TESTING HYPOTHESES IN UNBALANCED VARIANCE COMPONENTS MODELS
FOR COMPLETE TWO-WAY LAYOUTS

by

Ib Thomsen
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1. Introduction and summary

Graybill and Hultquist (1961) describe a variance components model as follows: An \((n \times 1)\) vector of observations \(Y\) is assumed to be a linear sum of \(k+2\) quantities,

\[
Y = J \beta_0 + \sum_{i=1}^{k} B_i \beta_i + \beta_{k+1}
\]

Here \(\beta_0\) is a fixed unknown constant. \(\beta_i\) is a \((p_i \times 1)\) vector of multinormally distributed random variables with mean \(0\) and covariance matrix \(\sigma_i^2 I_{p_i}\). \((I_{p_i}\) denotes a \(k\)-dimensional identity matrix and \(0\) a null matrix).

The vectors \(\beta_1, \beta_2, \ldots, \beta_{k+1}\) are stochastically independent. \(J_k\) is a \((k \times 1)\) vector with all elements equal to 1. \(B_i\) \((i = 1, 2, \ldots, k)\) a \((n \times p_i)\) matrix of known constants.

Some general theorems concerning this model have been derived by Graybill and Hultquist (1961) under one or both of the following assumptions

(i) \(A_i\) and \(A_j\) commute, where \(A_i = B_i B_i'\) \((i = 1, 2, \ldots, k)\)

(ii) The matrix \(B_i\) is such that \(J_i' B_i = r_i r_i'\) and \(B_i' J_n = J_n\).

where \(r_i\) is a positive integer.

The assumptions (i) are not satisfied in unbalanced models.

In this paper we will consider a special case of model (1.1) without assumption (i), viz. the common variance components model for a complete two-way layout. Spjøtvoll (1968) has treated the same model in a different manner.

In sections 2 and 3 we shall transform our model to a "semi-canonical" form and find a method for obtaining confidence intervals and testing hypotheses concerning the \(\sigma_i^2\). In section 4 these tests are compared with the corresponding tests in a fixed effects model. In section 5 the test statistics are expressed in terms of the original observations.

2. Modification of the model of Graybill and Hullquist

We consider the following model:

\[
y_{ijk} = \mu + a_i + b_j + \gamma_{ij} + e_{ijk};
\]

\(i = 1, 2, \ldots, r; j = 1, 2, \ldots, s,\) and \(k = 1, 2, \ldots, n_{ij}\). Here \(\mu\) is a constant, while \(a_i, b_j, \gamma_{ij},\) and \(e_{ijk}\) are independent normally distributed random
variables with means 0 and variances $\sigma^2_A$, $\sigma^2_B$, $\sigma^2_{AB}$, and $\sigma^2$, respectively.

Define $\bar{y}_{ij} = (1/n_{ij}) \sum_{k=1}^{n_{ij}} y_{ijk}; i = 1,2,\ldots,r; j = 1,2,\ldots,s$. Then

$$\bar{y}_{ij} = \mu + \bar{a}_i + \bar{b}_j + \bar{c}_{ij} + \bar{e}_{ij}. \tag{2.2}$$

With $\bar{e}_{ij} = (1/n_{ij}) \sum_{k=1}^{n_{ij}} e_{ijk}$.

For any set of variables $a_{ij}$ ($i = 1,2,\ldots,r; j = 1,2,\ldots,s$), let $\bar{a}$ be the vector $(a_{11}, a_{12}, \ldots, a_{1s}, a_{21}, \ldots, a_{rs})'$. Then $\bar{a}$ is multivariate normally distributed with mean 0 and covariance matrix $\Sigma(\bar{a}) = K \sigma^2$, where

$$\Sigma = \text{Diag}(n^{-1}_{11}, n^{-1}_{12}, \ldots, n^{-1}_{rs}). \tag{2.3}$$

Formula (2.2) may be written in matrix form as

$$\tilde{\chi} = J_{rs} \mu + B_1 \bar{a} + B_2 \bar{b} + B_3 \bar{c} + \bar{e}, \tag{2.4}$$

with $B_1 = \begin{pmatrix} J_{ss}, 0, \ldots, 0 \\ 0, J_{ss}, \ldots, 0 \\ \vdots \end{pmatrix}$, $B_2 = \begin{pmatrix} I_{ss} \\ I_{ss} \\ \vdots \\ I_{ss} \end{pmatrix}$, and $B_3 = I_{rs}$, which is of the same form as (1.1). The covariance matrix for $\tilde{\chi}$ turns out as

$$\Sigma(\tilde{\chi}) = B_1 \Sigma_{a} B_1' + B_2 \Sigma_{b} B_2' + B_3 \Sigma_{c} B_3' + K \sigma^2.$$

Lemma 1: $B_1 B_1'$ and $B_2 B_2'$ commute.

