MAXIMIN TESTS AND
LOCALLY MOST POWERFUL TESTS.

By

Emil Spjøtvoll
1. INTRODUCTION AND SUMMARY.

We shall be concerned with the parametric problem of testing hypotheses concerning the value of one parameter when the values of other parameters (nuisance parameters) are not specified. Neyman [6] derived under certain conditions a locally most powerful two-sided test for this problem i.e. he gave the form of the test maximizing the second derivative of the power function with respect to the parameter of interest at the point specified by the hypothesis. Generalizations of Neyman's results were given by Scheffe [7] and Lehmann [2], using the same technique as Neyman. They were also able to prove that the tests were UMP unbiased. A new technique for dealing with these problems were introduced by Sverdrup [9] and Lehmann and Scheffe [4] where the completeness of the sufficient statistics in an exponential family of densities is used to derive UMP unbiased tests. It is stated by Lehmann and Scheffe [4] that the conditions imposed earlier imply an exponential family of densities.

When no UMP unbiased test exists we have little general theory. The problem is both one of principle and of technique. Most stringent tests exist under general conditions but are difficult to derive in particular cases. Lehmann [3] proposed maximin tests and a local form of maximin tests called tests which maximize the minimum power locally. Definitions of these are given in Section 2. Spjøtvoll [8] has given an example of the form of a maximin test when no UMP unbiased
and invariant test exists.

This paper is an attempt to establish some general theory for testing hypotheses when the probability density of the observations does not constitute an exponential family under both the hypothesis and the alternative. The assumptions made in Section 3 is satisfied if we have an exponential family under the hypothesis, but do not say anything about the form of the density under the alternative. The results concern maximin tests and locally most powerful tests, and under certain conditions the form of these tests for the particular family of densities studied, is given in Section 3.

In Section 4 the theory in Section 3 is applied to the problem of testing serial correlation (not circular) in a first order autoregressive sequence. It is found that the usual tests is nearly UMP invariant.

In Section 5 the problem of testing the value of the ratio of variances in the one-way classification variance components model is considered. Some numerical results is given for the power functions of the maximin test, the locally most powerful test and the standard F-test. The results indicate that the standard F-test performs well compared with the other tests.
2. DEFINITIONS AND ELEMENTARY CONSEQUENCES.

Let $X$ be a random variable with distribution belonging to a family $\mathcal{F} = \{P_{\theta, \varphi} : (\theta, \varphi) \in \Omega\}$ of distributions. Let the parameter $\theta$ be real. We shall consider the problem of testing the hypotheses

$$H_1 : \theta = \theta_0 \quad \text{against} \quad \theta > \theta_0,$$
$$H_2 : \theta = \theta_0 \quad \text{against} \quad \theta \neq \theta_0.$$ 

The functions

$$d_1(\theta) = \max(\theta - \theta_0, 0),$$
$$d_2(\theta) = |\theta - \theta_0|,$$

will be used as measures of distance from the hypotheses $H_1$ and $H_2$ respectively.

In the following let $H$ and $d$ stand for either $H_1$ and $d_1$ or $H_2$ and $d_2$. Let $\varphi(\theta, \varphi)$ denote the power function of a test $\varphi$.

The conceptsimaxin tests and tests which maximize the minimum power locally have been introduced by Lehmann [3]. Definitions of these tests and of locally most powerful tests are now given in terms of the distance function $d$.

DEFINITION 2.1. A level $\alpha$ test $\varphi_0$ of $H$ is locally most powerful (LMP) with respect to the distance function
d if, given any other level $\alpha$ test $\varphi$, there exists
for each $\nu$ a $\Delta$ such that $\beta(\theta, \nu, \varphi_0) \geq \beta(\theta, \nu, \varphi)$ when
$0 < d(\theta) < \Delta$.

Define sets $\omega_\Delta$ by

$$\omega_\Delta = \{ (\theta, \nu) : d(\theta) \geq \Delta \}.$$

**DEFINITION 2.2.** A level $\alpha$ test $\varphi_0$ of $H$ maximizes the
minimum power over $\omega_\Delta$ ($\Delta$ fixed) if
$$\inf_{\omega_\Delta} \beta(\theta, \nu, \varphi_0) \geq \inf_{\omega_\Delta} \beta(\theta, \nu, \varphi)$$
for any other level $\alpha$ test $\varphi$.

The test $\varphi_0$ is said to be a maximin test over $\omega_\Delta$.

More generally we can speak of maximin tests over any subset $\omega$ of the set of alternatives by replacing $\omega_\Delta$ by $\omega$ in
Definition 2.2. In Section 3 we shall be concerned with
subsets of the form $\{ (\theta, \nu) : d(\theta) = \Delta \}$.

**DEFINITION 2.3.** A level $\alpha$ test $\varphi_0$ of $H$ is said to
maximize the minimum power locally with respect to the
distance function $d$ if, given any other level $\alpha$ test $\varphi$,
there exists a $\Delta'$ such that
$$\inf_{\omega_\Delta} \beta(\theta, \nu, \varphi_0) \geq \inf_{\omega_\Delta} \beta(\theta, \nu, \varphi)$$
when $0 < \Delta < \Delta'$.

A level $\alpha$ test $\varphi$ is similar if $\beta(\theta_0, \nu, \varphi) = \alpha$ for
all $\nu$, it is unbiased if $\beta(\theta, \nu, \varphi) \geq \alpha$ when $d(\theta) > 0$, and we shall say that it is unbiased at $\theta_0$ if for each $\nu$
there exists $\Delta$ such that $\beta(\theta_0, \nu, \varphi) \geq \alpha$ when $0 < d(\theta) < \Delta$.

The following two lemmas are easily proved by comparing
with the test \( \varphi(x) = \alpha \).

**Lemma 2.1.** If for each \( \varphi \) the power function \( \beta(\theta, \varphi, \varphi) \) is continuous in \( \Theta \) at \( \Theta_0 \), then a LMP test is similar.

**Lemma 2.2.** A test which maximizes the minimum power locally is unbiased, and if the power function of each test is continuous in \( \Theta \) at \( \Theta_0 \), then it is similar.

In view of Lemma 2.1 we introduce the following definition.

**Definition 2.4.** A level \( \alpha \) test \( \varphi_0 \) of \( H \) is locally most powerful similar (unbiased) with respect to the distance function \( d \) if, given any other similar (unbiased at \( \Theta_0 \)) level \( \alpha \) test \( \varphi \), there exists for each \( \nu \) a \( \Delta \) such that \( \beta(\theta, \nu, \varphi_0) \geq \beta(\theta, \nu, \varphi) \) when \( 0 < d(\theta) < \Delta \).

We shall use the abbreviations LMPS and LMPU. Note that a LMPU test need not be unbiased after the usual definition of unbiasedness, only unbiasedness near \( \Theta_0 \) is required.

The following lemma proves that in some cases a LMPU test maximizes the minimum power locally. (Compare Lehmann [3], p. 342).

**Lemma 2.3.** Let \( \varphi_0 \) be a LMPU level \( \alpha \) test of \( H \) and suppose that the power function \( \beta(\theta, \nu, \varphi_0) \) depends only
upon $d(\theta)$ and is continuous as a function of $d$. Then $\varphi_o$ maximizes the minimum power locally provided $\beta(\theta, \nu, \varphi_o)$ is bounded away from $\Delta$ for every set of alternatives which is bounded away from $H$ (measured by $d$).

**Proof.** If a test $\varphi$ is not unbiased, then for some $\Delta'$ we have $\inf_{\Delta' \setminus \Delta} (\beta(\theta, \nu, \varphi) < \alpha$ for $0 < \Delta < \Delta'$. Hence $\inf_{\Delta' \setminus \Delta} (\beta(\theta, \nu, \varphi) > \inf_{\Delta' \setminus \Delta} (\beta(\theta, \nu, \varphi)$ for $0 < \Delta < \Delta'$. Suppose then that $\varphi$ is unbiased. Let $\nu^k$ be a fixed value of $\nu$, since $\varphi_o$ is LMPS we have for some $\Delta$.

\[
(2.1) \quad \beta(\theta, \nu^k, \varphi_o) \geq \beta(\theta, \nu^k, \varphi) \quad \text{when} \quad 0 < d(\theta) < \Delta.
\]

Define $a(\Delta)$ by $\inf_{\Delta} \beta(\theta, \nu^k, \varphi_o) = \alpha + a(\Delta)$. By the condition of the lemma $a(\Delta)$ does not depend upon $\nu^k$ and $a(\Delta) > 0$ when $\Delta > 0$. Since $\beta(\theta, \nu^k, \varphi_o)$ is continuous as a function of $d$, there exists $\Delta' < \Delta^*$ such that $a(\Delta') < a(\Delta^*)$.

