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ON THE USE OF STANDARD TABLES TO OBTAIN DODGE-ROMIG LTPD SAMPLING INSPECTION PLANS

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Procedures are described which yield single and double sample Dodge-Romig [1] lot tolerance percent defective (LTPD) rectifying inspection plans. For the determination of such plans only a desk calculator and standard tables of the discrete probability distributions are required. Some advantages gained by using these procedures rather than the Dodge-Romig table include :(a) The Consumer's Risk is not limited to .10 (b) More choices of LTPD are available. (c) Smaller average total inspection is achieved by using a plan designed for specific "process average" and lot size rather than a compromise plan designed to cover intervals on these two parameters.

1. INTRODUCTION

A product that is mass produced is assembled at random into lots of size N. From each lot items are sampled at random and the number of defectives is observed. On the basis of the observation the lot is either accepted or rejected. If the lot is accepted all defective items found when sampling are replaced by non-defectives. If the lot is rejected all N items are examined and all defective items in the lot are replaced. The procedure just described is the special case of rectifying inspection which we are about to consider. Our goal is to determine reasonable sampling plans for the type of situation just described.

Let us assume that the process produces a defective with probability p. Each inspected lot will contain an unknown number of defectives, say k. Let Y be the number of defectives in a random sample of size n drawn from a lot. It is well known that the probability function of Y given k is the hypergeometric

$$p(N,n,k,y) = \frac{\binom{k}{y}\binom{N-k}{n-y}}{\binom{N}{n}}, \qquad a \leq y \leq b$$

where $a = \max[0, n-(N-k)]$, $b = \min[k, n]$ and the unconditional probability function of Y is the binomial

$$b(y;n,p) = \binom{n}{y} p^y (1-p)^{n-y}, \quad y=0,1,2,\ldots,n \tag{1.2}$$

For both single and double sampling we will minimize the average total inspection if $p = \bar{p}$, the "process average," subject to the condition that the operating characteristic(OC) of the sampling plan be no more than β_1 if the lot contains $k_1 = Np_1$ defectives. (In the language of Dodge-Romig [1] $p_1 = p_t$ = LTPD, β_1 = The Consumer's Risk.)

Binomial cumulative sums will be denoted by

$$E(r;n,p) = \sum_{y=r}^{n} b(y;n,p) \tag{ .3}$$

For our purposes we find that three tables are useful. The Ordnance Corps [7] table gives (1.3) to seven decimal places for $n = 1(1)150$, $p=.01(.01).50$. The Harvard [4] table gives (1 .3) to five decimal places for $n=1(1)50(2)100(10)200(20)500(50)1000$, $p=.01(.01).50$ (plus a few rational fractions). The Weintraub [8] table gives the same sum to ten decimal places for $n=1(1)100$, $p=.0001,.0001(.0001).001(.001).10$.

In the hypergeometric case we will use the tables of Lieberman and Owen [5] which gives both (1.1) and

$$P(N,n,k,r) = \sum_{y=a}^{r} p(N,n,k,y) \tag{1.4}$$

to six decimal places for $N=1(1)50(10)100$. In addition two approximations to(1.4) will be used. These are

$$P(N,n,k,r) \cong 1-E(r+1;n,\tfrac{k}{N}) \tag{1.5}$$

if $n/N \leq .10, k \geq n$

$$P(N,n,k,r) \cong 1-E(r+1;k,\tfrac{n}{N}) \tag{1.6}$$

if $k/N \leq .10, k<n$ Even when

neither condition $n/N \leq .10, k/N \leq .10$ is satisfied the approximation is usually surprisingly good if we use (1.5) when $k \geq n$, (1.6) when $k < n$(as suggested by Lieberman and Owen). The examples considered later in the paper suggest that the accuracy obtained using the binomial approximation *is sufficient for practical purposes.*

If sample sizes are larger than 150, it is usually convenient to use the Poisson approximation to the binomial. Two good tables, which together provide about all the entries that would ever be needed, are the ones prepared by General Electric [2] and Molina [6].

If more accuracy is desired than can be obtained from the approximations (which is unlikely in most applications), then a high speed computer can be used to obtain a solution by following the same procedure demonstrated in the examples.

2. THE SINGLE SAMPLE CASE

A sample of size n *is selected at random from a lot of size* N. *Let* X *be the number of defectives in the sample. If* $x \leq c$ *defective items are found in the sample, these items are replaced by non-defectives and the lot is accepted without further inspection. If* $x > c$ *the lot is totally inspected and all defective items in the lot are replaced by non-defectives. If the lot contains* k *defectives, then the operating characteristic is*

$$OC = P(N,n,k,c) \tag{2.1}$$

When $k = k_1 = Np_1$ *we wish to accept the lot with probability at most* β_1 *so that* n *and* c *must satisfy the inequality*

$$P(N,n,k_1,c) \leq \beta_1 \tag{2.2}$$

For a lot containing k *items the expected number of items inspected is*

$$J_S = n+(N-n)\left[1-P(N,n,k,c)\right] \tag{2.3}$$

However, if the process average is p, *then* k *is a random variable and the number of defective items in a sample of size* n *has an unconditional* *binomial distribution with parameters n and p. In other words J_S is a conditional expectation, the unconditional expected value being*

$$I_S = n+(N-n)E(c+1;n,p) \qquad (2.4)$$

Dodge and Romig [1] have minimized (2.4) at

$p=\bar{p}$ subject to (2.2) but if we prefer we could minimize (2.3) at $\bar{k} = N\bar{p}$ instead.

