A discrete energy account for long waves and its implication for the optical description.

G. Pedersen

November 20, 1995

Abstract

The shallow water equations, the corresponding energy conservation law and a standard finite difference method are reviewed. Then, as the first point of principal novelty, we derive the discrete counterpart to the energy equation. Invoking the assumption of a gently varying medium we then proceed to establish a discrete Greens law, that has previously been found by a WBKJ type approximation. Further, we demonstrate that the ratio between the averaged discrete energy flux and density equals the group velocity, as in the analytical case. Finally, corresponding results are presented for a somewhat wider spectrum of methods.

1 Introduction

Among researchers in wave theory a key question concerning numerical solution of wave equations is to what extent the numerical procedure define a virtual medium with properties that are analogous to those of the physical medium. Hence, the performance of numerical methods is often discussed in terms like numerical dispersion, numerical diffusion, spurious reflection etc. Pure mathematicians and numericists, on the other hand, generally prefer to stay within a framework of theoretical spaces while measuring the success of a method in a series of more or less obscure norms[1]. In the present paper, that definitely belong to the former tradition, we discuss the role of energy within the context of a finite difference technique for the long wave equations. Special emphasize is put on the link to physical optics.

The work herein is motivated by two preceding reports on amplification in shoaling water [2],[3]. The first of these contains a series of tests on the influence of grid effects on tsunami propagation with particular focus on amplification in shoaling water. In the latter report this theme was revisited with a more theoretical approach based on an optical theory derived from the numerical dispersion relation and a discrete generalization of the WBKJ expansion. One of the key results was a simple numerical counterpart to the well known Greens law. As an alternative to the WBKJ

[1]Even though the author has little affection for such an approach, it must be admitted that much of the work of this kind is good and sometimes even valuable.

method, or other formal perturbation expansions, the optics of *physical* waves may be derived from energy considerations; the basic assumption being that the wave locally is a single harmonic mode fulfilling the same dispersion relation and possessing the same averaged energy density and flux as in homogeneous medium. This line of advance was not attempted in [3] due to lacking definitions or undocumented properties of quantities like discrete energy density, but is the main issue of the present work. Starting with a non-trivial derivation of expressions for the discrete energy density and flux, we proceed to reproduce the optical results of [3]. A related question, that also is of principal interest in its own right, concerns the role of the discrete group velocity. The application of Fourier transform and the stationary phase approximation may provide us with one path to an answer. However, pursuing our main line of advance we will instead seek direct proof for the interpretation of the group velocity, formally obtained from the dispersion relation, as a measure of energy celerity.

2 Basic theory.

2.1 Scaling and equations.

Marking dimensional quantities by a star we introduce a coordinate system with horizontal axes $ox^\star$, $oy^\star$ in the undisturbed water level and $oz^\star$ pointing vertically upwards. Further we assume a bottom at $z^\star = -h^\star$ and denote the surface elevation and averaged horizontal particle velocity by and $\eta^\star$ and $\vec{v}^\star$ respectively. Applying the maximum depth, h_0, and a characteristic wavelength, L, as "vertical" and "horizontal" lengthscales we are then led to the following definition of non-dimensional variables

$$\left.\begin{array}{lll} x^\star = L^\star x, & y^\star = L^\star y, & t^\star = L^\star (g h_0^\star)^{-\frac{1}{2}} t, \\ \eta^\star = \alpha h_0^\star \eta, & z^\star = h_0^\star z, & \vec{v}^\star = \alpha (g h_0^\star)^{\frac{1}{2}} \vec{v}, \end{array}\right\} \qquad (1)$$

where g is the constant of gravity and α is an amplitude measure. Provided α and $\beta \equiv (h_0/L)^2$ are sufficiently small, the flow is governed by the long wave equations:

$$\frac{\partial \eta}{\partial t} = -\nabla \cdot (h\vec{v}), \quad \frac{\partial \vec{v}}{\partial t} = -\nabla \eta. \qquad (2)$$

We recognize the former as the depth integrated continuum equation, whereas the latter describes momentum conservation.