Hence

\[
(2.2) \quad \inf_{\Delta' \setminus \Delta} (\beta(\theta, \nu^k, \varphi_o) = \inf_{\Delta' \setminus \Delta} (\beta(\theta, \nu^k, \varphi) \quad \text{for} \quad 0 < \Delta < \Delta'.
\]

Then we have by (2.1), (2.2) and the fact that $\beta(\theta, \nu, \varphi_o)$ does not depend upon $\nu$.

$$
\inf_{\Delta' \setminus \Delta} (\beta(\theta, \nu, \varphi) \leq \inf_{\Delta' \setminus \Delta} (\beta(\theta, \nu^k, \varphi) \leq \inf_{\Delta' \setminus \Delta} (\beta(\theta, \nu^k, \varphi_o)
$$

$$
\leq \inf_{\Delta' \setminus \Delta} (\beta(\theta, \nu^k, \varphi_o) = \inf_{\Delta' \setminus \Delta} (\beta(\theta, \nu^k, \varphi)
$$
The next two lemmas give conditions under which a test is LMPS or LMPU. Let \( \beta'(\theta, \nu, \varphi) \) and \( \beta''(\theta, \nu, \varphi) \) denote the two first derivatives of the power function with respect to \( \Theta \).

**Lemma 2.4.** Suppose that the power function of any test \( \varphi \) has a continuous derivative with respect to \( \Theta \) at \( \Theta_0 \). If there exists a similar test \( \varphi_0 \) such that
\[
\beta'_\theta(\theta_0, \nu, \varphi_0) > \beta'_\theta(\theta_0, \nu, \varphi)
\]
for all \( \nu \) when \( \varphi \) is any other similar test, then \( \varphi_0 \) is the unique LMPS test of \( H_1 \) with respect to \( d_1 \).

**Lemma 2.5.** Suppose that the power function of any test has a continuous second derivative with respect to \( \Theta \) at \( \Theta_0 \). If there exists a test \( \varphi_0 \) unbiased at \( \Theta_0 \) such that
\[
\beta''(\theta_0, \nu, \varphi_0) > \beta''(\theta_0, \nu, \varphi)
\]
for all \( \nu \) when \( \varphi \) is any other test unbiased at \( \Theta_0 \), then \( \varphi_0 \) is the unique LMPU test of \( H_2 \) with respect to \( d_2 \).

**Proof.** We give only the proof of Lemma 2.5, since the proof of Lemma 2.4 is similar.

For any test \( \varphi \) we have
\[
\frac{\partial}{\partial \theta} \beta(\theta, \nu, \varphi) = \beta'(\theta, \nu, \varphi) + (\theta - \Theta_0)^2 \beta''(\theta_0 + t \varphi, t \nu, \varphi) + \frac{1}{2} (\theta - \Theta_0)^2 \beta''(\theta_0 + t \varphi, t \nu, \varphi)
\]
for some \( t \varphi \) with \( 0 < t \varphi < 1 \). If \( \varphi \) is unbiased at \( \Theta_0 \), we get
\[
\frac{\partial}{\partial \theta} \beta(\theta, \nu, \varphi) = + \frac{1}{2} (\theta - \Theta_0)^2 \beta''(\theta_0 + t \varphi, \nu, \varphi).
\]
Consider a fixed \( \nu \) and \( \varphi \). When \( |\Theta - \Theta_0| \) is small enough then...
\[ p''(\theta_0 + t \phi_0, \theta-\theta_0, \nu, \phi) > p''(\theta_0 + t \phi(\theta-\theta_0), \nu, \phi) \] since
\[ p''(\theta_0, \nu, \phi_0) > p''(\theta_0, \nu, \phi) \]. Hence
\[ p(\theta, \nu, \phi_0) > p(\theta, \nu, \phi) \] for \[ |\theta-\theta_0| \] small enough, and the lemma is proved.

A test \( \varphi \) of \( H_2 \) satisfying
(a) \( p(\theta_0, \nu, \psi) = \alpha \),
(b) \( p'(\theta_0, \nu, \psi) = 0 \) and
maximum among tests satisfying (a) and (b), was denoted test of type B by Neyman [6]. If there exists a unique test of type B, then by Lemma 2.5 it is a LMPU test of \( H_2 \). But in general we cannot be certain that a type B test is LMPU as defined above.

The reason for formulating the definitions of locally most powerful tests as above is to have a definition covering both one-sided and two-sided tests, and to ensure that the corresponding tests have optimum properties near the hypotheses to a greater extent than guaranteed by type B tests. This way of defining locally most powerful tests is not new, see e.g. Lehmann [3], p.342, where the one-parameter case is considered.

3. DERIVATION OF MAXIMIN TESTS AND LMP TESTS FOR A PARTICULAR FAMILY OF DISTRIBUTIONS.

We shall consider the case where the probability distribution of \( X \) belongs to a family \( g^X = \{ \rho_{\theta, \nu}^X \colon (\theta, \nu) \in \Omega \} \) where \( \rho_{\theta, \nu}^X \) is defined by
where \( \mu \) is a \( \sigma \)-finite measure over a Euclidean space.

We shall assume that there exists a value \( \nu_o \) of \( \nu \) such that the distribution \( P^X_{\theta, \nu_o} \) dominates the family \( \mathcal{D}^X \). In that case we may write

\[
dp_{\theta, \nu_o}^X(x) = a(x, \theta, \nu) b(t(x), \theta, \nu) \mu(x),
\]

(3.1)

where \( \mu \) is a \( \sigma \)-finite measure over a Euclidean space.

Further it is assumed that the statistic \( T = t(X) \) is sufficient when \( \theta = \theta_o \), and that the family of distributions for \( T \) when \( \theta = \theta_o \) is boundedly complete.

The assumptions stated above will be assumed to hold for the rest of this section.

Let \( P^X_{\theta_o} | t \) denote the conditional probability distribution of \( X \) given \( T = t \) when \( \theta = \theta_o \). Since \( T \) is sufficient, \( P^X_{\theta_o} | t \) can be chosen to be independent of \( \nu \).

Let \( E^X_{\theta_o} | t \) denote expectation taken with respect to \( P^X_{\theta_o} | t \).

Similarly let \( E^X_{\theta, \nu} \) and \( E^T_{\theta, \nu} \) denote expectations with respect to the distribution of \( X \) and the marginal distribution of \( T \) respectively.

A test \( \mathcal{Q} \) is similar if \( E^X_{\theta_o, \nu} (\mathcal{Q}(X)) = \alpha \) for all \( \nu \).

Since \( T \) is sufficient and complete when \( \theta = \theta_o \), a test is similar if and only if \( E^X_{\theta_o} | t \mathcal{Q}(X) = \alpha \) a.e. \( \mathcal{T}^T_{\theta_o} \) where \( \mathcal{T}^T_{\theta_o} \) denotes the family of distributions for \( T \) when \( \theta = \theta_o \).
The following lemma will be useful when establishing the uniqueness of tests.

**Lemma 3.1.** Let $X$ be a random variable, $T = t(X)$ a statistic and let $E^X, E^T$ and $E^X|T$ denote expectations with self-evident notation. Given a test function $\varphi_0$ such that

$$
\varphi_0(x) = \begin{cases} 
1 & \text{when } h(x) > \sum_{i=1}^m k_i(t(x)) f_i(x) \\
0 & \text{when } h(x) < \sum_{i=1}^m k_i(t(x)) f_i(x)
\end{cases}
$$

for some functions $h, f_1, f_2, \ldots, f_m, k_1, k_2, \ldots, k_m$. Then a test function $\varphi$ satisfying

$$
(3.2) \quad E^X|T \varphi(X)f_i(x) = E^X|T \varphi_0(X)f_i(x) \quad \text{a.e. } i = 1, 2, \ldots, m
$$

satisfies

$$
E^X \varphi(X)h(X) = E^X \varphi_0(X)h(X)
$$

if and only if $\varphi(x) = \varphi_0(x)$ a.e. on the set

$$
\{x : h(x) \neq \sum_{i=1}^m k_i(t(x)) f_i(x)\}.
$$

Otherwise $E^X \varphi(X)h(X) < E^X \varphi_0(X)h(X)$.

