Model development and simulation of tsunamis outside Portugal and Greece; Second GITEC status report.


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Preface

This manuscript is the contribution to the second GITEC progress report from the group at the Department of Mathematics, Mechanics Division, University of Oslo. The first progress report from the Oslo group is found in [1].

GITEC (Genesis and Impact of Tsunamis on European Coasts) is an European project under the Environment Program. The project, that started in 1993 and will be completed within May 1995, has involved 9 different research groups; two from Italy (Bologna and Genoa), one from France (Paris), one from Portugal (Lisbon), two from Greece (Athens) and two from Norway (Bergen and Oslo). The responsible coordinator for the project has been Prof. S. Tinti from Bologna.

1 Summary of work accomplished

This report summarizes the work done under the GITEC project at Department of Mathematics, University of Oslo during the period 1 Jan. - 31 Oct. 1994.

1.1 Personnel

The GITEC-group at University of Oslo includes following scientists;

Dr.philos Bjørn Gjevik, professor, project leader
Dr.philos Geir Pedersen, professor
Dr.scient Carl B. Harbitz, associated scientist
Cand.scient Elen Dybesland, research assistant

Gjevik and Pedersen are working part time for the project, the costs being paid by the University. Harbitz has been working part time and Dybesland full-time both being paid by the project.

Two other scientists at the institute, Dr. scient. Hans Petter Langtangen and cand scient Helge Jonsgaard have contributed to the project with their ongoing related activity on model development. Details of this work is described below.

1.2 Model development

An extensive upgrading of the basic hydrostatic tsunami model is undertaken. Thus we can now introduce complex and more realistic source shapes. In addition to the box-shaped dipole source used in the previous simulations we have also implemented the Okada source used by the Paris group. A detailed study of noise generation by moving under-water slide has also been undertaken.

The extensions also meet the need to refine and modify the depth matrix in certain regions.
Open boundary conditions are implemented with a flow relaxation scheme (FRS). Therefore, we are able to run the model in smaller domains with higher grid resolution. Open boundary conditions are for example essential for the refined Storegga case simulations.

Besides, the model is continuously upgraded, including pre- and postprocessor programming, to improve the model accessibility for other users, and to simplify the analysis of the model data.

The project work has also benefitted from related ongoing model development at the institute. A finite-element program package (DIFFPACK) developed by Dr. Hans Petter Langtangen has been used for modelling wave generation by underwater slides. A simulation model for 3-D run-up based on a Lagrangian coordinate description developed by cand scient Helge Johnsaard in cooperation with professor G. Pedersen has been extensively tested.

1.3 Simulations of the coastal response of the Storegga tsunami.

Refined simulations of the tsunami generated by the Second Storegga Slide (ca 7200 BP) are done with a new high resolution depth matrix (grid size 500 m) including both the open sea regions and the fjord systems of Mid-Norway.

The objectives of the new Storegga simulations are to gain information on tsunami propagation in fjord systems, and to compare calculated tsunami heights with run-up heights deduced from marine deposits by the GITEC group, University of Bergen. The simulations have also been used to produce a video animation of the wave, which has been shown in a program on the main Norwegian TV-channel focusing on the Storegga slide and tsunami.

1.4 Model simulations of tsunami waves from the area around Gorringe Bank southwest of Portugal

The work with simulation of tsunami generation and propagation by earthquake sources near Gorringe Bank has continued. In particular we have worked with refining the parameterization of the tsunami source and studied the sensitivity of the simulation to changes in the source parameters. The Okada source, used by the Paris group has also been included in this study.

We have also performed simulations where the source was located on the shelf slope west and south of Portugal. The purpose of these simulations have been to investigate possible source locations for the disastrous 1755 events with up to 10 m wave height in the Lisbon and the Algarve areas.

The Lisbon group has provided a new high resolution depth matrix for the model domain. We have rerun our computations for the 1969 event with the new depth matrix and are in the process of analyzing the results and making comparison with the tide gauge records from Cascais, Lagos, Faro, Cadiz and Horta provided by the Lisbon group.
Grid refinement studies have also been another major effort. These studies have enabled us to obtained better estimates of travel time when comparing the modelled travel times with observations from coastal tide gauges for the 1969 event.