Multiplying the momentum equation in (2) by $\vec{v}$ and invoking the continuum equation we readily derive the energy equation

$$\frac{\partial E}{\partial t} + \nabla \cdot \vec{F} = 0, \qquad (3)$$

where

$$E \equiv \frac{1}{2} h(u^2 + v^2) + \frac{1}{2}\eta^2, \quad \vec{F} \equiv \eta h \vec{v}. \qquad (4)$$

where $\vec{v} \equiv u\vec{\imath} + v\vec{\jmath}$. Naturally the two terms in E are kinetic and potential energy, while it is easily realized that $\vec{F}$ is the effect of the pressure work and the advection of potential energy[2]. This process of identifying each of the terms and assure that they inherit every physical transport mechanism is crucial when conservation laws are derived from another set of governing equations. In fact, for any twice differentiable vector field $\vec{G}$, (3) imply a new conservation law with the modified density $E_m = E + \nabla \cdot \vec{G}$ and the modified flux $\vec{F}_m = \vec{F} - \partial\vec{G}/\partial t$. Hence, to name (3) as an energy equation, for instance, we must justify that E and $\vec{F}$ correspond exactly to the energy density and flux. This is generally rather straightforward. On the other hand, when seeking discrete conservation laws we may encounter serious conceptual problems since we cannot rely on exact physical interpretations. We will encounter this problem below.

2.2 Discrete formalism.

In the present paper we perform a lot of arithmetics on discrete quantities. These manipulations are carried out within a simple formalism that the author has found convenient also on a series of other occasions. The approximation to a quantity f at a grid-point with coordinates $(\beta\Delta x,\ \gamma\Delta y,\ \kappa\Delta t)$ where Δx, Δy and Δt are the grid increments, is denoted by $f^{(\kappa)}_{\beta,\gamma}$. To improve the readability of the difference equations we introduce the symmetric difference operator, δ_x :

$$\delta_x f^{(\kappa)}_{\beta,\gamma} = \frac{1}{\Delta x}(f^{(\kappa)}_{\beta+\frac{1}{2},\gamma} - f^{(\kappa)}_{\beta-\frac{1}{2},\gamma}), \tag{5}$$

and the midpoint average operator $\overline{}^{x}$ by:

$$(\overline{f}^{x})^{(\kappa)}_{\beta,\gamma} = \frac{1}{2}(f^{(\kappa)}_{\beta-\frac{1}{2},\gamma} + f^{(\kappa)}_{\beta+\frac{1}{2},\gamma}). \tag{6}$$

We note that the differences and averages are defined at intermediate grid locations as compared to f. Difference and average operators with respect to the other coordinates y and t are defined correspondingly. It is easily shown that all combinations of these operators are commutative. To abbreviate the expressions further we also group terms of identical indices inside square brackets, leaving the super- and subscripts outside the right bracket. Using the above definitions we derive some relations that will be needed later. Omitting the dummy specifications of grid sites we may first state the product rules:

$$\delta_x(fg) = \overline{g}^{x}\delta_x f + \overline{f}^{x}\delta_x g, \quad \overline{fg}^{x} = \overline{f}^{x}\overline{g}^{x} + \frac{\Delta x^2}{4}(\delta_x f)(\delta_x g). \tag{7}$$

It is easily realized that successive application of two difference operators produce the standard three point approximation to the second derivative. For the second order

[2] The advection of kinetic energy is of higher order in α and does not appear in the present approximation.

average operator we then find

$$\overline{f}^{xx} = f + \frac{\Delta x^2}{4}\delta_x^2 f. \tag{8}$$

In the expressions for discrete energy it is convenient to introduce a geometrical mean for products:

$$[(f * g)_t]^{(\kappa)} = \frac{1}{2}\left(f^{(\kappa-\frac{1}{2})}g^{(\kappa+\frac{1}{2})} + f^{(\kappa+\frac{1}{2})}g^{(\kappa-\frac{1}{2})}\right). \tag{9}$$

Corresponding operators are again defined with respect to the other coordinates. To improve the legibility we use a shorter notation for the temporal geometrical mean of squares:

$$[f^{(t*2)}]^{(n)} \equiv (f * f)_t^{(n)} = f^{(n+\frac{1}{2})}f^{(n-\frac{1}{2})}. \tag{10}$$

We may relate the geometric mean to the arithmetic mean through relations like:

$$(f * g)_t = \overline{f}^t\overline{g}^t - \frac{\Delta t^2}{4}(\delta_t f)(\delta_t g), \quad (\overline{u}^x)^{(t*2)} + \frac{\Delta x^2}{4}(\delta_x u)^{(t*2)} = \overline{u^{(t*2)}}^x. \tag{11}$$

By means of (7) a product like $\overline{f}^x\delta_x f$ is easily rewritten as the total difference $\delta_x(\frac{1}{2}f^2)$. Invoking also (8) and (11) we derive the related formula:

$$f\delta_t(\overline{f}^t) = \delta_t\{\frac{1}{2}(\overline{f}^t)^2 - \frac{\Delta t^2}{8}(\delta_t f)^2\} = \delta_t(f^{(t*2)}), \tag{12}$$

that will prove itself very helpful.