**Proof.** We have by (3.2)
\[ E^X \varphi(x) k_i(t(x)) f_i(x) = E^T k_i(T) E^X|t \]
\[ \varphi(x) f_i(x) = E^T k_i(T) E^X|t \varphi_0(x) f_i(x) \]
\[ = E^X \varphi_0(x) k_i(t(x)) f_i(x). \]

It follows that
\[ E^X \varphi_0(x) h(x) - E^X \varphi(x) h(x) \]
\[ = E^X (\varphi_0(x) - \varphi(x))(h(x) - \sum_{i=1}^m k_i(t(x)) f_i(x)). \]

By the definition of \( \varphi_0 \) the above difference is \( \geq 0 \) for all \( \varphi \). It is \( = 0 \) if and only if \( \varphi(x) = \varphi_0(x) \) a.e. on the set \( \{ x : h(x) \neq \sum_{i=1}^m k_i(t(x)) f_i(x) \} \).

We have the following theorem

**THEOREM 3.1.** For the hypothesis \( \Theta = \Theta_0 \) against 
\( (\Theta, \psi) = (\Theta_1, \psi_1) \) there exists a most powerful similar level \( \alpha \) test \( \varphi_1 \) defined by

\[ \varphi_1(x) = \begin{cases} 
1 & \text{when } a(x, \Theta_1, \psi_1)/a(x, \Theta_0, \psi_0) > c(t), \\
\gamma(t) & \text{when } a(x, \Theta_1, \psi_1)/a(x, \Theta_0, \psi_0) = c(t), \\
0 & \text{when } a(x, \Theta_1, \psi_1)/a(x, \Theta_0, \psi_0) < c(t),
\end{cases} \]

where \( c(t) \) and \( \gamma(t) \) are determined by \( E^X_{\Theta_0} \varphi_1(x) = \alpha \) for all \( t \).
PROOF. By (3.1) the power of any test \( \varphi \) at \((\theta_1, \nu_1)\) is

\[
E^X_{\theta_1, \nu_1} \varphi(X) = E^X_{\theta_0, \nu_0} \varphi(X) a(X, \theta_1, \nu_1) b(t(X), \theta_1, \nu_1) / \\
(a(X, \theta_0, \nu_0) b(t(X), \theta_0, \nu_0))
\]

\[
= E^T_{\theta_0, \nu_0} (b(t, \theta_1, \nu_1)/b(t, \theta_0, \nu_0)) E^X_{\theta_0} \varphi(X) a(x, \theta_1, \nu_1)/a(x, \theta_0, \nu_0).
\]

A test maximizes the power under the condition of similarity if it for each \( t \) maximizes \( E^X_{\theta_0} \varphi(X) a(x, \theta_1, \nu_1)/a(x, \theta_0, \nu_0) \) under the condition of similarity i.e. under the condition \( E^X_{\theta_0} \varphi(X) = \alpha \). By the Neyman-Pearson fundamental lemma the test \( \varphi_1 \) has this property.

REMARK. The test is unique (a.e. \( \mathcal{P}^X \)) if

\[
P^X_{\theta_0, \nu_0} (a(x, \theta_1, \nu_1) = c(t(X)) a(x, \theta_0, \nu_0)) = 0.
\]

The Remark is proved by using Lemma 3.1 with \( h(x) = a(x, \theta_1, \nu_1) b(t(x), \theta_1, \nu_1)/(a(x, \theta_0, \nu_0) b(t(x), \theta_0, \nu_0)) \) and \( P^X_{\theta_0, \nu_0} \) as probability measure. The remarks following Theorem 3.3 and Theorem 3.5 later can be proved in a similar way.

It should be noted that Theorem 3.1 can be regarded as an application of Theorem 3 of Sverdrup [9] to the family \( \mathcal{P}^X \).
The following corollary is obvious.

**COROLLARY 3.1.** If \( Q_1 \) does not depend upon \( \nu_1 \) then it is the uniformly most powerful similar level \( \alpha \) test for testing \( \Theta = \Theta_0 \) against \( \Theta = \Theta_1 \).

This leads to

**THEOREM 3.2.** Suppose that \( Q_1 \) in Theorem 3.1 does not depend upon \( \nu_1 \) and that \( \beta(\theta, \nu, Q_1) \geq \beta(\theta_1, \nu, Q_1) \) for all \((\theta, \nu)\) with \( \Theta \geq \Theta_1 \). Then \( Q_1 \) is a maximin test among similar level \( \alpha \) tests over the set of alternatives with \( \Theta \geq \Theta_1 \), and \( Q_1 \) is the unique maximin test if it is the unique most powerful test.

**PROOF.** By Corollary 3.1 \( Q_1 \) is the uniformly most powerful test for \( \Theta = \Theta_0 \) against \( \Theta = \Theta_1 \). Hence any other test has less or equal minimum power when \( \Theta = \Theta_1 \), and hence less or equal minimum power over \( \Theta \geq \Theta_1 \). If \( Q_1 \) is unique any other test must have less power for some points with \( \Theta = \Theta_1 \), and hence less minimum power over the set of alternatives with \( \Theta \geq \Theta_1 \).

Let \( a'_{\Theta} \) and \( b'_{\Theta} \) denote the derivatives with respect to \( \Theta \) of the functions \( a \) and \( b \) respectively. The next theorem gives the form of the test that maximizes \( \beta'(\theta_0, \nu, Q) \) locally.
THEOREM 3.3. Suppose that for any test \( \mathcal{Q} \) the derivative with respect to \( \theta \) of the power function \( \beta(\theta, \nu, \mathcal{Q}) \) can be computed under the integral sign. Then among similar level \( \alpha \) tests the following test \( \mathcal{Q}_2 \) maximizes the derivative of the power function at \((\theta_0, \nu_0)\)

\[
\mathcal{Q}_2(x) = \begin{cases} 
1 & \text{when } a'(x, \theta_0, \nu_0)/a(x, \theta_0, \nu_0) > c(t) \\
\gamma(t) & \text{when } a'(x, \theta_0, \nu_0)/a(x, \theta_0, \nu_0) = c(t) \\
0 & \text{when } a'(x, \theta_0, \nu_0)/a(x, \theta_0, \nu_0) < c(t),
\end{cases}
\]

where \( c(t) \) and \( \gamma(t) \) are determined by \( E_{\theta_0}^{X} \mathcal{Q}_2(X) = \alpha \) for all \( t \).

PROOF. The derivative at \((\theta_0, \nu_0)\) with respect to \( \theta \) of the power function of a test \( \mathcal{Q} \) is

\[
\beta'(\theta, \nu, \mathcal{Q}) = \int \mathcal{Q}(x)\left(a'(x, \theta, \nu)b(t(x), \theta, \nu) + a(x, \theta, \nu)b'(t(x), \theta, \nu)\right)d\mu(x)
\]

\[
= E_{\theta_0, \nu_0}^{X} \mathcal{Q}(x)\left(a'(x, \theta_0, \nu_0)/a(x, \theta_0, \nu_0) + b'(t(x), \theta_0, \nu_0)/b(t(x), \theta_0, \nu_0)\right)
\]

\[
= E_{\theta_0, \nu_0}^{T} \mathcal{Q}(x)\left(b'(t, \theta_0, \nu_0)/b(t, \theta_0, \nu_0)\right) E_{\theta_0}^{X} \mathcal{Q}(x)
\]

\[
+ E_{\theta_0, \nu_0}^{T} E_{\theta_0}^{X} \mathcal{Q}(x)\left(a'(x, \theta_0, \nu_0)/a(x, \theta_0, \nu_0)\right).
\]

\( \beta'(\theta_0, \nu_0, \mathcal{Q}) \) is maximized under the condition of similarity if for each \( t \) the second expectation in the last expression is maximized under the condition of similarity. An appli-
cation of the Neyman-Pearson fundamental lemma gives the test $\varphi_2$.

**REMARK.** The test is unique (a.e. $P^X$) if

$$P^X_{\theta_0, \psi_0} (a'(x, \theta_0, \psi_0) = c(t(x)) a(x, \theta_0, \psi_0) ) = 0.$$ 

The next theorem gives a condition under which the test $\varphi_2$ of Theorem 3.3 is LMPS.