In cooperation with the Lisbon and the Paris groups we have started drafting a journal article with results of the simulations of Portuguese events. We plan to have the draft of the article ready by the end of this year.

1.5 Model simulation of tsunami events in the Aegean Sea

Through cooperation with Dr. A. Yalciner at the University of Ankara we have got access to his depth matrix for the Aegean Sea and the Greek Archipelagos with grid resolution 2.9 km. We have interpolated in this depth matrix and made simulations of the 1956 Amorgos Island tsunami with grid resolution down to 0.48 km. Dr. G. Papadopoulos has provided data for the source with estimates of the key source parameters. The results of these simulations show large discrepancy between model and observations. A report on the results of these simulations will be written before the end of the project.

1.6 Video production

Based on the simulations of tsunami waves from Gorringe Bank southwest of Portugal we have made a video animation of the wave propagation. This work has been done in cooperation with Computing Services at the University of Oslo (USIT). The animation is produced with the Explorer visualization tools on Silicon Graphics workstations and the video has been well received. It has been presented at several formal and unformal meetings and copies have been distributed to the Paris and Lisbon groups, NORSAR, Norway, and Oceanographic Commission, UNESCO, Paris. More recently we have also made animations of the waves generated by the Storegga slide and 3-D run-up simulations. The massive support we have received from USIT during the work with the film has been essential for the success.

1.7 Tsunami Catalogue

The University of Oslo has provided all accessible information on five major Norwegian tsunami events for the common tsunami catalogue of the GITEC project.

1.8 Presentations at meetings

A general overview of the status of the project work by the Oslo group was given by Gjevik at the GITEC project meeting in Grenoble, April 1993.

Results of the simulations of the wave generation and propagation from the Gorringe Bank area southwest of Portugal was presented by Pedersen at the EGS meeting in Grenoble, April 1993.

Results of the project have also been presented at the “Faglig-Pedagogisk Dag” at the University of Oslo, January 1994. Gjevik was also invited to give a presentation
of the project at the MORENA-project meeting in Oslo in June. The MORENA is a project under the MAST-program.

2 Motivation for further model activities

As described in the preceding sections we have performed a series of case studies of tsunamies due to earthquake faulting as well as submarine slides. Our main tool has been a simple, though very effective, finite difference technique for the linear hydrostatic equations. Using this method we have resolved large domains by a uniform grids that typically contain some millions of grid points and require a few hours CPU time on a modern work station. To a smaller extent we have also employed a primitive nesting of the model. Some of the case studies involve a detailed comparison with measurements and observations at, or close to, the shore. Subsequently, we have experienced some of the limitations of the standard model. Frequently the errors due to discretization and approximative equations are small or can be roughly corrected for by simple means. In other cases important features cannot be resolved by a simple global model and further development is needed. To meet with the requirements of improved techniques, as well as detecting the limitations of the existing tools, the work on model development at the moment follows three different paths.

A considerable effort has been put into the estimation of numerical and conceptual shortcomings of the standard finite difference tsunami model, based on an Arakawa C grid [3] and usually hydrostatic linear equations. We reported some preliminary results along this line, concerning the seamounts as those located at the Gorringe ridge, in the first GITEC report. This time we will refer results that concern coastal areas. Some of this studies are relevant to other numerical methods as well.

It is our belief that fairly accurate run-up estimates in many cases can be obtained by application of simple, semi-analytic and even linear models. However, other cases, as well as some features concerning currents and sedimentation in the beach zone etc., require more sophisticated models including a 3-D description of a moving shoreline.

We pursue the development of both complex 2-D models as well as long wave models for the 3-D case. The latter is briefly described below.

Our experiences concerning the difference models suggest that there are a number of problems that preferably should be treated by a finite element approach. As a start we have developed a Boussinesq solver that we at present evaluate through the simulation of different test cases, in addition to analysis of numerical dispersion relations etc. Due to limitations on the extent of the report we will give only a brief sketch of this work.

