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A NOTE ON A THEOREM OF BARANKIN AND GURLAND

by

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In their paper "On asymptotically normal, efficient estimators: I", [1], Barankin and Gurland have left a gap in their proof of Theorem 4.1 ([1; p.97]). We shall in this note restate the theorem and give a complete proof.

For convenience we shall repeat the necessary definitions and notations used in [1].

DEFINITION 1. A family $\mathcal{P}$ of probability distributions in the sample space Ω is said to belong to the class Π_0 if

(i) there is a 1-1 correspondence between the elements of $\mathcal{P}$ and the points of Θ, an open set in a k-dimensional Euclidean space;

(ii) elements of $\mathcal{P}$ corresponding to distinct points in Θ are distinct;

(iii) there is a non-negative function p on $\Omega \times \Theta$ and a measure μ in Ω such that the element of $\mathcal{P}$ corresponding to $\theta \in \Theta$ has the density function $p(\cdot,\theta)$ with respect to μ;

(iv) for each $x \in \Omega$ the function $p(x,\cdot)$ is continuously differentiable in Θ;

(v) the differentiations

$$\frac{\partial}{\partial \theta_\varkappa} \int_\Omega p(\cdot,\theta)\,d\mu \qquad \varkappa = 1, \ldots, k$$

can be taken under the integral sign;

(vi) the integrals

$$\int_\Omega \left(\frac{\partial \ln p}{\partial \theta_\varkappa}\right)^2 p(\cdot,\theta)\,d\mu \qquad \varkappa = 1, \ldots, k$$

are finite for all $\theta \in \Theta$

(vii) for each $\theta \in \Theta$ the matrix

$$\left\| \int_{\Omega} \frac{\partial \ln p}{\partial \theta_{\varkappa}} \frac{\partial \ln p}{\partial \theta_{\lambda}} p(\cdot,\theta) d\mu \right\| \varkappa, \lambda = 1, \ldots, k$$

is non-singular.

DEFINITION 2. The family of distributions $\mathcal{P} = \{ p(\cdot,\theta) d\mu, \theta \in \Theta \} \in \Pi_0$ is said to belong to the class Π_1 if there is a finite set of $s \geq k$ μ-measurable, real-valued functions on Ω, $\phi = (\phi_1, \ldots, \phi_s)$, such that

(i) the moments

$$A_i(\theta) = \int_{\Omega} \phi_i p(\cdot,\theta) d\mu$$

and

$$a_{ij}(\theta) = \int_{\Omega} (\phi_i - A_i(\theta))(\phi_j - A_j(\theta)) p(\cdot,\theta) d\mu$$

are finite for all $i,j = 1, \ldots, s$ and all $\theta \in \Theta$; and are twice continuously differentiable functions of θ;

(ii) the matrix

$$\left\| a_{ij}(\theta) \right\| \quad i,j = 1, \ldots, s$$

is non-singular for each $\theta \in \Theta$;

(iii) the matrix

$$\left\| \frac{\partial A_i}{\partial \theta_{\varkappa}} \right\| \quad \begin{array}{l} i = 1, \ldots, s \\ \varkappa = 1, \ldots, k \end{array}$$

is of rank k for each $\theta \in \Theta$;

(iv) the differentiations

$$\frac{\partial}{\partial \theta_{\varkappa}} \int_{\Omega} \phi_i p(\cdot,\theta) d\mu \qquad \begin{array}{l} i = 1, \ldots, s \\ \varkappa = 1, \ldots, k \end{array}$$

can be taken under the sign of integration

(v) if Γ^s denotes an s-dimensional Euclidean space, then the mapping A, of Θ into Γ^s defined by

$$A(\theta) = (A_1(\theta), \ldots, A_s(\theta))$$

is homeomorphic.

The above definitions reproduces the conditions of [1], however, not every regularity assumption stated in the definitions are needed for the proof below (e.g. we do not have to assume any regularity conditions on the probability function $p(\cdot,\theta)$ except for the existence of a set of functions Φ such that their first moments $A(\theta)$ are finite and satisfy Definition 2 (iii) and (v)).

Following Barankin and Gurland we shall need the following definitions.

