COMPARISON OF CONTINGENCY TABLES

by

Jan F. Bjørnstad
ABSTRACT

Multiple comparison methods for several independent two-way contingency tables are proposed. We consider especially the problem of ordering the tables after degree of association and the probability of making an error in the ordering. In addition simultaneous confidence intervals for all linear functions and power-products of measures of association are presented.

Key words: Simultaneous confidence intervals, contingency table.
<table>
<thead>
<tr>
<th>CONTENTS</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. Introduction</td>
<td></td>
</tr>
<tr>
<td>1. Assumptions and notations</td>
<td>1</td>
</tr>
<tr>
<td>2. Asymptotical distribution theory</td>
<td>3</td>
</tr>
<tr>
<td>II. Multiple inference methods</td>
<td>6</td>
</tr>
<tr>
<td>1. Multiple normal-tests</td>
<td>6</td>
</tr>
<tr>
<td>2. The LSD-test for contingency tables</td>
<td>11</td>
</tr>
<tr>
<td>3. Simultaneous confidence intervals for general linear functions and power-products of measures of association from K contingency tables</td>
<td>14</td>
</tr>
<tr>
<td>4. Comparison of 2×2 tables</td>
<td>25</td>
</tr>
<tr>
<td>5. A discussion of the proposed test methods for differences</td>
<td>26</td>
</tr>
<tr>
<td>6. An example</td>
<td>29</td>
</tr>
<tr>
<td>References</td>
<td>34</td>
</tr>
</tbody>
</table>
I. INTRODUCTION

In [1] the author discussed the association problem in a fixed two-way contingency table. In this paper we shall consider the problem of multiple comparison of measures of association in several two-way tables. Three different multiple test-procedures are developed.

We first give some notations and necessary assumptions.

I.1. Assumptions and notations

Consider $K$ two-way contingency tables. Let the number of A-classes and B-classes in table $k$ be respectively $v_k$ and $w_k$, and let $v_k = v \cdot w_k$, for $k = 1, \ldots, K$. (We refer to [1] for a precise definition of the situation in a two-way contingency table.)

Let $p_{ij}^k$ denote the cell-probabilities in the $k$-th table,

$$
\frac{v_k w_k}{\sum_{i=1}^{v_k} \sum_{j=1}^{w_k} p_{ij}^k} = 1
$$

for $k = 1, \ldots, K$.

Further we let $X_{ijk}$ be the number of observations in cell $(i,j)$ in the $k$-th table, for $k = 1, \ldots, K$. The relative frequencies are denoted by $q_{ijk} = X_{ijk}/n_k$, where $n_k$ is the number of observations in table $k$.

Let $n = \sum_{k=1}^{K} n_k$ and $n_k = n_k/n$. The $n_k$'s are considered as constants as $n$ tends to infinity. We shall use the following notations:

$$
P_k = (p_{11k}, p_{12k}, \ldots, p_{w_k k}, p_{21k}, \ldots, p_{v_k w_k k})
$$

$$
q_k = (q_{11k}, q_{12k}, \ldots, q_{1w_k k}, q_{21k}, \ldots, q_{v_k w_k k})
$$

$$
p = (p_1, \ldots, p_K)
$$

$$
q = (q_1, \ldots, q_K)
$$
It will be assumed that the sequences of trials in each table are mutually independent, so that the random variables $q_1, \ldots, q_K$ are stochastically independent.

Let now $d$ be a measure of association, with continuous partial derivatives as function of the cell-probabilities. For choice of measure of association we refer to the author's review in [1], part 1. (Note: The results given below will under certain conditions also hold for the measures $\lambda, \lambda_b$ and $\lambda_r$ in [1].)

Let further $d_k$ be the measure $d$ in the $k$-th table, for $k = 1, \ldots, K$. Then $d_k$ is a function in $v_k$ variables with continuous partial derivatives. I.e. $d_k = d_k(p_k)$. Let $\hat{d}_k = d_k(q_k)$.