For a trigonometric function $f_\alpha = A\exp(\imath k\alpha\Delta x)$, defined for any real α, application of the difference and average operators yields the simple results:

$$\delta_x f = \imath\tilde{k}f, \quad \overline{f}^x = C_x f, \tag{13}$$

where $\tilde{k} \equiv \frac{2}{\Delta x}\sin(\frac{1}{2}k\Delta x)$ and $C_x = \cos(\frac{1}{2}k\Delta x)$. These relations will be useful for calculating discrete dispersion relations as well as averaged energy quantities.

2.3 Difference equations.

For the set (2) we employ the standard Arakawa C-grid [1] and staggered differences in time. This choice yields the most widely used method for the actual equations, even though the time discretization often is regarded as a FB ("forward/backward") representation in a non-staggered temporal grid. However, apart from initial and nonhomogeneous boundary conditions, the rather awkward FB interpretation do not affect the computed result. With $\eta_{i,j}^{(n)}$, $u_{i+\frac{1}{2},j}^{(n+\frac{1}{2})}$ and $v_{i,j+\frac{1}{2}}^{(n+\frac{1}{2})}$ as primary unknowns we write the difference equations as:

$$[\delta_t\eta = -\delta_x(\overline{h}^x u) - \delta_y(\overline{h}^y v)]_{i,j}^{(n+\frac{1}{2})}; \quad [\delta_t u = -\delta_x\eta]_{i+\frac{1}{2},j}^{(n)}; \quad [\delta_t v = -\delta_y\eta]_{i,j+\frac{1}{2}}^{(n)}. \tag{14}$$

For constant h there exist harmonic solutions with amplitudes A, $\hat{u}$ and $\hat{v}$ for η, u and v respectively. Denoting the frequency by ω and the wavenumber by $\vec{k} = k\vec{\imath} + \ell\vec{\jmath}$ we then obtain the discrete dispersion relation:

$$\tilde{\omega}^2 = h^{\frac{1}{2}}(\tilde{k}^2 + \tilde{\ell}^2), \tag{15}$$

where the involved quantities are defined according to:

$$\tilde{\omega} \equiv \frac{2}{\Delta t}\sin(\frac{\omega \Delta t}{2}), \quad \tilde{k} \equiv \frac{2}{\Delta x}\sin(\frac{k\Delta x}{2}), \quad \tilde{\ell} \equiv \frac{2}{\Delta y}\sin(\frac{k\Delta y}{2}). \tag{16}$$

We note that $\tilde{k} = k + O(\Delta x^2)$ etc and that the dispersion relation of the differential equations, (2), is reproduced to second order in the grid increments . The stability criterion, that follows readily from (15), reads:

$$h\Delta t^2 < \frac{\Delta x^2 \Delta y^2}{\Delta x^2 + \Delta y^2}. \tag{17}$$

For the velocity amplitudes we obtain:

$$\hat{u} = \frac{\tilde{k}}{\tilde{\omega}}A, \quad \hat{v} = \frac{\tilde{\ell}}{\tilde{\omega}}A. \tag{18}$$

3 Discrete energy conservation.

We seek a discrete counterpart to (3) where the involved quantities, as far as possible, maintain their interpretation as energy density and flux. First the nodes for the discrete density and fluxes must be decided. Guided by the structure of the C-grid and (4) we denote the x and y components of the energy flux by U and V respectively and invoke the discrete quantities:

$$E_{i,j}^{(n)}, \quad U_{i+\frac{1}{2},j}^{(n+\frac{1}{2})}, \quad V_{i,j+\frac{1}{2}}^{(n+\frac{1}{2})}. \tag{19}$$

The numerical counterpart to (3) then reads

$$[\delta_t E + \delta_x U + \delta_y V = 0]_{i,j}^{(n+\frac{1}{2})}. \tag{20}$$

According to the discussion at the end of section 2.1 we must expect problems connected with the ambiguity of this equation. When $\Delta x, \Delta y, \Delta t \to 0$ the discrete fluxes and densities must approach those in (4). Beyond this requirement the physical identification of energy cannot lead us to the "correct" choice for E, U and V.