Proof: Multiplying $B_1 B_1'$ with $B_2 B_2'$, we get a symmetric matrix. When the product of two symmetric matrices is symmetric, the matrices commute. □
From lemma 1 it follows that there exists an orthogonal matrix \( P \) with the property that \( PA_1P' \) and \( PA_2P' \) are diagonal matrices with the eigenvalues on the diagonal (Herbach, 1959). \( P \) may be chosen so that the first row, in \( \hat{P} \) is \((rs)^{-\frac{1}{2}}(1,1,\ldots,1)\). \((A_1 = B_1B_1'; A_2 = B_2B_2')\).

If \( Z = \hat{P}Y \), the covariance matrix for \( Z \) is

\[
\Sigma(Z) = \hat{P}A_1\hat{P}'\sigma_A^2 + \hat{P}A_2\hat{P}'\sigma_B^2 + \sum_{i=rs}^{r+s} \sigma_{AB}^2 + \frac{\sigma_K^2}{\sigma_A^2} \sigma_A^2.
\]

**Lemma 2**: (i) \( \text{Rank}(\hat{P}_1) = r; \)
(ii) \( \text{Rank}(\hat{P}_2) = s; \)
(iii) \( \text{Rank}(\hat{P}_1\hat{P}_2) = r + s - 1; \)
(iv) \( \text{Rank}(A_1 + A_2) = \text{rank}(B_1B_2'). \)

**Proof**: (i), (ii), and (iii) are seen from (2.4). (iv) follows from the proof of Graybill and Hultquist's (1961) theorem 1. \( \square \)

From the fact that \( \text{rank}(A_1) = \text{rank}(B_1) = r \) and because \( A_1 \) has the eigenvalues \( r \) of multiplicity \( r \) and 0 of multiplicity \( (r + s - r) = r(s - 1) \), it follows that \( PA_1P' \) has \( r \) diagonal elements all equal to \( s \) and the rest equal to 0. In the same way it is seen that \( PA_2P' \) has \( s \) diagonal elements all equal to \( r \) and the other elements equal to 0.

From (iii) and (iv) it is seen that the matrix \( (PA_1P' + PA_2P') \) has \((r + s - 1)\) diagonal elements different from zero. Thus when the diagonal element in \( PA_1P' \) is different from zero, the corresponding element in \( PA_2P' \) is equal to zero except in one place (in the first row).

We now partition \( Z \) in the following way:

(i) \( Z_1 = (rs)^{\frac{1}{2}}y \ldots \), which is the first element in \( Z \).

(ii) \( Z_A \) consists of the \((r - 1)\) elements in \( Z \) whose covariance matrix is independent of \( \sigma_B^2 \).

(iii) \( Z_B \) consists of the \((s - 1)\) elements in \( Z \) whose covariance matrix is independent of \( \sigma_A^2 \).

(iv) \( Z_{AB} \) consists of the \((r - 1)(s - 1)\) elements in \( Z \) whose covariance matrix is independent of \( \sigma_A^2 \) and \( \sigma_B^2 \).
Lemma 3: \( \frac{EZ}{\gamma A} = \frac{EZ}{\gamma B} = \frac{EZ}{\gamma AB} = 0. \)

Proof: This follows from the fact that \( \gamma \) is orthogonal with a first row which is \((rs)^{-1}(1,...,1). \)