We note that all themes will be topics of subsequent articles or separate reports at least.
3 A run-up model

The model is based on Lagrangian coordinates and may be regarded as an 3-D extension of the method in [5].

3.1 Basic equations

Marking dimensional quantities by a star we introduce a coordinate system with horizontal axes \( oz^* \), \( oy^* \) in the undisturbed water level and \( oz^* \) pointing vertically upwards. Further we assume a bottom at \( z^* = -h^* \) and denote the surface elevation and averaged horizontal particle velocity by \( \eta^* \) and \( v^* \) respectively. Applying the maximum depth, \( h_0^* \), and a typical wave length, \( L^* \), as "vertical" and "horizontal" length scales we are then led to the following definition of non-dimensional variables

\[
\begin{align*}
x^* &= L^* x, & y^* &= L^* y, & t^* &= L^* (gh_0^*)^{-\frac{1}{2}} t, \\
\eta^* &= h_0^* \eta, & x^* &= h_0^* x, & v^* &= (gh_0^*)^{\frac{1}{2}} v,
\end{align*}
\] (1)

where \( g \) is the constant of gravity. Provided \( \beta \equiv (h_0^*/L^*)^2 \) is sufficiently small, the wave motions of the basin can be ruled by the long wave equations. To trace the moving coastline automatically we formulate the equations in a Lagrangian description. The Lagrangian enumeration coordinates, \( a \) and \( b \), are introduced through the transformation formulas:

\[
\begin{align*}
a_t + v \cdot \nabla a &= 0, \quad a(x, y, 0) = a_0(x, y) ; &

b_t + v \cdot \nabla b &= 0, \quad b(x, y, 0) = b_0(x, y),
\end{align*}
\] (2)

where \( v \) is the averaged horizontal velocity, \( \nabla \) the horizontal component of the gradient operator and indices denote partial differentiation. Usually the initial conditions \( a_0 \) and \( b_0 \) are replaced by \( x \) and \( y \) respectively. Another choice for \( a_0 \) and \( b_0 \) really corresponds to the definition of curvilinear Lagrangian coordinates. Already at this stage we note that the use of curvilinear Lagrangian coordinates do not lead to additional terms in the continuity or momentum equations since Lagrangian coordinates in any case can be regarded as set of time dependent curvilinear coordinates from an Eulerian point of view. The continuity equation now becomes:

\[
H \frac{\partial (x, y)}{\partial (a, b)} = V,
\] (3)

where \( H \equiv h + \eta \) is the total water depth and

\[
V = H(a, b, 0) \frac{\partial (x, y)}{\partial (a, b)}_{t=0}.
\] (4)

The quantity \( V \) has interpretation of volume density per area in the \((a, b)\) plane and we note that equation (3) explicitly express volume conservation for a vertical material fluid column. Using (3) we obtain momentum equations of the form

\[
\frac{\partial^2 x}{\partial t^2} = -\frac{H}{V} \frac{\partial (\eta, y)}{\partial (a, b)}; \quad \frac{\partial^2 y}{\partial t^2} = -\frac{H}{V} \frac{\partial (x, \eta)}{\partial (a, b)},
\] (5)
By further application of (3) we may recast the momentum equations into conservative form. We display only the $z$ component that reads:

$$V \frac{\partial^2 z}{\partial t^2} = -\frac{\partial}{\partial a} \left( \frac{1}{2} H^2 \frac{\partial y}{\partial b} \right) + \frac{\partial}{\partial b} \left( \frac{1}{2} H^2 \frac{\partial y}{\partial a} \right) + V \frac{d h}{d z}$$  \hspace{1cm} (6)

3.2 Numerical method.

We denote a discrete approximation to a function $f$, at $a = i \Delta a$, $b = j \Delta b$ and $t = p \Delta t$, by $f_i^{(p)}$. The computational domain is divided into material fluid cells with a $n$– node in the middle and $z$– and $y$– nodes at the corners, as shown in figure 1. This arrangement, that is well suited for calculations of Jacobi determinants, is often referred to as an Arakawa B grid [3]. The primary unknowns are:

$$\eta_i^{(p)}_{j}, \ H_i^{(p)}, \ z_i^{(p)}_{j+\frac{1}{2}}, \ y_i^{(p)}_{j+\frac{1}{2}}$$  \hspace{1cm} (7)