**THEOREM 3.4.** In addition to the assumption of Theorem 3.3 suppose that the derivative $\varphi'_1(x, \theta; \phi)$ is continuous in $\theta$ at $\theta_0$ for any test $\phi$, and suppose that the hypothesis testing problem $H_1$ is invariant under a group $G$ of transformations, and that $\theta$ is a maximal invariant under the induced group $\bar{G}$ of transformations of the parameter space. If $\varphi_2$ of Theorem 3.3 is unique and the power function is invariant, then $\varphi_2$ is the unique LMPS level $\alpha$ test of $H_1$ with respect to the distance function $d_1$.

**PROOF.** Since $\varphi_2$ is unique we have

$$(3.3) \quad \beta'_0(\theta_0, \nu_0; \varphi_2) > \beta'_0(\theta_0, \nu_0; \varphi)$$

for any other similar level $\alpha$ test $\varphi$. We shall show that $\beta'_0(\theta_0, \nu, \nu_0) > \beta'_0(\theta_0, \nu_0; \varphi)$ for all $\nu$. Then by Lemma 2.4 $\varphi_2$ is LMPS.
Suppose on the contrary that there exists a $\phi^x$ such that

$$\beta'_\phi(\theta_0, \phi^x, \varphi_2) \leq \beta'_\phi(\theta_0, \phi^y, \varphi).$$

Since $\theta$ is maximal invariant there exists a $g \in \bar{G}$ such that $g((\theta, \varphi_0)) = (\theta, \phi^x)$. Consider the test $\phi^x = \phi_0$ where $g$ corresponds to $g$. We have $E_{\theta, \varphi_0}^\varphi g(x) = E_{\theta, \varphi_0}^{\phi^x}(x) = E_{\theta, \varphi_0}^{\phi^x}(x)$, or in terms of power functions,

$$\beta_{\phi}(\theta_0, \phi^x, \varphi) = \beta_{\phi}(\theta_0, \phi^x, \varphi).$$

Hence by (3.4) $\beta'_\phi(\theta_0, \phi^x, \varphi_2) \leq \beta'_\phi(\theta_0, \phi^y, \varphi_2)$. But $\phi_2$ is invariant, hence $\beta'_\phi(\theta_0, \phi^x, \varphi_2) = \beta'_\phi(\theta_0, \phi^x, \varphi_2)$, and we get $\beta'_\phi(\theta_0, \phi^x, \varphi_2) \leq \beta'_\phi(\theta_0, \phi^y, \varphi_2)$ which contradicts (3.3).

**Corollary 3.2.** Under the assumptions of Theorems 3.3 and 3.4 the test $\phi^x$ maximizes the minimum power locally with respect to the distance function $d_1$, provided its power function is bounded away from $x$ when $\theta$ is bounded away from $\theta_0$.

**Proof.** Follows from Theorem 3.4 and Lemma 2.3, since here a LMPS test is LMPU.

Let $a''_\phi$ and $b''_\phi$ denote the second derivatives with respect to $\theta$ of the functions $a$ and $b$ respectively.

**Theorem 3.5.** Suppose that for any test $\phi$ the first and second derivative with respect to $\theta$ of $\beta(\theta, \varphi, \varphi)$ can be
computed under the integral sign and suppose that
\[ a'_\theta(x, \theta_0, \nu) / a(x, \theta_0, \nu) + b'_\theta(t(x), \theta_0, \nu) / b(t(x), \theta_0, \nu) = k(\nu) h(x) \]
for some functions \( k \) and \( h \), with \( k(\nu) > 0 \) for all \( \nu \).
Then among level \( \alpha \) tests unbiased at \( \Theta_0 \) (with respect to \( d_2 \)), the following test \( \varphi_3 \) maximizes the second derivative of the power function at \( (\Theta_0, \nu_0) \)
\[
\varphi_3(x) = \begin{cases} 
1 & \text{when } a''(x, \theta_0, \nu_0)/a(x, \theta_0, \nu_0) + c_1(t) a'_\theta(x, \theta_0, \nu_0)/a(x, \theta_0, \nu_0) > c_2(t) \\
\gamma(t) & \text{when } a''(x, \theta_0, \nu_0)/a(x, \theta_0, \nu_0) + c_1(t) a'_\theta(x, \theta_0, \nu_0)/a(x, \theta_0, \nu_0) = c_2(t) \\
0 & \text{when } a''(x, \theta_0, \nu_0)/a(x, \theta_0, \nu_0) + c_1(t) a'_\theta(x, \theta_0, \nu_0)/a(x, \theta_0, \nu_0) < c_2(t), 
\end{cases}
\]
where \( c_1(t), c_2(t) \) and \( \gamma(t) \) are determined by
\[
E_{\Theta_0} \varphi_3(x) = \alpha \quad \text{and} \quad E_{\Theta_0} \varphi_3(x)(a'_\theta(x, \theta_0, \nu_0)/a(x, \theta_0, \nu_0) + b'_\theta(t(x), \theta_0, \nu_0)/b(t(x), \theta_0, \nu_0)) = 0.
\]

**PROOF.** Unbiasedness in some neighbourhood of \( \Theta_0 \) implies \( \beta'_\theta(\theta_0, \nu_0, q) = 0 \), hence
\[
0 = E_{\Theta_0, \nu_0, \theta_0} \varphi(x)(a'_\theta(x, \theta_0, \nu_0)/a(x, \theta_0, \nu_0) + b'_\theta(t(x), \theta_0, \nu_0)/b(t(x), \theta_0, \nu_0)) \\
= E_{\Theta_0, \nu_0, \theta_0} \varphi(x)(a'_\theta(x, \theta_0, \nu_0)/a(x, \theta_0, \nu_0) + b'_\theta(t(x), \theta_0, \nu_0)/b(t(x), \theta_0, \nu_0)).
\]
We may choose the function $h(x)$ in the theorem equal to
\[ a'(x, \theta_0, \nu_0)/a(x, \theta_0, \nu_0) + b'(t(x), \theta_0, \nu_0)/b(t(x), \theta_0, \nu_0). \]
Hence by (3.5)
\[ 0 = E_{\theta_0}^{T} t(x)(a'(x, \theta_0, \nu_0)/a(x, \theta_0, \nu_0) + b'(t(x), \theta_0, \nu_0)/b(t(x), \theta_0, \nu_0)). \]
Completeness of $T$ implies
\[ E_{\theta_0}^{X|t} \phi(x)(a'(x, \theta_0, \nu_0)/a(x, \theta_0, \nu_0) + b'(t(x), \theta_0, \nu_0)/b(t(x), \theta_0, \nu_0)) = 0, \text{ a.e. } \mathcal{G}_{\theta_0}^{T}. \]
The test must be similar, hence as before
\[ E_{\theta_0}^{X|t} \phi(x) = \lambda \text{ a.e. } \mathcal{G}_{\theta_0}^{T}. \]
We have
\[ (3.7) \quad E_{\theta_0}^{X|t} \phi(x) = \lambda \text{ a.e. } \mathcal{G}_{\theta_0}^{T}. \]
Maximum is obtained if for each $t$ the expectation $E_{\theta_0}^{X|t}$ in the above expression is maximized under conditions (3.6) and (3.7). An application of the Neyman-Pearson fundamental lemma gives the test $\phi_3$. 
REMARK. The test is unique (a.e. \( \mathcal{F}^X \)) if

\[
P_{\theta_0, \nu_0}(a_{\theta}(x, \theta_0, \nu_0) + c_1(t(X)) + a_{\theta}(x, \theta_0, \nu_0)) =
\]

\[
c_2(t(X))a(x, \theta_0, \nu_0)) = 0.
\]

THEOREM 3.6. In addition to the assumptions of Theorem 3.5 suppose that \( \phi^n(\theta, \nu; \varphi) \) is continuous in \( \theta \) at \( \theta_0 \) for any test \( \varphi \) and suppose that the hypothesis testing problem \( H_2 \) is invariant under a group \( G \) of transformations, and that \( |\theta - \theta_0| \) is a maximal invariant under the induced group \( \overline{G} \) of transformations of the parameter space. If \( \varphi_3 \) of Theorem 3.5 is unique and the power function is invariant, then \( \varphi_3 \) is the unique LMPU level \( \alpha \) test of \( H_2 \) with respect to the distance function \( d_2 \).

PROOF. Analogous to the proof of Theorem 3.4.

A corollary analogous to Corollary 3.2 could also be formulated.

We have tacitly assumed measurability of the functions occurring in the theorems. We shall not prove this, but only note that in each specific case we may find (measurable) tests which is of the form given in the theorems. By Lemma 3.1 they will be most powerful.
4. TESTING FOR SERIAL CORRELATION.

