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MAXIMIN TESTS AND LOCALLY MOST POWERFUL TESTS.

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1. INTRODUCTION AND SUMMARY.

We shall be concerned with the parametric problem of testing hypotheses concerning the value of one parameter when the values of other parameters (nuisance parameters) are not specified. Neyman [6] derived under certain conditions a locally most powerful two-sided test for this problem i.e. he gave the form of the test maximizing the second derivative of the power function with respect to the parameter of interest at the point specified by the hypothesis. Generalizations of Neyman's results were given by Scheffe [7] and Lehmann [2], using the same technique as Neyman. They were also able to prove that the tests were UMP unbiased. A new technique for dealing with these problems were introduced by Sverdrup [9] and Lehmann and Scheffe [4] where the completeness of the sufficient statistics in an exponential family of densities is used to derive UMP unbiased tests. It is stated by Lehmann and Scheffe [4] that the conditions imposed earlier imply an exponential family of densities.

When no UMP unbiased test exists we have little general theory. The problem is both one of principle and of technique. Most stringent tests exist under general conditions but are difficult to derive in particular cases. Lehmann [3] proposed maximin tests and a local form of maximin tests called tests which maximize the minimum power locally. Definitions of these are given in Section 2. Spjøtvoll [8] has given an example of the form of a maximin test when no UMP unbiased and invariant test exists.

This paper is an attempt to establish some general theory for testing hypotheses when the probability density of the observations does not constitute an exponential family under both the hypothesis and the alternative. The assumptions made in Section 3 is satisfied if we have an exponential family under the hypothesis, but do not say anything about the form of the density under the alternative. The results concern maximin tests and locally most powerful tests, and under certain conditions the form of these tests for the particular family of densities studied, is given in Section 3.

In Section 4 the theory in Section 3 is applied to the problem of testing serial correlation (not circular) in a first order autoregressive sequence. It is found that the usual tests is nearly UMP invariant.

In Section 5 the problem of testing the value of the ratio of variances in the one-way classification variance components model is considered. Some numerical results is given for the power functions of the maximin test, the locally most powerful test and the standard F-test. The results indicate that the standard F-test performs well compared with the other tests.

2. DEFINITIONS AND ELEMENTARY CONSEQUENCES.

Let X be a random variable with distribution belonging to a family $\mathcal{P} = \{P_{\theta,\vartheta} : (\theta,\vartheta) \in \Omega\}$ of distributions. Let the parameter θ be real. We shall consider the problem of testing the hypotheses

$$H_1 : \theta = \theta_0 \quad \text{against} \quad \theta > \theta_0,$$

$$H_2 : \theta = \theta_0 \quad \text{against} \quad \theta \neq \theta_0.$$

The functions

$$d_1(\theta) = \max(\theta - \theta_0, 0),$$

$$d_2(\theta) = |\theta - \theta_0|,$$

will be used as measures of distance from the hypotheses H_1 and H_2 respectively.

In the following let H and d stand for either H_1 and d_1 or H_2 and d_2. Let $\beta(\theta,\vartheta,\varphi)$ denote the power function of a test φ.

The concepts maximin tests and tests which maximize the minimum power locally have been introduced by Lehmann [3]. Definitions of these tests and of locally most powerful tests are now given in terms of the distance function d.

DEFINITION 2.1. A level α test φ_0 of H is locally most powerful (LMP) with respect to the distance function d if, given any other level α test φ, there exists for each ϑ a Δ such that $\beta(\theta,\vartheta,\varphi_0) \geq \beta(\theta,\vartheta,\varphi)$ when $0 < d(\theta) < \Delta$.

Define sets ω_Δ by

$$\omega_\Delta = \{(\theta,\vartheta) : d(\theta) \geq \Delta\}.$$

<u>DEFINITION 2.2</u>. A level α test φ_0 of H maximizes the minimum power over ω_Δ (Δ fixed) if $\inf_{\omega_\Delta} \beta(\theta,\vartheta,\varphi_0) \geq \inf_{\omega_\Delta} \beta(\theta,\vartheta,\varphi)$ for any other level α test φ.

The test φ_0 is said to be a maximin test over ω_Δ. More generally we can speak of maximin tests over any subset ω of the set of alternatives by replacing ω_Δ by ω in Definition 2.2. In Section 3 we shall be concerned with subsets of the form $\{(\theta,\vartheta) : d(\theta) = \Delta\}$.

<u>DEFINITION 2.3</u>. A level α test φ_0 of H is said to maximize the minimum power locally with respect to the distance function d if, given any other level α test φ, there exists a Δ' such that $\inf_{\omega_\Delta} \beta(\theta,\vartheta,\varphi_0) \geq \inf_{\omega_\Delta} \beta(\theta,\vartheta,\varphi)$ when $0 < \Delta < \Delta'$.

A level α test φ is <u>similar</u> if $\beta(\theta_0,\vartheta,\varphi) = \alpha$ for all ϑ, it is <u>unbiased</u> if $\beta(\theta,\vartheta,\varphi) \geq \alpha$ when $d(\theta) > 0$, and we shall say that it is <u>unbiased at θ_0</u> if for each ϑ there exists Δ such that $\beta(\theta,\vartheta,\varphi) \geq \alpha$ when $0 < d(\theta) < \Delta$.

The following two lemmas are easily proved by comparing with the test $\varphi(x) = \alpha$.

LEMMA 2.1. If for each φ the power function $\beta(\theta,\vartheta,\varphi)$ is continuous in θ at θ_0, then a LMP test is similar.

LEMMA 2.2. A test which maximizes the minimum power locally is unbiased, and if the power function of each test is continuous in θ at θ_0, then it is similar.

In view of Lemma 2.1 we introduce the following definition.

DEFINITION 2.4. A level α test φ_0 of H is locally most powerful similar (unbiased) with respect to the distance function d if, given any other similar (unbiased at θ_0) level α test φ, there exists for each ϑ a Δ such that $\beta(\theta,\vartheta;\varphi_0) \geq \beta(\theta,\vartheta,\varphi)$ when $0 < d(\theta) < \Delta$.

We shall use the abbreviations LMPS and LMPU. Note that a LMPU test need not be unbiased after the usual definition of unbiasedness, only unbiasedness near θ_0 is required.

The following lemma proves that in some cases a LMPU test maximizes the minimum power locally. (Compare Lehmann [3], p.342).

LEMMA 2.3. Let φ_0 be a LMPU level α test of H and suppose that the power function $\beta(\theta,\vartheta,\varphi_0)$ depends only upon $d(\theta)$ and is continuous as a function of d. Then φ_0 maximizes the minimum power locally provided $\beta(\theta,\vartheta,\varphi_0)$ is bounded away from α for every set of alternatives which is bounded away from H (measured by d).