To abbreviate the expressions and improve their legibility we define a symmetric difference operator, $\delta_a$, and an midpoint average operator, $\overline{f}_a$, according to:

$$\delta_a f_i^{(p)}_{j+\frac{1}{2}} = \frac{f_{i+1,j}^{(p)} - f_{i,j}^{(p)}}{\Delta a} \hspace{1cm} \overline{f}_a f_i^{(p)}_{j+\frac{1}{2}} = \frac{f_{i+1,j}^{(p)} + f_{i,j}^{(p)}}{2}$$  \hspace{1cm} (8)

The difference and average operators with respect to the other coordinates are defined correspondingly. We also collect terms within square brackets leaving the common indices outside.
The numerical method is explicit in time and the computational cycle starts with the calculation of a new generation of $H$-values from the updated displacement field, $x$ and $y$, through a discrete version of (3):

$$[H = V/J]_{i,j}^{(p)}$$

where

$$[J = \delta_a \delta_b \delta_b \delta_a]_{i,j}^{(p)}$$

(9)

The quantity $V_{i,j}$, that represent the volume of the computational cell, is simply the product of $H$ and $J$ at $t = 0$. New values for the surface elevation, $\eta$, may now be obtained from:

$$[\eta = H - h(\bar{x}^{(b)}, \bar{y}^{(b)})]_{i,j}^{(p)}$$

(10)

As the last step in each time cycle we discretize the momentum equations. The simplest option is to represent (5) directly according to:

$$[\delta_i^2 x = - \frac{H}{V_0} \eta^{(b)} (\delta_a \delta_b x^{(b)} - \delta_b \delta_a x^{(b)})]_{i+\frac{1}{2},j+\frac{1}{2}}$$

(11)

supplemented by a corresponding discrete $y$-component. Alternatively we have also employed discrete versions of the conservative formulation (6):

$$[V \delta_i^2 x = - \delta_a \left( \frac{1}{2} \frac{H}{H^2} \delta_b x^{(b)} \right) + \delta_b \left( \frac{1}{2} \frac{H}{H^2} \delta_a x^{(b)} \right) + V \{h_x\}]_{i+\frac{1}{2},j+\frac{1}{2}}$$

(12)

where different representations of the bottom pressure term ($h_x$) have been attempted. The two first terms on the right hand side represent the vertically integrated pressure on the sides of the momentum conservation cell in figure 1.

We adapt a curvilinear grid, see discussion below (2), to make a possibly curved shoreline to pass through a line of $H$-nodes at $i = 0$, say. The choice (11) for the momentum equation then involves $\eta$ values at the shoreline. These values are found by invoking $H = 0$ and linearly extrapolated values for $x$ and $y$ in (10) for $i = 0$. In the conservative formulation we employ the boundary condition $H_{i,j}^{(n)} = 0$ directly, whereas the $h_x$ term is calculated either by differentiating the depth function or by finite differences, again involving some sort of extrapolation at the shoreline. We have performed a large number of tests of the two formulations with a variety of more or less sophisticated implementations of the bottom pressure term $h_x$. So far we have found no indications that the simpler non-conservative option is inferior in any respect. Hence, we employ (11) in the preceding sections.

Along boundaries with a non-vanishing fluid depth a no-flux condition apply. In this paper only straight boundaries are considered, and the no-flux condition is easily implemented through symmetry relations.