We shall use notations similar to those used in [1], part 2:

$$
\sigma_{d,k}^2 = \sum_{i=1}^{v_k} \sum_{j=1}^{w_k} p_{ijk}(d_{ij,k} - d_k^*)^2
$$

where

$$
d_{ij,k} = \frac{\partial d_k}{\partial p_{ijk}} \quad \text{for } i = 1, \ldots, v_k \text{ and } j = 1, \ldots, w_k
$$

and

$$
d_k^* = \sum_{i=1}^{v_k} \sum_{j=1}^{w_k} d_{ij,k} p_{ijk} \quad \text{for } k = 1, \ldots, K.
$$

A consistent estimator of $\sigma_{d,k}^2$ is given by

$$
S_{d,k}^2 = \sum_{i=1}^{v_k} \sum_{j=1}^{w_k} q_{ijk}(\hat{d}_{ij,k} - \hat{d}_k^*)^2
$$

where

$$
\hat{d}_{ij,k} = d_{ij,k}(q_k)
$$

and

$$
\hat{d}_k^* = d_k^*(q_k).
$$

It is assumed that no $p_{ijk}$ is zero, and further that for all
k = 1, \ldots, K there exists i, j so that

\[ d_{ij,k} \neq d_{i}^* \]  

(i.e. \( \sigma_{d,k}^2 > 0 \) for \( k = 1, \ldots, K \)).

The assumptions made above are supposed to hold throughout the paper, and will not be repeated. We use \( X_n \xrightarrow{D} X \) for convergence in distribution, and \( X_n \xrightarrow{P} X \) for convergence in probability.

I. 2. Asymptotical distribution theory

There are three basic results we need. We shall now state and prove them.

**Lemma 1.**

For \( i \neq j, \ i = 1, \ldots, K \) and \( j = 1, \ldots, K \):

\[ T_{ij} = \frac{\hat{d}_i - \hat{d}_j - (d_i - d_j)}{\left( \frac{\sigma_{d,i}^2}{n_i} + \frac{\sigma_{d,j}^2}{n_j} \right)^{\frac{1}{2}}} \xrightarrow{D} \mathcal{N}(0,1) \]

\( N(0,\sigma) \) denotes the normal distribution with mean 0 and variance \( \sigma^2 \).

**Proof.**

From ([1], theorem 1) we have that

\[ Z_{i,n} = \frac{\hat{d}_i - d_i}{\left( \frac{\sigma_{d,i}^2}{n_i} \right)^{\frac{1}{2}}} \xrightarrow{D} X_i \sim \mathcal{N}(0,1) \]  and  \[ Z_{j,n} = \frac{\hat{d}_j - d_j}{\left( \frac{\sigma_{d,j}^2}{n_j} \right)^{\frac{1}{2}}} \xrightarrow{D} X_j \sim \mathcal{N}(0,1). \]

Let \( T_{ij,n} = \frac{\hat{d}_i - \hat{d}_j - (d_i - d_j)}{\left( \frac{\sigma_{d,i}^2}{n_i} + \frac{\sigma_{d,j}^2}{n_j} \right)^{\frac{1}{2}}} \). \( T_{ij,n} \) can be expressed in the
following way:

\[ T_{ij,n} = \frac{1}{(S_{d,i}^2 + S_{d,j}^2)^{\frac{1}{2}}} \{ Z_{i,n} \sqrt{\pi_j S_{d,i}^2} - Z_{j,n} \sqrt{\pi_i S_{d,j}^2} \} \]

Let \( Y_{i,n} = Z_{i,n} \sqrt{\pi_j S_{d,i}^2} \) and \( Y_{j,n} = Z_{j,n} \sqrt{\pi_i S_{d,j}^2} \), \( Y_{i,n} \) and \( Y_{j,n} \) are stochastically independent for all \( n \), and

\[ S_{d,j}^2 \overset{P}{=} \sigma_{d,j}^2, \quad S_{d,i}^2 \overset{P}{=} \sigma_{d,i}^2. \]

Hence

\[ Y_{i,n} \overset{D}{=} Y_i \sim N(0, \pi_j \sigma_{d,i}) \]
\[ Y_{i,n} \overset{D}{=} Y_j \sim N(0, \pi_i \sigma_{d,j}). \]

Let now \( g_n(t) = E e^{itZ_{ij,n}} \) be the characteristic function of \( Z_{ij,n} = Y_{i,n} - Y_{j,n} \), \( f_{k,n}(t) \) is the characteristic function of \( Y_{k,n} \) and \( f_k(t) \) is the characteristic function of \( Y_k \). Then

\[ f_{k,n}(t) \to f_k(t) \]

and hence

\[ g_n(t) = f_{i,n}(t) \cdot f_{j,n}(-t) \to f_i(t) \cdot f_j(-t). \]