PROOF. If a test φ is not unbiased, then for some Δ' we have $\inf_{\omega_\Delta} \beta(\theta,\vartheta,\varphi) < \alpha$ for $0 < \Delta < \Delta'$. Hence $\inf_{\omega_\Delta} \beta(\theta,\vartheta,\varphi_0) > \inf_{\omega_\Delta} \beta(\theta,\vartheta,\varphi)$ for $0 < \Delta < \Delta'$. Suppose then that φ is unbiased. Let ϑ^* be a fixed value of ϑ, since φ_0 is LMPS we have for some Δ^*

(2.1) $$\beta(\theta,\vartheta^*,\varphi_0) \geq \beta(\theta,\vartheta^*,\varphi) \quad \text{when} \quad 0 < d(\theta) < \Delta^*.$$

Define $a(\Delta)$ by $\inf_{\omega_\Delta} \beta(\theta,\vartheta,\varphi_0) = \alpha + a(\Delta)$. By the condition of the lemma $a(\Delta)$ does not depend upon ϑ and $a(\Delta) > 0$ when $\Delta > 0$. Since $\beta(\theta,\vartheta,\varphi_0)$ is continuous as a function of d, there exists $\Delta' < \Delta^*$ such that $a(\Delta') < a(\Delta^*)$. Hence

(2.2) $$\inf_{\omega_\Delta} \beta(\theta,\vartheta,\varphi_0) = \inf_{\Delta \leq d(\theta) \leq \Delta^*} \beta(\theta,\vartheta,\varphi_0) \quad \text{for} \quad 0 < \Delta < \Delta'.$$

Then we have by (2.1), (2.2) and the fact that $\beta(\theta,\vartheta,\varphi_0)$ does not depend upon ϑ.

$$\inf_{\omega_\Delta} \beta(\theta,\vartheta,\varphi) \leq \inf_{\Delta \leq d(\theta) \leq \Delta^*} \beta(\theta,\vartheta,\varphi) \leq \inf_{\Delta \leq d(\theta) \leq \Delta^*} \beta(\theta,\vartheta^*,\varphi)$$

$$\leq \inf_{\Delta \leq d(\theta) \leq \Delta^*} \beta(\theta,\vartheta^*,\varphi_0) = \inf_{\omega_\Delta} \beta(\theta,\vartheta^*,\varphi_0)$$

$$= \inf_{\omega_\Delta} \beta(\theta,\vartheta,\varphi_0) \quad \text{for} \quad 0 < \Delta < \Delta'.$$

The next two lemmas gives conditions under which a test is LMPS or LMPU. Let β'_θ and β''_θ denote the two first derivatives of the power function with respect to θ.

LEMMA 2.4. Suppose that the power function of any test φ has a continuous derivative with respect to θ at θ_0. If there exists a similar test φ_0 such that $\beta'_\theta(\theta_0,\vartheta,\varphi_0) > \beta'_\theta(\theta_0,\vartheta,\varphi)$ for all ϑ when φ is any other similar test, then φ_0 is the unique LMPS test of H_1 with respect to d_1.

LEMMA 2.5. Suppose that the power function of any test has a continuous second derivative with respect to θ at θ_0. If there exists a test φ_0 unbiased at θ_0 such that $\beta''_\theta(\theta_0,\vartheta,\varphi_0) > \beta''_\theta(\theta_0,\vartheta,\varphi)$ for all ϑ when φ is any other test unbiased at θ_0, then φ_0 is the unique LMPU test of H_2 with respect to d_2.

PROOF. We give only the proof of Lemma 2.5, since the proof of Lemma 2.4 is similar.

For any test φ we have $\beta(\theta,\vartheta,\varphi) = \beta(\theta_0,\vartheta,\varphi)+(\theta-\theta_0)$ $\beta'_\theta(\theta_0,\vartheta,\varphi)+\frac{1}{2}(\theta-\theta_0)^2\beta''_\theta(\theta_0+t_\varphi(\theta-\theta_0),\vartheta,\varphi)$ for some t_φ with $0 < t_\varphi < 1$. If φ is unbiased at θ_0, we get $\beta(\theta,\vartheta,\varphi) =$ $+\frac{1}{2}(\theta-\theta_0)^2\beta''_\theta(\theta_0+t_\varphi(\theta-\theta_0),\vartheta,\varphi)$. Consider a fixed ϑ and φ. When $|\theta-\theta_0|$ is small enough then

$\beta''_\theta(\theta_0+t_{\varphi_0}(\theta-\theta_0),\vartheta,\varphi_0) > \beta''_\theta(\theta_0+t_\varphi(\theta-\theta_0),\vartheta,\varphi)$ since $\beta''_\theta(\theta_0,\vartheta,\varphi_0) > \beta''_\theta(\theta_0,\vartheta,\varphi)$. Hence $\beta(\theta,\vartheta,\varphi_0) > \beta(\theta,\vartheta,\varphi)$ for $|\theta-\theta_0|$ small enough, and the lemma is proved.

A test φ of H_2 satisfying (a) $\beta(\theta_0,\vartheta,\varphi) = \alpha$, (b) $\beta'_\theta(\theta_0,\vartheta,\varphi) = 0$ and $\beta''_\theta(\theta_0,\vartheta,\varphi) =$ maximum among tests satisfying (a) and (b), was denoted test of type B by Neyman [6]. If there exists a unique test of type B, then by Lemma 2.5 it is a LMPU test of H_2. But in general we cannot be certain that a type B test is LMPU as defined above.

The reason for formulating the definitions of locally most powerful tests as above is to have a definition covering both one-sided and two-sided tests, and to ensure that the corresponding tests have optimum properties near the hypotheses to a greater extent than guaranteed by type B tests. This way of defining locally most powerful tests is not new, see e.g. Lehmann [3], p.342, where the one-parameter case is considered.

3. DERIVATION OF MAXIMIN TESTS AND LMP TESTS FOR A PARTICULAR FAMILY OF DISTRIBUTIONS.

We shall consider the case where the probability distribution of X belongs to a family $\mathcal{P}^X = \{P_{\theta,\vartheta}^X : (\theta,\vartheta) \in \Omega\}$ where $P_{\theta,\vartheta}^X$ is defined by

$$dP^X_{\theta,\vartheta}(x) = a(x,\theta,\vartheta)b(t(x),\theta,\vartheta)d\mu(x),$$

where μ is a σ-finite measure over a Euclidean space.

We shall assume that there exists a value ϑ_0 of ϑ such that the distribution $P^X_{\theta_0,\vartheta_0}$ dominates the family $\mathcal{P}^X$. In that case we may write

$$(3.1)\qquad dP^X_{\theta,\vartheta}(x) = a(x,\theta,\vartheta)b(t(x),\theta,\vartheta)/(a(x,\theta_0,\vartheta_0)\, b(t(x),\theta_0,\vartheta_0))dP^X_{\theta_0,\vartheta_0}(x) \quad \text{a.e. } \mathcal{P}^X.$$

Further it is assumed that the statistic $T = t(X)$ is sufficient when $\theta = \theta_0$, and that the family of distributions for T when $\theta = \theta_0$ is boundedly complete.

The assumptions stated above will be assumed to hold for the rest of this section.

Let $P^{X|t}_{\theta_0}$ denote the conditional probability distribution of X given $T = t$ when $\theta = \theta_0$. Since T is sufficient, $P^{X|t}_{\theta_0}$ can be chosen to be independent of ϑ. Let $E^{X|t}_{\theta_0}$ denote expectation taken with respect to $P^{X|t}_{\theta_0}$. Similarily let $E^X_{\theta,\vartheta}$ and $E^T_{\theta,\vartheta}$ denote expectations with respect to the distribution of X and the marginal distribution of T respectively.

A test φ is similar if $E^X_{\theta_0,\vartheta}\varphi(X) = \alpha$ for all ϑ. Since T is sufficient and complete when $\theta = \theta_0$ a test is similar if and only if $E^{X|t}_{\theta_0}\varphi(X) = \alpha$ a.e. $\mathcal{P}^T_{\theta_0}$ where $\mathcal{P}^T_{\theta_0}$ denotes the family of distributions for T when $\theta = \theta_0$.

The following lemma will be useful when establishing the uniqueness of tests.

