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The linear cost model previously formalized by Hald [4], [5], [9] is reviewed. Techniques are described which permit easy determination of sampling plans based on that model. The degenerate, the two point, and the beta distributions are considered as prior distributions for p, the process fraction defective. For calculations only standard tables and a desk calculator are required.

1. INTRODUCTION

In recent years a number of papers have appeared concerned with sampling inspection models which are constructed under the assumption that both costs and a prior distribution of p, the process fraction defective, should be incorporated into the model. Hald [4], [5], [9] has done extensive work in this area and is responsible for much of the notation which we will use. The contribution of this paper is the presentation of elementary procedures by which such sampling plans can be determined and through which properties of such plans can be investigated.

In our discussions and evaluations we will need some of the well known discrete probability functions. For a binomial random variable X we will use the notation

$$b(x;n,p) = \binom{n}{x}p^x(1-p)^{n-x} \quad , \quad x = 0,1,2,\ldots,n \tag{1.1}$$

for the probability function and cumulative sums will be denoted by

$$B(r;n,p) = \sum_{x=0}^{r} b(x;n,p) \tag{1.2}$$

and

$$E(r;n,p) = \sum_{x=r}^{n} b(x;n,p) \tag{1.3}$$

The most convenient table to use is the one published by the Ordnance Corps [16] which gives (1.3) to seven decimal places for $p = .01(.01).50$ and $n = 1(1)\ 150$. The Harvard [12] table gives the same sum to five decemal places for n's up to 1000 but in jumps ranging from 2 units between $n = 50$ and $n = 100$ to 50 units between $n = 500$ and $n = 1000$. The probability function of a hypergeometric random variable will be denoted by

$$p(N,n,k,x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} \ , \quad a \leqq x \leqq b \tag{1.4}$$

where $a = \max[0,n-N+k]$, $b = \min[k,n]$ and cumulative sums by

$$P(N,n,k,r) = \sum_{x=a}^{r} p(N,n,k,x) \tag{1.5}$$

Lieberman and Owen [13] have given both (1.4) and (1.5) to six decimal places for $N = 1(1)50(10)100$. For other N we use

$$P(N,n,k,r) \cong B(r;n,\tfrac{k}{N}) \tag{1.6}$$

if $n \leqq k$, $n/N \leqq .10$ and

$$P(N,n,k,r) \cong B(r;k,\tfrac{n}{N}) \tag{1.7}$$

if $k < n$, $k/N \leqq .10$,

approximations which appear to be quite adequate. For the Poisson probability function we use

$$p(x;\mu) = \frac{e^{-\mu}\mu^{x}}{x!} \ , \quad x = 0,1,2,\ldots \tag{1.8}$$

and for cumulative sums

$$E(r;\mu) = \sum_{x=r}^{\infty} p(x;\mu) \tag{1.9}$$

Both Molina [14] and General Electric [3] have tabulated (1.8) and (1.9) to at least six decimal places for a considerable range of μ . The former table is probably better if $\mu > 2$ and the latter if $\mu \leqq 2$. Finally, we also need the negative binomial probability function

$$f(x;k,p) = \frac{\Gamma(x+k)}{x!\Gamma(k)} p^k(1-p)^x , \quad x = 0,1,2,... \tag{1.10}$$

with cumulative sums

$$F(r;k,p) = \sum_{x=0}^{r} f(x;k,p) \tag{1.11}$$

Williamson and Bretherton [17] have given both (1.10) and (1.11) to six decimal places for p mostly in steps of .02 and k mostly in steps of .1 , a table that is very adequate for our purposes.

Terminology for single sampling attribute plans is fairly standard. Items produced by a process are assembled at random into lots of size N . From each lot a random sample of size n is selected and X , the number of defective items in the sample, is observed. If $x \leqq c$, where c is called the acceptance number, the lot is accepted and if $x > c$ the lot is rejected. One course of action which is frequently followed calls for replacement of defective items found in sampling accepted lots and for total inspection of rejected lots with all defectives being replaced.

Let Y be the number of defective items in the lot. We assume that the behavior of Y is governed by a binomial distribution with parameters N and p . A given lot will have a fixed number of defectives, say $Y = y = k$, and the probability function of X given k is the hypergeometric (1.4). Further, if we let $U = Y - X$ be the number of defective items in the remaining $N - n$ items in a lot, it is well known (i.e., see Hald [3], p.401-402) that the unconditional distributions of X and U are independent and binomially distributed with parameters n , p and $N - n$, p respectively.

The probability of accepting a lot is called the operating characteristic (OC) and is

$$OC = P(N,n,k,c) \tag{1.12}$$

a sum of hypergeometric probabilities. Since (1.12) is a conditional probability depending upon a given k , we may be more interested in the average operating characteristic (AOC), the expected value of (1.12) taken over Y , given by

$$AOC = B(c;n,p) \tag{1.13}$$

From a practical standpoint usually it makes little difference which view is adoped since, according to (1.6), (1.13) approximates (1.12) with $p = k/N$.

2. THE LINEAR COST MODEL

We will assume that associated with a sampling plan is a cost function which can be expressed in the form

$$h(x;k;N,n,c,p) = \begin{cases} nS_1+xS_2+(N-n)A_1+(k-x)A_2 \ , & x \leqq c \\ nS_1+xS_2+(N-n)R_1+(k-x)R_2 \ , & x > c \end{cases} \tag{2.1}$$

Hald [4] has suggested the following interpretation for the constants:

S_1 Cost per item for sampling and testing

S_2 Repair cost for a defective item found in sampling

A_1 Cost per item associated with handling the $N - n$ items not inspected in an accepted lot (frequently is 0)

A_2 Cost associated with a defective item which is accepted (may be quite large)

R_1 Cost per item of inspecting the remaining $N - n$ items in a rejected lot

R_2 Repair cost associated with a defective item in the remaining $N - n$ items of a rejected lot

Logically we would expect that $S_1 \geqq R_1$, $S_2 \geqq R_2$ (with equality frequently holding) since it should be no more expensive to sample or repair on a large scale than on a small scale.

As an example suppose that a manufacturer of boys jackets assembles lots of size 100 for shipment. An item is regarded as a defective if it is judged to require repair before it would be acceptable to a customer. Let us assume that $R_1 = S_1 = .10$, $R_2 = S_2 = 2.00$, $A_1 = 0$, $A_2 = 4.00$, figures determined on the basis of accounting records. The cost A_2 arises from the fact that the buyers also use sampling inspection procedures and may return individual items or entire lots.