Different procedures can be employed for the determination of E, U and V. Herein we present the most illustrative one, which is also closest to the standard approach for differential equations. Starting with multiplying the discrete components of the momentum equation with the volume flux components, we then use (14) and the identities of section 2.2 to change the terms into total differences. Performing our calculations as straightforward as possible and putting our trust in providence we may then hope to obtain a useful result. In view of the problem with non-uniqueness and the somewhat uncommon arithmetics involved, we will present the calculations in some detail.

Due to the location of the energy node we average the x-component of the momentum equation with respect to both x and t before multiplication with the averaged x component of the volume flux. Applying the product rule (7) we then obtain:

$$\overline{\overline{h}^x u}^x \delta_t \overline{u}^{xt} = -\overline{\overline{h}^x u}^x \delta_x \overline{\eta}^{xt} = -\delta_x(\overline{h}^x u \overline{\eta}^{xt}) + \overline{\eta}^{xxt} \delta_x(\overline{h}^x u). \tag{21}$$

The first term at the rightmost side is immediately recognized as a discrete representation of the x-component of the flux term in (4). Next we apply (7), (8) and the x-component of the discrete momentum equation to find:

$$\overline{\eta}^{xxt} - \overline{\eta}^t = \frac{\Delta x^2}{4}\delta_x^2 \overline{\eta}^t = -\frac{\Delta x^2}{4}\delta_x \delta_t \overline{u}^t, \quad \overline{(\overline{h}^x u)}^x = h\overline{u}^x + \frac{\Delta x^2}{4}\delta_x(u\delta_x h). \tag{22}$$

Substitution into (21) and application of (12) then yield:

$$\delta_t \left(\frac{1}{2} h (\overline{u}^x)^{(t*2)} \right) = -\delta_x(\overline{h}^x u \overline{\eta}^{xt}) + \overline{\eta}^t \delta_x(\overline{h}^x u) + \frac{\Delta x^2}{4} R, \tag{23}$$

where the last term, R, may be rewritten in several steps:

$$\begin{aligned} R &= -\delta_x(\overline{h}^x u)\delta_x \delta_t \overline{u}^t - \delta_x(u\delta_x h)\delta_t \overline{u}^t \\ &= -\delta_x(u \delta_x h \delta_t \overline{u}^t) + \left(\overline{u\delta_x h}^x - \delta_x(\overline{h}^x u) \right) \delta_x \delta_t \overline{u}^t \\ &= -\delta_x \delta_t \left(\frac{1}{2} \delta_x h \, u^{(t*2)} \right) - \delta_t \left(\frac{1}{2} h (\delta_x u)^{(t*2)} \right) \end{aligned}$$

The first term in the last line may be included in the flux as well as in the density. In fact we possess no conclusive means for deciding the true nature of the term. However, judging from its mere appearance the flux option seems most appropriate. We note that the questionable term disappears for constant depth. Employing (11) we may then write:

$$\delta_t \left(\frac{1}{2} h \overline{u^{(t*2)}}^x \right) = -\delta_x \left(u \left(h * \overline{\eta}^t \right)_x \right) + \overline{\eta}^t \delta_x(\overline{h}^x u). \tag{24}$$

Naturally, the corresponding expression from the y-component may be obtained simply by replacing u and x by v and y, respectively, in the above equation. Adding the two together and invoking the discrete continuity equation we obtain a conservation law of type (20) with

$$E = \frac{1}{2} h \left(\overline{u^{(t*2)}}^x + \overline{v^{(t*2)}}^y \right) + \frac{1}{2}\eta^2, \tag{25}$$

$$U = u \left(h * \overline{\eta}^t \right)_x, \quad V = v \left(h * \overline{\eta}^t \right)_y. \tag{26}$$

As mentioned above, the nonuniqueness vanish on constant depth. Still, as shown in section 5, there are other expressions for the energy quantities that fulfill the conservation law (20) also when h is constant.

The physical energy density as defined in (4) is obviously positive definite, which implies that the presence of a wave always leads to an increased energy relative to the equilibrium state. On the other hand, the discrete E given in (25) is *not* positive definite due to the geometrical averaging in the kinetic energy. This is most easily demonstrated for plane waves propagating parallel to the x-axis. For a given energy node the right hand side of (25) then involves the corresponding η node and the four neighbouring u nodes. An inspection of the numerical scheme reveals that all of those may be regarded as independent in the sense that no internal constraints on the five node values are imposed by the difference equations (14). This, as well as the consequence for positive definiteness, is easily demonstrated through an example. We specify initial conditions for u and η at respectively $t = -\frac{1}{2}\Delta t$ and $t = 0$, say. We are then free to choose, for instance,

$$u_{I-\frac{1}{2}}^{(-\frac{1}{2})} = u_{I+\frac{1}{2}}^{(-\frac{1}{2})} = 1, \quad \eta_I^{(0)} = 0.$$