LEMMA 3.1. Let X be a random variable, $T = t(X)$ a statistic and let E^X, E^T and $E^{X|t}$ denote expectations with self-evident notation. Given a test function φ_o such that

$$\varphi_o(x) = \begin{cases} 1 & \text{when} \quad h(x) > \sum_{i=1}^{m} k_i(t(x))f_i(x) \\ 0 & \text{when} \quad h(x) < \sum_{i=1}^{m} k_i(t(x))f_i(x) \end{cases}$$

for some functions $h, f_1, f_2, \ldots, f_m, k_1, k_2, \ldots, k_m$. Then a test function φ satisfying

$$(3.2) \quad E^{X|t}\varphi(X)f_i(X) = E^{X|t}\varphi_o(X)f_i(X) \quad \text{a.e.} \quad i = 1,2,\ldots,m$$

satisfies

$$E^X\varphi(X)h(X) = E^X\varphi_o(X)h(X)$$

if and only if $\varphi(x) = \varphi_o(x)$ a.e. on the set $\{x : h(x) \neq \sum_{i=1}^{m} k_i(t(x))f_i(x)\}$. Otherwise $E^X\varphi(X)h(X) <$ $E^X\varphi_o(X)h(X)$.

PROOF. We have by (3.2)

$$E^X\varphi(X)k_i(t(X))f_i(X) = E^T k_i(T)E^{X|t}$$

$$\varphi(X)f_i(X) = E^T k_i(T)E^{X|t}\varphi_o(X)f_i(X)$$

$$= E^X\varphi_o(X)k_i(t(X))f_i(X).$$

It follows that

$$E^X\varphi_o(X)h(X) - E^X\varphi(X)h(X)$$

$$= E^X(\varphi_o(X)-\varphi(X))(h(X) - \sum_{i=1}^{m} k_i(t(X))f_i(X)).$$

By the definition of φ_o the above difference is $\geqq 0$ for all φ. It is $= 0$ if and only if $\varphi(x) = \varphi_o(x)$ a.e. on the set $\{x : h(x) \neq \sum_{i=1}^{m} k_i(t(x))f_i(x)\}$.

We have the following theorem

<u>THEOREM 3.1</u>. For the hypothesis $\theta = \theta_o$ against $(\theta,\vartheta) = (\theta_1,\vartheta_1)$ there exists a most powerful similar level α test φ_1 defined by

$$\varphi_1(x) = \begin{cases} 1 & \text{when } a(x,\theta_1,\vartheta_1)/a(x,\theta_o,\vartheta_o) > c(t) \\ \gamma(t) & \text{when } a(x,\theta_1,\vartheta_1)/a(x,\theta_o,\vartheta_o) = c(t) \\ 0 & \text{when } a(x,\theta_1,\vartheta_1)/a(x,\theta_o,\vartheta_o) < c(t), \end{cases}$$

where $c(t)$ and $\gamma(t)$ are determined by $E^{X|t}_{\theta_o}\varphi_1(X) = \alpha$ for all t.

PROOF. By (3.1) the power of any test φ at (θ_1, ϑ_1) is

$$E^X_{\theta_1,\vartheta_1}\varphi(X) = E^X_{\theta_o,\vartheta_o}\varphi(X)a(X,\theta_1,\vartheta_1)b(t(X),\theta_1,\vartheta_1)/ (a(X,\theta_o,\vartheta_o)b(t(X),\theta_o,\vartheta_o))$$

$$= E^T_{\theta_o,\vartheta_o}(b(T,\theta_1,\vartheta_1)/b(T,\theta_o,\vartheta_o))E^{X|t}_{\theta_o}(\varphi(X) a(X,\theta_1,\vartheta_1)/a(X,\theta_o,\vartheta_o)).$$

A test maximizes the power under the condition of similarity if it for each t maximizes $E^{X|t}_{\theta_o}(\varphi(X)a(X,\theta_1,\vartheta_1)/a(X,\theta_o,\vartheta_o))$ under the condition of similarity i.e. under the condition $E^{X|t}_{\theta_o}\varphi(X) = \alpha$. By the Neyman-Pearson fundamental lemma the test φ_1 has this property.

REMARK. The test is unique (a.e. $\mathcal{P}^X$) if $P^X_{\theta_o,\vartheta_o}(a(X,\theta_1,\vartheta_1) = c(t(X))a(X,\theta_o,\vartheta_o)) = 0$.

The Remark is proved by using Lemma 3.1 with $h(x) = a(x,\theta_1,\vartheta_1)b(t(x),\theta_1,\vartheta_1)/(a(x,\theta_o,\vartheta_o)b(t(x),\theta_o,\vartheta_o))$ and $P^X_{\theta_o,\vartheta_o}$ as probability measure. The remarks following Theorem 3.3 and Theorem 3.5 later can be proved in a similar way.

It should be noted that Theorem 3.1 can be regarded as an application of Theorem 3 of Sverdrup [9] to the family $\mathcal{P}^X$.

The following corollary is obvious.

COROLLARY 3.1. If φ_1 does not depend upon ϑ_1 then it is the uniformly most powerful similar level α test for testing $\theta = \theta_0$ against $\theta = \theta_1$.

This leads to

THEOREM 3.2. Suppose that φ_1 in Theorem 3.1 does not depend upon ϑ_1 and that $\beta(\theta, \vartheta, \varphi_1) \geq \beta(\theta_1, \vartheta, \varphi_1)$ for all (θ, ϑ) with $\theta \geq \theta_1$. Then φ_1 is a maximin test among similar level α tests over the set of alternatives with $\theta \geq \theta_1$, and φ_1 is the unique maximin test if it is the unique most powerful test.

PROOF. By Corollary 3.1 φ_1 is the uniformly most powerful test for $\theta = \theta_0$ against $\theta = \theta_1$. Hence any other test has less or equal minimum power when $\theta = \theta_1$, and hence less or equal minimum power over $\theta \geq \theta_1$. If φ_1 is unique any other test must have less power for some points with $\theta = \theta_1$, and hence less minimum power over the set of alternatives with $\theta \geq \theta_1$.

Let a'_θ and b'_θ denote the derivatives with respect to θ of the functions a and b respectively. The next theorem gives the form of the test that maximizes $\beta'_\theta(\theta_0, \vartheta, \varphi)$ locally.

THEOREM 3.3. Suppose that for any test φ the derivative with respect to θ of the power function $\beta(\theta,\vartheta,\varphi)$ can be computed under the integral sign. Then among similar level α tests the following test φ_2 maximizes the derivative of the power function at (θ_0,ϑ_0)

$$\varphi_2(x) = \begin{cases} 1 & \text{when } a'_\theta(x,\theta_0,\vartheta_0)/a(x,\theta_0,\vartheta_0) > c(t) \\ \gamma(t) & \text{when } a'_\theta(x,\theta_0,\vartheta_0)/a(x,\theta_0,\vartheta_0) = c(t) \\ 0 & \text{when } a'_\theta(x,\theta_0,\vartheta_0)/a(x,\theta_0,\vartheta_0) < c(t), \end{cases}$$

where $c(t)$ and $\gamma(t)$ are determined by $E^{X|t}_{\theta_0}\varphi_2(X) = \alpha$ for all t.