The plans which we intend to discuss are based upon the average cost per lot. To obtain that average we take the expected value of the random variable associated with (2.1). Recall that U and X are independent so that $E(U|x) = (N-n)p$. Hence if we take the conditional expectation with respect to U given x followed by taking the expectation with respect to X we get (for fixed p) that the average cost is

$$K(N,n,c,p) = nK_s(p) + (N-n)[K_a(p)P(p) + K_r(p)Q(p)] \qquad (2.2)$$

where $K_s(p) = S_1 + S_2p$, $K_a(p) = A_1 + A_2p$, $K_r(p) = R_1 + R_2p$, $P(p) = B(c;n,p)$, $Q(p) = 1 - P(p)$. Alternatively we may write (2.2) using only one binomial sum as

$$K(N,n,c,p) = nK_s(p) + (N-n)\{K_a(p) + [K_r(p)-K_a(p)]Q(p)\} \qquad (2.3)$$

If $K_r(p) - K_a(p) = R_1 - A_1 - (A_2-R_2)p = 0$ has a solution in the interval (0,1), denote that solution by p_r . We note that if

$$\begin{array}{l} p < p_r \ , \ K_r(p) - K_a(p) > 0 \\ p > p_r \ , \ K_r(p) - K_a(p) < 0 \end{array} \qquad (2.4)$$

3. PRIOR DISTRIBUTION OF p DEGENERATE

We first consider the case in which the distribution of p concentrates all its probability at one point. In other words,

p does not really have a distribution but assumes some unknown value with probability 1 (the situation of classical statistics). If in (2.2) we let $R_1 = S_1 = 1$ and all other cost constants are set equal to zero, then

$$K(N,n,c,p) = n + (N-n)E(c+1;n,p) \tag{3.1}$$

the cost function for rectifying inspection proposed by Dodge and Romig [1], [2] as early as 1929. Hald [7] has already mentioned the generalization of (3.1) considered in this section.

If we knew p, then we could always minimize (2.2). If $p < p_r$, then we would always accept without sampling so that $P(p) = 1$, $n = 0$ and

$$K(N,n,c,p) = NK_a(p) \tag{3.2}$$

On the other hand if $p > p_r$ we would always reject without sampling so that $P(p) = 0$, $n = 0$ and

$$K(N,n,c,p) = NK_r(p) \tag{3.3}$$

Since p is unknown, a reasonable type problem is to minimize $K(N,n,c,\overline{p})$, where $\overline{p}$ is our guess for the true p, subject to one or two conditions on the OC (or AOC) curve. For example, we could require

$$P(N,n,k_1,c) \leqq \beta_1 \tag{3.4}$$

(as Dodge and Romig have done) or

$$P(N,n,k_0,c) \geqq 1 - \alpha_0 \tag{3.5}$$

or both (3.4) and (3.5).

For the minimization procedure it is convenient to rewrite (2.2) as

$$K(N,n,c,p) = n[K_s(p)-K_r(p)] + (N-n)[K_a(p) - K_r(p)]P(p) + NK_r(p) \tag{3.6}$$

If $R_1 = S_1$, $R_2 = S_2$ so that $K_s(p) - K_r(p) \equiv 0$, then let

$$R(N,n,c,p) = [K(N,n,c,p) - NK_r(p)]/[K_a(p)-K_r(p)] \tag{3.7}$$

$$= (N-n)P(p) \tag{3.8}$$

To minimize (3.6), maximize or minimize (3.8) according to whether the denominator of (3.7) is negative or positive. If either $S_1 > R_1$ or $S_2 > R_2$ (or both), let

$$R(N,n,c,p) = [K(N,n,c,p) - NK_r(p)]/[K_s(p)-K_r(p)] \tag{3.9}$$

$$= n + (N-n)\gamma P(p) \tag{3.10}$$

where $\gamma = [K_a(p)-K_r(p)]/[K_s(p)-K_r(p)]$ and minimize (3.10). In all situations either the minimum is obvious or minimization is easily accomplished by first finding the minimum with $c = 0$, then the minimum with $c = 1$, then with $c = 2$, etc. terminating when it is obvious that the absolute minimum has been found. We illustrate by examples.

Example 3.1

Using the cost constants of the jacket example of Section 2, minimize the average cost when $\bar{p} = .02$ subject to the condition (a) OC $\leqq .10$ if $k_1 = Np_1 = 10$, (b) OC $\geqq .95$ if $k_0 = Np_0 = 1$, (c) both (a) and (b).

Solution

(a) Since $R_1 = S_1 = .10$, $R_2 = S_2 = 2.00$, $K_a(.02) - K_r(.02) = -.10 + 2(.02) = -.06$, we maximize $R(100,n,c,.02)$ $= (100-n)[1-E(c+1;n,.02)]$ subject to $P(100,n,10,c) \leqq .10$. Using the Lieberman and Owen [13] table we observe that the side condition is satisfied if $n \geqq 20$ with $c = 0$, if $n \geqq 33$ with $c = 1$, if $n \geqq 44$ with $c = 2$, if $n \geqq 55$ with $c = 3$. With each value of c we need consider only the smallest value of n since both $100 - n$ and $[1-E(c+1;n,.02)]$ decrease with increasing n.

Using the Ordnance Corps [16] table for binomial sums we obtain

$R(100,20,0,.02) = 80(.66761) = 53.41$
$R(100,33,1,.02) = 67(.85917) = 57.56$
$R(100,44,2,.02) = 56(.94223) = 52.76$
$R(100,55,3,.02) = 45(.97567) = 43.91$

and the desired plan is $n = 33$, $c = 1$ with
$K(100,33,1,.02) = (-.06)(57.56) + 100(.14) = 10.55$

(b) The side condition $P(100,n,1,c) \geqq .95$ is satisfied for $n \leqq 5$ with $c = 0$ and for all $n \leqq 100$ with $c \geqq 1$ (obviously). Now $R(100,n,c,.02)$ is maximized if we always accept with no sampling and $K(100,0,c,.02) = 8.00$.

(c) With both conditions we see from (a) and (b) that solutions are possible for $c = 1$, $33 \leqq n \leqq 100$, for $c = 2$, $44 \leqq n \leqq 100$, for $c = 3$, $55 \leqq n \leqq 100$ and the desired plan is $n = 33$, $c = 1$.

Example 3.2

Repeat Example 3.1 with $N = 1000$

Solution

(a) Now the side condition $P(1000,n,100,c) \leqq .10$ is expressed in terms of the binomial approximation obtaining $E(c+1;n,.10) \geqq .90$. From the Ordnance Corps table we find by observation that $n \geqq 22$ with $c = 0$, $n \geqq 38$ with $c = 1$, $n \geqq 52$ with $c = 2$, $n \geqq 65$ with $c = 3$, $n \geqq 78$ with $c = 4$, $n \geqq 91$ with $c = 5$, etc. Then

$R(1000,22,0,.02) = 978(.64117) = 627.1$

$R(1000,38,1,.02) = 962(.82397) = 792.7$

$R(1000,52,2,.02) = 948(.91407) = 866.6$

$R(1000,65,3,.02) = 935(.95862) = 896.3$

$R(1000,78,4,.02) = 922(.97972) = 903.3$

$R(1000,91,5,.02) = 909(.98994) = 899.9$

and the desired plan is $n = 78$, $c = 4$ with
$K(1000,78,4,.02) = 95.80$.

(b) The side condition $P(1000,n,10,c) \geqq .95$ is replaced by $E(c+1;n,.01) \leqq .05$ an inequality satisfied for $n \leqq 5$

with $c = 0$, $n \leqq 35$ with $c = 1$, $n \leqq 82$ with $c = 2$. However, once more we accept with no sampling and $K(1000,0,c,.02) = 80$.