In general the result would not be too different since the binomial is used to approximate the hypergeometric and thus (2.4) approximates (2.3). The minimum values will be denoted by $\bar{J}_S$ and $\bar{I}_S$.

The minimization is accomplished by trial starting with $c = 0$ and increasing c a unit at a time. For each c the minimum n satisfying (2.2) is found and I_S (or J_S) is computed. Calculations cease when the minimum is observed. We will demonstrate by examples.

Example 2.1

If $N=50, k_1=12, \beta_1 = .20$ find the plans which minimize I_S and J_S when $p = \bar{p} = .06$. Find the OC if $\bar{k} = N\bar{p} = 3$.

Solution

Condition (2.2) becomes $P(50,n,12,c)\leq .20$. With the Lieberman and Owen [5] table we verify that if $c=0, n\geq 6$, if $c=1, n\geq 11$, if $c=2, n\geq 16$, if $c=3$, $n\geq 20$, etc. Obviously we chose the minimum n in each interval. Using the Ordnance Corps [7] (or Harvard [4]) and the Lieberman and Owen tables we find if $c=0, n=6$

$$I_S = 6+44\ E(1;6,.06) = 6+44(.310130)=19.65$$

$$J_S = 6+44[1-P(50,6,3,0)] = 6+44(.324286) = 20.27$$

if $c=1, n=11$

$$I_S = 11+39\ E(2;11,.06) = 11+39(.138216)=16.39$$

$$J_S = 11+39[1-P(50,11,3,1)]=11+39(.117857)=15.60$$

$$\text{if } c=2, n=16 \quad I_S=16+34\ E(3;16,.06)=16+34(.067280)=18.29$$

$$J_S=16+34[1-P(50,16,3,2)]=16+34(.028571)=16.97$$

Further calculations are obviously unnecessary and the plan which minimizes both I_S and J_S is $n=11, c=1$ with $\bar{I}_S=16.39$, $\bar{J}_S=15.60$.

The OC at $\bar{k}=3$ is

$$OC = P(50,11,3,1)=.882143$$

Example 2.2

If $N=1000, k_1=100(p_1=.10), \beta_1=.10$ find the plan which minimizes the average amount of inspection if $\bar{p}=.02$. Find the OC when $k=\bar{k}=N\bar{p}$.

Solution

With $N=1000$ we are out of the range of the Lieberman-Owen hypergeometric table and we use the approximation

$$OC = P(1000,n,100,c) \cong 1-E(c+1;n,.10)$$

(with $n > 100$ approximation (1.6) is slightly better) so that (2.2) becomes

$$E(c+1;n,.10) \geq .90$$

With the Ordnance Corps [7] table we verify that if $c=0, n \geq 22$, if $c=1$, $n \geq 38$, if $c=2$, $n \geq 52$, if $c=3, n \geq 65$, if $c=4, n \geq 78$, if $c=5, n \geq 91$, if $c=6, n \geq 104$, if $c=7, n \geq 116$, etc. Since the value of J_S would have to be computed by using the binomial approximation we calculate only I_S which is the approximation for J_S. We get

$$\text{if } c=0, n=20 \quad I_S=20+980\ E(1;20,.02)=20+980(.33239)=345.7$$
$$c=1, n=38 \quad I_S=38+962\ E(2;38,.02)=38+962(.17603)=207.3$$
$$c=2, n=52 \quad I_S=52+948\ E(\;;52,.02)=52+948(.08593)=133.5$$
$$c=3, n=65 \quad I_S=65+935\ E(4;65,.02)=65+935(.04138)=103.7$$
$$c=4, n=78 \quad I_S=78+922\ E(5;78,.02)=78+922(.02028)=96.7$$
$$c=5, n=91 \quad I_S=91+909\ E(6;91,.02)=91+909(.01006)=100.9$$

Obviously further calculations are unnecessary and the desired plan is $n=78, c=4$.

Dodge and Romig [1] give $n=65, c=3$ but their plan is designed to cover intervals on both N and $\bar{p}$.

The OC at $k=\bar{k}=1000(.02)=20$ is $OC \cong 1-E(5;78,.02)=.97972$.

Hald [3] has derived assymptotic formulas which, together with some auxillary tables, can be used to obtain sampling plans of the type we have considered. Considerable calculation seems to be required. His paper has one numerical example which we will now work for a comparison of results.

Example 2.3

If $N=280, p_1=.10(k_1=28), \beta_1=.10$, find the plan which minimizes the average amount of inspection if $\bar{p}=.045$.