\( f_i(t) \cdot f_j(-t) \) is the characteristic function of \( N(0, \sqrt{\pi_j \sigma_{d,i}^2 + \pi_i \sigma_{d,j}^2}) \), so we have that \( Z_{ij,n} \overset{D}{=} Y \sim N(0, \sqrt{\pi_j \sigma_{d,i}^2 + \pi_i \sigma_{d,j}^2}) \) and hence:

\[ T_{ij,n} \overset{D}{=} \frac{Z_{ij,n}}{\sqrt{\pi_j S_{d,i}^2 + \pi_i S_{d,j}^2}} \sim N(0, 1). \]

Q.E.D.

**Lemma 2.**

Assume \( d_1 = d_2 = \cdots = d_k \). Then

\[ U = n \sum_{k=1}^{K} \frac{\pi_k (\hat{a}_k - \bar{d})^2}{S_{d,k}^2} \overset{D}{=} \chi^2_{K-1}, \]

where \( \chi^2_{K-1} \) is a chi-squared distribution with \( K-1 \) degrees of freedom.
where \[ \hat{d} = \frac{K}{\Sigma} \frac{n_i \hat{d}_i}{S_{d,i}} \Bigg/ \frac{K}{\Sigma} \frac{n_i}{S_{d,i}} = \frac{\hat{K}}{\Sigma} \frac{n_i \hat{d}_i}{S_{d,i}} \]

and \( \chi^2_{K-1} \) is the chi-square distribution with \( K - 1 \) degrees of freedom.

Proof.

The result follows directly from, [7], result 6a 2. (v). The only difference here is that \( S^2_{d,k} \) is not a continuous function of \( d_k \).

The property we need is however \( S^2_{d,k} \sim \sigma^2_{d,k} \) which is true and hence the result follows.

Q.E.D.

**Lemma 3.**

\[ \frac{K}{\Sigma} \frac{n_k (\hat{a}_k - \hat{a}_k)^2}{S^2_{d,k}} \overset{D}{\sim} \chi^2_K. \]

Proof.

Let \( X^k_n = \frac{(\hat{a}_k - \hat{a}_k)^2 n_k}{S^2_{d,k}} \), for \( k = 1, \ldots, K \).

\( X^1_n, \ldots, X^K_n \) are stochastically independent, and from [1], theorem 1 we have: \( (X^1_n, \ldots, X^K_n) \overset{D}{\sim} (X^1, \ldots, X^K) \) where \( X^1, \ldots, X^K \) are independent and \( X^k \) have a \( \chi^2_1 \)-distribution for \( k = 1, \ldots, K \). Now for each continuous function \( g \) we have that \( g(X^1_n, \ldots, X^K_n) \overset{D}{\sim} g(X^1, \ldots, X^K) \).

Especially

\[ \frac{K}{\Sigma} X^k_n \overset{D}{\sim} \frac{K}{\Sigma} X^k \sim \chi^2_K. \]

Q.E.D.
II. MULTIPLE INference METHODS

II.1. Multiple normal-tests

Let us assume that $d$ is a suitable measure of association for the contingency tables we want to compare. Let $x(p)$ be the upper $p$-fractile in $N(0,1)$. Simultaneous confidence intervals for the differences $d_i - d_j$, $i < j$ are obtained from the following.

$$P(T_{ij} \leq x\left(\frac{a}{K(K-1)}\right) \text{ for all } i, j, i \neq j)$$

$$= 1 - P\left( \bigcup_{i \neq j} T_{ij} > x\left(\frac{a}{K(K-1)}\right) \right)$$

$$= 1 - P\left( \bigcup_{i < j} |T_{ij}| > x\left(\frac{a}{K(K-1)}\right) \right)$$

$$\geq 1 - \sum_{i < j} P(|T_{ij}| > x\left(\frac{a}{K(K-1)}\right)) \rightarrow 1 - \sum_{i < j} \frac{2a}{K(K-1)}$$

$$= 1 - \frac{2a}{K(K-1)} \left(\frac{K}{2}\right) = 1 - \alpha .$$

Hence

$$\lim_{n \to \infty} P\left(\hat{d}_i - \hat{d}_j - x\left(\frac{a}{K(K-1)}\right) \sqrt{\frac{S_{d,i}^2}{n_i} + \frac{S_{d,j}^2}{n_j}} \leq d_i - d_j \leq \hat{d}_i - \hat{d}_j + x\left(\frac{a}{K(K-1)}\right) \sqrt{\frac{S_{d,i}^2}{n_i} + \frac{S_{d,j}^2}{n_j}} \right) \geq 1 - \alpha . \quad (2)$$