The model for the observations $x_1, x_2, \ldots, x_n$ is

$$x_i = \rho x_{i-1} + u_i \quad i = 2, 3, \ldots, n$$

where $u_2, u_3, \ldots, u_n$ are independent $N(0, \sigma^2)$, and $x_1, x_2, \ldots, x_n$ have a multinormal distribution with $E x_i = 0$, $\text{Var} x_i = \sigma^2/(1-\rho^2)$ and $\text{Cov}(x_i, x_j) = \rho^{|i-j|}\sigma^2/(1-\rho^2)$.

The parameters $\sigma$ and $\rho$ are unknown.

We shall consider the problem of testing the hypotheses

$$H_1 : \rho = \rho_0 \quad \text{against} \quad \rho > \rho_0,$$

$$H_2 : \rho = 0 \quad \text{against} \quad \rho \neq 0.$$

The hypothesis testing problem $H_1$ is invariant under a common (positive) change of scale of $x_1, x_2, \ldots, x_n$. A maximal invariant is

$$S' = \left[ \frac{X_1}{\sqrt{\sum_{i=1}^{n} x_i^2}}, \frac{X_2}{\sqrt{\sum_{i=1}^{n} x_i^2}}, \ldots, \frac{X_n}{\sqrt{\sum_{i=1}^{n} x_i^2}} \right].$$

The distribution of $S$ depends only upon $\rho$, hence any invariant test is similar. When considering invariant tests it is therefore no restriction to restrict attention to similar tests.
The probability density of $X_1, X_2, \ldots, X_n$ is

$$(2\pi)^{-\frac{1}{2}} \sigma^{-n}(1-\varrho^2)^{\frac{1}{2}} \exp\left(-\frac{1}{2} \sigma^2 \left(\sum_{i=1}^{n} x_i^2 - 2\sum_{i=2}^{n} x_i x_{i-1} + \varrho^2 \sum_{i=2}^{n} x_i^2 \right)\right),$$

which can be written in the form $a(x, \varrho, \sigma) b(t(x), \varrho, \sigma)$ with

$$a(x, \varrho, \sigma) = \exp\left(-\frac{1}{2} \sigma^2 \left(\varrho^2 \sum_{i=2}^{n} x_i^2 - 2(\varrho - \varrho_0) \sum_{i=2}^{n} x_i x_{i-1}\right)\right)$$

and

$$b(t(x), \varrho, \sigma) = (2\pi)^{-\frac{1}{2}} \sigma^{-n}(1-\varrho^2)^{\frac{1}{2}} \exp\left(-\frac{1}{2} \sigma^2 t(x)\right)$$

where

$$t(x) = \sum_{i=1}^{n} x_i^2 + \varrho_0^2 \sum_{i=2}^{n} x_i^2 - 2\varrho_0 \sum_{i=2}^{n} x_i x_{i-1}. $$

The probability measure for $\varrho = \varrho_0$ and $\sigma = \sigma_0$ can be used as a dominating measure for any $\sigma_0$. We have $a(x, \varrho_0, \sigma) = 1$. $T = t(x)$ is sufficient and complete when $\varrho = \varrho_0$. Applying Theorem 3.1, the most powerful similar test against an alternative $(\varrho_1, \sigma_1)$ is found to have the rejection region

$$2 \left(\sum_{i=2}^{n} x_i x_{i-1} - (\varrho_1 + \varrho_0) \sum_{i=2}^{n} x_i^2\right) > c(T).$$
Introduce
\[ W_1 = \frac{2 \sum_{i=2}^{n} X_i^2 X_{i-1} - (\phi_1 + \phi_0) \sum_{i=2}^{n-1} X_i^2}{\sum_{i=1}^{n} X_i^2 + \phi_0^2 \sum_{i=2}^{n-1} X_i^2 - 2 \phi_0 \sum_{i=2}^{n} X_i X_{i-1}}. \]

The distribution of \( W_1 \) does not depend upon \( \sigma \). \( T \) is sufficient and complete when \( \phi = \phi_0 \). Then by a theorem of Basu [1] \( W_1 \) and \( T \) are independent when \( \phi = \phi_0 \).

The rejection region may be written \( W_1 > c(T)/T \) where \( c(T) \) is determined by \( P(W_1 > c(T)/T | T) = \alpha \) when \( \phi = \phi_0 \). But since \( W_1 \) and \( T \) are independent when \( \phi = \phi_0 \) we must have \( c(t)/t \) equal to a constant. Hence the rejection region is \( W_1 > c \) where \( c \) is determined by \( P(W_1 > c) = \alpha \) when \( \phi = \phi_0 \).

Since \( W_1 \) does not depend upon \( \sigma_1 \) it is the most powerful similar test for \( \phi = \phi_0 \) against \( \phi = \phi_1 \). Since here invariance implies similarity and \( W_1 \) is invariant, it is also the most powerful invariant test for \( \phi = \phi_0 \) against \( \phi = \phi_1 \). By the Hunt-Stein theorem ([3], p.336) the test also maximizes the minimum power over the set of alternatives with \( \phi = \phi_1 \). If we could prove that the power function of the test increases with \( \phi \), then it is proved that it maximizes the minimum power over the set of alternatives with \( \phi \geq \phi_1 \).

The following argument will show that the test based on \( W_1 \) is almost a UMP invariant test. We have
If we neglect the term \((X_1^2 + X_n^2) \sum_{i=1}^{n} x_i^2\) which is small even for moderately large values of \(n\), we find that to reject when \(W_1 > c\) is equivalent to reject when \(W_0 > c'\) where

\[
W_0 = \frac{\sum_{i=2}^{n} x_i x_{i-1}}{\sum_{i=1}^{n} x_i^2}.
\]

For each \(\mathcal{G}_1\) this is an approximation to the most powerful invariant test for \(\mathcal{G} = \mathcal{G}_0\) against \(\mathcal{G} = \mathcal{G}_1\). It does not depend upon \(\mathcal{G}_1\). Hence it is almost a UMP invariant test for \(\mathcal{G} = \mathcal{G}_0\) against \(\mathcal{G} > \mathcal{G}_0\).

Using Theorem 3.3 and reasoning as above it is found that the test which maximizes the derivative of the power function with respect to \(\mathcal{G}\) at the point \((\mathcal{G}_0, \sigma_0)\) is based on the statistic

\[
W_2 = \frac{\sum_{i=2}^{n} x_i x_{i-1} - \mathcal{G}_0 \sum_{i=2}^{n-1} x_i^2}{\sum_{i=1}^{n} x_i^2 + \mathcal{G}_0^2 \sum_{i=1}^{n-1} x_i^2 - 2\mathcal{G}_0 \sum_{i=2}^{n} x_i x_{i-1}}
\]

with rejection region \(W_2 > \text{constant}\).
Since \( \mathcal{G} \) is maximal invariant in the parameter space and the distribution of \( W_2 \) depends only upon \( \mathcal{G} \), it is seen from Theorem 3.4 that the test based on \( W_2 \) is LMPS.

If we in \( W_2 \) neglect the term \((X_1^2 + X_n^2)/\sum_{i=1}^{n} X_i^2\) it is seen as for \( W_1 \) that the test based on \( W_2 \) reduces to the test based on \( W_0 \). Hence the test based on \( W_0 \) may be regarded as an approximation to the LMPS test.

The statistics \( W_1 \) and \( W_2 \) do not, of course, uniquely reduce to \( W_0 \) when we neglect terms of the form \((X_1^2 + X_n^2)/\sum_{i=1}^{n} X_i^2\). Another possible statistic is

\[
W_0' = \frac{\sum_{i=2}^{n} (X_i - X_{i-1})^2}{\sum_{i=1}^{n} X_i^2} = 2 \frac{X_1^2 + X_n^2}{\sum_{i=1}^{n} X_i^2} - 2W_0.
\]

See for example [5]. The test with rejection region \( W_0' < \) constant can also be regarded as both a nearly UMP invariant test and LMPS test. The difference between the power functions of the two tests can be expected to be small.

If we set \( \mathcal{Q}_0 = 0 \), then the test based upon \( W_2 \) reduces exactly to the test based upon \( W_0 \), hence in this case the latter is LMPS. If we set \( \mathcal{Q}_0 = 0 \) and \( \mathcal{Q}_1 = 1 \), then the test based upon \( W_1 \) reduces exactly to the test based upon \( W_0' \), hence the latter is most powerful invariant against the alternative \( \mathcal{A}_1 = 1 \). This should give an indication of the difference between the two tests. The test based upon \( W_0 \) is a little more powerful than the test based upon \( W_0' \) near the hypothesis, and the latter is a
little more powerful at alternative near $\sigma = 1$.