PROOF. The derivative at (θ_0,ϑ_0) with respect to θ of the power function of a test φ is

$$\begin{aligned}\beta'_\theta(\theta_0,\vartheta_0,\varphi) &= \int\varphi(x)(a'_\theta(x,\theta_0,\vartheta_0)b(t(x),\theta_0,\vartheta_0) \\ &\qquad +a(x,\theta_0,\vartheta_0)b'_\theta(t(x),\theta_0,\vartheta_0))d\mu(x) \\ &= E^X_{\theta_0,\vartheta_0}\varphi(X)(a'_\theta(X,\theta_0,\vartheta_0)/a(X,\theta_0,\vartheta_0)+b'_\theta(t(X),\theta_0,\vartheta_0)/ \\ &\qquad b(t(X),\theta_0,\vartheta_0)) \\ &= E^T_{\theta_0,\vartheta_0}(b'_\theta(T,\theta_0,\vartheta_0)/b(T,\theta_0,\vartheta_0))\; E^{X|t}_{\theta_0}\varphi(X) \\ &\quad + E^T_{\theta_0,\vartheta_0}E^{X|t}_{\theta_0}(\varphi(X)a'_\theta(X,\theta_0,\vartheta_0)/a(X,\theta_0,\vartheta_0)).\end{aligned}$$

$\beta'_\theta(\theta_0,\vartheta_0,\varphi)$ is maximized under the condition of similarity if for each t the second expectation in the last expression is maximized under the condition of similarity. An application of the Neyman-Pearson fundamental lemma gives the test φ_2.

<u>REMARK</u>. The test is unique (a.e. $\mathcal{P}^X$) if

$$P^X_{\theta_0,\vartheta_0}(a'_\theta(X,\theta_0,\vartheta_0) = c(t(X))a(X,\theta_0,\vartheta_0)) = 0.$$

The next theorem gives a condition under which the test φ_2 of Theorem 3.3 is LMPS.

<u>THEOREM 3.4</u>. In addition to the assumption of Theorem 3.3 suppose that the derivative $\beta'_\theta(\theta,\vartheta,\varphi)$ is continuous in θ at θ_0 for any test φ, and suppose that the hypothesis testing problem H_1 is invariant under a group G of transformations, and that θ is a maximal invariant under the induced group $\bar{G}$ of transformations of the parameter space. If φ_2 of Theorem 3.3 is unique and the power function is invariant, then φ_2 is the unique LMPS level α test of H_1 with respect to the distance function d_1.

<u>PROOF</u>. Since φ_2 is unique we have

$$(3.3) \qquad \beta'_\theta(\theta_0,\vartheta_0,\varphi_2) > \beta'_\theta(\theta_0,\vartheta_0,\varphi)$$

for any other similar level α test φ. We shall show that $\beta'_\theta(\theta_0,\vartheta,\varphi_2) > \beta'_\theta(\theta_0,\vartheta,\varphi)$ for all ϑ. Then by Lemma 2.4 φ_2 is LMPS.

Suppose on the contrary that there exists a ϑ^* such that

$$(3.4) \qquad \beta'_\theta(\theta_0,\vartheta^*,\varphi_2) \leq \beta'_\theta(\theta_0,\vartheta^*,\varphi).$$

Since θ is maximal invariant there exists a $g \in \bar{G}$ such that $\bar{g}((\theta,\vartheta_0)) = (\theta,\vartheta^*)$. Consider the test $\varphi^* = \varphi g$ where g corresponds to $\bar{g}$. We have $E^X_{\theta,\vartheta_0}\varphi g(X) = E^X_{\bar{g}((\theta,\vartheta_0))}\varphi(X) = E^X_{\theta,\vartheta^*}\varphi(X)$, or in terms of power functions, $\beta(\theta,\vartheta_0,\varphi g) = \beta(\theta,\vartheta^*,\varphi)$. Hence by (3.4) $\beta'_\theta(\theta_0,\vartheta^*,\varphi_2) \leq \beta'_\theta(\theta_0,\vartheta^*,\varphi) = \beta'_\theta(\theta_0,\vartheta_0,\varphi g)$. But φ_2 is invariant, hence $\beta'_\theta(\theta_0,\vartheta^*,\varphi_2) = \beta'_\theta(\theta_0,\vartheta_0,\varphi_2)$, and we get $\beta'_\theta(\theta_0,\vartheta_0,\varphi_2) \leq \beta'_\theta(\theta_0,\vartheta_0,\varphi g)$ which contradicts (3.3).

<u>COROLLARY 3.2</u>. Under the assumptions of Theorems 3.3 and 3.4 the test φ_2 maximizes the minimum power locally with respect to the distance function d_1, provided its power function is bounded away from α when θ is bounded away from θ_0.

<u>PROOF</u>. Follows from Theorem 3.4 and Lemma 2.3, since here a LMPS test is LMPU.

Let a''_θ and b''_θ denote the second derivatives with respect to θ of the functions a and b respectively.

<u>THEOREM 3.5</u>. Suppose that for any test φ the first and second derivative with respect to θ of $\beta(\theta,\vartheta,\varphi)$ can be

computed under the integral sign and suppose that

$a'_\theta(x,\theta_0,\vartheta)/a(x,\theta_0,\vartheta)+b'_\theta(t(x),\theta_0,\vartheta)/b(t(x),\theta_0,\vartheta) = k(\vartheta)h(x)$

for some functions k and h, with $k(\vartheta) > 0$ for all ϑ. Then among level α tests unbiased at θ_0 (with respect to d_2), the following test φ_3 maximizes the second derivative of the power function at (θ_0,ϑ_0)

$$\varphi_3(x) = \begin{cases} 1 & \text{when } a''_\theta(x,\theta_0,\vartheta_0)/a(x,\theta_0,\vartheta_0)+c_1(t)a'_\theta(x,\theta_0,\vartheta_0)/a(x,\theta_0,\vartheta_0) > c_2(t) \\ \gamma(t) & \text{when } a''_\theta(x,\theta_0,\vartheta_0)/a(x,\theta_0,\vartheta_0)+c_1(t)a'_\theta(x,\theta_0,\vartheta_0)/a(x,\theta_0,\vartheta_0) = c_2(t) \\ 0 & \text{when } a''_\theta(x,\theta_0,\vartheta_0)/a(x,\theta_0,\vartheta_0)+c_1(t)a'_\theta(x,\theta_0,\vartheta_0)/a(x,\theta_0,\vartheta_0) < c_2(t), \end{cases}$$

where $c_1(t), c_2(t)$ and $\gamma(t)$ are determined by $E^{X|t}_{\theta_0}\varphi_3(X) = \alpha$ and $E^{X|t}_{\theta_0}\varphi_3(X)(a'_\theta(X,\theta_0,\vartheta_0)/a(X,\theta_0,\vartheta_0)+b'_\theta(t(X),\theta_0,\vartheta_0)/b(t(X),\theta_0,\vartheta_0)) = 0$.