Solution

We need

$$OC = P(280,n,28,c) \cong 1-E(c+1;28,\frac{n}{280}) \leqq .10$$

or

$$E(c+1;28,\frac{n}{280}) \geqq .90$$

and

$$I_S=n+(280-n)E(c+1;n,.045)$$

Without interpolating on p in the binomial table we find

if $c=0,\ n/280\geqq .08, n\geqq 26, I_S=26+254(.69794)=203$

$c=1,\ n/280\geqq .14, n\geqq 40, I_S=40+240(.54265)=170$

$c=2,\ n/280\geqq .18, n\geqq 51, I_S=51+229(.40442)=144$

$c=3,\ n/280\geqq .23, n\geqq 65, I_S=65+215(.33554)=137$

$c=4, n/280 \geqq .27, n \geqq 76, I_S=76+204(.25699)=128$
$c=5, n/280 \geqq .31, n \geqq 87, I_S=87+193(.19784)=125$
$c=6, n/280 \geqq .35, n \geqq 98, I_S=98+182(.15299)=126$
$c=7, n/280 \geqq .39, n \geqq 110, I_S=110+170(.1283)=132$

where $E(c+1;n,.045)$ was found from the Weintraub [8] table except for $n=110$ for which the Poisson approximation was used. Recomputing the three smallest I_S's using linear interpolation in the binomial table yields

if $c=4, n/280 \geqq .2655, n \geqq 75, I_S=75+205(.24845)=125.9$
$c=5, n/280 \geqq .3065, n \geqq 86, I_S=86+194(.19090)=123.0$
$c=6, n/280 \geqq .3460, n \geqq 97, I_S=97+183(.14740)=124.0$

The plan with minimum I_S is $n=86$, $c=5$ with $\bar{I}_S=123.0$. Hald gives $n=84, c=5, \bar{I}_S=119.6$

but his Consumer's Risk is slightly larger than .10 while ours (within the limits of the approximation) has Consumer's Risk slightly less than .10.

The average outgoing quality for the single sample case is

$$AOQ = (1- \frac{n}{N})p\left[1-E(c+1;n,p)\right] \qquad (2.5)$$

The maximum of (2.5) taken over p is called the average outgoing quality limit (AOQL). Dodge and Romig [1,pp.37-39] describe a method of approximating AOQL. We observe that AOQL can also be found by trial using Weintraub's [8] table. For Example 2.2 in which $N=1000$, $n=78$, $c=2$

we find

if $p=.045$ $AOQ=.922(.045)(.72575)=.03011$
$p=.046$ $AOQ=.922(.046)(.71065)=.03014$
$p=.047$ $AOQ=.922(.047)(.69538)=.03013$

so that AOQL=.030. The Dodge-Romig solution also gives AOQL=.030, occurring at $p=.0467$.

3. THE DOUBLE SAMPLE CASE

A sample of size n_1 is selected at random from a lot of size N. Let X_1 be the number of defective items in the sample. If $x_1 \leq c_1$ defective items are found in the sample, these items are replaced by non-defectives and the lot is accepted without further inspection. If $c_1 < x_1 \leq c_2$ a second sample of size n_2 is selected at random from the remaining $N-n_1$ items and X_2 the number of defective items in the second sample is observed. If $c_1 < x_1+x_2 \leq c_2$ the lot is accepted without further inspection but all defective items found in both samples are replaced by good ones. If either $x_1 > c_2$ or $c_1 < x_1 \leq c_2$ and $x_1+x_2 > c_2$ the lot is totally inspected and all defective items in the lot are replaced by non-defectives. If the lot contains k defectives, then

$$OC=H(k;N,n_1,n_2,c_1,c_2)$$

$$=P(N,n_1,k,c_1)+\sum_{j=1}^{c_2-c_1} p(N,n_1,k,c_1+j)\ P(N-n_1,n_2,k-c_1-j,c_2-c_1-j) \quad (3.1)$$

The counterparts of (2.3) and (2.4) are

$$J_D=n_1+n_2\left[1-P(N,n_1,k,c_1)\right]+(N-n_1-n_2)\left[1-H(k;N,n_1,n_2,c_1,c_2)\right] \quad (3.2)$$

and

$$I_D=n_1+n_2E(y_1;\ n_1,p)+(N-n_1-n_2)K(p;n_1,n_2,y_1,y_2) \quad (3.3)$$

where $y_1=c_1+1, y_2=c_2+1$ and

$$K(p;n_1,n_2,y_1,y_2)=E(y_2;n_1,p)+\sum_{j=0}^{y_2-y_1-1} b(y_1+j;n_1,p)\ E(y_2-y_1-j;n_2,p) \quad (3.4)$$

As in the single sample case we will minimize the average amount of inspection at $\bar{p}$ subject to the condition that

$OC \leq \beta_1$ if $k=k_1$

or

$$H(k_1;N,n_1,n_2,c_1,c_2) \leq \beta_1 \quad (3.5)$$

The minimum values of (3.2) and (3.3) will be denoted by $\bar{J}_D$ *and* $\bar{I}_D$ *respectively. If* $N > 50$ *so that it is not practical to use the table of Lieberman and Owen* [5] *, then we will use binomial approximations for hypergeometric sums, power instead of OC, and condition (3.5) is replaced by*

$$K(p_1;n_1,n_2,y_1,y_2) \geq 1-\beta_1 \tag{3.6}$$

where $p_1=k_1/N$*. Of course, then we may minimize only* I_D *but, as in the single sample case,* $\bar{I}_D$ *and* $\bar{J}_D$ *and the resulting plans will not be enough different to be of practical importance. We will consider both cases in numerical examples.*