For constant depth, $h \equiv 1$, the linearized version of the difference equations (9) through (11) yields the CFL stability criterion $\Delta t < \min(\Delta a, \Delta b)$. To the leading order the numerical dispersion relation may be written

$$\omega^2 = k^2 + \frac{\Delta t(k^2 + l^2)}{12} - \Delta a^2 \left( \frac{k^4}{12} + \frac{k^2 l^2}{4} \right) - \Delta b^2 \left( \frac{l^4}{12} + \frac{k^2 l^2}{4} \right) + \text{h.o.}$$

(13)
3.3 Geometry and incident wave

We have chosen a simple set up for our calculations. The incident wave is a plane wave with a trigonometric hat shape that is specified in a region of constant depth. As an idealized model of the coastline we employ a symmetric bell shaped point adjoined at each side by a straight coastline. The bottom is sloping monotonously from the coast until the constant depth of the offshore region is attained. Since the geometry as well as the incident wave are symmetric we perform calculations in half the domain only. In the present context it is convenient to employ a horizontal length scale, $L$, linked to the slope, while the constant depth, $h_0$, of the offshore region is used to make the surface elevation dimensionless. Accordingly the time scale becomes $L/\sqrt{gh_0}$. We may the express the incident wave through requiring zero velocity and imposing an initial surface elevation of the form:

$$
\eta_{in} = A(1 + \cos(k(x - x_i))) \quad \text{for} \quad |k(x - x_i)| < \pi
$$

where $k = 2\pi/\lambda$. In the linear approximation this initial condition leads to an incident wave of amplitude $A$.

A bottom profile at constant $y$ contains a section of constant slope adjoined to a flat bottom section by a second order polynomial. The computational domain is confined to $0 < y < y_p$, where we at $y = y_p$ introduce on half of a Gaussian shaped point implicitly completed by the application of a symmetry condition. The bottom function can be written:

$$
h(x, y) = q(x - s(y)) ; s(y) = Be^{-\frac{(x-y_p)^2}{r}}
$$

where the depth profile is given by the continuously differentiable function.

$$
q(\xi) = \begin{cases} 
\tan \theta \xi & \text{if } \xi < \cot \theta - \ell \\
1 - \frac{\tan \theta}{4\ell} (\xi - \cot \theta - \ell)^2 & \text{if } -\ell < \xi - \cot \theta < \ell \\
1 & \text{if } \xi > \cot \theta + \ell 
\end{cases}
$$

3.4 An example

For illustration we will display some results for one case only, corresponding to the parameters: $\lambda \tan \theta = 2$, $A = 0.0175$, $B \tan \theta = r \tan \theta = 0.5$. These parameters may correspond to a slide generated wave in a fjord or lake, but the incident wave are too high and long (in relation to the bottom slope) as compared with most tsunamis originating in deep sea. In figure 2 we have displayed the displacement field and surface elevations at selected times. We notice in particular the large amplitudes and deformations due to the generated edge waves at $t = 250$. 

9
Figure 2: Contour plots of the surface, increment 0.002, and wireplot for the displacement field, displaying one out of four points in each direction.
4 Grid effects in nearshore regions

4.1 Introduction

There are several techniques available for solving partial differential equations in geometries with curved boundaries. The most powerful alternative is probably the finite element technique, while comparatively simple geometries, at least, can be treated effectively through invocation of curvilinear grids. However, for linear shallow water equations the straightforward method of replacing the true boundary by a “staircase” boundary is still extensively used. Even though this approach yields a rather crude first order representation of the coastline this is generally regarded as outweighed by the simple implementation and effectiveness of the resulting code.

At first glance a staircase boundary might be expected to create a large amount of noise and possibly a series of additional crucial spurious effects. However, experience shows that for quite a number of applications this is not the case. On the other hand, a closer scrutiny sometimes reveals that staircase boundaries may introduce qualitatively new features even in very simple contexts. We will refer a simple and relevant example. A vertical impermeable wall situated in deep water and aligned at an angle 45° to the grid axes, will reflect incident waves without creation of any appreciable noise. Still, as shown in [9] and [8] such a boundary may act as a guide for trapped waves even in constant depth.