<u>PROOF</u>. Unbiasedness in some neighbourhood of θ_0 implies $\beta'_\theta(\theta_0,\vartheta,\varphi) = 0$, hence

$$(3.5)\quad \begin{aligned} 0 &= E^X_{\theta_0,\vartheta}\varphi(X)(a'_\theta(X,\theta_0,\vartheta)/a(X,\theta_0,\vartheta) \\ &\quad +b'_\theta(t(X),\theta_0,\vartheta)/b(t(X),\theta_0,\vartheta)) \\ &= E^T_{\theta_0,\vartheta}E^{X|t}_{\theta_0}\varphi(X)(a'_\theta(X,\theta_0,\vartheta)/ \\ &\quad a(X,\theta_0,\vartheta)+b'_\theta(t(X),\theta_0,\vartheta)/b(t(X),\theta_0,\vartheta)). \end{aligned}$$

We may choose the function $h(x)$ in the theorem equal to $a'_\theta(x,\theta_0,\vartheta_0)/a(x,\theta_0,\vartheta_0)+b'_\theta(t(x),\theta_0,\vartheta_0)/b(t(x),\theta_0,\vartheta_0)$. Hence by (3.5)

$$0 = E^T_{\theta_0,\vartheta} E^{X|t}_{\theta_0} \varphi(X)(a'_\theta(X,\theta_0,\vartheta_0)/a(X,\theta_0,\vartheta_0)+ b'_\theta(t(X),\theta_0,\vartheta_0)/b(t(X),\theta_0,\vartheta_0)).$$

Completeness of T implies

$$E^{X|t}_0 \varphi(X)(a'_\theta(X,\theta_0,\vartheta_0)/a(X,\theta_0,\vartheta_0)+b'(t(X),\theta_0,\vartheta_0)/ b(t(X),\theta_0,\vartheta_0)) = 0, \quad \text{a.e. } \mathcal{P}^T_{\theta_0}. \tag{3.6}$$

The test must be similar, hence as before

$$E^{X|t}_{\theta_0} \varphi(X) = \alpha \quad \text{a.e. } \mathcal{P}^T_{\theta_0}. \tag{3.7}$$

We have

$$\beta''_\theta(\theta_0,\vartheta_0,\varphi) = E^T_{\theta_0,\vartheta_0} E^{X|t}_{\theta_0} \varphi(X)(a''_\theta(X,\theta_0,\vartheta_0)/ a(X,\theta_0,\vartheta_0)+2a'_\theta(X,\theta_0,\vartheta_0)b'_\theta(t(X),\theta_0,\vartheta_0)/(a(X,\theta_0,\vartheta_0)b(t(X),\theta_0,\vartheta_0)) +b''_\theta(t(X),\theta_0,\vartheta_0)/b(t(X),\theta_0,\vartheta_0)).$$

Maximum is obtained if for each t the expectation $E^{X|t}_{\theta_0}$ in the above expression is maximized under conditions (3.6) and (3.7). An application of the Neyman-Pearson fundamental lemma gives the test φ_3.

REMARK. The test is unique (a.e. $\mathcal{P}^X$) if

$$P_{\theta_o,\vartheta_o}(a''_\theta(X,\theta_o,\vartheta_o)+c_1(t(X))a'_\theta(X,\theta_o,\vartheta_o) = c_2(t(X))a(X,\theta_o,\vartheta_o)) = 0.$$

THEOREM 3.6. In addition to the assumptions of Theorem 3.5 suppose that $\beta''_\theta(\theta,\vartheta;\varphi)$ is continuous in θ at θ_o for any test φ and suppose that the hypothesis testing problem H_2 is invariant under a group G of transformations, and that $|\theta-\theta_o|$ is a maximal invariant under the induced group $\bar{G}$ of transformations of the parameter space. If φ_3 of Theorem 3.5 is unique and the power function is invariant, then φ_3 is the unique LMPU level α test of H_2 with respect to the distance function d_2.

PROOF. Analoguous to the proof of Theorem 3.4.

A corollary analogous to Corollary 3.2 could also be formulated.

We have tacitly assumed measurability of the functions occuring in the theorems. We shall not prove this, but only note that in each specific case we may find (measurable) tests which is of the form given in the theorems. By Lemma 3.1 they will be most powerful.

4. TESTING FOR SERIAL CORRELATION.

The model for the observations $X_1,X_2,\ldots,X_n$ is

$$X_i = \rho X_{i-1}+U_i \qquad i = 2,3,\ldots,n$$

where $U_2,U_3,\ldots,U_n$ are independent $N(0,\sigma^2)$, and $X_1,X_2,\ldots,X_n$ have a multinormal distribution with $EX_i = 0$, $\text{Var } X_i = \sigma^2/(1-\rho^2)$ and $\text{Cov}(X_i,X_j) = \rho^{|i-j|}\sigma^2/(1-\rho^2)$. The parameters σ and ρ are unknown.

We shall consider the problem of testing the hypotheses

$$H_1 : \rho = \rho_0 \quad \text{against} \quad \rho > \rho_0,$$

$$H_2 : \rho = 0 \quad \text{against} \quad \rho \neq 0.$$

The hypothesis testing problem H_1 is invariant under a common (positive) change of scale of $X_1,X_2,\ldots,X_n$. A maximal invariant is

$$S' = \left[\frac{X_1}{\sqrt{\sum_{i=1}^{n} X_i^2}}, \frac{X_2}{\sqrt{\sum_{i=1}^{n} X_i^2}}, \ldots, \frac{X_n}{\sqrt{\sum_{i=1}^{n} X_i^2}}\right].$$

The distribution of S depends only upon ρ, hence any invariant test is similar. When considering invariant tests it is therefore no restriction to restrict attention to similar tests.

The probability density of $X_1, X_2, \ldots, X_n$ is

$$(2\pi)^{-\frac{1}{2}n}\sigma^{-n}(1-\rho^2)^{\frac{1}{2}}\exp(-\tfrac{1}{2}\sigma^{-2}(\sum_{i=1}^{n}x_i^2 -$$

$$2\rho\sum_{i=2}^{n}x_ix_{i-1} + \rho^2\sum_{i=2}^{n-1}x_i^2)),$$

which can be written in the form $a(x,\rho,\sigma)b(t(x),\rho,\sigma)$ with

$$a(x,\rho,\sigma)=\exp(-\tfrac{1}{2}\sigma^{-2}((\rho^2-\rho_0^2)\sum_{i=2}^{n-1}x_i^2-2(\rho-\rho_0)\sum_{i=2}^{n}x_ix_{i-1}))$$

and

$$b(t(x),\rho,\sigma)=(2\pi)^{-\frac{1}{2}n}\sigma^{-n}(1-\rho^2)^{\frac{1}{2}}\exp(-\tfrac{1}{2}\sigma^{-2}t(x))$$

where

$$t(X) = \sum_{i=1}^{n}X_i^2 + \rho_0^2\sum_{i=2}^{n-1}X_i^2 - 2\rho_0\sum_{i=2}^{n}X_iX_{i-1}.$$

The probability measure for $\rho=\rho_0$ and $\sigma=\sigma_0$ can be used as a dominating measure for any σ_0. We have $a(x,\rho_0,\sigma) = 1$. $T = t(X)$ is sufficient and complete when $\rho=\rho_0$. Applying Theorem 3.1, the most powerful similar test against an alternative (ρ_1,σ_1) is found to have the rejection region

$$2\sum_{i=2}^{n}X_iX_{i-1} - (\rho_1+\rho_0)\sum_{i=2}^{n-1}X_i^2 > c(T).$$

Introduce

$$W_1 = \frac{2\sum_{i=2}^{n} X_i X_{i-1} - (\rho_1+\rho_o)\sum_{i=2}^{n-1} X_i^2}{\sum_{i=1}^{n} X_i^2 + \rho_o^2 \sum_{i=2}^{n-1} X_i^2 - 2\rho_o \sum_{i=2}^{n} X_i X_{i-1}} .$$

The distribution of W_1 does not depend upon σ. T is sufficient and complete when $\rho = \rho_o$. Then by a theorem of Basu [1] W_1 and T are independent when $\rho = \rho_o$.