DEFINITION 3. E and F are two spaces. An (E,F)-separator, D, is a real-valued function on $E \times F$, having the properties

(i) for each point $e \in E$, $\inf_{f \in F} D(e,f) > -\infty$, and

(ii) this infimum is attained in F.

DEFINITION 4. A (Γ^s, Θ)-separator, D, is said to be $(\mathcal{P}, \Phi)$-regular if

(i) for each $\theta \in \Theta$ and each $\theta' \neq \theta$ we have $D(A(\theta),\theta') > D(A(\theta),\theta)$

(ii) for each $\theta' \in \Theta$, there is a neighborhood $N_{A(\theta')} \subset \Gamma^s$ of $A(\theta')$, and a neighborhood $N_{\theta'} \subset \Theta$ of θ', such that in $N_{A(\theta')} \times N_{\theta'}$, the partial derivatives

$$\frac{\partial D(z,\theta)}{\partial \theta_{\varkappa}}, \quad \frac{\partial^2 D(z,\theta)}{\partial \theta_{\varkappa} \partial \theta_{\lambda}}, \quad \frac{\partial^2 D(z,\theta)}{\partial z_i \partial \theta_{\varkappa}}$$

$$\varkappa, \lambda = 1, \ldots, k \qquad i = 1, \ldots, s$$

are continuous

(iii) for each $\theta' \in \Theta$, there exist spherical neighborhoods $N^o_{A(\theta')} \subset N_{A(\theta')}$, of $A(\theta')$ and $N^o_{\theta'} \subset N_{\theta'}$ of θ' such that

$$\left| \frac{\partial^2 D(z,\theta)}{\partial \theta_{\varkappa} \partial \theta_{\lambda}} \right|_{\varkappa,\lambda = 1,\ldots,k} \neq 0$$

in $N^o_{A(\theta')} \times N^o_{\theta'}$, and for each $z \in N^o_{A(\theta')}$

$$\inf_{\theta \in N^o_{\theta'}} D(z,\theta) \leq \inf_{\theta \in \Theta - N^o_{\theta'}} D(z,\theta)$$

(iv) there exists at least one Borel-measurable function H_0 on Γ^s to Ⓗ such that

$$D(z,H_0(z)) = \inf_{\theta \in Ⓗ} D(z,\theta) \;, \quad z \in \Gamma^s$$

We shall also need the <u>tubular neighborhood theorem</u> (see [2; 5.5 Theorem]) which essentially states:

<u>Let f be a homeomorhic continuously differentiable mapping from an open set M to Γ^m. The Jacobian of f has maximal rank in every point of M. Then there is a neighborhood W of $f(M)$ and a continuously differentiable mapping r of W to $f(M)$ such that $r(y) = y$ for every $y \in f(M)$.</u>

We now state Theorem 4.1 of [1] in a slightly changed form:

THEOREM. <u>Let D be a $(\mathcal{P}, \Phi)$-regular $(\Gamma^s, Ⓗ)$-separator. Then there is an open neighborhood S of R and a mapping G of S to Ⓗ having the following properties</u>:

(a) <u>G is continuously differentiable on S</u>

(b) <u>G is the inverse mapping of A on R</u>

(c) $\inf_{\theta \in Ⓗ}$ <u>$D(z,\theta) = D(z,G(z))$ when $z \in S$</u>.

<u>Finally, G is the unique mapping with the properties (a), (b) and (c)</u>

<u>REMARK</u>. Where Barankin and Gurland are sound we sketch the argument for completeness and convenience. Where Barankin and Gurland are incomplete we shall be quite explicit, hoping that we are not having any gap.

<u>PROOF</u>. 1) <u>Local inverse</u>. Let θ' be a fixed but arbitrary point in Ⓗ. For every $z \in N^0_{A(\theta')}$ there is a $\theta^0 \in N^0_{\theta'}$, such that $\inf_{\theta \in Ⓗ} D(z,\theta) = D(z,\theta^0)$ (Definition 4 (iii)). Define then

$$G_{\theta'}(z) = \theta^0 \quad \text{for} \quad z \in N^0_{A(\theta')} \;.$$

$G_{\Theta'}$ is shown to be unique because of the existence of continuous derivatives (ii) and because

$$\left| \frac{\partial^2 D(z,\Theta)}{\partial \Theta_{\varkappa} \partial \Theta_{\lambda}} \right| \neq 0$$

on $N^o_{A(\Theta')} \times N^o_{\Theta'}$.