We have

\[
\Sigma (Z_{\gamma A}) = s \gamma^{-1} \sigma^2_A + I \gamma^{-1} \sigma^2_{AB} + K_1 \sigma^2,
\]

\[
\Sigma (Z_{\gamma B}) = r \gamma^{-1} \sigma^2_B + I \gamma^{-1} \sigma^2_{AB} + K_2 \sigma^2,
\]

and

\[
\Sigma (Z_{\gamma AB}) = I \gamma^{-1} (r^{-1})(s^{-1}) \sigma^2_{AB} + K_3 \sigma^2.
\]

Here \( K_1, K_2, K_3 \) are the corresponding submatrices of \( PKP'. \)

In what follows, \( Z_A, Z_B \) and \( Z_{AB} \) will be used for testing hypotheses concerning \( \sigma^2_A/\sigma^2, \sigma^2_B/\sigma^2, \) and \( \sigma^2_{AB}/\sigma^2. \)

2.a Test for \( \sigma^2_{AB}/\sigma^2 \)

\( \Sigma (Z_{\gamma AB}) \) may be written as \((I \gamma^{-1} (r^{-1})(s^{-1}) \Delta_{AB} + K_3) \sigma^2, \) where \( \Delta_{AB} = \sigma^2_{AB}/\sigma^2. \)

Then

\[
Q_{AB} = Z_{AB}^!(I \gamma^{-1} (r^{-1})(s^{-1}) \Delta_{AB} + K_3)^{-1} Z_{AB}/\sigma^2
\]

has a \( X^2 \)-distribution with \((r-1)(s-1)\) degrees of freedom. There exists an orthogonal matrix \( A \) such that \( \Delta_{AB} = \Delta_{AB} \) is a diagonal matrix. Introduce \( Z_{AB}^* = A Z_{AB}. \) The covariance matrix for \( Z_{AB}^* \) is \((I \gamma^{-1} (r^{-1})(s^{-1}) \Delta_{AB} + D) \) and

\[
Z_{AB}^!(I \gamma^{-1} (r^{-1})(s^{-1}) \Delta_{AB} + K_3)^{-1} Z_{AB} = Z_{AB}^*(I \gamma^{-1} (r^{-1})(s^{-1}) \Delta_{AB} + D)^{-1} Z_{AB}^*
\]

\[
= \sum_{j=1}^{(r-1)(s-1)} (Z_{AB}^*)^2/(\Delta_{AB} + d_j).
\]

Here \( d_1, \ldots, d_{(r-1)(s-1)} \) are the diagonal elements of \( D. \) We see that \( Q_{AB} \) is a decreasing function of \( \Delta_{AB}. \)

Define \( Q = \sum_{i,j,k} (y_{ijk} - \bar{y}_{ij})^2. \) Then \( Q/\sigma^2 \) has a \( X^2 \)-distribution with \((n-rs)\) degrees of freedom. \( Q \) is stochastically independent of \( Q_{AB}. \) Thus \( F(\Delta_{AB}) = (n-rs) Q_{AB}/(r-1)(s-1) Q \) has an \( F \)-distribution. Since \( Q_{AB} \) decreases with \( \Delta_{AB}, F(\Delta_{AB}) \) decreases with \( \Delta_{AB}. \) Hence a confidence interval can be obtained in the usual way.
When testing the hypothesis

\[ \Delta_{AB} \leq \Delta_0 \text{ against } \Delta_{AB} > \Delta_0, \]

we reject when \( F(\Delta_c) \) is larger than the upper \( \alpha \)-quantile, \( f_{1-\alpha} \), of the corresponding \( F \)-distribution. The power function is

\[ \beta(\Delta_{AB}) = \Pr((n-rs) \left[ \sum_{i=1}^{s} Z_{iAB}^2/(\Delta_0 + d) \right]/[(r-1)(s-1) \nu] > f_{1-\alpha} \]

\[ = \Pr((n-rs) \left[ \sum_{i=1}^{s} (\Delta_{AB} + d) \right]/[(r-1)(s-1)]) > f_{1-\alpha} \]

where \( R_1, \ldots, R_{(r-1)(s-1)} \) are independent \( \chi^2 \)-distributed random variables with 1 degree of freedom. \( \beta(\Delta_{AB}) \) decreases with \( \Delta_{AB} \).