Then, if we may select the rest of the initial values as to give

$$u_{I-\frac{1}{2}}^{(\frac{1}{2})} = u_{I+\frac{1}{2}}^{(\frac{1}{2})} = -1,$$

it follows that $E_I^{(0)} = -\frac{1}{2}h$. This is obtained when

$$\eta_{I-1}^{(0)} = -\frac{2\Delta x}{\Delta t}, \quad \eta_{I+1}^{(0)} = \frac{2\Delta x}{\Delta t}.$$

4 Energies of single harmonics.

4.1 Stable modes.

When h is constant and the Courant criterion (17) is fulfilled we may consider a wave mode of the form:

$$[\eta = A\cos\theta]_{i,j}^{(n)}, \quad [u = A\hat{u}\cos\theta]_{i+\frac{1}{2},j}^{(n+\frac{1}{2})}, \quad [v = A\hat{v}\cos\theta]_{i,j+\frac{1}{2}}^{(n+\frac{1}{2})}, \tag{27}$$

where the amplitudes are related by (18) and the phase, $\theta \equiv kx + \ell y - \omega t + \theta_0$, is defined at all grid sites. The energy density then becomes:

$$E = \frac{1}{2}A^2\left\{\frac{\tilde{k}^2C_x^2 + \tilde{\ell}^2C_y^2}{\tilde{k}^2 + \tilde{\ell}^2}\cos(2\theta) + C_t^2\right\}, \tag{28}$$

where $C_t = \cos(\frac{1}{2}\omega\Delta t)$ etc. From the dispersion relation (15) it immediately follows that $C_t \geq C_x$ for stable modes. Hence, E as given above is clearly non-negative.

However, as demonstrated in the preceding section a combination of modes may lead to negative values for E.

4.2 Unstable modes.

The discrete energy conservation law (20) with involved quantities given by (25) and (26) is valid regardless of the stability criterion (17). When the criterion is violated there exist modes with imaginary ω given by:

$$\omega = \frac{\pi}{\Delta t} + \mathrm{i}\omega_i\,, \quad \cosh^2 \omega_i = \frac{\Delta t^2}{4}(\tilde{k}^2 + \tilde{\ell}^2). \tag{29}$$

Confining the discussion to plane waves, $\ell = 0$, the discrete field is given by:

$$\left.\begin{array}{lcl} u_{j+\frac{1}{2}}^{(n+\frac{1}{2})} & = & (-1)^{n+1} h^{-\frac{1}{2}} A e^{-\omega_i (n+\frac{1}{2})\Delta t} \cos(k(j+\frac{1}{2})\Delta x + \theta_0), \\ \eta_j^{(n)} & = & (-1)^n A e^{-\omega_i n \Delta t} \cos(kj\Delta x + \theta_0), \end{array}\right\} \tag{30}$$

where ω_i may attain positive as well as negative values. The energy density now becomes:

$$E_j^{(n)} = \frac{\Delta x^2}{8} A^2 \tilde{k}^2 e^{-2\omega_i n \Delta t} \cos(2kj\Delta x + 2\theta_0), \tag{31}$$

We observe that instable modes inherit an energy density that is proportional with the square of the instantaneous amplitude, $A \exp^{-\omega_i t}$, and changes sign periodically in space. The latter is a necessity for reconciling the growth of instable modes with global conservation of discrete energy. This is most clearly demonstrated for eigenoscillations in a closed basin with noflux conditions applied at the side walls. For modes with $\omega_i \neq 0$ the total (summed) discrete energy then equals zero.

4.3 Averaged energy quantities and physical optics.

We define an average of a discrete variable by:

$$< F > = \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=n_0}^{n_0+N} F^{(n)}. \tag{32}$$

For products of harmonics it is readily shown that the result is independent of n_0 and spatial location and that the square of a cosine yields $\frac{1}{2}$ for the average etc. Substitution of a single stable harmonic mode, still defined as in section 4.1, that obeys the relations (16) and (18), into (25) and (26) and averaging give:

$$< E > = \frac{1}{2} C_t^2 A^2, \quad < U > = \frac{1}{2} \frac{C_t C_x \tilde{k}}{\tilde{\omega}} h A^2, \quad < V > = \frac{1}{2} \frac{C_t C_y \tilde{\ell}}{\tilde{\omega}} h A^2. \tag{33}$$