The rejection region may be written $W_1 > c(T)/T$ where $c(T)$ is determined by $P(W_1 > c(T)/T \mid T) = \alpha$ when $\rho = \rho_o$. But since W_1 and T are independent when $\rho = \rho_o$ we must have $c(t)/t$ equal to a constant. Hence the rejection region is $W_1 > c$ where c is determined by $P(W_1 > c) = \alpha$ when $\rho = \rho_o$.

Since W_1 does not depend upon σ_1 it is the most powerful similar test for $\rho = \rho_o$ against $\rho = \rho_1$. Since here invariance implies similarity and W_1 is invariant, it is also the most powerful invariant test for $\rho = \rho_o$ against $\rho = \rho_1$. By the Hunt-Stein theorem ([3], p.336) the test also maximizes the minimum power over the set of alternatives with $\rho = \rho_1$. If we could prove that the power function of the test increases with ρ, then it is proved that it maximizes the minimum power over the set of alternatives with $\rho \geq \rho_1$.

The following argument will show that the test based on W_1 is almost a UMP invariant test. We have

$$W_1 = \frac{2\sum_{i=2}^{n} X_i X_{i-1} / \sum_{i=1}^{n} X_i^2 + (\rho_1+\rho_o)((X_1^2+X_n^2)/\sum_{i=1}^{n} X_i^2 - 1)}{1+\rho_o^2(1-(X_1^2+X_n^2)/\sum_{i=1}^{n} X_i^2) - 2\rho_o \sum_{i=2}^{n} X_i X_{i-1}/\sum_{i=1}^{n} X_i^2} .$$

If we neglect the term $(X_1^2+X_n^2)/\sum_{i=1}^{n} X_i^2$ which is small even for moderately large values of n, we find that to reject when $W_1 > c$ is equivalent to reject when $W_o > c'$ where

$$W_o = \frac{\sum_{i=2}^{n} X_i X_{i-1}}{\sum_{i=1}^{n} X_i^2} .$$

For each ρ_1 this is an approximation to the most powerful invariant test for $\rho = \rho_o$ against $\rho = \rho_1$. It does not depend upon ρ_1. Hence it is almost a UMP invariant test for $\rho = \rho_o$ against $\rho > \rho_o$.

Using Theorem 3.3 and reasoning as above it is found that the test which maximizes the derivative of the power function with respect to ρ at the point (ρ_o, σ_o) is based on the statistic

$$W_2 = \frac{\sum_{i=2}^{n} X_i X_{i-1} - \rho_o \sum_{i=2}^{n-1} X_i^2}{\sum_{i=1}^{n} X_i^2 + \rho_o^2 \sum_{i=2}^{n-1} X_i^2 - 2\rho_o \sum_{i=2}^{n} X_i X_{i-1}}$$

with rejection region $W_2 >$ constant.

Since ρ is maximal invariant in the parameter space and the distribution of W_2 depends only upon ρ, it is seen from Theorem 3.4 that the test based on W_2 is LMPS.

If we in W_2 neglect the term $(X_1^2+X_n^2)/\sum_{i=1}^{n} X_i^2$ it is seen as for W_1 that the test based on W_2 reduces to the test based on W_o. Hence the test based on W_o may be regarded as an approximation to the LMPS test.

The statistics W_1 and W_2 do not, of course, uniquely reduce to W_o when we neglect terms of the form $(X_1^2+X_n^2)/\sum_{i=1}^{n} X_i^2$. Another possible statistic is

$$W_o' = \frac{\sum_{i=2}^{n}(X_i-X_{i-1})^2}{\sum_{i=1}^{n} X_i^2} = 2 - \frac{X_1^2+X_n^2}{\sum_{i=1}^{n} X_i^2} - 2W_o \ .$$

See for example [5]. The test with rejection region $W_o' <$ constant can also be regarded as both a nearly UMP invariant test and LMPS test. The difference between the power functions of the two tests can be expected to be small.

If we set $\rho_o = 0$, then the test based upon W_2 reduces exactly to the test based upon W_o, hence in this case the latter is LMPS. If we set $\rho_o = 0$ and $\rho_1 = 1$, then the test based upon W_1 reduces exactly to the test based upon W_o', hence the latter is most powerful invariant against the alternative $\rho_1 = 1$. This should give an indication of the difference between the two tests. The test based upon W_o is a little more powerful than the test based upon W_o' near the hypothesis, and the latter is a little more powerful at alternative near $\rho = 1$.

Finally we shall find a test of the hypothesis $H_2 : \rho = 0$ against $\rho \neq 0$. $[X_1/X_n, X_2/X_n, \ldots, X_{n-1}/X_n]$ is a maximal invariant under a common change of scale of all variables, and $|\rho|$ is a maximal invariant in the parameter space.

If we apply Theorem 3.5 it is found that the test which maximizes the second derivative of the power function at $(0,\sigma_0)$ subject to the restriction of unbiasedness and similarity, rejects when

$$-\sum_{i=2}^{n-1} X_i^2 + \frac{1}{\sigma_0^2}\Big(\sum_{i=2}^{n} X_i X_{i-1}\Big)^2 + c_1(T)\sum_{i=2}^{n} X_i X_{i-1} > c_2(T)$$

where in this case $T = \sum_{i=1}^{n} X_i^2$. This can be written as

$$\frac{X_1^2 + X_n^2}{\Big(\sum_{i=1}^{n} X_i^2\Big)^2} + \frac{1}{\sigma_0^2}\left(\frac{\sum_{i=2}^{n} X_i X_{i-1}}{\sum_{i=1}^{n} X_i^2} + c_3(T)\right)^2 > c_4(T).$$

Neglecting the term $(X_1^2 + X_n^2)/(\sum_{i=1}^{n} X_i^2)^2$ and reasoning as before we get the rejection region

$$W_0 < -c \quad \text{and} \quad W_0 > c$$

where c is determined from the condition of level α.

This test is an approximation to the test which maximizes the second derivative of the power function at $(0,\sigma_0)$. Since the power function of the former test depends only upon $|\rho|$ it is by Theorem 3.6 an approximation to the LMPU test at $\rho = 0$.

5. VARIANCE COMPONENTS MODELS.

In a previous paper [8] the author has studied the unbalanced one-way classification variance components model.

$$X_{ij} = \mu + U_i + V_{ij} \quad j = 1,2,\ldots,n_i, \; i = 1,2,\ldots,r,$$

where μ is an unknown constant, and where the U_i and V_{ij} are all independently normally distributed with expectations zero and variances τ^2 and σ^2 respectively.

The hypothesis to be tested is

$$H : \Delta = \Delta_0 \quad \text{against} \quad \Delta > \Delta_0$$

where $\Delta = \tau^2/\sigma^2$.

In [8] it is shown that a maximal invariant under a group of translations, changes of scale and orthogonal transformations is

$$\left[\frac{Z_1}{Q^{\frac{1}{2}}}, \frac{Z_2}{Q^{\frac{1}{2}}}, \ldots, \frac{Z_{r-1}}{Q^{\frac{1}{2}}} \right]$$

where $Q = \sum_{i=1}^{r} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2$ and $Z_i = \sqrt{n_i}(\bar{X}_i - \bar{X}_r)$ $(i = 1,2,\ldots,r-1)$.

The family of probability distributions of $Z' = [Z_1, Z_2, \ldots, Z_{r-1}]$ and Q can be written in the form (3.1) with

$$a(z,q,\Delta,\sigma) = \exp(-(2\sigma^2)^{-1}(z'A(\Delta)^{-1}z - z'A(\Delta_o)^{-1}z))q^{\frac{1}{2}(n-r)-1}$$

for any σ_o, where $A(\Delta)\sigma^2$ is the covariance matrix of Z and $n = \sum_{i=1}^{r} n_i$.