Simultaneous confidence intervals with asymptotical confidence level $P \geq 1 - \alpha$, for all differences $d_i - d_j$ are herewith given by (2).

Let us consider the hypothesis

$$H: d_1 = d_2 = \cdots = d_K \quad (3)$$

and let

$$H_{ij}: d_i = d_j . \quad (4)$$

We reject $H_{ij}$ if the interval (2) for $d_i - d_j$ does not cover 0.
I.e. if

\[ |T_{ij}^0| = \frac{|\hat{d}_i - \hat{d}_j|}{\sqrt{\frac{S^2_{d,i}}{n_i} + \frac{S^2_{d,j}}{n_j}}} > x\left(\frac{\alpha}{K(K-1)}\right) \]  

(5)

The test criterion (5) implies the following natural test for comparison of the \( d_i \)'s:

**Multiple normal-tests (N-tests).**

State \( d_i > d_j \) if \( H_{ij} \) is rejected and \( \hat{d}_i > \hat{d}_j \), i.e.

\[ T_{ij}^0 > x\left(\frac{\alpha}{K(K-1)}\right) \]

(6)

This comparison procedure has the following properties:

**Lemma 4.**

a) \[ \lim_{n \to \infty} P(\text{at least one false statement } \ "d_i > d_j" \ | H) \leq \alpha \]

b) \[ \lim_{n \to \infty} P(\text{state } d_i > d_j \ | d_i < d_j) = 0 \]

c) \[ \lim_{n \to \infty} P(\text{state } d_i > d_j \ | d_i = d_j) = \frac{\alpha}{K(K-1)} . \]

**Proof.**

a) The event \{at least one false statement under \( H\)\} equals

\[ \bigcup_{i \neq j} \{ \text{state } d_i > d_j \} = \bigcup_{i < j} (\text{reject } H_{ij}) , \text{ and} \]

\[ \lim_{n \to \infty} P\left( \bigcup_{i < j} \text{reject } H_{ij} \ | H \right) = 1 \lim_{n \to \infty} P_{H} \left( \text{ every interval (2) covers } 0 \right) \]

\[ = 1 \lim_{n \to \infty} P_{H} \left( \text{ every interval (2) covers } d_i - d_j \right) \leq 1 - (1 - \alpha) = \alpha . \]

b) This result follows from the fact that \( \hat{d}_i - \hat{d}_j \geq d_i - d_j \),

giving: \[ \lim_{n \to \infty} P(\text{state } d_i > d_j \ | d_i < d_j) \leq \lim_{n \to \infty} P(\hat{d}_i - \hat{d}_j > 0 \ | d_i - d_j < 0) = 0 . \]

c) follows from lemma 1.

Q.E.D.
We shall now consider the probability of at least one false statement, generally. For this purpose we use the same approach as Spjøtvoll, [8], and define the index sets \( V_i \) for \( i = 1, \ldots, t \) as follows:

\[
V_i \subset \{1,2,\ldots,K\}, \quad V_i \cap V_j = \emptyset \quad \text{for} \quad i \neq j \quad \text{and} \\
\bigcup_{i=1}^{t} V_i = \{1,2,\ldots,K\}
\]

Let further \( v_i \) be the number of elements in \( V_i \), \( \sum_{i=1}^{t} v_i = K \).