Because of condition (3.5) it is necessary for any c_1,c_2 *that*

$$P(N,n_1,k_1,c_1) \leq \beta_1 \tag{3.7}$$

When the binomial approximation is used (3.7) becomes

$$E(y_1;n_1,p_1) \geq 1-\beta_1 \tag{3.8}$$

These inequalities provide a lower bound on n_1*, say* n_1^1 *. Let* n_0 *be the minimum value of* n *which satisfies (2.2) when* $c=c_2$*, that is, the value of* n *for the corresponding single sample plan found with* $c=c_2$ *(a double sample plan with* $n_1=n_0$, $n_2=0$*). Then for each pair* c_1,c_2 *we consider only sampling plans for which* $n_1^1 \leq n_1 \leq n_0$ *(for larger* n_1 *the value of* n_2 *is 0 and* $I_D=I_S$ *is larger).*

Further, given n_1,c_1,c_2 *we next consider only the minimum value of* n_2*, say* n_2^1 *which satisfies (3.6), or (3.5), since larger values just increase* I_D*. Although intuitively obvious, this is true because*

$K(p;n_1,n_2,y_1,y_2)<E(y_1;n_1,p)$*, a result which follows from (3.4) and the fact that*

$E(y_2-y_1-j;n_2,p) \leq 1$*. Finally, only if* $n_1<\bar{I}_S$ *do we need to consider a plan*

since otherwise $I_D > \bar{I}_S$ *and the object of double sampling (to reduce average total inspection) would be defeated. Plans which satisfy the above three conditions, namely*

1. For chosen c_1, c_2 we have $n_1^1 \leq n_1 \leq n_0$.
2. For chosen c_1, c_2, and $n_1^1 \leq n_1 \leq n_0$ we have n_2 a minimum.
3. The value of n_1 is such that $n_1 < \bar{I}_S$.

Will be called acceptable plans. Obviously the candidates for a plan which yields $\bar{I}_D$ can be limited to the class of acceptable plans.

Two facts useful in determining the minimum n_2 for a given n_1 are :

1. We must have $n_1 + n_2 \geq n_0$. To see this assume the converse is true, that is there exist n_1, n_2 such that $n_1 + n_2 < n_0$. Then the power at $k = k_1$ is made $\geq 1 - \beta_1$ by taking n_1 observations all of the time and n_2 observations part of the time. The power is not decreased if the second sample is taken with probability 1. But this means that a single sample plan with the given c_2 exists with $n = n_1 + n_2 < n_0$ contrary to the definition of n_0.

2. As n_1 increases $n_1 + n_2$ is non-increasing and has as its minimum value n_0 (attainable at least when $n_1 = n_0, n_2 = 0$). This sum may be considerably greater than n_0 when $n_1 = n_1^1$ but gets close to n_0 after n_1 has been increased by relatively few units. This is explained by observing that when $n = n_1^1$, $E(y_1; n_1, p)$, which is greater than the power, is very nearly $1 - \beta_1$ and to satisfy (3.6) the terms $E(y_2 - y_1 - j;\ n_2, p)$ must be large so that n_2 is large. As n_1 increases the difference between power and $E(y_1; n_1, p)$ grows at a relatively rapid pace permitting the $E(y_2 - y_1 - j; n_2, p)$ and n_2 to be much smaller.

We now consider the organization of a numerical problem. As a first step we should calculate $\bar{I}_S$ (or $\bar{J}_S$) since, as we have already mentioned, it is not necessary to consider plans for which $I_D > \bar{I}_S$. Then

1. With $c_1 = 0$ determine n_1^1 from (3.7) or (3.8).

(a) With $c_2 = 1$

  (1) Find n_0.

  (2) By trial find n_2^1 using the fact that $n_1 + n_2 \geq n_0$.

  (3) With n_1^1, n_2^1 find I_D (or J_D).

(b) Repeat (a) with $c_2 = 2$. As a first guess for the new n_2^1 increase the old n_2^1 by the same amount that n_1^1 has increased.

(c) Repeat (a) with $c_2 = 3$.

etc.

Terminate when it is obvious that I_D must increase with further increase in c_2.

2. *Repeat Step 1. but with n_1^1 replaced by $n_1^1 +1$. Then repeat Step 1 with n_1^1 replaced by $n_1^1 +2$ etc., terminating when it is obvious that a minimum has been found for each c_2 which it has been necessary to consider with $c_1=0$.*

3. *Repeat Steps 1 and 2 with $c_1=1$, then with $c_1=2$, then with $c_1=3$, etc., terminating when the I_D get too large. This happens at worst when $n_1>\bar{I}_S$.*

4. *By observation select the minimum I_D (or J_D).*

Although the procedure outlined in the previous paragraph may require a number of calculations, it goes rather quickly using a de k calculator which has accumulative multiplication. When using the hypergeometric table it is probably advisable to copy down all figures before going to the calculator (because of the format of the table). In the binomial case it is advisable to copy down $E(y_2; n_1,p)$ and the $b(y_1+j;n_1,p)$ however, the $E(y_2-y_1-j;n_2,p)$ may be transferred directly from the binomial table to the calculator and need not be copied. The major advantage of proceding as suggested in the previous paragraph is that all the previous $b(y_1+j;n_1,p)$ are used plus one more as y_2 is increased by a unit. We now consider examples.