(c) It is impossible to satisfy both conditions with $c = 0$ $(n \leqq 5, n \geqq 22)$ and with $c = 1$ $(n \leqq 35, n \geqq 38)$. With $c = 2$ we can have $52 \leqq n \leqq 82$ and in that interval $n = 52$ yields the minimum. Since $K(1000,n,c,p)$ will be minimized at the smallest n in each interval it is not necessary to find an upper bound for $c = 3,4,$etc. and the desired solution is again the same as in (a).

Example 3.3

Rework part (a) of Example 1 if S_1 is increased to .15, S_2 is increased to 2.5 and all other constants remain the same.

Solution

Now we use (3.10) with $\gamma = -1$ when $p = .02$. Thus we minimize $R(100,n,c,.02) = n - (100-n)[1-E(c+1;n,.02)]$. We find

$R(100,20,0,.02) = 20 - 80(.66761) = -35.41$

$R(100,33,1,.02) = 33 - 67(.85916) = -24.56$

$R(100,44,2,.02) = 44 - 56(.94223) = -8.76$

and the desired plan is $n = 20$, $c = 0$ with

$K(100,20,0,.02) = .06(-35.41) + 100(.14) = 11.88$

4. THE TWO POINT PRIOR DISTRIBUTION

In many situations it is probably realistic to assume that values of p are determined according to some probability distribution. For example, suppose that a machine is used to produce a particular item. After every lot the machine setting is checked and reset at the correct value. We could assume that during the production of a particular lot p remains constant (and the other binomial conditions are satisfied) but for each lot p is determined according to a prior distribution. A simple prior distribution which has been given considerable attention by Hald [6], [7], [8], [9] is the two point prior with probability function

$$f(p) = w_1 \quad , \; p = p_1 \qquad (4.1)$$
$$\quad = w_2 \quad , \; p = p_2$$

where $w_2 = 1 - w_1$ and the w's and p's are assumed to be known.

The prior (4.1) is admittedly difficult to justify in practical situations and the beta prior considered in Section 5 has considerably more appeal. Let us attempt to obtain a setting for the current model in terms of the jacket example. Suppose that three shifts, two day shifts and one night shift, have made the same number of lots. The lots have been mixed together at random in a warehouse. Later it was determined that the day shifts make jackets of which .01 are defective on the average while the night shift has .10 on the average of their items defective. Under these assumptions the prior distribution of p is

$$f(p) = 2/3 \quad , \; p = .01 \qquad (4.2)$$
$$\quad = 1/3 \quad , \; p = .10$$

Now that p has a distribution (2.2) generates a random variable with an average value, say $K(N,n,c)$. If we let

$$K_s = w_1 K_s(p_1) + w_2 K_s(p_2) \qquad (4.3)$$
$$K_m = w_1 K_a(p_1) + w_2 K_r(p_2) \qquad (4.4)$$

and define

$$R(N,n,c) = [K(N,n,c) - NK_m]/(K_s - K_m) \qquad (4.5)$$

then after a little algebra, we get Hald's [i.e., see 9] "standardized" form

$$R(N,n,c) = n + (N-n)[\gamma_1 Q(p_1) + \gamma_2 P(p_2)] \qquad (4.6)$$

where

$$\gamma_1 = w_1[K_r(p_1) - K_a(p_1)]/(K_s-K_m) \tag{4.7}$$

$$\gamma_2 = w_2[K_a(p_2) - K_r(p_2)]/(K_s-K_m) \tag{4.8}$$

For our purposes the advantage of using $R(N,n,c)$ rather than $K(N,n,c)$ is that less writing and calculating are involved in the minimization procedure.

There are, of course, different types of sampling plans which can be determined incorporating the average cost function $K(N,n,c)$. If n and c are determined so that $K(N,n,c)$ is a minimum, then (n,c) is called a Bayesian sampling plan. We shall also consider plans determined by minimizing $K(N,n,c)$ under one or two side restrictions (on either the OC or AOC curve). The latter type sampling plans are called restricted Bayesian plans.

We first consider the Bayesian case. If $p_1 < p_r < p_2$ (see (2.4)), it can be shown that $\gamma_1 > 0$, $\gamma_2 > 0$. Under these circumstances $R(N,n,c)$ may be minimized by always rejecting without sampling in which case

$$R(N,n,c) = N\gamma_1 \tag{4.9}$$

or by always accepting without sampling so that

$$R(N,n,c) = N\gamma_2 \tag{4.10}$$

or by finding a pair (n,c) which makes (4.6) as small as possible. The procedure for minimizing (4.6) consists of determining the smallest $R(N,n,0)$, then the smallest $R(N,n,1)$, then the smallest $R(N,n,2)$, etc. terminating when the absolute minimum has been found. If $p_r < p_1$ or $p_r > p_2$, it can be shown that $R(N,n,c)$ is minimized by (4.9) and (4.10) respectively.

Before we demonstrate the minimization process with examples, several comments seem to be in order. Extensive numerical investigation conducted by Hald [6] indicates that for fixed c

and N $R(N,0,c) = \min(N\gamma_1, N\gamma_2)$ and either $R(N,n,c)$ (a) is a monotonic increasing function of n, (b) rises to a relative maximum, then drops to a relative minimum below $\min(N\gamma_1, N\gamma_2)$, then increases monotonically, or (c) behaves as in (b) except that the relative minimum is above $\min(N\gamma_1, N\gamma_2)$. Further, these minima over n when plotted against c have a unique minimum which may or may not occur at $c = 0$. Our calculations verify these condusions. It can be added that any such "curves" (actually a set of isolated points) can be drawn as accurately as desired by using more values of n or c, whichever is appropriate.

Example 4.1

Assuming that the two point prior (4.2) is appropriate find the Bayesian sample plan for the jacket example.

Solution

We have $K_r(p) = .10 + 2p$, $K_a(p) = 4p$, and $K_r(p) - K_a(p) = .10 - 2p = 0$ has a solution $p_r = .05$ between $p_1 = .01$ and $p_2 = .10$. After a little arithmetic we find $R(100,n,c) = n + (100-n)[E(c+1;n,.01) + .625B(c;n,.10)]$ and $\min[N\gamma_1, N\gamma_2] = 62.5$.

With $c = 0$ we obtain

$R(100,10,0) = 10 + 90[.09562 + .625(.34868)] = 38.22$

$R(100,15,0) = 15 + 85[.13994 + .625(.20589)] = 37.88$

$R(100,20,0) = 20 + 80[.18209 + .625(.12158)] = 40.65$

so that the minimum has been cornered. Further calculations yield $R(100,14,0) = 37.58$, $R(100,13,0) = 37.48$, $R(100,12,0)=37.53$ and $n = 13$ **yields the minimum with** $c = 0$.

With $c = 1$ we find $R(100,20,1) = 40.94$, $R(100,25,1) = 39.65$, $R(100,30,1) = 40.57$, $R(100,24,1) = 39.71$, $R(100,26,1) = 39.67$ and $n = 25$ yields the minimum with $c = 1$.