In tsunami calculations we want to extend the calculations of our models as close to the shoreline as possible, even though the actual runup motion generally is achievable only through application of nonlinear fine scale models including a moving shoreline description. As a wave propagates in shoaling water its length will decrease. According to geometrical optics the wavelength will be proportional to \( h^{3/2} \) (\( h \): depth) for waves of approximately normal incidence. On the other hand, any second order finite difference scheme will possess a dispersion relation of the form

\[
c_n = c_a(1 + k^2(q_1 \Delta x^2 + q_2 \Delta t^2 + q_3 c_a \Delta x \Delta t)) + O(\Delta x^3, \ldots)
\]

where \( c_n \) and \( c_a \) are numerical and analytical wave speed respectively, \( k \) is the wave number, \( \Delta x \) and \( \Delta t \) are the grid increments, and \( q_1, q_2, q_3 \) are coefficient specific for the actual method. It is then readily deduced that the relative importance of numerical dispersion is proportional to \( h^{-1} \) for a fixed \( \Delta x \). In the limit \( h \to 0 \) results from geometrical optics become completely inappropriate and the wavelength, defined in some sense or another, approach a finite value. Additional problems may, however, be anticipated due to the singularity at the shoreline point that may give rise to a logarithmic singularity in the surface elevation and a first order pole in the velocity. Still, it has been verified that even the simplest linear numerical schemes may provide good run up estimates in 2D calculations, as long as a noflux condition is properly applied at the shore [2]. When discussing 3D calculations we must consider

\footnote{It is noteworthy that the importance of the \( \Delta t^2 \) term remains unaltered due to the invariance of the period.}
the combined effects of staircase boundaries and the singularity and further difficulties may arise. In this report we intend to shed some light on these problems through some simple numerical experiments and observations.

4.2 Plane waves in shoaling water – a 2D test case

We have generated a two dimensional geometry by extracting an E-W cross section roughly midway between Lisabon and Cabo de Sao Vincente, as shown in figure 3, from the depth matrix for the Iberian sea made available to us by P. Miranda from the Lisbon university in 1993. The figure also displays an initial surface elevation corresponding to an uplift zone of the sea bottom, running South-North, of width \( a = 25 \text{km} \) and maximum height \( \Delta h = 1 \text{m} \). The source is situated at a depth of 5000m, just outside the continental shelf. Both the geometry and the source inherit characteristic dimensions which we believe to be typical for the Iberian tsunami incidents. We have performed computations using the linear hydrostatic equations and the Boussinesq equations, with and without the nonlinear terms. There are three main intentions behind the simulation: (i) estimate the nonlinear and dispersive effects, (ii) study the grid dependence in shallow regions, (iii) determine the amplification with special emphasize on the validity of Greens law.

Applying spline interpolation to the original depth data, we have defined a twice differentiable bottom function that enables the construction of grids of arbitrary resolution and distribution of grid points. Computations have been carried out for grid increments ranging from \( \Delta z = 10 \text{km} \) to \( \Delta z = 208 \text{m} \) (5km/24). In a related study of wave propagation across a seamount as well as in runup calculations in an idealized geometry we have employed an optimal, nonuniform, distribution of grid points with very good results. However, while this technique may be applicable to 3D finite element methods it has no really practical extension to 3D finite difference simulations.

In addition to snapshots of the surface elevation and time series the solutions are investigated through the secondary unknown \( \eta_{\text{max}} \). This quantity is the maximum surface elevation at any instant and is displayed as function of time or the depth at which it occurs.

4.2.1 Nonlinear and dispersive effects

For weakly dispersive waves we have the dispersion relation \( c = c_0(1 - \frac{1}{2}(kh)^2 + ...) \) where \( c_0 \) is the phase speed of infinitely long waves. Assuming again that the wavelength is determined by geometrical optics we find that the relative importance of dispersion is proportional to \( h \).

To demonstrate the effect of dispersion on disturbances of different shapes we have computed wave propagation from the three initial conditions in figure 4(a). In addition to the shape from figure 3 (marked “tilt” or “tiltslab”) we have employed

\(^2\)The simulation was originally designed as a first study of wave generation from a fault parallel to the Portuguese coast, but this is not essential in the present context.
a smooth shape (marked bell) and finally a shape resembling the displacement fields obtained from Okada's formula [4] (marked Okada). After 8 minutes the waves are still in deep water and the effect of dispersion is noticeable for all shapes. In addition to the production of a residual wave train the dispersion effect reduces the height of the leading pulse by roughly 20-30% as shown in figure 4(b-d).