Example 3.1

If $N=50, k_1=12, \beta_1=.20$ find the double sampling plans which minimize I_D and J_D when $p=\bar{p}=.06(\bar{k}=3)$.

Solution

In Example 2.1 we already found that $\bar{I}_S=16.39, \bar{J}_S=15.60$. Also we had that if $c=0, n_0=6$, if $c=1, n_0=11$, if $c=2, n_0=16$, if $c=3, n_0=20$.

We begin by selecting $c_1=0$. Then possible values for c_2 are 1, 2, 3, 4, etc. and the OC is

$$H(k;50,n_1,n_2,0,c_2) = P(50,n_1,k,0) + \sum_{j=1}^{c_2} p(50,n_1,k,j)\, P(50-n_1,n_2,k-j,c_2-j)$$

Condition (3.7) is $P(50,n_1,12,0) \leqq .20$ which requires $n_1 \geqq 6$.

If $c_2 = 1$, then $n_1 + n_2 \geqq 11$. With $n_1 = 6$ the OC is

$$H(k; 50,6,n_2,0,1) = P(50,6,k,0) + p(50,6,k,1)\ P(44,n_2,k-1,0)$$

and

$$H(12; 50,6,n_2,0,1) = P(50,6,12,0 + p(50,6,12,1)\ P(44,n_2,11,0)$$

By trial we find

$$H(12; 50,6,9,0,1) = .194350,\ H(12; 50,6,8,0,1) = .203423$$

so that $n_1 = 6, n_2 = 9, c_1 = 0, c_2 = 1$ is an acceptable plan. Then

$$H(3; 50,6,9,0,1) = P(50,6,3,0) + p(50,6,3,1)\ P(44,9,2,0)$$

$$= .675714 + (.289592)(.628964) = .857857$$

$$K(.06; 6,9,1,2) = E(2; 6,.06) + b(1; 6,.06)\ E(1; 9,.06)$$

$$= .04592 + (.26421)(.42701) = .15874$$

and when $\bar{k} = 3, \bar{p} = .06$

$$J_D = 6 + 9[1-P(50,6,3,0)] + 35[1-H(3; 50,6,9,0,1)]$$

$$= 6 + 9(.324286) + 35(.142143) = 13.90$$

$$I_D = 6 + 9E(1; 6,.06) + 35\ K(.06; 6,9,1,2)$$

$$= 6 + 9(.31013) + 35(.15875) = 14.35$$

We next take $c_2 = 2$ with $c_1 = 0$, $n_1 = 6$. Now $n_1 + n_2 \geqq 16$ and the OC is

$$H(k; 50,6,n_2,0,2) = P(50,6,k,0) + p(50,6,k,1)\ P(44,n_2,k-1,1)$$
$$+ p(50,6,k,2)\ P(44,n_2,k-2,0)$$

and

$$H(12; 50,6,n_2,0,2) = .173729 + .379046\ P(44,n_2,11,1)$$
$$+ .306581\ P(44,n_2,10,0)$$

By trial we find (a good first guess is $n_2 = 14$)

$$H(12; 50,6,15,0,2) = .192763,\ H(12; 50,6,14,0,2) = .200929$$

so that $n_1 = 6$, $n_2 = 15$, $c_1 = 0$, $c_2 = 2$ is an acceptable plan. Then

$$H(3; 50,6,15,0,2) = P(50,6,3,0) + p(50,6,3,1)\ P(44,15,2,1)$$
$$+ p(50,6,3,2)\ P(44,15,1,0)$$
$$= .675714 + (.289592)(.889006)$$
$$+ (.257449)(.659091) = .955357$$

$$K(.06;\ 6,15,1,3,) = E(3;\ 6,.06) + b(1;\ 6,.06)\ \ E(2;\ 15,.06)$$
$$+ b'(2;\ 6,.06)\ \ E(1;\ 15,.06)$$
$$= .00376 + (.26421)(.22624)$$
$$+ .(04226)(.60471) = .08909$$

and when $\bar{k} = 3,\ \bar{p} = .06$

$$J_D = 6 + 15[1 - P(50,6,3,0)] + 29[1 - H(3;\ 50,6,15,0,2)]$$
$$= 6 + 15(.324286) + 29(.044643) = 12.16$$
$$I_D = 6 + 15\ E(1;\ 6,.06) + 29\ K(.06;\ 6,15,1,3)$$
$$= 6 + 15(.31013) + 29(.08909) = 13.24$$

We next take $c_2 = 3$ with $c_1 = 0,\ n_1 = 6.$ Now $n_1 + n_2 \geq 20$ and the OC is

$$H(k;\ 50,6,n_2,0,3) = P(50,6,k,0) + p(50,6,k,1)\ \ P(44,n_2,k-1,2)$$
$$+ p(50,6,k,2)\ \ P(44,n_2,k-2,1)$$
$$+ p(50,6,k,3)\ \ P(44,n_2,k-3,0)$$

and

$$H(12;\ 50,6,n_2,0,3) = .173729 + (.379046)\ \ P(44,n_2,11,2)$$
$$+ (.306581)\ \ P(44,n_2,10,1)$$
$$+ (.116793)\ \ P(44,n_2,9,0)$$