Since the minima are increasing we can terminate calculations. The desired plan is $n = 13$, $c = 0$ with

$K(100,13,0) = (.16/3)(37.48) + 100(.38/3) = 14.67$

In case we are interested we quickly find the OC at $p = .01$ and $p = .10$. We get if

$k_1 = .01(100)$, $OC = P(100,13,1,0) = .870000$

$k_2 = .10(100)$, $OC = P(100,13,10,0) = .231120$

Example 4.2

If $p_1 = .01$, $p_2 = .05$, $w_1 = .85$, $w_2 = .15$, $S_1 = .40$, $S_2=0$, $R_1 = .30$, $R_2 = 0 = A_1$, $A_2 = 10.00$ (Hald's [9] figures), find the Bayesian sample plan for $N = 1000$.

Solution

We verify that $p_r = .03$, $\gamma_1 = .6296$, $\gamma_2 = .1111$ so that

$R(1000,n,c) = n + (1000-n)[.6296E(c+1;n,.01) + .1111B(c;n,.05)]$

Since $\min(N\gamma_1,N\gamma_2) = 111.1$, obviously $n \leqq 111$.

With $c = 0$ we get $R(1000,5,0) = 121.2$, $R(1000,10,0) = 135.5$, $R(1000,15,0) = 152.5$, $R(1000,20,0) = 171.4$ These values indicate that $R(1000,n,0)$ is a monotonic increasing function of n . A few more calculations verify this. We add $R(1000,3,0) = 116.6$, $R(1000,1,0) = 112.7$. Thus with $c = 0$ the minimum occurs with $n = 0$ and is 111.1 .

With $c = 1$ we get $R(1000,5,1) = 113.7$, $R(1000,10,1)=113.2$, $R(1000,20,1) = 110.52$, $R(1000,30,1) = 111.7$, $R(1000,24,1)=110.31$ $R(1000,23,1) = 110.29$, $R(1000,22,1) = 110.30$. Hence the minimum occurs at $n = 23$.

Next try $c = 2$. We get $R(1000,40,2) = 116.7$, $R(1000,50,2) = 115.32$, $R(1000,60,2) = 116.9$, $R(1000,51,2) = 115.33$, $R(1000,49,2) = 115.33$. Here the relative minimum is larger than $\min(N\gamma_1,N\gamma_2) = 111.1$.

There is, of course, no reason to find a minimum with $c = 3$. However, if we had to do this, we might guess (on the basis of the n which produced the previous minima) that the minimum would occur at about $n = 79$. Hence $n = 70$ or $n = 75$ would be a good starting point.

Since sampling plans can be found so quickly, it is easy to investigate the effects of using an incorrect prior distribution. For example, suppose that in Example 4.1 the correct weights are $w_1 = .8$, $w_2 = .2$. Now the Bayesian plan is $n = 6$, $c = 0$ with $R(100,6,0) = 27.11$, $K(100,6,0) = 10.94$. Using the old plan $n = 13$, $c = 0$ and the correct cost function yields $R(100,13,0) = 30.57$, $K(100,13,0) = 11.16$ so that the average cost is increased by $.22$ by using the incorrect weights. As a second example suppose that the weights are correct but that we should have used $p_1 = .02$, $p_2 = .08$. Now the Bayesian plan is $n = 7$, $c = 0$ with $R(100,7,0) = 45.20$, $K(100,7,0) = 15.81$. This time the old plan $n = 13$, $c = 0$ and the correct cost function yields $R(100,13,0) = 47.81$, $K(100,13,0) = 15.91$ and an additional average cost of $.10$. Many more such comparisons can be made with relative ease. One obvious fact is that if Nk_m is much larger than $(K_s-K_m)R(N,n,c)$, then small changes in the prior distribution do not change $K(N,n,c)$ substantially.

Hald [6 , p.43] has defined efficiency of a sampling plan as
$$e(N,n,c) = R_0(N)/R(N,n,c)$$
where $R_0(N)$ and $R(N,n,c)$ denote respectively the standardized cost of the optimum plan and the plan in question. His paper gives some numerical results, some comments about efficiency for large lot sizes, and some suggestions. One of his recommendations is that if the choice of p_1 and p_2 is in doubt,

then choose p_1 too large and p_2 too small. His calculations indicate that small changes in either the p's or the w's do not materially reduce the efficiency.

Since a Bayesian solution requiring no sampling may be unsatisfactory, it is worthwhile to consider restricted Bayesian plans. That is, we will minimize $K(N,n,c)$ subject to side conditions on either the OC or the AOC curve. For example we may require that the AOC be no more than β_2 if $p = p_2$, a restriction which produces the inequality

$$E(c+1;n,p_2) \geqq 1 - \beta_2 \tag{4.11}$$

Alternatively, we may require that if $p = p_1$ the AOC be at least $1 - \alpha_1$ which yields the inequality

$$E(c+1;n,p_1) \leqq \alpha_1 \tag{4.12}$$

Finally, we may insist that $K(N,n,c)$ be minimized subject to both conditions. For each c Inequalities (4.11) and (4.12) yield lower and upper bounds on n. Since we already know how to minimize the average cost function over all n, we obviously can use the same procedure to perform the minimization over a subset of these n.

Example 4.3

Minimize the cost function of Example 4.1 subject to the condition that AOC be no more than .10 if $p = p_2 = .10$.

Solution

We minimize

$R(100,n,c) = n + (100-n)[E(c+1;n,.01) + .625B(c;n,.10)]$ subject to the restriction $E(c+1;n,.10) \geqq .90$. From the binomial table we observe that we must have $n \geqq 22$ with $c = 0$, $n \geqq 38$ with $c = 1$, $n \geqq 52$ with $c = 2$, etc.

With $c = 0$ we find $R(100,22,0) = 42.27$, $R(100,23,0) = 43.16$ and conclude that the minimum occurs at

$n = 22$.

With $c = 1$ we find $R(100,38,1) = 45.13$, $R(100,39,1) = 45.88$ and the minimum is achieved at $n = 38$.

Since the minima are increasing we terminate calculations. With $c = 2$ we would have $n \geqq 52$ which is obviously too large. Thus the desired plan is $n = 22$, $c = 0$ and $K(100,22,0) = 14.92$. (Recall that the Bayesian plan gave 14.67.)

Example 4.4

Rework Example 4.3 if the weights are changed to $w_1 = w_2 = \frac{1}{2}$ and $N = 1000$.

Solution

Now $\gamma_1 = 1$, $\gamma_2 = 1.25$ and

$$R(1000,n,c) = n + (1000-n)[\bar{B}(c+1;n,.01) + 1.25B(c;n,.10)]$$

With $c = 0$, $n \geqq 22$ we find that the minimum occurs at $n = 26$ and $R(1000,26,0) = 328.64$.

With $c = 1$, $n \geqq 38$ the minimum is achieved at $n = 49$ and $R(1000,49,1) = 175.05$.

With $c = 2$, $n \geqq 52$ we might guess that the minimum occurs at $n = 72$. Actually it occurs at $n = 71$ and $R(1000,71,2) = 129.02$.

With $c = 3$, $n \geqq 65$ we might guess that the minimum occurs at $n = 93$. We find that $n = 90$ does the job and $R(1000,90,3) = 120.98$.

With $c = 4$, $n \geqq 78$ we might guess that the minimum occurs at $n = 108$. Actually $n = 107$ produces a minimum with $R(1000,107,4) = 127.28$.