In [8] it is shown that the test which maximizes the minimum power over the set of alternatives with $\Delta \geq \Delta_1$, has a rejection region of the form $W_1 >$ constant where

$$W_1 = \frac{Z'A(\Delta_o)^{-1}Z - Z'A(\Delta_1)^{-1}Z}{Z'A(\Delta_o)^{-1}Z + Q} .$$

A limiting form of W_1 is obtained when $\Delta_1 \to \infty$. Then we shall reject when $T >$ constant where

$$T = \frac{Z'A(\Delta_o)^{-1}Z}{Q} .$$

From the identity

$$(5.1) \qquad Z'A(\Delta)^{-1}Z = \sum_{i=1}^{r} \frac{n_i}{n_i\Delta+1} (\bar{X}_i - \bar{X})^2$$

where $\bar{X} = (\sum_{i=1}^{r} \frac{n_i}{n_i\Delta+1})^{-1} \sum_{i=1}^{r} \frac{n_i}{n_i\Delta+1} \bar{X}_i$ the statistics T and W_1 may be computed by the observations X_{ij} [8].

Since the distribution of the invariant statistic depends only upon Δ, any invariant test has constant power when $\Delta = \Delta_0$. Hence similarity represents no restriction when considering invariant tests. We shall now find the form of the locally most powerful invariant (LMPI) test. Derivation gives

$$a'_{\Delta}(z,q,\Delta_0,\sigma) = (2\sigma^2)^{-1} z' A^*(\Delta_0) z \; q^{\frac{1}{2}(n-r)-1}$$

where $A^*(\Delta) = -\frac{\partial A(\Delta)^{-1}}{\partial \Delta}$. The statistic $Z'A(\Delta_0)^{-1}Z+Q$ is sufficient and complete when $\Delta = \Delta_0$ [8]. Using Theorem 3.3 and Theorem 3.4 and arguing as in Section 4 it is found that the LMPI test has rejection region $W_2 >$ constant where

$$W_2 = \frac{Z'A^*(\Delta_0)Z}{Z'A(\Delta_0)^{-1}Z+Q} .$$

From the identity (5.1) it is found by derivation that

$$Z'A^*(\Delta)Z = \sum_{i=1}^{r} \left(\frac{n_i}{n_i\Delta+1}\right)^2 (\bar{X}_i - \bar{X})^2 .$$

It is seen that the LMPI test puts more weight to the group means with many observations than the other tests. It should be noted that the tests based on T, W_1 and W_2 reduce to the usual test when $n_1 = n_2 = \ldots = n_r$. The same is the case if $r = 2$.

It is of interest to compare the three tests by means of their power functions. In [8] it is proved that W_1 and T are distributed as ratios of linear combinations of chi-square distributed random variables. The exact distribution is not known. In the case $r = 3$ the following lemma can be used to obtain a relatively simple expression for the cumulative distribution of the three statistics.

LEMMA 5.1. Let X_1, X_2, X_3 be independently distributed chi-square random variables with ν_1, ν_2, ν_3 degrees of freedom respectively, and let a_1 and a_2 be two constants. Then

$$U = \frac{a_1 X_1 + a_2 X_2}{X_1 + X_2 + X_3}$$

is distributed as $Y_1 Y_2$ where Y_1 and Y_2 are independent and Y_1 has a beta distribution with $\nu_1 + \nu_2$ and ν_3 degrees of freedom and $(Y_2 - a_1)/(a_2 - a_1)$ has a beta distribution with ν_2 and ν_1 degrees of freedom.

PROOF. Define Y_1 and Y_2 by $Y_1 = (X_1 + X_2)/(X_1 + X_2 + X_3)$ and $Y_2 = (a_1 X_1 + a_2 X_2)/(X_1 + X_2)$. Then Y_2 is independent of $X_1 + X_2$, and hence independent of Y_1. Also $(Y_2 - a_1)/(a_2 - a_1) = X_2/(X_1 + X_2)$.

We shall use Lemma 5.1 with $\nu_1 = \nu_2 = 1$ and $\nu_3 = n-3$ where $n = \sum_{i=1}^{3} n_i$. By integration it is found that if $0 < a_1 < a_2$ then

$$P(U > u) = \tfrac{1}{2}(1-u/a_1)^{\frac{1}{2}(n-3)} + \tfrac{1}{2}(1-u/a_2)^{\frac{1}{2}(n-3)}$$

$$(5.2) \qquad +2\pi^{-1}(n-3)^{-1} \int_{u/a_2}^{u/a_1} (1-x)^{\frac{1}{2}(n-5)} \operatorname{Arcsin}\left(1-2\,\frac{u/x-a_1}{a_2-a_1}\right)dx$$

for $u \leq a_1$, and for $a_1 \leq u \leq a_2$

$$P(U > u) = \tfrac{1}{2}(1-u/a_2)^{\frac{1}{2}(n-3)}$$

$$(5.3) \qquad +2\pi^{-1}(n-3)^{-1} \int_{u/a_2}^{1} (1-x)^{\frac{1}{2}(n-5)} \operatorname{Arcsin}\left(1-2\,\frac{u/x-a_1}{a_2-a_1}\right)dx.$$

To avoid complicated formulaes we shall in the following consider only the case $\Delta_o = 0$.

Let β_o, β_1 and β_2 denote the power functions of the tests based upon T, W_1 and W_2 respectively, and let c_o, c_1 and c_2 denote the corresponding constants used in the tests. In [8] it is shown that W_1 is distributed as

$$W_1(\Delta) = \frac{\sum_{i=1}^{r-1}(\Delta\lambda_i+1-(\Delta\lambda_i+1)/(\Delta_1\lambda_i+1))S_i^2}{\sum_{i=1}^{r-1}(\Delta\lambda_i+1)S_i^2+Q}$$

where $S_1^2, S_2^2, \ldots, S_{r-1}^2$ are independently chi-square distributed with 1 degree of freedom, independent of Q

which has a chi-square distribution with $n-r$ degrees of freedom. The λ_i are the roots of the equation $|B-\lambda C| = 0$ where B and C are determined from $A(\Delta) = B\Delta+C$.

We find

$$\beta_1(\Delta) = P(W_1(\Delta) > c_1)$$

$$= P\left(\frac{\sum_{i=1}^{r-1}\lambda_i(\Delta_1(\Delta\lambda_i+1)/(\Delta_1\lambda_i+1)-\Delta c_1)S_i^2}{\sum_{i=1}^{r-1}S_i^2+Q} > c_1\right)$$

where the statistic is in the form of U in Lemma 5.1.