Finally we shall find a test of the hypothesis

\[ H_2 : \sigma = 0 \text{ against } \sigma \neq 0. \]

\[ \left[ X_1/X_n, X_2/X_n, \ldots, X_{n-1}/X_n \right] \text{ is a maximal invariant under a common change of scale of all variables, and } |\sigma| \text{ is a maximal invariant in the parameter space.} \]

If we apply Theorem 3.5 it is found that the test which maximizes the second derivative of the power function at $(0, \sigma_0)$ subject to the restriction of unbiasedness and similarity, rejects when

\[ -\sum_{i=2}^{n-1} x_i^2 + \frac{1}{\sigma_0^2} \left( \sum_{i=2}^{n} x_i x_{i-1} \right)^2 + c_1(T) \sum_{i=2}^{n} x_i x_{i-1} > c_2(T) \]

where in this case $T = \sum_{i=1}^{n} x_i^2$. This can be written as

\[ \frac{x_1^2 + x_n^2}{\left( \sum_{i=1}^{n} x_i^2 \right)^2} + \frac{1}{\sigma_0^2} \frac{\sum_{i=2}^{n} x_i x_{i-1}}{\sum_{i=1}^{n} x_i^2} + c_3(T)^2 > c_4(T). \]

Neglecting the term $(x_1^2 + x_n^2)/(\sum_{i=1}^{n} x_i^2)$ and reasoning as before we get the rejection region

\[ w_0 < -c \text{ and } w_0 > c \]

where $c$ is determined from the condition of level $\alpha$.

This test is an approximation to the test which maximizes the second derivative of the power function at $(0, \sigma_0)$. Since the power function of the former test depends
only upon \( |\phi| \) it is by Theorem 3.6 an approximation to the LMPU test at \( \phi = 0 \).

5. VARIANCE COMPONENTS MODELS.

In a previous paper [8] the author has studied the unbalanced one-way classification variance components model.

\[
X_{ij} = \mu + U_i + V_{ij} \quad j = 1, 2, \ldots, n_i, \quad i = 1, 2, \ldots, r,
\]

where \( \mu \) is an unknown constant, and where the \( U_i \) and \( V_{ij} \) are all independently normally distributed with expectations zero and variances \( \tau^2 \) and \( \sigma^2 \) respectively.

The hypothesis to be tested is

\[
H : \Delta = \Delta_0 \quad \text{against} \quad \Delta > \Delta_0
\]

where \( \Delta = \frac{\tau^2}{\sigma^2} \).

In [8] it is shown that a maximal invariant under a group of translations, changes of scale and orthogonal transformations is

\[
\left[ \frac{Z_1}{Q^2}, \frac{Z_2}{Q^2}, \ldots, \frac{Z_{r-1}}{Q^2} \right]
\]

where \( Q = \sum_{i=1}^{r} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2 \) and \( Z_i = \sqrt{\frac{n_i}{r}} (\bar{X}_i - \bar{X}) \)

(\( i = 1, 2, \ldots, r-1 \)).
The family of probability distributions of
\[ Z' = [Z_1, Z_2, \ldots, Z_{r-1}] \]
and \( Q \) can be written in the form (3.1) with
\[
a(z, q, \Delta, \sigma) = \exp\left(-\frac{1}{2}(2\sigma^2)^{-1}\left(z'^\top A(\Delta)^{-1}z - z'^\top A(\Delta_0)^{-1}z\right)\right)q^{\frac{1}{2}(n-r)-1}
\]
for any \( \sigma_0 \), where \( A(\Delta)\sigma^2 \) is the covariance matrix of \( Z \) and \( n = \sum_{i=1}^{r} n_i \).

In [8] it is shown that the test which maximizes the minimum power over the set of alternatives with \( \Delta \leq \Delta_1 \), has a rejection region of the form \( W_1 > \text{constant} \) where
\[
W_1 = \frac{Z'^\top A(\Delta_0)^{-1}z - z'^\top A(\Delta_1)^{-1}z}{Z'^\top A(\Delta_0)^{-1}z + Q}.
\]

A limiting form of \( W_1 \) is obtained when \( \Delta_1 \to \infty \). Then we shall reject when \( T > \text{constant} \) where
\[
T = \frac{Z'^\top A(\Delta_0)^{-1}z}{Q}.
\]

From the identity
\[
(5.1) \quad Z'^\top A(\Delta)^{-1}z = \sum_{i=1}^{r} \frac{n_i}{n_i\Delta + 1} (\bar{X}_i - \bar{X})^2
\]
where \( \bar{X} = \left(\sum_{i=1}^{r} \frac{n_i}{n_i\Delta + 1}\right)^{-1} \sum_{i=1}^{r} \frac{n_i}{n_i\Delta + 1} \bar{X}_i \) the statistics \( T \) and \( W_1 \) may be computed by the observations \( X_{ij} \). Since the distribution of the invariant statistic depends only upon \( \Delta \), any invariant test has constant
power when \( \Delta = \Delta_0 \). Hence similarity represents no restriction when considering invariant tests. We shall now find the form of the locally most powerful invariant (LMPI) test. Derivation gives

\[
a'(z,q,\Delta_0; \sigma^2) = (2\pi)^{-1/2} z' A' \sigma^2 (\Delta_0) z \frac{1}{q^2(n-r^{-1})}.
\]

where \( A' (\Delta) = -\frac{\partial A (\Delta)}{\partial \Delta} \). The statistic \( z' A' (\Delta_0)^{-1} Z + Q \) is sufficient and complete when \( \Delta = \Delta_0 \) \([8]\). Using Theorem 3.3 and Theorem 3.4 and arguing as in Section 4 it is found that the LMPI test has rejection region \( W_2 > \text{constant} \) where

\[
W_2 = \frac{z' A' (\Delta_0) Z}{z' A' (\Delta_0)^{-1} Z + Q}.
\]

From the identity (5.1) it is found by derivation that

\[
z' A' (\Delta) Z = \sum_{i=1}^{r} \left( \frac{n_i}{n_i \Delta + 1} \right)^2 (\bar{x}_i - \bar{x})^2.
\]

It is seen that the LMPI test puts more weight to the group means with many observations than the other tests. It should be noted that the tests based on \( T, W_1 \) and \( W_2 \) reduce to the usual test when \( n_1 = n_2 = \ldots = n_r \). The same is the case if \( r = 2 \).

It is of interest to compare the three tests by means of their power functions. In \([8]\) it is proved that \( W_1 \) and \( T \) are distributed as ratios of linear combinations of
chi-square distributed random variables. The exact distribution is not known. In the case \( r = 3 \) the following lemma can be used to obtain a relatively simple expression for the cumulative distribution of the three statistics.

**Lemma 5.1.** Let \( X_1, X_2, X_3 \) be independently distributed chi-square random variables with \( \nu_1, \nu_2, \nu_3 \) degrees of freedom respectively, and let \( a_1 \) and \( a_2 \) be two constants. Then

\[
U = \frac{a_1 X_1 + a_2 X_2}{X_1 + X_2 + X_3}
\]

is distributed as \( Y_1 Y_2 \) where \( Y_1 \) and \( Y_2 \) are independent and \( Y_1 \) has a beta distribution with \( \nu_1 + \nu_2 \) and \( \nu_3 \) degrees of freedom and \( (Y_2 - a_1)/(a_2 - a_1) \) has a beta distribution with \( \nu_2 \) and \( \nu_1 \) degrees of freedom.

**Proof.** Define \( Y_1 \) and \( Y_2 \) by

\[
Y_1 = \frac{(X_1 + X_2)}{(X_1 + X_2 + X_3)} \quad \text{and} \quad Y_2 = \frac{(a_1 X_1 + a_2 X_2)}{(X_1 + X_2)}.
\]

Then \( Y_2 \) is independent of \( X_1 + X_2 \), and hence independent of \( Y_1 \). Also \( (Y_2 - a_1)/(a_2 - a_1) = X_2/(X_1 + X_2) \).