By trial we find (a good first guess is $n_2 = 19$)

$$H(12;\ 50,6,19,0,3) = .199822,\ H(12;\ 50,6,18,0,3) = .210583$$

so that $n_1 = 6,\ n_2 = 19,\ c_1 = 0,\ c_2 = 3$ is an acceptable plan. Then

$$H(3;\ 50,6,19,0,3) = 1\ \text{(obviously)}$$

$$K(.06;\ 6,19,1,4) = E(4;\ 6,.06) + b(1;\ 6,.06)\ \ E(3;\ 19.06)$$
$$+ b'(2;\ 6,.06)\ E(2;\ 19,.06)$$
$$+ b(3;\ 6,.06)\ E(1;\ 19,.06)$$

$$= .00018 + (.26421)(.10207)$$
$$+ (.04226)(.31709)$$
$$+ (.00358)(.69138) = .04302$$

and when $\bar{k} = 3$, $\bar{p} = .06$

$J_D = 6 + 19(.324286) + 25(0) = 12.16$

$I_D = 6 + 19(.31031) + 25(.04302) = 12.97$

We next take $c_2 = 4$ *with* $c_1 = 0$, $n_1 = 6$. *Now* $n_1 + n_2 \geq 25$ *and the OC is*

$$H(k; 50,6,n_2,0,4) = P(50,6,k,0) + p(50,6,k,1)\ P(44,n_2,k-1,3)$$
$$+ p(50,6,k,2)\ P(44,n_2,k-2,2)$$
$$+ p(50,6,k,3)\ P(44,n_2,k-3,1)$$
$$+ p(50,6,k,4)\ P(44,n_2,k-4,0)$$

and

$$H(12; 50,6,n_2,0,4) = .173729 + (.379046)\ P(44,n_2,11,3)$$
$$+ (.306581)\ P(44,n_2,10,2)$$
$$+ (.116793)\ P(44,n_2,9,1)$$
$$+ (.021899)\ P(44,n_2,8,0)$$

By trial we find (a good first guess is $n_2 = 24$*)*

$H(12; 50,6,24,0,4) = .194144$, $H(12; 50,6,23,0,4) = .203719$ *so that* $n_1 = 6$, $n_2 = 24$, $c_1 = 0$, $c_2 = 4$ *is an acceptable plan. There is obviously no point in computing* J_D *since the last term remains* 0 *while the second term increases thus increasing* J_D.

We find

$$K(.06;\ 6,24,1,5) = E(5;\ 6,.06) + b(1;\ 6,.06)\ E(4;\ 24,.06)$$
$$+ b(2;\ 6,.06)\ E(3;\ 24,.06)$$
$$+ b(3;\ 6,.06)\ E(2;\ 24,.06)$$
$$+ b(4;\ 6,.06)\ E(1;\ 24,.06)$$
$$= 0 + (.26421)(.03413)$$
$$+ (.04226)(.12845)$$
$$+ (.00358)(.36176)$$
$$+ (.00017)(.72730) = .01586$$

and when $\bar{p} = .06$

$$I_D = 6 + 24\ (.31031) + 20\ (.01586) = 13.76$$

Since the last term can decrease at most .3172 an increase of only 2 units in n_2 will more than overcome this figure. Hence there is no point in calculating further I_D with $c_1 = 0$, $n_1 = 6$.

We next repeat all of the above steps with $c_1 = 0$, $n_2 = 7$, then with $c_1 = 0$, $n_2 = 8$, etc., Continuing until it is obvious that a minimum has been found for each value of c_2 which it is necessary to consider. With $c_1 = 0$ we get the following (n_1, n_2) and values of J_D:

$c_2 = 1$	$c_2 = 2$	$c_2 = 3$
(6,9), 13.90	(6,15), 12.16	(6,19), 12.16
(7,5), 13.98	(7,11), 12.14	(7,16), 12.93
(8,4), 14.40	(8,9), 12.73	(8,14), 13.80

For I_D we get:

$c_2 = 1$	$c_2 = 2$	$c_2 = 3$	$c_2 = 4$
(6,9), 14.35	(6,15), 13.24	(6,19), 12.97	(6,24), 13.76
(7,5), 14.04	(7,11), 12.46	(7,16), 13.63	(7,20), 14.42
(8,4), 15.16	(8,9), 13.82	(8.14), 14.44	(8,18), 15.40

Now we repeat all the preceding steps with $c_1 = 1$. This time c_2 can take on the values 2,3,4,etc. We get the following (n_1, n_2) and values of J_D:

$c_2 = 2$	$c_2 = 3$
(11,9), 13.07	(11,14), 12.77
(12.5), 13.62	(12,10), 13.39
(13.3), 14.39	(13,8), 14.29

For I_D we get:

$c_2 = 2$	$c_2 = 3$	$c_2 = 4$
(11,9), 14.72	(11,15), 14.08	(11,19), 14.04
(12,5), 14.98	(12,10), 14.49	(12.15), 14.71
(13.3), 15.70	(13,8), 15.53	(13,13), 15.72

We need not consider $c_1 = 3$ since now $n_1 \geq 16$ and $\bar{I}_s$ and $\bar{J}_s$ are obviously exceeded. Hence calculations are terminated.