We terminate calculations and the desired plan is $n = 90$, $c = 3$ with $R(1000,90,3) = 120.98$.

Example 4.5

Minimize the cost function of Example 4.1 subject to the condition that the AOC be at least .95 if $p = p_1 = .01$.

Solution

The side condition is $E(c+1;n,.01) \leqq .05$. From the binomial we observe that $n \leqq 5$ with $c = 0$, $n \leqq 35$ with $c = 1$, $n \leqq 82$ with $c = 2$.

With $c = 0$ we find $R(100,5,0) = 44.72$, $R(100,4,0) = 47.15$ and the minimum occurs at $n = 5$.

With $c = 1$, $n \leqq 35$ we find that the minimum occurs at $n = 26$ and $R(100,26,1) = 39.67$.

With $c = 2$ we must have $n \leqq 82$ to satisfy the side condition. However, to compete with plans obtained with $c = 1$ we must have n less than about 39 . Actually $n = 34$ is the right choice and $R(100,34,2) = 47.74$.

We terminate calculations and the desired plan is $n = 26$, $c = 1$ and $R(100,26,1) = 39.67$, $K(100,26,1) = 14.78$.

Example 4.6

Minimize the cost function of Example 4.4 subject to the side condition $E(c+1;n,.01) \leqq .05$.

Solution

With $c = 0$, $n \leqq 5$ $R(1000,5,0) = 788.2$, $R(1000,4,0) = 860.1$ and $n = 5$ produces the minimum.

With $c = 1$, $n \leqq 35$ $R(1000,35,1) = 228.6$, $R(1000,34,1) = 238.3$ and $n = 35$ produces the minimum.

With $c = 2$, $n \leqq 82$. After trying $n = 82,80,70,60,71,72$ we find that $n = 71$ yields the minimum and $R(1000,71,2) = 129.02$.

With $c = 3$, the side condition requires $n \leqq 137$.

Observing the n's which produced the minima with $c = 1$, $c = 2$ we might guess that $n = 107$. After trying $n = 105, 100, 90, 80, 89, 91$ we find that $n = 90$ yields a minimum and $R(1000,90,3) = 120.98$.

With $c = 4$ we must solve $E(5;n,.01) \leqq .05$. This can be done with the Poisson approximation using $E(5;.01n) \leqq .05$. The Molina [4] table reveals that $.01n \leqq 9.43$ or $n \leqq 943$, a bound which is not too helpful. Observing the results for $c = 2$ and $c = 3$ would suggest that $n = 109$. After trying $n = 120, 110, 100, 109, 108, 107, 106$, we find that $n = 107$ yields the minimum and $R(1000,107,4) = 127.28$.

We terminate calculations and conclude that $n = 90$, $c = 3$ is the desired plan with $R(1000,90,3) = 120.98$.

Example 4.7

Minimize the cost function of Example 4.4 subject to the two conditions $E(c+1;n,.01) \leqq .05$ and $E(c+1;n,.10) \geqq .90$.

Solution

We have only to put together the results of Example 4.4 and Example 4.6. To satisfy the side conditions we found that

if $c = 0$ $\quad n \leqq 5$, $n \geqq 22$

$c = 1$ $\quad n \leqq 35$, $n \geqq 38$

$c = 2$ $\quad n \leqq 82$, $n \geqq 52$

$c = 3$ $\quad n \leqq 137$, $n \geqq 65$

$c = 4$ $\quad n \leqq 943$, $n \geqq 78$

Obviously we must have $c \geqq 2$.

With $c = 2$ we seek the minimum with $52 \leqq n \leqq 82$. We already know from the previous examples that this occurs at $n = 71$ and $R(1000,71,2) = 129.02$.

With $c = 3$, $65 \leqq n \leqq 137$. The previous examples show that $n = 90$ produces the minimum and $R(1000,90,3) = 120.98$.

With $c = 4$, $82 \leqq n \leqq 943$. The previous examples show that $n = 107$ produces the minimum and $R(1000,107,4) = 127.28$.

Thus, the desired plan is $n = 90$, $c = 3$.

Example 4.8

Minimize the cost function of Example 4.1 subject to the two side conditions of Example 4.7 .

Solution

We already know from Example 4.7 the values of c which permit a solution and the corresponding ranges of n . Although we used each side condition separately in Examples 4.3 and 4.5 the minima previously obtained are not applicable.

With $c = 2$, $52 \leqq n \leqq 82$ we find $R(100,52,2) = 55.64$, $R(100,53,2) = 56.40$ and the minimum occurs at $n = 52$, a fact we would have known from Examples 4.3 and 4.5 .

With $c = 3$, $65 \leqq n \leqq 137$ we find $R(100,65,3) = 67.32$, $R(100,66,3) = 68.15$.

The desired solution is $n = 52$, $c = 2$.

Hald [9] has advocated the use of consumer and (or) producer risks which decreases with lot size. He suggests that one should take

$$P(p_2) = \begin{cases} \beta_2 , & N \leqq N_0 \\ \dfrac{\beta_2 N_0}{N} , & N > N_0 \end{cases} \qquad (4.13)$$

and (or)

$$Q(p_1) = \begin{cases} \alpha_1 , & N \leqq N_0 \\ \dfrac{\alpha_1 N_0}{N} , & N > N_0 \end{cases} \qquad (4.14)$$

Intuitively this makes sense since the consequences of wrong decision are apt to be more severe with a large lot than with a small one. If both (4.13) and (4.14) are used, $R(N,n,c)$

will be considerably less for large lots than if one or both risks remain fixed regardless of lot size. Another good reason for using two decreasing risks is that for large N the cost function will be roughly the same as the cost function of the Bayesian plan. Finally we note that if N, N_0, α_1, β_2 are specified, the problem reduces to one of the types already considered.

5. THE BETA PRIOR DISTRIBUTION

Perhaps one of the most reasonable choices for the prior distribution of p is the beta distribution with density

$$f(p;a_1,a_2) = \frac{1}{\beta(a_1,a_2)} p^{a_1-1}(1-p)^{a_2-1}, \quad 0 < p < 1 \qquad (5.1)$$

where $a_1 > 0$, $a_2 > 0$ are constants suitably chosen to fit a given problem (probably, as mentioned later, by the method of moments). We will be interested in distrubutions which concentrate the probability about small values of p. Since the expected value of a beta random variable is $a_1/(a_1+a_2)$, this means that a_2 will be somewhat larger than a_1. No matter how estimates of the a's are obtained we will round a_2 to a near integer and a_1 to the nearest .1 if the estimate is between 0 and 1 and to the nearest integer if the estimate is greater than 1, keeping the expectation of p as close to the desired average as possible. This convention allows us to use standard tables for the determination of sampling plans and, due to the apparent insensitivity to small changes in the parameters, without undue restriction on the choice of priors available. Even with these limitations we still have available three types of beta distributions:

