\bar{v}})$.

Let L denote the length scale of the mean motion. We then have

$$|\nabla^2\vec{\bar{v}}| \sim \frac{|\vec{\bar{v}}|}{L^2} << \frac{n|\vec{\bar{v}}|}{k} \tag{25}$$

utilizing that k/d^2 is according to (15) very small.

Correspondingly,

$$|\vec{\bar{v}}\cdot\nabla\vec{\bar{v}}| \sim \frac{|\vec{\bar{v}}|d}{\nu}\cdot\frac{\nu|\vec{\bar{v}}|}{Ld} << \mathrm{Re}\,\frac{\nu n|\vec{\bar{v}}|}{k} \tag{26}$$

where ν is the kinematic viscosity and Re is the Reynolds number. By a similar argument also the local acceleration term may be cancelled such that the time derivative only enters in (24).

We then end up with the following set of equations

$$\nabla p + \rho_o \alpha \theta \vec{g} + \frac{\mu}{k'} \vec{v} = 0 \tag{27}$$

$$\nabla \cdot \vec{v} = 0 \tag{28}$$

$$\left(1 + \frac{(1-n)\rho_1 c_1}{n\rho_o c_o}\right) \frac{\partial \theta}{\partial t} + \vec{v} \cdot \nabla \theta = \kappa_m \nabla^2 \theta \tag{29}$$

where we have omitted the bar and

$$k' = k/n.$$

These equations will be utilized in a forthcoming paper on nonlinear porous convection.

References

Rumer, R.R. & Drinker, P. 1966 J. Hydraulics. Div., Am.Soc. Civ. Engrs. 92 (5), 155.

Katto, J. & Masuoka, T. 1967 Intern. J. Heat & Mass Transfer 10, 297 (1967)