We observe that the plan $n_1 = 7$, $n_2 = 11$, $c_1 = 0$, $c_2 = 2$ minimizes both I_D and J_D with $\bar{I}_D = 12.46$, $\bar{J}_D = 12.14$.

No comparison with Dodge-Romig [1] is possible since $\beta_1 = .20$ is not an entry in their table.

We can make a further comparison with the single sample plan. Recall that at $\bar{k} = 3$ the OC had value .882143. For the double sample plan this is increased to .966785.

Example 3.2

If $N = 1000$, $k_1 = 100$ $(p_1 = .10)$, $\beta_1 = .10$ *find the double sampling plan which minimizes the average amount of inspection of* $\bar{p} = .02$. *Find the OC when* $k = \bar{k} = N\bar{p}$.

Solution

In example 2.2 we already found that $\bar{I}_s = 96.7$. *Also we had that if* $c = 0$, $n_0 = 22$, *if* $c = 1$, $n_0 = 38$, *if* $c = 2$, $n_0 = 52$, *if* $c = 3$, $n_0 = 65$, *if* $c = 4$, $n_0 = 78$, *etc. Now we minimize only* I_D *(since* J_D *would be approximated by* I_D*).*

We will omit the calculations and results for $c_1 = 0$ *and* $c_1 = 2$, *demonstrating the procedure with* $c_1 = 1$, *the value which yields* $\bar{I}_D$. *Now* c_2 *can be* 2, 3, 4, *etc. Condition (3.8) is* $E(2;\ n_1, .10) \geq .90$ *which requires that* $n_1 \geq 38$.

If $c_2 = 2$ *(or* $y_2 = 3$*) we must have* $n_1 + n_2 \geq 52$. *With* $n_1 = 38$ *the power is*

$$K(p;\ 38, n_2, 2, 3) = E(3;\ 38, p) + b(2;\ 38, p)\ \ E(1;\ n_2, p)$$

and

$$K(.10;\ 38, n_2, 2, 3) = E(3;\ 38, .10) + b(2;\ 38, .10)\ \ E(1;\ n_2, .10)$$

$$= .74633 + .15837\ \ E(1;\ n_2, .10)$$

By trial we find

$$K(.10;\ 38, 34, 2, 3) = .90030,\ \ K(.10;\ 38, 33, 2, 3) = .89981$$

so that $n_1 = 38$, $n_2 = 34$, $c_1 = 1$, $c_2 = 2$ *is an acceptable plan. Then*

$$K(.02;\ 38, 34, 2, 3) = E(3;\ 38, .02) + b(2;\ 38, .02)\ \ E(1;\ 34, .02)$$

$$= .04015 + (.13588)(.49686) = .10766$$

and when $\bar{p} = .02$

$$I_D = 38 + 34\ \ E(2;\ 38, .10) + 972\ K(.02;\ 38, 34, 2, 3)$$

$$= 38 + 34\ (.17603) + 972\ (.10766) = 143.89$$

We next take $c_2 = 3$ with $c_1 = 1$, $n_1 = 38$. Now $n_1 + n_2 \geq 65$ and the power is

$$K(p;\ 38,n_2,2,4) = E(4;\ 38,p) + b(2;\ 38,p)\ \ E(2;\ n_2,p)$$
$$+ b(3;\ 38,p)\ \ E(1;\ n_2,p)$$

and

$$K(.10;\ 38,n_2,2,4) = .53516 + (.15837)\ \ E(2;\ n_2,.10)$$
$$+ (.21117)\ \ E(1;\ n_2,.10)$$

By trial we find (we might first guess $n_2 = 47$)

$$K(.10;\ 38,54,2,4) = .90024,\ K(.10;\ 38,53,2,4) = .89980$$

so that $n_1 = 38$, $n_2 = 54$, $c_1 = 1$, $c_2 = 3$ is an acceptable plan. Then

$$K(.02;\ 38,54,2,4) = E(4;\ 38,.02) + b(2;\ 38,.02)\ \ E(2;\ 54,.02)$$
$$+ b(3;\ 38,.02)\ \ E(1;\ 54,.02)$$
$$= .00687 + (.13588)(.29393)$$
$$+ (.03327)(.66410) = .06890$$

and when $\bar{p} = .02$

$$I_n = 38 + 54\ \ E(2;\ 38,.10) + 908\ K(.02;\ 38,54,2,4)$$
$$= 38 + 54\ (.17603) + 908\ (.06890) = 100.07$$

We next take $c_2 = 4$ with $c_1 = 1$, $n_1 = 38$. Now $n_1 + n_2 \geq 78$ and the power is

$$K(p;\ 38,n_2,2,5) = E(5;\ 38,p) + b(2;\ 38,p)\ \ E(3;\ n_2,p)$$
$$+ b(3;\ 38,p)\ \ E(2;\ n_2,p)$$
$$+ b(4;\ 38,p)\ \ E(1;\ n_2,p)$$

and

$$K(.10;\ 38,n_2,2,5) = .32986 + (.15837)\ \ E(3;\ n_2,.10)$$
$$+ (.21117)\ \ E(2;\ n_2,.10)$$
$$+ (.20530)\ \ E(1;\ n_2,.10)$$