**Definition 1.**

\( w(V_1, \ldots, V_t) \) is the set of all \( d_1, \ldots, d_K, \sigma_{d_1}, \ldots, \sigma_{d_K} \), such that \( d_i = d_j \) if \( i,j \in V_h \) and \( d_i \neq d_j \) if \( i \) and \( j \) belong to different \( V_h \)'s. Let \( d = (d_1, \ldots, d_K) \) and \( \sigma_d = (\sigma_{d_1}, \ldots, \sigma_{d_K}) \). Let further \( \alpha(d, \sigma_d) \) be the probability of at least one false statement "\( d_i > d_j \)". Then we have

**Theorem 1.**

For \( (d, \sigma_d) \in w(V_1, \ldots, V_t) \):

\[ a) \quad \text{For} \quad t < K: \quad \lim_{n} \alpha(d_1, \ldots, d_K, \sigma_d) = \lim_{n} P\left( \bigcup_{h=1}^{t} \max_{i,j \in V_h} T_{ij}^0 > x\left(\frac{\sigma_d}{K(K-1)}\right)\right) = 1 - \prod_{h=1}^{t} \lim_{n} P\left( \max_{i,j \in V_h} T_{ij}^0 \leq x\left(\frac{\sigma_d}{K(K-1)}\right)\right) \]

\[ b) \quad \lim_{n} \alpha(d_1, \ldots, d_K, \sigma_d) \leq (1 - \frac{t-1}{K})(1 - \frac{t-1}{K-1})^\alpha \quad (7) \]

**Proof.**

From lemma 4b) it follows that for \( (d, \sigma_d) \in w(V_1, \ldots, V_t) \):

\[ \lim_{n} P\left( \bigcup_{h \neq h} \bigcup_{i \in V_h} \bigcup_{j \in V_h} (\text{false statement "}d_i > d_j\text{"})\right) = 0 \quad (8) \]
Hence b) is proved for \( t = K \). Let us now assume \( t < K \).

\[
\lim \alpha(d_1, \ldots, d_K, g_d) = \lim \left( \bigcup_{n} \bigcup_{g \in A} \bigcup_{i \in G} \bigcup_{j \in V} d_i > d_j \right) \quad \text{(false statement: \( U U U \))}
\]

\[
\bigcup_{h=1}^{t} \bigcup_{i \neq j}^{i, j \in V_h} \quad \text{(state: \( d_i > d_j \))}. \quad \text{Hence from 8)}
\]

\[
\lim \alpha(d, g_d) = \lim \left( \bigcup_{n} \bigcup_{h=1}^{t} \bigcup_{i < j}^{i, j \in V_h} \bigcup_{i, j \in V} (d_i \neq d_j) \right)
\]

\[
= \lim \left( \bigcup_{n} \bigcup_{h=1}^{t} \bigcup_{i < j}^{i, j \in V_h} \left[ T_{ij}^o > x\left( \frac{\alpha}{K(K-1)} \right) \right] \right) \quad \text{(9)}
\]

\[
= \lim \left( \bigcup_{n} \bigcup_{h=1}^{t} \bigcup_{i, j \in V_h} \left[ \max T_{ij}^o \leq x\left( \frac{\alpha}{K(K-1)} \right) \right] \right) = 1 - \lim \left( \bigcup_{n} \bigcup_{h=1}^{t} \bigcup_{i, j \in V_h} \left[ \max T_{ij}^o \leq x\left( \frac{\alpha}{K(K-1)} \right) \right] \right)
\]

\[
= 1 - \lim \left( \bigcup_{n} \bigcup_{h=1}^{t} \bigcup_{i, j \in V_h} \left[ \max T_{ij}^o \leq x\left( \frac{\alpha}{K(K-1)} \right) \right] \right), \quad \text{since } Y_1, \ldots, Y_t \text{ are independent, where } Y_h = \max_{i, j \in V_h} T_{ij}^o. \quad \text{Hence, a) is proved.}
\]

From (9) it follows that

\[
\lim \alpha(d_1, \ldots, d_K, g_d) \leq \sum_{h=1}^{t} \lim \left( \bigcup_{n} \bigcup_{i < j}^{i, j \in V_h} \left[ T_{ij}^o > x\left( \frac{\alpha}{K(K-1)} \right) \right] \right)
\]

\[
= \sum_{h=1}^{t} \sum_{i < j}^{i, j \in V_h} \frac{2\alpha}{K(K-1)} = \frac{2\alpha}{K(K-1)} \sum_{h=1}^{t} \frac{v_h(v_h-1)}{2}
\]

\[
= \frac{\alpha}{K(K-1)} \left[ \sum_{h=1}^{t} v_h^2 - K \right]
\]