2.b. Test for \( \sigma_A^2/\sigma^2 \) assuming \( \sigma_{AB} = 0 \)

When \( \sigma_{AB} = 0 \) the covariance matrix for \( \begin{pmatrix} Z_A \\ \gamma_A \\ Z_{AB} \end{pmatrix} \) is equal to

\[ \Sigma = \begin{pmatrix} \Sigma_A & 0 & 0 \\ 0 & \gamma_A & 0 \\ 0 & 0 & \gamma_{AB} \end{pmatrix} \]

\[ \Sigma = \sigma_A^2 \begin{pmatrix} I_{(r-1)} & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \nu \end{pmatrix} \]

where \( E[Z_A Z_A'] = \gamma_A \begin{pmatrix} I_{(r-1)} & 0 \\ 0 & \nu \end{pmatrix} \) is positive semi-definite, and \( \begin{pmatrix} K_1 & K_3 \\ K_4 & K_5 \end{pmatrix} \) is positive definite, so we can find a non-singular matrix \( H \) such that

\[ H = \begin{pmatrix} K_1 & K_3 \\ K_4 & K_5 \end{pmatrix}, \quad H' = I_{\nu}, \quad \text{and } H' \begin{pmatrix} I_{(r-1)} & 0 \\ 0 & \nu \end{pmatrix} H = \lambda = \text{diag}(\lambda_1, \ldots, \lambda_{r-1}, 0, \ldots, 0). \]

Define \( U = \begin{pmatrix} Z_A \\ U_{AB} \\ Z_{AB} \end{pmatrix} \). If \( \Delta_A = \sigma_A^2/\sigma^2 \), \( Q_A = U_A^T(\Delta_A A + I_{(r-1)})^{-1} U_A/\nu \)

has a \( \chi^2 \)-distribution with \( (r-1) \) degrees of freedom, and \( Q_{AB} = U_{AB}^T I_{(r-1)(s-1)} U_{AB}/\nu \)

has a \( \chi^2 \)-distribution with \( (r-1)(s-1) \) degrees of freedom. \( Q_A, Q_{AB} \) and \( Q \) are stochastically independent.

To test the hypothesis \( \Delta_A \leq \Delta_0 \) against \( \Delta_A > \Delta_0 \), we reject when

\[ (2.5) \quad G(\Delta_A) = Q_A \{(n-rs) + (r-1)(s-1)\}/(Q + Q_{AB})(r-1) \]

is larger than the upper \( \alpha \)-quantile, \( f_{1-\alpha} \), of the corresponding \( F \)-distribution.
In the same way as above it may be proved that the test is unbiased. A similar test exists concerning \( \sigma_B^2/\sigma^2 \).

3. On the possibility of testing hypotheses concerning \( \sigma_A^2/\sigma^2 \) without assuming \( \sigma_{AB} = 0 \)

In balanced experimental design models we know that

\[
(r-1)(s-1)Z_A^2 \sigma_A^2 + (r-1)\sigma_{AB}^2 + \sigma^2)^{-1} Z_A^2 (r-1)Z_{AB}^2 (r-1)(s-1)\sigma_{AB}^2 + \sigma^2)^{-1} Z_{AB}
\]

is F-distributed. This is not always the case in unbalanced models because \( Z_A \) and \( Z_{AB} \) may not be stochastically independent. Let us now assume that \( Z_A \) and \( Z_{AB} \) are stochastically independent (this may happen even in an unbalanced model). Define two orthogonal matrices \( \mathbf{M}_1 \) and \( \mathbf{M}_2 \) such that

\[
\mathbf{M}_1 \mathbf{M}_1^T = \mathbf{I}_r \quad \text{and} \quad \mathbf{M}_2^T \mathbf{M}_2^T = \mathbf{I}_s
\]

are diagonal. Let \( \mathbf{V}_A = \mathbf{M}_1^T \mathbf{Z}_A \) and \( \mathbf{V}_{AB} = \mathbf{M}_2^T \mathbf{Z}_{AB} \). Then (3.1) may be written as

\[
(r-1)(s-1) \left( \sum_{i=1}^{r-1} \frac{V_{IA}^2}{(s \Delta_A^2 + \Delta_{AB}^2 + \ell_1)} \right) \left( \sum_{j=1}^{s-1} \frac{V_{jAB}^2}{(s \Delta_A^2 + \Delta_{AB}^2 + \ell_2)} \right)
\]

where \( \ell_1 \) and \( \ell_2 \) are the diagonal elements of \( \mathbf{M}_1^T \) and \( \mathbf{M}_2^T \). The quantity in (3.2) has an F-distribution, but the assumption that \( Z_A \) and \( Z_{AB} \) are stochastically independent is not sufficient to give a test for the hypothesis \( \Delta_A = \Delta_0 \) against \( \Delta_A > \Delta_0 \).