We shall use Lemma 5.1 with \( \nu_1 = \nu_2 = 1 \) and \( \nu_3 = n-3 \) where \( n = \sum_{i=1}^{3} n_i \). By integration it is found that if \( 0 < a_1 < a_2 \) then
\[ P(U > u) = \frac{1}{2}(1-u/a_1)\frac{1}{2}(n-3) + \frac{1}{2}(1-u/a_2)\frac{1}{2}(n-3) \]

(5.2) \[ + 2 \Pi^{-1}(n-3)^{-1} \int_{u/a_2}^{1} (1-x)\frac{1}{2}(n-5) \]
\[ \text{Arcsin}(1-2 \frac{u-x-a_1}{a_2-a_1})dx \]

for \( u \neq a_1 \), and for \( a_1 \neq u \neq a_2 \)

\[ P(U > u) = \frac{1}{2}(1-u/a_2)\frac{1}{2}(n-3) \]

(5.3) \[ + 2 \Pi^{-1}(n-3)^{-1} \int_{u/a_2}^{1} (1-x)\frac{1}{2}(n-5) \]
\[ \text{Arcsin}(1-2 \frac{u-x-a_1}{a_2-a_1})dx. \]

To avoid complicated formulae we shall in the following consider only the case \( \Delta_0 = 0 \).

Let \( \beta_0, \beta_1 \) and \( \beta_2 \) denote the power functions of the tests based upon \( T, W_1 \) and \( W_2 \) respectively, and let \( c_0, c_1 \) and \( c_2 \) denote the corresponding constants used in the tests. In [8] it is shown that \( W_1 \) is distributed as

\[ W_1(\Delta) = \frac{\sum_{i=1}^{r-1}(\Delta \lambda_i + 1 - (\Delta \lambda_i + 1)/(\Delta_1 \lambda_i + 1))S_i^2}{\sum_{i=1}^{r-1}(\Delta \lambda_i + 1)S_i^2 + Q} \]

where \( S_1^2, S_2^2, \ldots, S_{r-1}^2 \) are independently chi-square distributed with 1 degree of freedom, independent of \( Q \)
which has a chi-square distribution with \( n-r \) degrees of freedom. The \( \lambda_i \) are the roots of the equation
\[
|B-\lambda C| = 0 \quad \text{where} \quad B \text{ and } C \text{ are determined from}
\]
\[
A(\Delta) = B\Delta + C.
\]

We find
\[
\beta_1(\Delta) = P(W_1(\Delta) > c_1)
\]
\[
= P\left( \frac{\sum_{i=1}^{r-1} \frac{\lambda_i(A_1(\Delta \lambda_i + 1) + 1)/(\Delta \lambda_i + 1) - \Delta c_1)}{S_i^2}}{\sum_{i=1}^{r-1} S_i^2 + Q} > c_1 \right)
\]

where the statistic is in the form of \( U \) in Lemma 5.1.

Regarding the test \( T \) we may use the fact that \( T \) is distributed as
\[
T(\Delta) = \frac{\sum_{i=1}^{r-1} (\Delta \lambda_i + 1) S_i^2}{Q}
\]

to write
\[
\beta_0(\Delta) = P\left( \frac{\sum_{i=1}^{r-1} (\Delta \lambda_i + 1 + c_0) S_i^2}{\sum_{i=1}^{r-1} S_i^2 + Q} > c_0 \right).
\]

The power function of the test based on \( W_2 \) is
where the $\mu_i$ are the roots of

$$|A^*(0) - c_2A(0)^{-1} + c_2A(\Delta)^{-1}\mu_i(\Delta)^{-1}| = 0.$$
It is seen from the tables that the difference between \( \beta_0 \) and \( \beta_1 \) is small, and very little is gained by the LMPI test near the hypothesis as compared with the loss of power for moderate values of \( \Delta \). It is also seen that we may have a serious loss of power compared with the situation where \( n_1 = n_2 = n_3 = n/3 \).
<table>
<thead>
<tr>
<th>Δ</th>
<th>$\delta_0$</th>
<th>$\delta_1^{-1}$</th>
<th>$\delta_2^{-1}$</th>
<th>$\theta_0$</th>
<th>$\theta_1^{-1}$</th>
<th>$\theta_2^{-1}$</th>
<th>$\phi_0$</th>
<th>$\phi_1^{-1}$</th>
<th>$\phi_2^{-1}$</th>
<th>$\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.010</td>
<td>0.000</td>
<td>0.000</td>
<td>0.010</td>
<td>0.000</td>
<td>0.000</td>
<td>0.010</td>
<td>0.000</td>
<td>0.000</td>
<td>0.010</td>
</tr>
<tr>
<td>0.01</td>
<td>0.011</td>
<td>0.001</td>
<td>0.000</td>
<td>0.014</td>
<td>0.000</td>
<td>0.000</td>
<td>0.014</td>
<td>0.000</td>
<td>0.000</td>
<td>0.014</td>
</tr>
<tr>
<td>0.02</td>
<td>0.013</td>
<td>0.001</td>
<td>0.000</td>
<td>0.019</td>
<td>0.001</td>
<td>0.000</td>
<td>0.019</td>
<td>0.000</td>
<td>0.000</td>
<td>0.019</td>
</tr>
<tr>
<td>0.03</td>
<td>0.015</td>
<td>0.001</td>
<td>0.000</td>
<td>0.024</td>
<td>0.001</td>
<td>0.000</td>
<td>0.024</td>
<td>0.000</td>
<td>0.000</td>
<td>0.025</td>
</tr>
<tr>
<td>0.04</td>
<td>0.017</td>
<td>0.001</td>
<td>0.000</td>
<td>0.030</td>
<td>0.002</td>
<td>0.000</td>
<td>0.030</td>
<td>0.000</td>
<td>0.000</td>
<td>0.032</td>
</tr>
<tr>
<td>0.05</td>
<td>0.018</td>
<td>0.002</td>
<td>0.000</td>
<td>0.036</td>
<td>0.003</td>
<td>0.000</td>
<td>0.036</td>
<td>0.000</td>
<td>0.000</td>
<td>0.039</td>
</tr>
<tr>
<td>0.06</td>
<td>0.021</td>
<td>0.002</td>
<td>0.000</td>
<td>0.043</td>
<td>0.003</td>
<td>0.000</td>
<td>0.043</td>
<td>0.000</td>
<td>0.000</td>
<td>0.047</td>
</tr>
<tr>
<td>0.07</td>
<td>0.023</td>
<td>0.002</td>
<td>0.000</td>
<td>0.051</td>
<td>0.003</td>
<td>0.000</td>
<td>0.051</td>
<td>0.000</td>
<td>0.000</td>
<td>0.055</td>
</tr>
<tr>
<td>0.08</td>
<td>0.025</td>
<td>0.003</td>
<td>0.000</td>
<td>0.059</td>
<td>0.003</td>
<td>0.000</td>
<td>0.059</td>
<td>0.000</td>
<td>0.000</td>
<td>0.064</td>
</tr>
<tr>
<td>0.09</td>
<td>0.027</td>
<td>0.004</td>
<td>0.000</td>
<td>0.067</td>
<td>0.004</td>
<td>0.000</td>
<td>0.067</td>
<td>0.000</td>
<td>0.000</td>
<td>0.073</td>
</tr>
<tr>
<td>0.1</td>
<td>0.030</td>
<td>0.003</td>
<td>0.000</td>
<td>0.075</td>
<td>0.004</td>
<td>0.000</td>
<td>0.075</td>
<td>0.000</td>
<td>0.000</td>
<td>0.082</td>
</tr>
<tr>
<td>0.2</td>
<td>0.058</td>
<td>0.006</td>
<td>0.000</td>
<td>0.163</td>
<td>0.004</td>
<td>0.000</td>
<td>0.163</td>
<td>0.000</td>
<td>0.000</td>
<td>0.180</td>
</tr>
<tr>
<td>0.3</td>
<td>0.091</td>
<td>0.007</td>
<td>0.000</td>
<td>0.246</td>
<td>0.002</td>
<td>0.000</td>
<td>0.246</td>
<td>0.000</td>
<td>0.000</td>
<td>0.271</td>
</tr>
<tr>
<td>0.4</td>
<td>0.125</td>
<td>0.007</td>
<td>0.000</td>
<td>0.319</td>
<td>0.002</td>
<td>0.000</td>
<td>0.319</td>
<td>0.000</td>
<td>0.000</td>
<td>0.348</td>
</tr>
<tr>
<td>0.5</td>
<td>0.160</td>
<td>0.004</td>
<td>0.000</td>
<td>0.381</td>
<td>0.005</td>
<td>0.000</td>
<td>0.381</td>
<td>0.000</td>
<td>0.000</td>
<td>0.413</td>
</tr>
<tr>
<td>0.6</td>
<td>0.193</td>
<td>0.002</td>
<td>0.000</td>
<td>0.433</td>
<td>0.006</td>
<td>0.000</td>
<td>0.433</td>
<td>0.000</td>
<td>0.000</td>
<td>0.466</td>
</tr>
<tr>
<td>0.7</td>
<td>0.225</td>
<td>0.001</td>
<td>0.000</td>
<td>0.478</td>
<td>0.008</td>
<td>0.000</td>
<td>0.478</td>
<td>0.000</td>
<td>0.000</td>
<td>0.512</td>
</tr>
<tr>
<td>0.8</td>
<td>0.255</td>
<td>0.004</td>
<td>0.000</td>
<td>0.516</td>
<td>0.008</td>
<td>0.000</td>
<td>0.516</td>
<td>0.000</td>
<td>0.000</td>
<td>0.551</td>
</tr>
<tr>
<td>0.9</td>
<td>0.283</td>
<td>0.007</td>
<td>0.000</td>
<td>0.550</td>
<td>0.010</td>
<td>0.000</td>
<td>0.550</td>
<td>0.000</td>
<td>0.000</td>
<td>0.584</td>
</tr>
<tr>
<td>1</td>
<td>0.310</td>
<td>0.010</td>
<td>0.000</td>
<td>0.579</td>
<td>0.010</td>
<td>0.000</td>
<td>0.579</td>
<td>0.000</td>
<td>0.000</td>
<td>0.613</td>
</tr>
<tr>
<td>2</td>
<td>0.503</td>
<td>0.032</td>
<td>0.000</td>
<td>0.746</td>
<td>0.011</td>
<td>0.000</td>
<td>0.746</td>
<td>0.000</td>
<td>0.000</td>
<td>0.772</td>
</tr>
<tr>
<td>3</td>
<td>0.614</td>
<td>0.040</td>
<td>0.000</td>
<td>0.818</td>
<td>0.009</td>
<td>0.000</td>
<td>0.818</td>
<td>0.000</td>
<td>0.000</td>
<td>0.839</td>
</tr>
<tr>
<td>4</td>
<td>0.685</td>
<td>0.042</td>
<td>0.000</td>
<td>0.859</td>
<td>0.008</td>
<td>0.000</td>
<td>0.859</td>
<td>0.000</td>
<td>0.000</td>
<td>0.875</td>
</tr>
<tr>
<td>5</td>
<td>0.734</td>
<td>0.042</td>
<td>0.000</td>
<td>0.884</td>
<td>0.006</td>
<td>0.000</td>
<td>0.884</td>
<td>0.000</td>
<td>0.000</td>
<td>0.898</td>
</tr>
<tr>
<td>10</td>
<td>0.850</td>
<td>0.032</td>
<td>0.000</td>
<td>0.940</td>
<td>0.004</td>
<td>0.000</td>
<td>0.940</td>
<td>0.000</td>
<td>0.000</td>
<td>0.947</td>
</tr>
</tbody>
</table>
TABLE 2

