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THE JOINT DISTRIBUTION OF TWO STUDENTIZED REGRESSION COEFFICIENTS

by

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1. Introduction. We consider the binormal model: $X = (X_1, X_2)'$ is $N(\mu, \Sigma)$ where $\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$.
The regression coefficient of X_1 given X_2 is $\beta_1 = \rho\,\sigma_1/\sigma_2$ and the regression coefficient of X_2 given X_1 is $\beta_2 = \rho\,\sigma_2/\sigma_1$.

Let $X_1, \ldots, X_N$ be N independent observations of X. Put $n = N-1$. Let $a_{ij} = \frac{1}{n}\sum_{\alpha=1}^{N}(X_{i\alpha}-\bar{X}_i)(X_{j\alpha}-\bar{X}_j)$ and let $r = a_{12}/\sqrt{a_{11}a_{22}}$ be the sample correlation coefficient. The maximum likelihood estimates of β_1 and β_2 are $\hat{\beta}_1 = a_{12}/a_{22}$ and $\hat{\beta}_2 = a_{12}/a_{11}$ and it is shown by M.S. Bartlett [2] that the studentized regression coefficients

$$U = \sqrt{\frac{(n-1)a_{22}}{(1-r^2)a_{11}}}(\hat{\beta}_1-\beta_1), \quad V = \sqrt{\frac{(n-1)a_{11}}{(1-r^2)a_{22}}}\,(\hat{\beta}_2-\beta_2)$$

are distributed according to the Student distribution with $n-1$ degrees of freedom.

We are going to show that the joint distribution of U and V is a two-dimensional Student distribution, (introduced by C.W. Dunnett and M. Sobel in [3]). We will assume that $0 < \rho^2 < 1$. If $\rho = 0$ then $U = V$ and if $\rho^2 = 1$ then X_1 and X_2 are linearly dependent.

2. A transformation. Introduce two new variables, Y and Z, by

$$(2.1) \qquad Y = \sqrt{a_{11}a_{22}}, \quad Z = \sqrt{a_{11}/a_{22}} .$$

Then we may write

$$(2.2) \quad U = \sqrt{\frac{n-1}{1-r^2}}(r-\beta_1/Z), \quad V = \sqrt{\frac{n-1}{1-r^2}}(r-\beta_2 Z)$$

LEMMA. The transformation (2.2) is one-to-one.

PROOF. Naturally, when r and Z are given, U and V are uniquely determined. Now, let $U = u$ and $V = v$ and put $p = u/\sqrt{n-1}$, $q = v/\sqrt{n-1}$. We have to show that the equations

$$(2.3) \qquad \frac{r-\beta_1/z}{\sqrt{1-r^2}} = p, \quad \frac{r-\beta_2 z}{\sqrt{1-r^2}} = q$$

have a unique solution. Assume $p \neq q$. Then we easily find that r is given by

$$(2.4) \qquad r = \frac{1}{q-p}(\beta_1 q/z - \beta_2 pz).$$

Subtracting the equations in (2.3) we get

$$(2.5) \qquad \frac{\beta_2 z - \beta_1/z}{\sqrt{1-r(z)^2}} = p-q$$

where $r(z)$ is given by (2.4). Let $h(z) = \dfrac{\beta_2 z - \beta_1/z}{\sqrt{1-r(z)^2}}$.

If $h(z)$ is a monotone function, (2.5) will have a unique solution. After some algebra we find

$$h'(z) = \frac{\beta_1/z^2 - 2\rho^2 r(z)/z + \beta_2}{[1 - r(z)^2]^{3/2}}$$

Consider the numerator, $l(z)$. We see that

$$l_1(z) = \frac{\beta_1}{z^2} - \frac{2\rho^2}{z} + \beta_2 \leqq l(z) \leqq \frac{\beta_1}{z^2} + \frac{2\rho^2}{z} + \beta_2 = l_2(z).$$

But $l_j(z) = 0$ implies $\frac{1}{z} = \frac{(-1)^{j-1}\rho^2 \pm \rho\sqrt{\rho^2-1}}{\beta_1}$ and it follows that $l_j(z)$, $j = 1,2$, have the same sign for all z. Putting $z = 1$ (say) we get

$$l_j(z) = \beta_1 + (-1)^j 2\rho^2 + \beta_2 = \rho\,\frac{\sigma_1{}^2 + (-1)^j 2\rho\sigma_1\sigma_2 + \sigma_2{}^2}{\sigma_1\sigma_2}$$

and we see that the sign is determined by ρ only. Therefore, $l_1(z)$ and $l_2(z)$ have the same sign, which means that $l(z)$ has the same sign for all z. Thus, $h(z)$ is a monotone function.

Let us consider the case $p = q$. From (2.3) we get $z = \sigma_1/\sigma_2$ and then $r - \rho = p\sqrt{1-r^2}$. Plotting the left and the right side of this equation for all 4 combinations of signs of ρ and p we see that the solution is unique.