In cases where \( \ell_1 = \ell_2 \) for all \( i \) and \( j \), formula (3.2) is reduced to

\[
(r-1)(s-1) \sum_{i=1}^{r-1} \frac{V_{IA}^2}{(s \Delta_A^2 + \Delta_{AB}^2 + \ell_1)} \frac{V_{jAB}^2}{(s \Delta_A^2 + \Delta_{AB}^2 + \ell_2)}
\]

If the null hypothesis is \( \Delta_A = 0 \), we have that \( g(\Delta_A) = (s-1)(r-1) \sum \frac{V_{iA}^2}{(s-1)(r-1)} \sum \frac{V_{jAB}^2}{(s-1)(r-1)} \) is F-distributed under the null hypothesis. Hence we reject if \( g(0) \) is larger than the upper \( \alpha \)-quantile of the corresponding F-distribution.

In the case \( r = s = 2 \) assumption (3.2) is always fullfilled.
4. Comparison with corresponding tests in fixed effects models

A two-way layout in fixed effects models may be described as

\[ y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk}; \]

\( i = 1,2,\ldots,r; \ j = 1,2,\ldots,s; \ k = 1,2,\ldots,n_{ij}, \)

where \( \mu, \alpha_i, \beta_j, \) and \( \gamma_{ij} \) are unknown constants such that

\[ \sum_{i} \alpha_i = \Sigma \beta_j = \Sigma \gamma_{ij} = 0, \]

and the \( e_{ijk} \) have a joint normal distribution with mean 0 and covariance matrix \( \Sigma^2. \)

The null hypothesis \( \gamma_{ij} = 0 \) \((i = 1,2,\ldots,r; \ j = 1,2,\ldots,s)\) is tested by minimizing the sum of squares

\[ Q = \sum_{i,j,k} (y_{ijk} - \mu - \alpha_i - \beta_j - \gamma_{ij})^2 \]

under the null hypothesis and under the a priori specifications. Let the two minima of \( Q \) be \( Q_w \) and \( Q_\Omega \), respectively. The null hypothesis is rejected when

\[ (Q_w - Q_\Omega)/(n-rs)/Q_\Omega(r-1)(s-1) \]

is larger than the upper \( \alpha \)-quantile \( f_{1-\alpha} \) of the corresponding F-distribution.

We will prove that the quantity in (4.2) is equal to the test-statistic \( F(0) \) in section 2a.

If as in section 2 we introduce \( \bar{y} \) we have that

\[ \bar{y} = \bar{y} + \begin{bmatrix} \bar{\alpha}_1 \\ \bar{\beta}_1 \\ \bar{\gamma}_{1j} \\ \bar{e}_{1j} \end{bmatrix} = \begin{bmatrix} \bar{\alpha}_1 \\ \bar{\beta}_1 \\ \bar{\gamma}_{1j} \\ \bar{e}_{1j} \end{bmatrix}. \]

The only difference from the random effects model (2.4) is that \( \alpha_i, \beta_j, \) and \( \gamma_{ij} \) here are fixed constants with the side conditions (4.1). We write the side conditions in the form

\[ \bar{\alpha}_r = - \sum_{i=1}^{r-1} \alpha_i; \bar{\alpha}_s = - \sum_{j=1}^{s-1} \beta_j; \]

\[ \bar{\gamma}_{is} = - \sum_{j=1}^{s-1} \gamma_{ij}; \bar{\gamma}_{sj} = - \sum_{i=1}^{r-1} \gamma_{ij}; \]

and

\[ \bar{\gamma}_{rs} = \sum_{i=1}^{r-1} \sum_{j=1}^{s-1} \gamma_{ij}. \]