1. Those with a density which is 0 at $p = 0$ and $p = 1$ and rises to a single maximum in between $(a_1>1, a_2>1)$.

2. Those with a density which is 1 at $p = 0$ and decreases monotonically $(a_1=1, a_2>1)$.

3. Those with a density which is monotonic decreasing with no maximum $(0<a_1<1, a_2>1)$.

We first assume that both a_1 and a_2 are integers. For the purpose of finding the expected value of the cost function, say $K(N,n,c)$, rewrite (2.2) as

$$\begin{aligned} K(N,n,c,p) &= n(S_1+S_2p) \\ &\quad + (N-n)\{[(A_1-R_1)+(A_2-R_2)p]P(p)+(R_1+R_2p)\} \end{aligned} \tag{5.2}$$

The four expectations arising from (5.2) are

$$E[n(S_1+S_2p)] = n\left[\frac{(a_1+a_2)S_1 + a_1S_2}{a_1 + a_2}\right] \tag{5.3}$$

$$E[(N-n)(R_1+R_2p)] = (N-n)\left[\frac{(a_1+a_2)R_1 + a_1R_2}{a_1 + a_2}\right] \tag{5.4}$$

$$E[P(p)] = \sum_{y=0}^{c} \binom{a_1+y-1}{a_1-1}\binom{n+a_2-y-1}{a_2-1}\Big/\binom{n+a_1+a_2-1}{a_1+a_2-1} \tag{5.5}$$

$$E[pP(p)] = \frac{a_1}{a_1 + a_2} \sum_{y=0}^{c} \binom{a_1+y}{a_1}\binom{n+a_2-y-1}{a_2-1}\Big/\binom{n+a_1+a_2}{a_1+a_2} \tag{5.6}$$

We next show that (5.5) and (5.6) can be expressed in terms of hypergeometric sums. To do this consider the inverse hypergeometric distribution (the terminology of Patil and Joshi [15]). From N items, k of which are defective, drawings are made one at a time at random and without replacement. The probability that u draws are required to obtain d defectives is

$$g(u) = \frac{\binom{k}{d-1}\binom{N-k}{u-1-d+1}}{\binom{N}{u-1}} \frac{k-d+1}{N-(u-1)}, \quad u = d,\ d+1, \ldots, N-k+d$$

$$= \binom{u-1}{d-1}\binom{N-u}{k-d}\Big/\binom{N}{k}$$

and the probability that r or less draws are required to obtain the d th defective is

$$\sum_{u=d}^{r} g(u) \tag{5.7}$$

Now suppose we set as a goal the obtaining of d defectives but agree to limit the number of draws to r. Then the goal will be achieved if the d th defective is obtained on the d th draw, the $d+1$ th draw,..., or the r th draw. Hence the probability of achieving the goal is (5.7). Alternatively, the goal is achieved if we make r draws and obtain d, $d+1$,..., or r defectives. The probability of the latter event is

$$\sum_{x=d}^{r} \binom{k}{x}\binom{N-k}{r-x}\Big/\binom{N}{r} = 1 - P(N,r,k,d-1)$$

Hence we have

$$\sum_{u=d}^{r} \binom{u-1}{d-1}\binom{N-u}{k-d}\Big/\binom{N}{k} = 1 - P(N,r,k,d-1) \tag{5.8}$$

which enables us to evaluate inverse hypergeometric probabilities from a hypergeometric table.

In (5.5) let $u = y + a_1$. Then, using (5.8) and identifying $d = a_1$, $k = a_1 + a_2 - 1$, $r = c + a_1$, $N = n + a_1 + a_2 - 1$ we get

$$E[P(p)] = 1 - P(n+a_1+a_2-1,\ c+a_1,\ a_1+a_2-1,\ a_1-1) \tag{5.9}$$

Similarly, letting $u = y + a_1 + 1$ in (5.6) yields

$$E[pP(p)] = \frac{a_1}{a_1+a_2}\,[1-P(n+a_1+a_2,\ c+a_1+1,\ a_1+a_2,\ a_1)] \tag{5.10}$$

Now define

$$R(N,n,c) = [K(N,n,c) - NK_m]/(K_s-K_m) \tag{5.11}$$

where

$$K_s = \frac{(a_1+a_2)S_1 + a_1S_2}{a_1 + a_2} \tag{5.12}$$

$$K_m = \frac{(a_1+a_2)A_1 + a_1R_2}{a_1 + a_2} \tag{5.13}$$

Then, using (5.3), (5.4), (5.9), (5.10) to evaluate the expectation associated with (5.2) yields

$$R(N,n,c) = n + (N-n)\{\gamma_1[1-P(n+a_1+a_2,\ c+a_1+1,\ a_1+a_2,\ a_1)] + \gamma_2 P(n+a_1+a_2-1, c+a_1,\ a_1+a_2-1,\ a_1-1)\} \tag{5.14}$$

where

$$\gamma_1 = \frac{(A_2-R_2)[a_1/(a_1+a_2)]}{K_s - K_m} \tag{5.15}$$

$$\gamma_2 = \frac{R_1 - A_1}{K_s - K_m} \tag{5.16}$$

(We note that if $S_2 = R_2$, then $K_s - K_m = S_1 - A_1$.)

If the coefficient of $Q(p)$ in (2.3) is always negative, as it is if $p_r < 0$, then $K(N,n,c)$ is minimized by taking $Q(p) = 1$, $n = 0$ giving

$$K(N,n,c) = N\left[\frac{(a_1+a_2)R_1 + a_1R_2}{a_1 + a_2}\right] = NK_r \tag{5.17}$$

On the other hand, if the coefficient is always positive, as it is if $p_r > 1$, the minimum is obtained by taking $Q(p) = 0$, $n = 0$ and is

$$K(N,n,c) = N\left[\frac{(a_1+a_2)A_1 + a_1A_2}{a_1 + a_2}\right] = NK_a \tag{5.18}$$

Corresponding to (5.18) is

$$R(N,n,c) = N\gamma_1 \tag{5.19}$$

and to (5.17) is

$$R(N,n,c) = N\gamma_2 \tag{5.20}$$

If $0 < p_r < 1$, $R(N,n,c)$ may achieve its smallest value at min $[N\gamma_1, N\gamma_2]$. Intuitively, we would expect that the minimum is (5.19) if $Pr(p>p_r)$ is small and (5.20) if $Pr(p<p_r)$ is small.

The constants a_1, a_2 can be estimated in various ways. One possibility is to equate the mean and variance of the beta distribution

$$\mu = a_1/(a_1+a_2) \tag{5.21}$$

$$\sigma^2 = \mu(1-\mu)/(a_1+a_2+1) \tag{5.22}$$

to estimates based upon sampling and solve for a_1 and a_2. Thus, if records are available for m samples of size n each of which yields an estimate $\hat{p}_i$, then we can calculate

$$\bar{p} = \sum_{i=1}^{m} \hat{p}_i/m \tag{5.23}$$

and

$$s^2 = \sum_{i=1}^{m} (\hat{p}_i-\bar{p})^2/(m-1) \tag{5.24}$$

Setting (5.21) equal to $\bar{p}$, (5.22) equal to s^2 yields (with $\bar{q}=1-\bar{p}$)

$$a_1 = \bar{p}(\bar{p}\bar{q}-s^2)/s^2 \tag{5.25}$$

$$a_2 = (1-\bar{p})(\bar{p}\bar{q}-s^2)/s^2 \tag{5.26}$$

Then, we would round these numbers according to the convention given in the first paragraph of this section.