3. The theorem. We are going to prove

THEOREM. The joint distribution of the two studentized regression coefficients U and V from a binormal sample of $N = n+1$ independent observations is the two-dimensional Student distribution with $n-1$ degrees of freedom and correlation coefficient $1-2\rho^2$, i.e. the joint density of U and V is

$$g(u,v;\rho) = \frac{1}{4\pi\sqrt{\rho^2(1-\rho^2)}}\left(1+\frac{u^2-2(1-2\rho^2)uv+v^2}{4(n-1)\rho^2(1-\rho^2)}\right)^{-\frac{1}{2}(n+1)}$$

PROOF. According to § 2 we may write

$$U = \sqrt{\frac{n-1}{1-r^2}}(r-\beta_1/Z), \quad V = \sqrt{\frac{n-1}{1-r^2}}(r-\beta_2 Z)$$

where $Z = \sqrt{a_{11}/a_{22}}$. The joint distribution of a_{11}, a_{22} and a_{12} is the Wishart-distribution, i.e. a_{11}, a_{22}, a_{12} have density

$$K(a_{11}a_{22}-a_{12}^2)^{\frac{1}{2}(n-3)} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{a_{11}}{\sigma_1^2}+\frac{a_{22}}{\sigma_2^2}-\frac{2\rho a_{12}}{\sigma_1\sigma_2}\right)\right]$$

where $K = \left(2^n\sqrt{\pi}\,(\sigma_1\sigma_2)^n(1-\rho^2)^{\frac{n}{2}}\,\Gamma(\frac{n}{2})\Gamma(\frac{n-1}{2})\right)^{-1}$, see e.g. [1], p.67. Now, $a_{12} = ry$ and from (2.1) we get $a_{11} = yz$, $a_{22} = y/z$. The Jacobian of this transformation is $-2y^2/z$, and the joint distribution of Y,Z and r is

$$2K(1-r^2)^{\frac{1}{2}(n-3)} z^{-1} y^{n-1} \exp\Big[- \frac{1}{2(1-\rho^2)\sigma_1{}^2\sigma_2{}^2} \cdot$$

$$\cdot(\sigma_2{}^2 z + \frac{\sigma_1{}^2}{z} - 2\rho\sigma_1\sigma_2 r)y\Big].$$

Integrating with respect to y over the range 0 to ∞ we get the joint density of Z and r

$$\frac{(n-1)2^{n-1}(\sigma_1\sigma_2)^n(1-\rho^2)^{\frac{1}{2}n}}{\pi}(1-r^2)^{\frac{1}{2}(n-3)} z^{-1}(\sigma_2^2 z + \frac{\sigma_1^2}{z} - 2\rho\sigma_1\sigma_2 r)^{-n}.$$

The Jacobian of the transformation (2.2) is

$$\frac{\partial(r,z)}{\partial(u,v)} = \frac{\sigma_1\sigma_2}{(n-1)\rho} z(1-r^2)^2(\sigma_2{}^2 z + \frac{\sigma_1{}^2}{z} - 2\rho\sigma_1\sigma_2 r)^{-1}$$

and thus the joint density of U and V is

$$(3.1)\quad g(u,v;\rho)=\left[\frac{2^{n-1}(\sigma_1\sigma_2)^{n+1}(1-\rho^2)^{\frac{1}{2}n}}{\pi\rho}\cdot\right.$$

$$\left.\cdot\left(\frac{(1-r^2)^{\frac{1}{2}}}{\sigma_2{}^2 z + \frac{\sigma_1{}^2}{z} - 2\rho\sigma_1\rho_2 z}\right)^{n+1}\right]_{\substack{r=\psi_1(u,v)\\ z=\psi_2(u,v)}}$$

where ψ is the inverse transformation of (2.2). According to the lemma, ψ is unique.

Now, using (2.2), we may write

$$u^2-2(1-2\rho^2)uv+v^2 = (u-v)^2+4\rho^2 uv$$

$$=\left(\frac{\rho}{\sigma_1\sigma_2}\right)^2 \frac{n-1}{(1-r^2)z^2}\Big[(\sigma_2{}^2z^2-\sigma_1{}^2)^2+4\sigma_1{}^2\sigma_2{}^2 z(rz-\beta_1)(r-\beta_2 z)\Big]$$

and after some calculations we get

$$4(n-1)\rho^2(1-\rho^2)+u^2-2(1-2\rho^2)uv+v^2 =$$

$$= \left(\frac{\rho}{\sigma_1\sigma_2}\right)^2 \frac{n-1}{1-r^2}\left(\sigma_2{}^2 z + \frac{\sigma_1{}^2}{z} - 2\rho\sigma_1\sigma_2 r\right)^2 .$$

Hence

$$\frac{1-r^2}{\left(\sigma_2{}^2 z + \frac{\sigma_1{}^2}{z} - 2\rho\sigma_1\sigma_2 r\right)^2} =$$

$$= \left(\frac{\rho}{\sigma_1\sigma_2}\right)^2 \frac{n-1}{4(n-1)\rho^2(1-\rho^2)+u^2-2(1-2\rho^2)uv+v^2}$$

Putting this into (3.1) the result follows.

4. Acknowledgments. The joint distribution of the studentized regression coefficients was found by the author in his master's thesis (Oslo 1964) by use of differential-equation methods. The transformation used in the present approach was proposed by professor O. Reiersøl.

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