For a discrete wave mode in a slowly varying bathymetry we may now assume that (33) is locally valid. When we restrict the discussion to plane waves in a two dimensional bathymetry conservation of discrete energy then leads to:

$$h^{\frac{1}{2}} A^2 C_x = 2\frac{U}{C_t} = \text{const.} \tag{34}$$

Employing the discrete dispersion relation (15) we reproduce the discrete counterpart of Greens law that was reported in [3]:

$$A = \text{const.} \times (h - h_c)^{\frac{1}{4}}. \tag{35}$$

The quantity $h_c \equiv \frac{\Delta x^2 \tilde{\omega}^2}{4}$ is the stopping depth at which the incident wave is totally reflected and the optical theory collapses.

We may obtain the formal group velocity by differentiation of the numerical dispersion relation (15). Introducing the components of the group velocity according to $\vec{c}_g = c_{gx}\vec{\imath} + c_{gy}\vec{\jmath}$ the result can be given as:

$$c_{gx} = \frac{hC_x\tilde{k}}{C_t\tilde{\omega}}, \quad c_{gy} = \frac{hC_y\tilde{\ell}}{C_t\tilde{\omega}}. \tag{36}$$

A trivial substitution will confirm that the identities

$$< U > = c_{gx} < E >, \quad < V > = c_{gy} < E >, \tag{37}$$

are valid also for the discrete quantities as defined above. We may now recognize the stopping depth, h_c, as the depth at which the group velocity becomes zero, with total reflection as the obvious result.

5 An ill defined conservation law.

We will demonstrate an alternative to the preceding definitions of discrete energy quantities. For simplicity we confine the discussion to constant depth and waves that propagate in the x-direction. The derivation in section 3 is followed until the last term on the right hand side of (21) is rewritten by means of (22). Instead we now write:

$$h\overline{\eta}^{xxt}\delta_x u = -\overline{\eta}^t\delta_t\eta + \frac{\Delta x^2}{4}\delta_x^2\overline{\eta}^t\delta_t\eta. \tag{38}$$

A few manipulations then yield the identity:

$$\delta_x^2\overline{\eta}^t\delta_t\eta = \delta_x(\delta_t\overline{\eta}^t\delta_x\overline{\eta}^x) - \delta_t\left(\frac{1}{2}\overline{(\delta_x\eta)^2}^x\right). \tag{39}$$

Collecting the terms we now obtain the modified energy density and flux:

$$E_1 = \frac{1}{2}h\left(\overline{u}^x\right)^{(t*2)} + \frac{1}{2}\eta^2 - \frac{\Delta x^2}{8}\overline{(\delta_x\eta)^2}^x, \quad U_1 = hu\overline{\eta}^{xt} + \frac{\Delta x^2}{4}\delta_t\overline{\eta}^x\delta_x\overline{\eta}^t. \tag{40}$$

As compared to those in (25) the above expressions are aesthetically inferior; we have been unable to group the terms of (40) to form the same nice and compact discrete counterpart to the analytical formulas. A more important objection against (40) is that

$$< U_1 > \neq c_g < E_1 >, \tag{41}$$

and that U_1 =const. does not reproduce (35). We are thus led to disregard E_1 and U_1 as proper definitions of discrete energy quantities. Still, we can not rule out the possibility that there do exist more than one set of discrete energy quantities that fulfill $< U > = c_g < E >$.

6 Related difference equations.

We have generalized the preceding results to a few other discrete equations related to those in (14). For simplicity we again confine the discussion to constant depth and no variation in the y-direction.

6.1 The linearized Boussinesq equations.

Taking into account terms of order β (see section 2.1) while assuming h =const. and neglecting nonlinear terms we obtain:

$$\frac{\partial \eta}{\partial t} = -h\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial t} = -\frac{\partial \eta}{\partial x} + \frac{1}{3}\beta h^2 \frac{\partial^3 u}{\partial x^2 \partial t}, \tag{42}$$

where a dispersion term has been introduced in the momentum equation and u must be interpreted as the depth averaged horizontal velocity component.

From (42) we obtain:

$$E \equiv \frac{1}{2}hu^2 + \frac{1}{2}\eta^2 + \frac{\beta}{6}h\left(\frac{\partial \eta}{\partial t}\right)^2, \quad U \equiv \eta h u - \frac{\beta}{3}h^3 u \frac{\partial^2 u}{\partial x \partial t}. \tag{43}$$

The $O(\beta)$ term in the density stems from the vertical motion of the fluid while the higher order term in the flux is due to the non hydrostatic part of the pressure. On the other hand, the effects of the vertical variations of the horizontal velocity cancel to the present order in β.