Finally, we consider the case $0 < a_1 < 1$, a_2 an integer, (Actually a_2 does not have to be an integer but rounding has little effect.) If $\bar{p}$ is very small, this may be exactly the type of prior which seems to characterize the behavior of p. Hald [11] has suggested that the binomial be approximated by the Poisson and the beta prior be replaced by a gamma prior. Thus,

$$P(p) \cong \sum_{y=0}^{c} e^{-np}(np)^y/y! \tag{5.27}$$

and the prior is

$$f(p;b_1,b_2) = b_2^{b_1} p^{b_1-1} e^{-b_2 p}/\Gamma(b_1), \ p > 0 \tag{5.28}$$

A random variable with density (5.28) has mean b_1/b_2. This leads to the suggestion that we set $b_1 = a_1$, $b_2 = a_1 + a_2$. We observe that for $0 < b_1 < 1$ the gamma density has the same shape as the beta with corresponding $0 < a_1 < 1$. Using (5.27) we find that (5.14) should be replaced by

$$R(N,c,p) = n + (N-n)\{\gamma_1 F(c;b_1+1,\frac{b_2}{n+b_2})$$

$$+ \gamma_2[1-F(c;b_1,\frac{b_2}{n+b_2})]\} \quad (5.29)$$

where the F symbol is the negative binomial sum defined by (1.11) and

$$\gamma_1 = \frac{A_2-R_2}{K_s-K_m}\left(\frac{b_1}{b_2}\right) \quad (5.30)$$

$$\gamma_2 = \frac{R_1-A_1}{K_s-K_m} \quad (5.31)$$

$$K_s = \frac{b_2S_1+b_1S_2}{b_2} \quad (5.32)$$

$$K_m = \frac{b_2A_1+b_1R_2}{b_2} \quad (5.33)$$

(The last four formulas are the same as (5.15), (5.16), (.12), (5.13) if a_1 is replaced by b_1, $a_1 + a_2$ by b_2.) Here the derivation is straight forward and no special device, as was required to obtain (5.8), is necessary.

As in the case of the two point prior, we are again interested in Bayesian and restricted Bayesian plans. The minimization procedure is unchanged. We now consider some examples.

Example 5.1

Let the cost constants be those of the jacket example, $R_1 = S_1 = .10$, $R_2 = S_2 = 2.00$, $A_1 = 0$, $A_2 = 4.00$. Suppose that it is decided that p has a beta distribution with $a_1 = 1$, $a_2 = 19$ (so that $\mu = .05$). Using $N = 100$ find the Bayesian sampling plan.

Solution

We get $K_m = .10$, $K_s - K_m = .10$, $\gamma_1 = 1$, $\gamma_2 = 1$, $N\gamma_1 = N\gamma_2 = 100$, $p_r = .05$ and

$$R(100,n,c) = n + (100-n)[1-P(n+20,c+2,20,1) + P(n+19,c+1,19,0)]$$

We first try $c = 0$. Then with the table of Lieberman and Owen [13] we find

$$R(100,5,0) = 5 + 95[1-P(25,20,2,1) + P(24,19,1,0)]$$
$$= 5 + 95[.633333 + .208333] = 84.96$$

Similarly $R(100,10,0) = 80.34, R(100,15,0) = 79.64$, $R(100,20,0) = 80.51$, $R(100,14,0) = 79.61$, $R(100,13,0) = 79.65$, $R(100,12,0) = 79.76$ and the minimum occurs at $n = 14$.

With $c = 1$ we get $R(100,20,1) = 80.51$, $R(100,25,1) = 79.91$, $R(100,30,1) = 80.32$, $R(100,26,1) = 79.93$, $R(100,24,1) = 79.93$. The minimum occurs at $n = 25$. Since this is larger than the minimum with $c = 0$, we terminate calculations.

Thus, the desired plan is $n = 14$, $c = 0$ with $R(100,14,0) = 79.61$, $K(100,14,0) = 17.96$.

Example 5.2

Rework Example 5.1 if $a_1 = 3$, $a_2 = 57$ (so that we again have $\mu = .05$).

Solution

Since K_m , $K_s - K_m$ and $a_1/(a_1+a_2)$ are unchanged, we still have $\gamma_1 = \gamma_2 = 1$. The standardized cost function is

$$R(100,n,c) = n + (100-n)[1-P(n+60,c+4,60,3) + P(n+59,c+3,59,2)]$$
$$= n + (100-n)[P(n+60,n,c+4,c) + 1 - P(n+59,n,c+3,c)]$$

To evaluate hypergeometric sums we use the binomial approximation and linear interpolation on p in the binomial table. More precise evaluations seem to be unwarrented.

With $c = 0$ we find

$$R(100,5,0) = 5 + 95[P(65,5,4,0) + 1 - P(64,5,3,0)]$$

and

$$P(65,5,4,0) \cong B(0,4,.077) = .726$$
$$P(64,5,3,0) \cong B(0,3,.078) = 1 - .216$$

where $5/65 = .077$, $5/64 = .078$. Finally we get $R(100,5,0) = 94.5$. Similar calculations produce $R(100,10,0) = 92.4$, $R(100,15,0) = 91.8$, $R(100,20,0) = 91.9$, $R(100,25,0) = 92.7$, $R(100,18,0) = 91.9$, $R(100,17,0) = 91.8$, $R(100,16,0) = 91.6$, $R(100,14,0) = 91.8$, $R(100,13,0) = 91.8$, $R(100,12,0) = 92.0$, $R(100,11,0) = 92.1$. (Because the approximation was used, a few extra values of $R(100,n,0)$ were computed.) Thus the minimum occurs at $n = 16$.

With $c = 1$ we get $R(100,20,1) = 92.0$, $R(100,25,1) = 91.4$, $R(100,30,1) = 91.3$, $R(100,35,1) = 91.6$, $R(100,29,1) = 91.2$, $R(100,28,1) = 91.2$, $R(100,27,1) = 91.3$, $R(100,26,1) = 91.3$. The minimum occurs at $n = 28$ or 29 . (We arbitrarily take $n = 28$.)

With $c = 2$ we find $R(100,30,2) = 92.7$, $R(100,35,2) = 92.3$, $R(100,40,2) = 92.1$, $R(100,45,2) = 92.3$, $R(100,41,2) = 92.2$, $R(100,39,2) = 92.1$, $R(100,38,2) = 92.1$, $R(100,37,2) = 92.2$, $R(100,36,2) = 92.2$. Hence the minimum occurs at $n = 38,39,40$ and is larger than the minimum with $c = 1$.

The desired plan has $n = 28$, $c = 1$, $R(100,28,1) = 91.2$, $K(100,28,1) = 19.12$.

If we use the plan of Example 5.1 when actually $a_1 = 3$, $a_2 = 57$, then $R(100,14,0) = 91.8$, $K(100,14,0) = 19.18$ and the erroneous assumption cost .06 more per lot.

Example 5.3

Rework Example 5.1 if $a_1 = .4$, $a_2 = 7.6$ (so that we again have $\mu = .05$).