The (4.3) takes the form

\[ \bar{y} = \bar{y} + \begin{bmatrix} \bar{\alpha}_r \\ \bar{\beta}_r \\ \bar{\gamma}_{rs} \\ \bar{e}_r \end{bmatrix}. \]
where $\mathbf{a}^x = (a_1, \ldots, a_{r-1})'$, $\mathbf{b}^x = (b_1, \ldots, b_{s-1})'$, $\mathbf{y}^x = (y_1, \ldots, y_{(r-1)(s-1)})'$; $\mathbf{Z}$ is a quadratic, non-singular $(rs \times rs)$-matrix and $\mathbf{e}$ is normally distributed with mean $\mathbf{0}$ and covariance matrix $\mathbf{K}^2$, with $\mathbf{K}$ given as above (2.3). (It is possible to write (4.1) in several other ways. This will lead to formally different $\mathbf{Z}$ matrices, and formally different $\mathbf{a}^x$, $\mathbf{b}^x$ and $\mathbf{y}^x$ in (4.5)). Define $\mathbf{V} = \mathbf{K}^{-1/2} \mathbf{Y}$. Then

\[
\mathbf{V} = \mathbf{K}^{-1/2} \mathbf{Y} = \begin{bmatrix}
\mathbf{a}^x
\mathbf{b}^x
\mathbf{y}^x
\end{bmatrix} + \mathbf{e}^x,
\]

where $\mathbf{e}^x$ is normally distributed with mean $\mathbf{0}$ and covariance matrix $\mathbf{I}_{rs} \sigma^2$.

The form (4.6) is very convenient because to minimize $Q$ is equivalent to minimize $(\mathbf{V} - \mathbf{EY})'(\mathbf{V} - \mathbf{EY})$. This is seen as follows: With the side conditions (4.4) on the parameters, $Q$ may be written

\[
Q = \sum_{i,j,k} (y_{ijk} - y_{ij})^2 + \sum_{i=1}^{r-1} \sum_{j=1}^{s-1} \sum_{k=1}^{s-1} n_{ijk} (y_{ijk} - \mu - a_i - b_j - y_{ij})^2 +
\]

\[
\sum_{j=1}^{s-1} \sum_{i=1}^{r-1} \sum_{k=1}^{s-1} n_{rj} (y_{rj} - \mu + a_i - b_j + y_{ij})^2 +
\]

\[
\sum_{i=1}^{r-1} \sum_{j=1}^{s-1} \sum_{k=1}^{s-1} n_{is} (y_{is} - \mu - a_i + b_j + y_{ij})^2 +
\]

\[
\sum_{i=1}^{r-1} \sum_{j=1}^{s-1} \sum_{k=1}^{s-1} n_{rs} (y_{rs} - \mu + a_i + b_j + y_{ij})^2
\]

The part of $Q$ which depends on the parameters, equals

\[
Q_p = (\mathbf{V} - \mathbf{EY})'(\mathbf{V} - \mathbf{EY}).
\]

The minimum of $Q$ is then equal to the minimum of $Q_p$ plus $\sum_{i,j,k} (y_{ijk} - y_{ij})^2$.

Define $Q_{p^a}$ and $Q_{p^0}$ as the minima of $Q_p$ under the a priori specifications and under the null hypothesis, respectively. We then have

**Lemma 4:** $Q_{p^0} - Q_{p^a} = Q_{p^0} - Q_{p^0}$.

The a priori specifications are (4.4), and the null hypothesis is $y_{ij} = 0$ ($i = 1, 2, \ldots, r-1$; $j = 1, 2, \ldots, s-1$)
From the general theory for linear models we know that

\[(4.9) \quad Q_{pw} - Q_{p0} = \hat{\gamma}^X (\Sigma_y)^{-1} \hat{\gamma}^X,\]

where \(\hat{\gamma}^X\) is the least squares estimate for \(\gamma^X\), and \(\Sigma_y\) is the covariance matrix for \(\gamma^X\).