The effect of nonlinearity is appreciable only in the vicinity of the shoreline.

4.2.2 Discretization errors, amplification

A scrutiny of the computed surface elevations reveals that the linear hydrostatic solutions are subjected to a significant numerical dispersion in shallow water, even for the finest grid $\Delta x = 208m$. The result is a trailing noise that is very different from the effect of real dispersion.

Denoting the depth in the source region by $h_m$ (5000m) we display the dimensionless quantity $f = \eta_{\text{max}}(h/h_m)^{1/2}/\Delta h$ as a function of $h$ in figure 5. As long as Greens law ($\eta_{\text{max}} \sim h^{-1/2}$) is fulfilled the depicted quantity then remains constant. For the linear hydrostatic solution, obtained from the finest grid, we observe a constant value of $f$ close to one half down to about 5m. The value of one half is due to a nearly uniform splitting of the initial disturbance in one outgoing and one in-going wave. Correspondingly, the Boussinesq solution yields a constant $f$, slightly less than 0.4, that can be observed down to $h = 10m$, say. The reduction from 0.5 to 0.4 is due to dispersion during the early stages of the wave evolution in deep water, while the interval with a constant $f$ indicates that dispersion effects are less important for $h$ less than 1000m, say, since the water is shallower and the different wave components already have become separated.

The most striking feature of the results in 5 is the systematic underestimation of amplitudes for the coarser grids. In shallow water we may even recognize a factor of more than 2.

4.3 Waves impinging on a Gaussian headland

We have performed 3-D simulations with a linear hydrostatic model and the bathymetry described in 3.3. The scale factor $L$ is chosen as to give $\tan \theta = \frac{1}{2}$, while the other parameters read: $A = 1$, $\lambda = 4$, $B = 3$, $r = 3$, $y_p = 80$ and $\ell = 0.25$. The total computational domain, $-1 < x < 47.241$, $39.9 < y < 80$, is large enough to prevent reflections from the lower and right boundaries to reach the vicinity of the peninsula. Simulations have been performed with 6 different resolutions. Employing the grid $\Delta x_1 = 0.241208$, $\Delta y_1 = 0.2005$ as a reference, we may write $\Delta x_\alpha = \alpha \Delta x_1$ etc. for $\alpha = 4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$. We note that $\Delta x_1$ corresponds roughly to having the slope covered by 8 points, the incident wave resolved by 16 points and to have 15 points along half the curved part of the coastline.

The incident wave is specified at a distance 10 from the straight part of the coastline and the equations are integrated over a time span of 75 dimensionless units. We have investigated the solutions by contour plots, time series and the surface
We will demonstrate a few key results only. In figure 6(a) we have depicted time series at a point near the tip of the headland. For the coarsest grid we observe a prevailing artificial oscillation, due to spurious edge wave generation. In figure 6(b) \eta values adjacent to the shore is shown. The results for the larger grid increments is clearly contaminated by noise. However, this noise does not spread to the offshore region. Both the spurious effects disappear if we choose to stop our calculations at a minimum depth \( h = 0.15 \), say.

5 A finite element technique for the Boussinesq equations.

5.1 Introduction

In the vicinity of geometrically complicated coastlines or in cases with large and abrupt depth variations it is generally convenient to apply the finite element method because of its flexibility in handling adaptive grids. According to section 4.3 even linear hydrostatic solutions become severely infected by noise when a staircase boundary is located in very shallow water. It is hardly conceivable that nonlinearities or the higher order derivatives in dispersion terms can be included in a model of this kind without introduction of a substantial artificial damping or filtering. The result will then be an unrealistic representation of the nearshore region. Even though small,
Figure 4: (a): The three different initial conditions. (b-d): The surface elevation at \( t = 8 \text{min.} \) for the different source shapes as obtained by hydrostatic and dispersive theory. The grid increment is \( \Delta x = 417 \text{m} \).
Figure 5: Amplification, Greens law. (a-b): Hydrostatic linear equations, (c-d): Boussinesq equations.
or at least simple domains, can be handled by curvilinear grids the finite element method provides the best prospects for a general treatment of shallow regions with irregular boundaries.