<table>
<thead>
<tr>
<th>Δ</th>
<th>$\beta_0$</th>
<th>$\beta_1 - \beta_0$</th>
<th>$\beta_2 - \beta_0$</th>
<th>$\beta_0$</th>
<th>$\beta_1 - \beta_0$</th>
<th>$\beta_2 - \beta_0$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.010</td>
<td>.000</td>
<td>.000</td>
<td>.010</td>
<td>.000</td>
<td>.000</td>
<td>.010</td>
</tr>
<tr>
<td>.01</td>
<td>.014</td>
<td>.001</td>
<td>.001</td>
<td>.023</td>
<td>.005</td>
<td>.005</td>
<td>.027</td>
</tr>
<tr>
<td>.02</td>
<td>.020</td>
<td>.002</td>
<td>.002</td>
<td>.042</td>
<td>.009</td>
<td>.010</td>
<td>.053</td>
</tr>
<tr>
<td>.03</td>
<td>.026</td>
<td>.003</td>
<td>.003</td>
<td>.065</td>
<td>.012</td>
<td>.013</td>
<td>.083</td>
</tr>
<tr>
<td>.04</td>
<td>.033</td>
<td>.004</td>
<td>.004</td>
<td>.088</td>
<td>.016</td>
<td>.016</td>
<td>.115</td>
</tr>
<tr>
<td>.05</td>
<td>.041</td>
<td>.005</td>
<td>.004</td>
<td>.113</td>
<td>.017</td>
<td>.017</td>
<td>.149</td>
</tr>
<tr>
<td>.06</td>
<td>.049</td>
<td>.006</td>
<td>.005</td>
<td>.137</td>
<td>.019</td>
<td>.018</td>
<td>.182</td>
</tr>
<tr>
<td>.07</td>
<td>.058</td>
<td>.006</td>
<td>.005</td>
<td>.161</td>
<td>.019</td>
<td>.017</td>
<td>.214</td>
</tr>
<tr>
<td>.08</td>
<td>.067</td>
<td>.007</td>
<td>.006</td>
<td>.184</td>
<td>.019</td>
<td>.016</td>
<td>.244</td>
</tr>
<tr>
<td>.09</td>
<td>.077</td>
<td>.006</td>
<td>.005</td>
<td>.206</td>
<td>.019</td>
<td>.015</td>
<td>.274</td>
</tr>
<tr>
<td>.1</td>
<td>.086</td>
<td>.007</td>
<td>.006</td>
<td>.227</td>
<td>.019</td>
<td>.013</td>
<td>.301</td>
</tr>
<tr>
<td>.2</td>
<td>.184</td>
<td>.007</td>
<td>.000</td>
<td>.391</td>
<td>.010</td>
<td>-.008</td>
<td>.502</td>
</tr>
<tr>
<td>.3</td>
<td>.272</td>
<td>.003</td>
<td>-.010</td>
<td>.498</td>
<td>.001</td>
<td>-.028</td>
<td>.617</td>
</tr>
<tr>
<td>.4</td>
<td>.346</td>
<td>-.001</td>
<td>-.020</td>
<td>.572</td>
<td>-.005</td>
<td>-.042</td>
<td>.689</td>
</tr>
<tr>
<td>.5</td>
<td>.408</td>
<td>-.005</td>
<td>-.028</td>
<td>.627</td>
<td>-.009</td>
<td>-.052</td>
<td>.739</td>
</tr>
<tr>
<td>.6</td>
<td>.460</td>
<td>-.008</td>
<td>-.034</td>
<td>.670</td>
<td>-.012</td>
<td>-.060</td>
<td>.775</td>
</tr>
<tr>
<td>.7</td>
<td>.504</td>
<td>-.010</td>
<td>-.039</td>
<td>.703</td>
<td>-.013</td>
<td>-.065</td>
<td>.802</td>
</tr>
<tr>
<td>.8</td>
<td>.541</td>
<td>-.011</td>
<td>-.042</td>
<td>.731</td>
<td>-.014</td>
<td>-.069</td>
<td>.824</td>
</tr>
<tr>
<td>.9</td>
<td>.573</td>
<td>-.012</td>
<td>-.044</td>
<td>.753</td>
<td>-.014</td>
<td>-.071</td>
<td>.841</td>
</tr>
<tr>
<td>1</td>
<td>.602</td>
<td>-.013</td>
<td>-.046</td>
<td>.773</td>
<td>-.015</td>
<td>-.074</td>
<td>.855</td>
</tr>
<tr>
<td>2</td>
<td>.761</td>
<td>-.014</td>
<td>-.046</td>
<td>.872</td>
<td>-.013</td>
<td>-.071</td>
<td>.923</td>
</tr>
<tr>
<td>3</td>
<td>.829</td>
<td>-.012</td>
<td>-.039</td>
<td>.911</td>
<td>-.010</td>
<td>-.062</td>
<td>.948</td>
</tr>
<tr>
<td>4</td>
<td>.867</td>
<td>-.010</td>
<td>-.033</td>
<td>.932</td>
<td>-.009</td>
<td>-.053</td>
<td>.961</td>
</tr>
<tr>
<td>5</td>
<td>.891</td>
<td>-.008</td>
<td>-.028</td>
<td>.945</td>
<td>-.007</td>
<td>-.047</td>
<td>.968</td>
</tr>
<tr>
<td>10</td>
<td>.943</td>
<td>-.005</td>
<td>-.017</td>
<td>.972</td>
<td>-.004</td>
<td>-.028</td>
<td>.984</td>
</tr>
</tbody>
</table>
REFERENCES