Solution

We will use the approximation based upon (5.27) and (5.28) with $b_1 = .4$, $b_2 = 8$. Again $\gamma_1 = \gamma_2 = 1$ and the standardized cost function is

$$R(100,n,c) = n + (100-n)[F(c;1.4,\frac{8}{n+8}) + 1 - F(c;.4,\frac{8}{n+8})]$$

For evaluations we use the table of Williamson and Bretherton [17] and linear interpolation on p (when necessary).

With $c = 0$ we find

$$R(100,8,0) = 8 + 92[F(0;1.4,.50) + 1 - F(0;.4,.50)]$$

$$= 8 + 92[.3789 + .2421] = 65.1$$

Similarly we get $R(100,12,0) = 63.4$, $R(100,16,0) = 63.9$, $R(100,11,0) = 63.5$, $R(100,13,0) = 63.4$, $R(100,14,0) = 63.6$, $R(100,10,0) = 63.8$. Thus the minimum occurs at $n = 12$ or 13 .

With $c = 1$ we get $R(100,20,1) = 65.4$, $R(100,21,1) = 65.6$, $R(100,22,1) = 65.4$, $R(100,23,1) = 65.5$, $R(100,24,1) = 65.6$, $R(100,25,1) = 65.9$, $R(100,19,1) = 65.5$, $R(100,18,1) = 65.7$. The minimum occurs at $n = 20$, 21 , 22 and is larger than the minimum with $c = 0$. Hence we terminate calculations.

The desired plan is $n = 12$ (or 13), $c = 0$, with $R(100,12,0) = 63.4$, $K(100,12,0) = 16.34$.

<u>Example 5.4</u>

Rework Example 5.1 using (5.29) with $b_1 = 1$, $b_2 = 20$. (This will allow us to compare the approximate solution with the exact solution.)

<u>Solution</u>

Again $\gamma_1 = \gamma_2 = 1$. Now we minimize

$$R(100,n,c) = n + (100-n)[F(c;2,\frac{20}{n+20}) + 1 - F(c;1,\frac{20}{n+20})]$$

With $c = 0$ we get $R(100,5,0) = 84.8$, $R(100,10,0) = 81.2$, $R(100,15,0) = 79.2$, $R(100,20,0) = 80.0$, $R(100,14,0) = 79.2$, $R(100,13,0) = 79.2$, $R(100,12,0)=79.4$ $R(100,16,0) = 79.2$, $R(100,17,0) = 79.4$. The minimum occurs with $n = 13, 14, 15, 16$ (which includes the correct $n = 14$).

With $c = 1$ we get $R(100,20,1) = 80.0$, $R(100,25,1) = 79.4$, $R(100,30,1) = 79.8$, $R(100,26,1) = 79.5$, $R(100,24,1) = 79.5$, $R(100,23,1) = 79.5$. The minimum occurs at $n = 25$, same as with the exact solution.

The minimum is 79.2 compared with 79.61 for the exact solution. We now get $K(100,14,0) = 17.92$ as compared with 17.96 before.

The approximation appears to work quite well in this example and should be even better when the prior has smaller mean value.

No further examples are necessary to demonstrate the procedure for restricted Bayesian plans. As we know from our discusion of the two point prior, conditions like (4.11) and (4.12) merely limit the range of n for each c . Hence, as with the previous model, we can certainly minimize over a subset of n if we can minimize over all n .

Although we limited our use of the approximation based upon (5.27) and (5.28) to the case $0 < a_1 < 1$, the Williamson and Bretherton [17] tables permit a wider range of a_1 . For many values of p the parameter k of (1.11) is used as an entry for $k = .1(.1)2.5(.5)5$. The primary criterion for the use of the approximate procedure is not the value of a_1 involved but the behavior of the prior. If most of the probability of the prior distribution is $\leqq .10$, the range for which the Poisson approximation is regarded as good, then we might expect that the approximate procedure would give very accurate answers. In our examples we had $u = .05$. If we had used $u = .01$, the values given by (5.29) would have been considerably more accurate.

Examination of Examples 5.1, 5.2, 5.3 illustrate that for a fixed mean μ , the greater the variance of p the smaller the average costs. This implies that as the variance increases the more urgent it becomes to incorporate the prior into the model.

6. COMMENTS

The linear cost model which we have considered seems to be very general and quite reasonable. Potential users may decide that some modification of (2.1) or the interpretation of the constants is necessary. If the material presented in this paper is available, perhaps such changes will be of a routine nature.

Our evaluations have been subject to the limitations of current standard tables. In the examples no serious handicap was encountered. If trouble should arise working with the ranges of parameters which have been tabulated, this would be good justification for extending the tables in their next editions.

REFERENCES

1. DODGE, H.F. and ROMIG, H.G., 1929. A method for sampling inspection. Bell Syst. Tech. J., 8, 613-631.
2. DODGE, H.F. and ROMIG, H.G., 1959. Sampling Inspection Tables. John Wiley and Sons, New York.
3. GENERAL ELECTRIC COMPANY, 1962. Tables of Poisson Distribution: Individual and Cumulative Terms. D. Van Nostrand, Princeton, New Jersey.
4. HALD, A., 1960. Statistical Concepts and models in the theory of sampling inspection by attributes. Nordisk Tidsskrift for Industriel Statistik, 5, 144-168.

5. HALD, A., 1960. The compound hypergeometric distribution and a system of single sampling plans based on prior distributions and costs. Technometrics, 2, 275-340.

6. HALD, A., 1965. Bayesian single sampling attribute plans for discrete prior distributions. Mat. Fys. Skr. Dan. Vid. Selsk., 3, 1965.

7. HALD, A., 1965. Single sampling inspection plans with specified acceptance probability and minimum average costs, Part I. Skand. Aktuartidskr., 48, 22-64.

8. HALD, A., 1965. Single sampling inspection plans with specified acceptance probability and minimum average costs, Part II. Skand. Aktuartidskr., 48, 143-183.

9. HALD, A., 1967. On the theory of single sampling inspection by attributes based on two quality levels. Rev. Int. Statist. Inst., 35, 1-29.

10. HALD, A., 1967. The determination of single sampling attribute plans with given producer's and consumer's risk. Technometrics, 9, 401-415.

11. HALD, A., 1968. Bayesian single sampling attribute plans for continouus prior distributions. Technometrics, 10, 667-683.

12. HARVARD UNIVERSITY COMPUTATION LABORATORY, 1955. Tables of the Cumulative Binomial Probability Distribution. Harvard University Press, Cambridge, Mass.

13. LIEBERMAN, G.J. and OWEN, D.B., 1961. Tables of the Hypergeometric Probability Distribution. Stanford University Press, Stanford, Calif.

14. MOLINA, E.C., 1949. Poisson's Binomial Exponential Limit. D. Van Nostrand, Princeton, New Jersey.

15. PATIL, G.G. and JOSHI, S.W., 1968. A Dictionary and Bibliography of Discrete Distributions. Oliver and Boyd, Edinburgh.

16. U.S.ARMY ORDNANCE CORPS, 1952. Tables of Cumulative Binomial Probabilities. ORDP 20-1. Office of Technical Services, Washington, D.C.

17. WILLIAMSON, ERIC and BRETHERTON, MICHAEL, 1963. Tables of the Negative Binomial Probability Distribution. John Wiley and Sons, New York.