Then Definition 4 (i) gives $G_{\Theta'}(A(\Theta')) = \Theta'$.

At last, the implicit function theorem might be used to prove that $G_{\Theta'}$ is continuously differentiable on $N^o_{A(\Theta')}$.

Summing up, we have for every $\Theta' \in \text{Ⓗ}$ found a mapping $G_{\Theta'}$, which locally has the properties (a), (b), (c) and is unique.

2) <u>Global inverse</u>. We shall here be more explicit. One easily verifies that the mapping A (Definition 2 (v)) has the properties of the mapping f in the tubular neighborhood theorem. Thus there is a neighborhood S_o of $R = A(\text{Ⓗ})$ and a continuously differentiable mapping r of S_o to R, such that $r(z) = z$ for every $z \in R$.

Let for every Θ the neighborhood N^o_Θ be replaced by a reduced neighborhood N'_Θ with radius equal to 1/3 of the radius of N^o_Θ. The neighborhoods N'_Θ have the property that:

<u>$N'_{\Theta'} \cap N'_{\Theta''} \neq \emptyset$ implies $N'_{\Theta'} \cup N'_{\Theta''}$ is either contained in $N^o_{\Theta'}$ or $N^o_{\Theta''}$</u>

The neighborhoods $N^o_{A(\Theta)}$ will be replaced by neighborhoods $N'_{A(\Theta)}$ constructed in the following way:

(i) for every $\Theta \in \text{Ⓗ}$, let $N''_{A(\Theta)} \subset N^o_{A(\Theta)}$ be a neighborhood of $A(\Theta)$ such that G_Θ maps $N''_{A(\Theta)}$ into N'_Θ . (This is possible since G_Θ is continuous.)

(ii) set

$$N'_{A(\Theta)} = N''_{A(\Theta)} \cap \left\{ z \in S_o \, , \, r(z) \in N''_{A(\Theta)} \cap R \right\}$$

One may verify that $N'_{A(\Theta)}$ really are neighborhoods, and that (i) also applies for $N'_{A(\Theta)}$ as well as for $N''_{A(\Theta)}$.

The neighborhoods $N'_{A(\Theta)}$ now have the nice property that:

<u>$N'_{A(\Theta')} \cap N'_{A(\Theta'')} \neq \emptyset$ implies that there is a $\Theta \in \text{Ⓗ}$ such that</u>

$A(\Theta) \in N'_{A(\Theta')} \cap N'_{A(\Theta'')}$.

For, if $z \in N'_{A(\theta')} \cap N'_{A(\theta'')}$, then $r(z) \in N'_{A(\theta')} \cap N'_{A(\theta')}$ and there is a $\theta \in Ⓗ$ such that $A(\theta) = r(z)$.

Define $S = \bigcup_{\theta \in Ⓗ} N'_{A(\theta)}$. We are going to show that if $z \in S$, $z \in N'_{A(\theta')} \cap N'_{A(\theta'')}$ then $G_{\theta'}(z) = G_{\theta''}(z)$.

Let $z \in N'_{A(\theta')} \cap N'_{A(\theta'')}$, then there is a $\theta^o \in Ⓗ$ such that $A(\theta^o) \in N'_{A(\theta')} \cap N'_{A(\theta'')}$. Hence, $\theta_o = G_{\theta'}(A(\theta^o)) = G_{\theta''}(A(\theta^o)) \in N'_{\theta'} \cap N'_{\theta''}$ because of property (i) of the neighborhoods $N'_{A(\theta)}$. Thus $N'_{\theta'} \cap N'_{\theta''} \neq \emptyset$ and $N'_{\theta'} \cup N'_{\theta''}$ is contained in for example $N^o_{\theta'}$. Therefore, since both $G_{\theta'}(z)$ and $G_{\theta''}(z)$ minimize $D(z,\theta)$ and both lie in $N^o_{\theta'}$, they have to be equal because of the locally uniqueness of G_θ.