By trial we find (we might guess $n_2 = 74$)

$$K(.10;\ 38,72,2,5) = .90037\ ,\ K(.10;\ 38,71,2,5) = .89999$$

so that $n_1 = 38$, $n_2 = 72$, $c_1 = 1$, $c_2 = 4$ *is an acceptable plan. Then*

$$K(.02; 38,72,2,5) = .00093 + (.13588)(.17484)$$
$$+ (.03327)(.42341)$$
$$+ (.00594)(.76651) = .04333$$

and when $\bar{p} = .02$

$$I_D = 38 + 72\ (.17603) + 890\ (.04333) = 93.14$$

Similarly with $c_2 = 5$ *we find*

$$I_D = 38 + 88\ (.17603) + 874\ (.02615) = 76.35$$

with $c_2 = 6$ *we get*

$$I_D = 38 + 103\ (.17603) + 859\ (.01533) = 69.30$$

with $c_2 = 7$ *we get*

$$I_D = 38 + 119\ (.17603) + 843\ (.00922) = 66.72$$

With $c_2 = 8$ *we get*

$$I_D = 38 + 133\ (.17603) + 829\ (.00522) = 65.74$$

It appears that if c_2 *is increased to 9 the increase in the second term of* I_D *will be roughly the same as the decrease of the third term. Thus, for the moment at least, further calculations with* $n_1 = 38$ *seem unnecessary.*

Next we repeat all the above steps for $c_1 = 1$ *with* $n_1 = 39$, *then* $n_1 = 40$, *etc., until it is obvious that we have a minimum for each* c_2. *The results in Table 1 are obtained. From the table it is observed that* $\bar{I}_D = 62.43$ *(given that the minimum does not occur with* $c_1 = 0$ *or* $c_1 = 2$*) and the desired plan is* $n_1 = 40$, $n_2 = 96$, $c_1 = 1$, $c_2 = 7$. *We note that it is unnecessary to consider* $c_1 = 3$ *(or greater)*

Table 1.

(n_1,n_2) and I_D for $c_1 = 1$

$c_2 = 2$	$c_2 = 3$	$c_2 = 4$
(38,34), 143.89	(38,54), 100.07	(38,72), 93.14
(39,24), 134.09	(39,43), 99.98	(39,59), 79.86
(40,19), 130.02	(40,37), 95.41	(40,53), 76.90
(41,16), 128.82	(41,33), 92.64	(41,49), 75.68
(42,14), 129.24	(42,30), 92.02	(42,45), 75.61
(43,11), 126.75	(43,28), 92.19	(43,43), 74.82
(44,10), 128.88	(44,26), 92.27	(44,40), 74.22
(45,8), 128.07	(45,24), 92.17	(45,38), 74.55
(46,7), 129.82	(46,22), 91.71	
(47,5), 128.30	(47,21), 93.01	

$c_2 = 5$	$c_2 = 6$	$c_2 = 7$	$c_2 = 8$
(38,88), 76.35	(38,103), 69.30	(38,119), 66.72	(38,133), 65.74
(39,75), 69.91	(39,89), 64.69	(39,104), 63.24	(39,119), 63.71
(40,68), 63.71	(40,82), 63.78	(40,96), 62.43	(40,110), 63.17
(41,63), 66.40	(41,77), 63.11	(41,92), 62.97	(41,105), 63.02
(42,60), 66.42	(42,74), 63.66	(42,87), 63.12	(42,101), 64.44
(43,57), 66.81	(43,71), 64.18	(43,84), 64,26	(43,97), 65.33
(44,54), 67.30	(44,68), 64.64	(44,81), 64.66	(44,94), 66.10
(45,52), 67.50	(45,66), 65.53	(45,79), 65.59	(45,92), 67.29

since the condition $E(y_1; n_1,.10) \geq .90$ requires $n_1 \geq 65$ and we have already found a number of plans with $I_D < 65$.

The OC at $\bar{p} = .02$ for the plan which minimizes I_D has value .99649. Recall that for the single sample plan of Example 2.2 we had .97972.

Dodge and Romig give for the solution to our problem $n_1 = 28$, $n_2 = 72$, $c_1 = 0$, $c_2 = 5$ for which $I_D = 70.89$, $K(.10; 28, 72, 1, 6) = .904$.

The average outgoing quality for the double sample case can be written in various forms but perhaps the one most convenient for use with tables is

$$AOQ = \frac{n_2}{N} p[1-E(y_1; n_1, p)] + (1-\frac{n_1+n_2}{N})p[1-K(p; n_1, n_2, y_1, y_2)] \quad (3.9)$$

Again the AOQL, the maximum taken over p, can be found by trial using the Weintraub [8] table. For the plan found in Example 3.2 which had $\bar{I}_D = 62.43$ we get

if $p = .046$ $AOQ = (.096)(.046)(.99959) + (.864)(.046)(.76707) = .03490$

$p = .047$ $AOQ = (.096)(.047)(.99953) + (.864)(.047)(.75139) = .03502$

$p = .048$ $AOQ = (.096)(.048)(.99946) + (.864)(.048)(.73338) = .03482$

so that AOQL = .035. The AOQL for the corresponding single sample case, found at the end of Section 2, was .030. Intuitively we might expect a larger AOQL for a plan which on the average requires less inspection.

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