The least squares estimate for \(Q^X\) is

\[
\begin{pmatrix}
\mu \\
\alpha^X \\
\beta^X \\
\gamma^X
\end{pmatrix} = (Z' \Sigma^{-1} \Sigma^{-1} Z)^{-1} Z \Sigma^{-1} \gamma,
\]

which reduces to

\[
\begin{pmatrix}
\mu \\
\alpha^X \\
\beta^X \\
\gamma^X
\end{pmatrix} = Z^{-1} \gamma.
\]

The covariance matrix for this estimator is \(\Sigma = (Z' \Sigma^{-1} \Sigma^{-1} Z)^{-1} \sigma^2\).

By introducing the transformation \(P\), where \(P\) is the orthogonal matrix with which the cell mean values were transformed in the corresponding random effect model, we will now prove that \(Q_{pw} - Q_{p0}\) is independent of the choice of \(Z, \alpha^X, \beta^X, \gamma^X\) and that \(\sigma^2(Q_{pw} - Q_{p0}) = Q_{AB}\) when \(\Delta_{AB} = 0\), where \(Q_{AB}\) is defined as in section 2.

The following lemma is useful:

Lemma 5: Partition \(Z\) into submatrices corresponding to the partitioning \((\hat{\mu}, \hat{\alpha}^X, \hat{\beta}^X, \hat{\gamma}^X)'\). Thus

\[
Z = \begin{bmatrix}
J_{rs} & Z_{1} (rs \times (r-1)) & Z_{2} (rs \times (s-1)) & Z_{3} (rs \times (r-1)(s-1))
\end{bmatrix}.
\]

Partition \(P\) likewise into

\[
P = \begin{bmatrix}
P_{11} (l \times rs) & P_{12} (l \times (r-1) \times rs) \\
P_{21} (l \times rs) & P_{22} ((s-1) \times rs) \\
P_{31} (l \times (r-1) \times rs) & P_{32} ((s-1) \times rs) \\
P_{41} (l \times rs) & P_{42} ((s-1) \times rs)
\end{bmatrix}.
\]
For any choice of $Z$ we then have:

(i) The rows of $P_2$ are orthogonal to the columns in $Z_2$.

(ii) The rows of $P_3$ are orthogonal to the columns in $Z_1$.

(iii) The rows of $P_4$ are orthogonal to the columns in $Z_1$ and $Z_2$.

Proof: By section 2 we can find a matrix $P$ such that $PA_1 P' = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}$ and $PA_2 P' = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}$. By the partitioning of $P$ introduced in the proof of lemma 3, $P_1 B_1 B_1 P_1 = S_1$, $P_2 B_2 B_2 P_2 = S_2$, $P_3 B_3 B_3 P_3 = S_3$, $P_4 B_4 B_4 P_4 = S_4$. By the partitioning of $P$ introduced in the proof of lemma 3, $P_1 B_1 B_1 P_1 = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}$, $P_2 B_2 B_2 P_2 = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}$, $P_3 B_3 B_3 P_3 = \begin{bmatrix} S_3 & 0 \\ 0 & S_4 \end{bmatrix}$, $P_4 B_4 B_4 P_4 = \begin{bmatrix} S_3 & 0 \\ 0 & S_4 \end{bmatrix}$.

It is always possible to find matrices $A$, $B$, $C$ such that $\beta^{r \times 1} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \gamma^{r \times 1}$, $\gamma^{s \times 1} = \begin{bmatrix} 0 & B \end{bmatrix} \gamma^{s \times 1}$. Formula (2.4) may now be written

$$\begin{bmatrix} r \times 1 \\ s \times 1 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \gamma^{r \times 1} + \begin{bmatrix} 0 & C \end{bmatrix} \gamma^{s \times 1}$$

$B_1 A$ and $B_2 B$ equal $Z_1$ and $Z_2$ in lemma 5, respectively, and $C$ equals $Z_2$. The columns in $B_1 A$ are linear combinations of the columns in $B_1$, so that $\mathcal{C}(B_1 A) \subset \mathcal{C}(B_1)$, where $\mathcal{C}(U)$ denotes the vector space spanned by the columns in any matrix $U$.

Thus $\mathcal{C}(Z_1) \subset \mathcal{C}(B_1)$ and $\mathcal{C}(Z_2) \subset \mathcal{C}(B_2)$. Then since $P_2 B_2 B_2 P_2 = 0$, $P_2 B_2 = 0$ and thus $P_2 P_2 = 0$, so the rows in $P_2$ are orthogonal to the columns in $Z_2$. The rest of the lemma now follows by treating $P_3$ and $P_4$ in a similar way.