As a natural extension of the discretization in section 2.3 we obtain the difference equations:

$$[\delta_t \eta = -h\delta_x u]_i^{(n+\frac{1}{2})}, \quad [\delta_t u = -\delta_x \eta + \frac{\beta}{3}h^2 \delta_x^2 \delta_t u]_{i+\frac{1}{2}}^{(n)}. \tag{44}$$

The dispersion relation now becomes:

$$\tilde{\omega}^2 = \frac{h\tilde{k}^2}{1 + \frac{\beta}{3}h^2\tilde{k}^2}, \tag{45}$$

while the amplitudes for u and η fulfill $\hat{u} = \tilde{\omega}A/(h\tilde{k})$.

When calculating the energy quantities we follow the procedure in section 3 as close as possible. Two new and laboursome terms appear during the derivation. One comes directly from the dispersion term, while another appear when the momentum equation is invoked in the counterpart to (22). For illustration we display the transformation of the first only:

$$\begin{aligned} \overline{u}^x \delta_x^2 \delta_t \overline{u}^{xt} &= (\overline{u}^{xxx} - \frac{\Delta x^2}{4}\delta_x^2 \overline{u}^x)\delta_x^2 \delta_t \overline{u}^{xt} \\ &= \delta_x(\overline{u}^{xx}\delta_x \delta_t \overline{u}^{xt}) - \frac{1}{2}\delta_t(\delta_x \overline{u}^{xx})^{(t*2)} - \frac{\Delta x^2}{8}\delta_t(\delta_x^2 \overline{u}^x)^{(t*2)}. \end{aligned} \tag{46}$$

At the end the result may be organized in the compact formulas

$$E = \frac{1}{2}h\overline{u^{(t*2)}}^x + \frac{1}{2}\eta^2 + \frac{\beta h}{6}\overline{(\delta_t\eta)^{(t*2)}}^{xx}, \quad U = hu\overline{\eta}^{xt} - \frac{\beta h^3}{3}\overline{\overline{u}^x\delta_x\delta_t\overline{u}^t}^x, \tag{47}$$

which correspond term by term to the analytical expressions in (43).

The averaged density and flux for a single harmonic now become:

$$< E > = \frac{1}{2}C_t^2A^2, \quad < U > = \frac{1}{2}\frac{\tilde{\omega}}{\tilde{k}}C_xC_t\left(1 - \frac{\beta h^3}{6}\tilde{\omega}^2\right)A^2. \tag{48}$$

Noting that c_g can be written as

$$c_g \equiv \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{C_x\tilde{\omega}}{C_t\tilde{k}}\left(1 - \frac{\beta h^3}{6}\tilde{\omega}^2\right), \tag{49}$$

we may again observe that $< U > = c_g < E >$.

6.2 Finite element discretizations.

In [3] the discrete Greens law was established for two of the most relevant finite element formulations for the long wave equations.

In one element approach we approximate η and u by piecewise constant and linear functions respectively, which yields a staggered grid. Correspondingly, we invoke constant weight functions for the continuity equation and linear weight functions for the momentum equation. Invoking the same enumeration of nodes as in the difference method we find the assembled equations:

$$[\delta_t\eta = -h\delta_x u]_j^{(n+\frac{1}{2})}; \quad [(1 + \frac{1}{6}\Delta x^2\delta_x^2)\delta_t u = -\delta_x\eta]_{j+\frac{1}{2}}^{(n)}. \tag{50}$$

We note that these are identical to the discrete Boussinesq equations with β replaced by $-\frac{1}{6}\Delta x^2$. The results of the preceding subsection will then apply directly.