The equations that are discretized by the element technique are, apart from minor details, identical to those given in [1]. Herein we only note that the actual Boussinesq equations are formulated with the velocity potential and surface elevation as unknowns. For a more detailed discussion of the equations as well as relevant finite difference techniques we refer to [8].

### 5.2 Discussion of numerical method

The main idea in the finite difference methods for the Boussinesq equations, reported in [8], [10] etc., is to apply centered differences in space and time, but with a staggered temporal grid. The surface elevation and the velocity potential are hence unknown at different time levels, the difference being equal to one half of the time step length. For the spatial discretization in the finite element formulation we apply the Galerkin method, which will result in discrete equations of the same nature as those obtained from centered finite differences.

Due to the temporal discretization the coupled system of nonlinear partial differential equations give rise to two linear systems of algebraic equations that can be solved sequentially. For the linear hydrostatic case the coefficient matrix in these linear systems is the well-known mass matrix, and lumping the mass matrix results in an explicit scheme similar to the one widely used in the case studies described elsewhere in this report. In fact, if one applies bilinear elements on a rectangular, uniform grid and the integrals that appear in the Galerkin formulation are computed
by the trapezoidal rule, our finite element method becomes equivalent to the standard
difference scheme.

When the nonlinear and/or the dispersive effects are included in the equations,
two non-trivial linear systems must be solved at each time level. This will usually
be the "bottleneck" in the simulation code. Therefore, a vital part of the numerical
method is to find efficient methods for the solution of these linear systems. The
coefficient matrices are sparse, but non-symmetric. An efficient class of methods
well suited for sparse non-symmetric systems from general finite element grids is the
Krylov subspace methods in combination with suitable preconditioners. Numerical
experiments with the current type of linear systems show that for example the BiCG-
STEP scheme combined with a relaxed incomplete LU factorization preconditioner
performs well. Other successful basic iterative methods for the present problem in­clude Orthomin, GCR and GMRES. The linear systems are dependent on previously
computed \( \eta \) and \( \phi \) values so the systems must be re-computed at each time level. In
the present implementation, the assembly of the linear systems is more time consum­ing than the iterative solution process.

On a rectangular, uniform grid divided into bilinear elements, our numerical
method has a local truncation error of second order in space and time. Moreover,
the stability criterion for linear waves on constant depth is of the form

\[
\Delta t^2 \leq C \left( \frac{\Delta x^2}{h} + \frac{4}{3} \beta h \right)
\]

(18)

where \( \Delta t \) is the time step length, \( \Delta x \) is a spatial discretization measure, \( C \) is a
constant of order unity and \( \beta \) is as defined below (1).

Higher order elements improve the spatial discretization order. A similar improve­ment of the temporal discretization can be obtained by including correction terms in
the partial differential equations. For example, if numerical correction terms akin to
the dispersion terms are introduced, the temporal discretization error will be of or­der \( O(\Delta t^4, \alpha \Delta t^2, \beta \Delta t^2) \) where \( \alpha \) is the nondimensional amplitude. Such an improved
time discretization procedure in combination with higher order finite elements, of size
comparable to \( h \), then yields numerical errors of order \( O(\alpha \beta, \beta^2) \), which is the same
magnitude as the terms that are neglected when the Boussinesq equations are de­rived. In other words, the numerical method will be as accurate as any analytical
solution procedure for the same governing partial differential equations.

5.3 Numerical examples

The finite element method for the Boussinesq equations is constructed to give ap­proximately the same behavior as well-tested finite difference methods on rectangular,
uniform grids if elements of first order (linear triangle, bilinear quadrilateral) are used.
We can confirm this by a simple analysis in case of the assembled equations with a
constant depth. In the more general case, numerical experiments can be used to
empirically derive error estimates. For the present method, the main interest is re­lated to the performance on distorted grids, especially in combination with higher
order elements and the improved time discretization procedure. At the moment we are working on the test case in section 3.3. So far the hydrostatic results are very promising, while the dispersive solutions seem to be more sensitive to large grid distortions.

References