Regarding the test T we may use the fact that T is distributed as

$$T(\Delta) = \frac{\sum_{i=1}^{r-1}(\Delta\lambda_i+1)S_i^2}{Q}$$

to write

$$\beta_o(\Delta) = P\left(\frac{\sum_{i=1}^{r-1}(\Delta\lambda_i+1+c_o)S_i^2}{\sum_{i=1}^{r-1}S_i^2+Q} > c_o\right).$$

The power function of the test based on W_2 is

$$\beta_2(\Delta) = P\left(\frac{Z'A^*(0)Z}{Z'A(0)^{-1}Z+Q} > c_2\right)$$

$$= P\left(\frac{Z'(A^*(0)-c_2A(0)^{-1}+c_2A(\Delta)^{-1})Z}{Z'A(\Delta)^{-1}Z+Q} > c_2\right)$$

$$= P\left(\frac{\sum_{i=1}^{r-1} \mu_i s_i^2}{\sum_{i=1}^{r-1} s_i^2+Q} > c_2\right)$$

where the μ_i are the roots of

$$|A^*(0)-c_2A(0)^{-1}+c_2A(\Delta)^{-1}-\mu A(\Delta)^{-1}| = 0.$$

By means of Lemma 5.1 and the expressions (5.2) and (5.3) the power of the tests can be computed for $r = 3$. The results for some combinations of n_1, n_2 and n_3 are given in Table 1 and Table 2. For fixed n_1, n_2 and n_3 the second and third column show how much must be added to the power function β_0 to get the power functions β_1 and β_2 respectively. The last column in each table gives the power of the F-test when $n_1 = n_2 = n_3 = n/3$. The level is 1 % and the value of Δ_1 is chosen to be 0.1. The reason for choosing $\Delta_1 = 0.1$ is that for larger values of Δ_1 the difference between β_0 and β_1 vanishes. For $\Delta_1 = 3.0$ for example the two power functions were identical to three decimal places. The power functions were also computed for 5 % level, but the results did not in tendency differ much from those for 1 % level, though the differences in power were smaller.

It is seen from the tables that the difference between β_0 and β_1 is small, and very little is gained by the LMPI test near the hypothesis as compared with the loss of power for moderate values of Δ. It is also seen that we may have a serious loss of power compared with the situation where $n_1 = n_2 = n_3 = n/3$.

TABLE 1

Δ	$n_1=n_2=2, n_3=26$			$n_1=5, n_2=10, n_3=15$			$n_1=8, n_2=10, n_3=12$			$n_1=n_2=n_3=10$
	β_0	$\beta_1-\beta_0$	$\beta_2-\beta_0$	β_0	$\beta_1-\beta_0$	$\beta_2-\beta_0$	β_0	$\beta_1-\beta_0$	$\beta_2-\beta_0$	β
0	.010	.000	.000	.010	.000	.000	.010	.000	.000	.010
.01	.011	.001	.001	.014	.000	.000	.014	.000	.000	.014
.02	.013	.001	.001	.019	.001	.001	.019	.000	.000	.019
.03	.015	.001	.001	.024	.001	.002	.025	.000	.000	.025
.04	.017	.001	.001	.030	.002	.002	.032	.000	.000	.032
.05	.018	.002	.002	.036	.003	.003	.039	.000	.000	.039
.06	.021	.002	.002	.043	.003	.003	.046	.001	.001	.047
.07	.023	.002	.002	.051	.003	.003	.055	.000	.000	.055
.08	.025	.003	.003	.059	.003	.003	.063	.001	.000	.064
.09	.027	.004	.003	.067	.004	.003	.072	.000	.000	.073
.1	.030	.003	.003	.075	.004	.003	.081	.001	.000	.082
.2	.058	.006	.006	.163	.004	-.003	.177	.001	-.001	.180
.3	.091	.007	.006	.246	.002	-.012	.267	.000	-.003	.271
.4	.125	.007	.005	.319	.002	-.021	.343	.000	-.004	.348
.5	.160	.004	.001	.381	-.005	-.031	.408	.001	-.005	.413
.6	.193	.002	-.003	.433	-.006	-.033	.462	-.001	-.006	.466
.7	.225	-.001	-.007	.478	-.008	-.037	.507	-.001	-.006	.512
.8	.255	-.004	-.012	.516	-.008	-.039	.546	-.001	-.006	.551
.9	.283	-.007	-.016	.550	-.010	-.041	.579	-.001	-.006	.584
1	.310	-.010	-.021	.579	-.010	-.042	.608	-.001	-.006	.613
2	.503	-.032	-.054	.746	-.011	-.041	.768	-.001	-.005	.772
3	.614	-.040	-.068	.818	-.009	-.034	.836	-.001	-.004	.839
4	.685	-.042	-.073	.859	-.008	-.029	.873	-.001	-.003	.875
5	.734	-.042	-.073	.884	-.006	-.025	.896	.000	-.002	.898
10	.850	-.032	-.059	.940	-.004	-.015	.946	.000	-.001	.947

TABLE 2

	$n_1=n_2=5, n_3=80$			$n_1=5, n_2=30, n_3=55$			$n_1=n_2=n_3=30$
Δ	β_0	$\beta_1-\beta_0$	$\beta_2-\beta_0$	β_0	$\beta_1-\beta_0$	$\beta_2-\beta_0$	β
0	.010	.000	.000	.010	.000	.000	.010
.01	.014	.001	.001	.023	.005	.005	.027
.02	.020	.002	.002	.042	.009	.010	.053
.03	.026	.003	.003	.065	.012	.013	.083
.04	.033	.004	.004	.088	.016	.016	.115
.05	.041	.005	.004	.113	.017	.017	.149
.06	.049	.006	.005	.137	.019	.018	.182
.07	.058	.006	.005	.161	.019	.017	.214
.08	.067	.007	.006	.184	.019	.016	.244
.09	.077	.006	.005	.206	.019	.015	.274
.1	.086	.007	.006	.227	.019	.013	.301
.2	.184	.007	.000	.391	.010	-.008	.502
.3	.272	.003	-.010	.498	.001	-.028	.617
.4	.346	-.001	-.020	.572	-.005	-.042	.689
.5	.408	-.005	-.028	.627	-.009	-.052	.739
.6	.460	-.008	-.034	.670	-.012	-.060	.775
.7	.504	-.010	-.039	.703	-.013	-.065	.802
.8	.541	-.011	-.042	.731	-.014	-.069	.824
.9	.573	-.012	-.044	.753	-.014	-.071	.841
1	.602	-.013	-.046	.773	-.015	-.074	.855
2	.761	-.014	-.046	.872	-.013	-.071	.923
3	.829	-.012	-.039	.911	-.010	-.062	.948
4	.867	-.010	-.033	.932	-.009	-.053	.961
5	.891	-.008	-.028	.945	-.007	-.047	.968
10	.943	-.005	-.017	.972	-.004	-.028	.984

REFERENCES

[1] BASU, D. (1955). On statistics independent of a complete sufficient statistic. Sankhyā 15 377-380.

[2] LEHMANN, E.L. (1947). On optimum tests of composite hypotheses with one constraint. Ann.Math.Statist. 20 473-494.

[3] LEHMANN, E.L. (1959). Testing Statistical Hypotheses. Wiley, New York.

[4] LEHMANN, E.L. and SCHEFFÉ, H. (1955). Completeness, similar regions, and unbiased estimation. Sankhyā 15 219-236.

[5] von NEUMAN, J., KENT, R.H., BELLINSON, H.R. and HART, B.I. (1941). The mean square successive difference. Ann.Math.Statist. 12 153-162.

[6] NEYMAN, J. (1935). Sur la vérification des hypothèses statistiques composées. Bull.soc.math. France 63 246-266.

[7] SCHEFFÉ, H. (1942). On the theory of testing composite hypotheses with one constraint. Ann.Math. Statist. 13 280-293.

[8] SPJØTVOLL, E. (1967). Optimal invariant tests in unbalanced variance components models. To be published in Ann.Math.Statist.

[9] SVERDRUP, E. (1953). Similarity, unbiasedness, minimaxibility and admissibility of statistical test procedures. Skand.Aktuarietidskr. 36 64-86.