Because $P_2 J = P_3 J = P_4 J = 0$, it follows by lemma 5 that $PZ$ has the form

$$PZ = \begin{bmatrix} S_1 & 0 & 0 & 0 \\ 0 & S_3 & 0 & 0 \\ 0 & 0 & S_5 & 0 \\ 0 & 0 & 0 & S_7 \end{bmatrix}$$.
We then see that \((PZ)^{-1}\) also is a triangular matrix with zeroes to the left of the diagonal. The \((r-1)(s-1) \times (r-1)(s-1)\) submatrix in the lower, right hand corner of \((PZ)^{-1}\) equals \((P_4 Z_3)^{-1}\).

Introduce \(P\) into the expression for the least squares estimate and its covariance matrix, we obtain:

\[
\begin{bmatrix}
\hat{\alpha} \\
\hat{\beta} \\
\hat{\gamma}
\end{bmatrix} = \Sigma^{-1} \bar{y} = (PZ)^{-1} P \bar{y}
\]

and \(\Sigma = (\xi' P_{\nu} (P_4 Z_3)^{-1} \xi')^{-1} \sigma^2 = (P_4 Z_3)^{-1} P_{\nu} P' (P_4 Z_3)^{-1} \sigma^2\). From what we found about \((PZ)^{-1}\), it follows that the \((r-1)(s-1)\) lower elements of \((PZ)^{-1}\) are \(\gamma_{ij} = (P_4 Z_3)^{-1} P_{\nu} \gamma_{ij}' (P_4 Z_3)^{-1}\), where \((P_4 P_4')_4\) is the \((r-1)(s-1) + (r-1)(s-1)\) submatrix in the lower right hand corner of \(P K P'\). (4.9) may then be written in the form

\[
\bar{y}' P_4 (P_4 Z_3)^{-1} (P_4 Z_3)^{-1} P_4 \bar{y} \sigma^2
\]

(4.10) \(= \bar{y}' P_4 (P K P')_4 (P_4 Z_3)^{-1} P_4 \bar{y} \sigma^2\).

This quadratic form is independent of \(Z_{\nu} \gamma_{ij}' \), \(\beta_j\) and \(\gamma_{ij}'\), and is the same as \(Q_{AB}\) in (2.4) when \(\Delta_{AB} = 0\), because \(Z_{\nu} \gamma_{ij}' = P_{\nu} \gamma_{ij}'\) and \(K_3 = (P K P')_4\). We have then proved that \((n-rs)(Q_0 - Q_1)/Q_1(r-1)(s-1) = F(0)\).

5. The test statistics expressed by the original observations

**Lemma 6:** With the choice of \(Z\) made in section 4, the least squares estimates for \((\mu, \alpha_{ij}', \beta_j, \gamma_{ij}')\) are \(\hat{\mu} = y\ldots\{\hat{\alpha}_{i} = \{y_{i.} - y\ldots\, \}, \{\hat{\beta}_j = \{y_{..j} - y\ldots\, \}, \{\hat{\gamma}_{ij}' = \{y_{ij} - y_{i.} - y_{.j} + y\ldots\, \, \} (i = 1, 2, \ldots, r-1; j = 1, 2, \ldots, s-1).\)

**Proof:** If we insert \(\mu, \{\alpha_{ij}\}, \{\beta_j\}\) and \(\{\gamma_{ij}\}\) for \(\mu, \{a_i\}, \{b_j\}\) and \(\gamma_{ij}\) in (4.7), \(Q\) reduces to \(\Sigma_{i,j,k} (y_{ijk} - y_{ij})^2\).

When testing the null hypothesis \(\Delta_{AB} < 0\) against \(\Delta_{AB} > 0\), we reject when

\[
(n-rs) (\Sigma_{ij,k} (\Sigma_{ij,k})^{-1} \Sigma_{ij,k} (y_{ijk} - y_{ij})^2 (r-1)(s-1)
\]

is larger than the upper -quantile of the corresponding F-distribution. This test is the same as the one suggested by Spjøtvoll (1968).

It should be noted that the test statistic reduces to the usual one when the model is balanced.
References