Define G to be

$$G(z) = G_\theta(z) \quad \text{for every} \quad z \in N'_{A(\theta)}$$

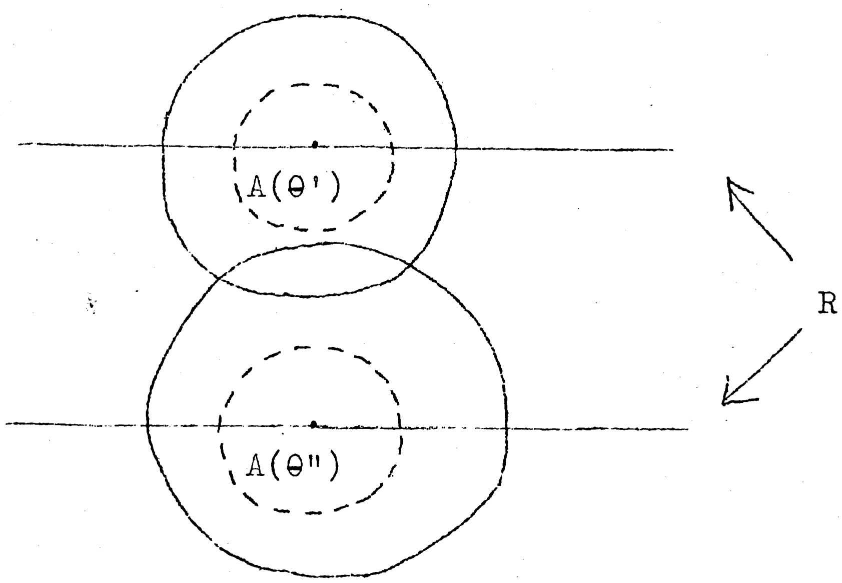
G is a mapping of S to Ⓗ and G has the properties (a), (b) and (c) in the theorem. It remains to prove that

3) <u>G is the unique mapping</u> wiht these properties. Assume that also G' has the properties (a), (b) and (c), but that there is a point $z_o \in S$ such that $G(z_o) \neq G'(z_o)$. Let $z_o \in N'_{A(\theta_o)}$. Because of the locally uniqueness of G, $G'(z_o) \notin N'_{\theta_o}$. Let C be the straight line between z_o and $A(\theta_o)$, then $C \subset N'_{A(\theta_o)}$. Since G and G' are continuous and N'_θ is open, there is a point $\theta' \in N'_\theta$ on G'(C) (the image of C by G') which is not on G(C). But then there must also exist a point $\theta'' \in N'_{\theta_o}$ on G(C) such that θ' and θ'' are the images of the same $z' \in N'_{A(\theta_o)}$. But this is impossible, since $D(z',\theta)$ cannot have two minimizing points in N'_{θ_o}. ||

We have in part 2) of the proof explicitly constructed the neighborhoods $N'_{A(\theta)}$ by means of the tubular neighborhood theorem. In Barankin and Gurland it is not evident how they construct the neighborhoods such that they have the property:

$N'_{A(\theta')} \cap N'_{A(\theta'')} \neq \emptyset$ implies that $R \cap (N'_{A(\theta')} \cup N'_{A(\theta'')})$ is connected.

If two neighborhoods $N_{A(\theta')}$ and $N_{A(\theta'')}$ intersect without the intersection having common points with R, it is not difficult to replace these neighborhoods by others $N'_{A(\theta')}$ and $N'_{A(\theta'')}$ having an empty intersection as might be seen from the following picture:



The difficulty arises when given an $A(\theta')$ there is an <u>infinite</u> number of points $A(\theta'')$ such that $N_{A(\theta')} \cap N_{A(\theta'')} \neq \emptyset$ but $N_{A(\theta')} \cap N_{A(\theta'')} \cap R = \emptyset$. If one carries through a similar reduction of the neighborhoods as above for each $A(\theta'')$, the reduced $N_{A(\theta')}$ may in the end turn out to be empty!

REFERENCES

[1] <u>E.W. Barankin and J. Gurland</u> (1951): "On asymptotically normal, efficient estimators: I", University of California Publications in Statistics, Vol.1, No.6, pp.89-130.

[2] <u>J.R. Munkres</u> (1963): "Elementary differential topology", Annals of Mathematical Studies, Number 54, Princeton University Press.

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