The other formulation is derived from the long wave equations expressed in terms of the velocity potential ϕ:

$$\frac{\partial\eta}{\partial t} = -\frac{\partial}{\partial x}\left(h\frac{\partial\phi}{\partial x}\right), \quad \frac{\partial\phi}{\partial t} = -\eta. \tag{51}$$

We note that differentiation of the right equation, with respect to x, and use of the identity $\partial\phi/\partial x \equiv u$ reproduce the equations employed previously. In the residual formulation the right hand side of the continuity equation is integrated by parts and both form and weight functions are chosen to be piecewise linear. Assuming a uniform grid we obtain assembled equations of the form:

$$[(1 + \frac{1}{6}\Delta x^2\delta_x^2)\delta_t\eta = -h\delta_x^2\phi]_j^{(n+\frac{1}{2})}; \quad [(1 + \frac{1}{6}\Delta x^2\delta_x^2)(\delta_t\phi + \eta) = 0]_j^{(n)}. \tag{52}$$

We now introduce the velocity as $u \equiv \delta_x \phi$ and apply the operator δ_x to the right (Bernoulli) equation. Noting that the mass matrix, represented by the operator $1 + \frac{1}{6}\Delta x^2 \delta_x^2$, is non singular we then arrive at:

$$[(1 + \frac{1}{6}\Delta x^2 \delta_x^2)\delta_t \eta = -h\delta_x u]_j^{(n+\frac{1}{2})}; \quad [\delta_t u = -\delta_x \eta]_j^{(n)}. \tag{53}$$

Proceeding in the same manner as before we then find:

$$E = \frac{1}{2} h\overline{u^{(t*2)}}^x + \frac{1}{2}\eta^2 - \frac{\Delta x^2}{12}\overline{(\delta_x \eta)^2}^x, \quad U = hu\overline{\eta}^{xt} + \frac{\Delta x^2}{6}\overline{\eta}^{xt}\delta_x \delta_t \eta. \tag{54}$$

It is now convenient to express the averaged quantities in terms of the velocity amplitude $\hat{u}$:

$$< E > = \frac{1}{2} h\hat{u}^2 C_t^2, \quad U = \frac{1}{2}\frac{\tilde{\omega}}{\tilde{k}} C_x C_t \left(h + \frac{\Delta x^2}{6}\tilde{\omega}^2\right)\hat{u}^2. \tag{55}$$

Again we find $< U > = c_g < E >$ and reproduce the result of physical optics in [3].

7 Discussion.

Guided by the analytical expressions for the energy density and flux we have obtained discrete counterparts from the difference equations and the particular discrete arithmetics that was outlined. Even though we have performed the calculations only for a few discretization, corresponding results may be expected for other methods inheriting a dispersion relation allowing neutrally stable modes.

The most complete analysis, involving variable depth, was performed only for a midpoint difference method based on the C-grid. As might be expected the actual expressions for energy density and flux were ambiguous. Following a procedure a simple as possible and making the most "natural" choices we obtained very compact formulas in perfect correspondence to the analytical expressions. However, this, in itself, is certainly an insufficient justification for the notations discrete energy and flux. More substance is given to these phrases when we observe that the expression for the flux under the assumption of a slowly varying medium leads to the same discrete version of Greens law that has previously been found by a discrete WBKJ technique [3]. Further confirmation is then allotted by the fact that the averaged density and flux for a single harmonic obey the usual relation $< \vec{F} > = \vec{c}_g < E >$ with $\vec{c}_g$ extracted from the numerical dispersion relation. Hence, we have found strong support for the existence of a proper and unique discrete energy concept in homogeneous medium. In the case of variable depth, on the other hand, the only clue to avoid ambiguity is the look of the dubious terms, that points to fluxes rather than densities.

In the case of plane waves on constant depth We have also calculated the discrete energy quantities for the linearized Boussinesq equations and FEM formulations for the hydrostatic equations. Again we find the proper relation between density, flux and group velocity and the optical results of [3] are reproduced.

Finally we give an important example of inference from the new "insight". Compared to the analytical Greens law the discrete version yields over-amplification. As a consequence of the results described above we may now present a quasi physical explanation for this feature. Claiming $c_g < E >$ constant we now observe that $< E >$ is reduced relative to the analytical value by a uniform factor (see (33)), while the corresponding ratio for the group velocities decreases in shoaling water due to the decreasing wavelength. Thus, to maintain a constant flux the discrete amplification must be larger than the analytical one.

The author wish to thank Mr. H. Johnsgard for helpful discussions and valuable suggestions concerning the energy expressions. The presented work has been carried out under the GITEC-TWO (Genesis and Impact of Tsunamis on European Coasts) project that is funded by the European Commission.

References

[1] Mesinger F., Arakawa A. 1976 Numerical methods used in atmospheric models. *GARP, Publ. Ser. WMO* **17** 64 pp.

[2] Pedersen G. 1995 Grid effects on tsunamis in nearshore regions. *University of Oslo, Research Report in Mechanics* **1**

[3] Pedersen G. 1995 An optical description of finite difference approximations to long waves. *University of Oslo, Research Report in Mechanics* **1**