Let us now consider \( \sum_{h=1}^{t} v_h^2 \). We shall show that the maximum value appears when \( (t-1) v_h \)'s equals 1. I.e.

\[
\sum_{h=1}^{t} v_h^2 \leq (K-t+1)^2 + t - 1 = M \quad \text{(10)}
\]

Generally \( t - r \) of the \( v_h \)'s equals 1. For \( r = 1 \) we have
equality in (10). Since \( t < K \) we cannot have \( r = 0 \), so let us therefore assume that \( 2 \leq r \leq t \). The other values of the \( v_h \)’s are denoted by: \( p_1 \leq p_2 \leq \ldots \leq p_r \); \( p_i \geq 2 \) for \( i = 1, \ldots, r \) and \( p_r = K - \sum_{i=1}^{r} p_i - t + r \).

The general expression for \( \Sigma v_h^2 \) becomes now:

\[
\sum_{h=1}^{t} v_h^2 = t - r + \sum_{i=1}^{r-1} p_i^2 + (K - \sum_{i=1}^{r} p_i - t + r)^2.
\]

After some calculation we find that

\[
M - \Sigma v_h^2 \geq 2 \sum_{i=1}^{r-2} p_i p_j + r(r-1) - 2r \sum_{i=1}^{r-1} p_i + 2p_{r-1} + 2(p_{r-1} + 1) \sum_{i=1}^{r-1} p_i = A.
\]

For \( r = 2 \) then \( A = 2(p_1 - 1)^2 \geq 2 > 0 \) so (10) is proved for \( r = 2 \).

Assume now \( r \geq 3 \). We find that

\[
A \geq (\sum_{i=1}^{r} p_i^2 + r - 1)^2 + r - 2 \sum_{i=1}^{r-1} p_i (p_{r-1} - p_i) + r - 1 + 2r p_{r-1} - 2(p_{r-1} + 1) p_i = (p_{r-1} - 1)^2 + 2r p_{r-1} - 2(p_{r-1} - 2)^2 > r - 1
\]

and (10) follows.

From (10) we now get

\[
\lim_{n \to \infty} \alpha(d_1, \ldots, d_K, \sigma_d) \leq \frac{\alpha}{\frac{K}{K-1}} \left[ (K-t+1)^2 + t - 1 - K \right]
\]

\[
= \frac{\alpha}{\frac{K}{K-1}} (K-t+1) [K - t + 1 - 1]
\]

\[
= \frac{(K-t+1)(K-t)}{K(K-1)} \alpha = (1 - \frac{t-1}{K})(1 - \frac{t-1}{K-1}) \alpha
\]

Q.E.D.

The upper bound in (7) decreases when \( t \) increases and has maximum for \( t = 1 \), which implies that for all values of \( (d, \sigma_d) \) we have
\[ \lim_{n \to \infty} \alpha(d_{ij}) \leq \alpha \]  

(11)

Some values of \((1 - \frac{t-1}{K})(1 - \frac{t-1}{K-1})\) are listed in the table below.

<table>
<thead>
<tr>
<th>K</th>
<th>t</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>...</th>
<th>9</th>
<th>...</th>
<th>19</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.333</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0.5</td>
<td>0.167</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0.6</td>
<td>0.3</td>
<td>0.1</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>0.8</td>
<td>0.622</td>
<td>0.467</td>
<td>0.333</td>
<td>0.022</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>1</td>
<td>0.9</td>
<td>0.805</td>
<td>0.716</td>
<td>0.623</td>
<td>0.347</td>
<td>0.005</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

II.2. The LSD-test for contingency tables

The title of the test is chosen because of the analogy with Fisher's LSD-test (least significant difference test for ranking of means in the analysis of variance, see [4], p.90).

We consider again the hypothesis (3):

\[ H: d_1 = d_2 = \ldots = d_K. \]

From lemma 2 it follows that the following test has asymptotic level equal to \(\alpha\). Reject \(H\) if

\[ U = n \sum_{i=1}^{K} \frac{\hat{a}_i \hat{d} - \hat{d}}{S_{d,i}^2} > z(K-1,\alpha) \]  

(12)

where \(z(K-1,\alpha)\) is the upper \(\alpha\)-fractile in the chi-square distribution with \(K-1\) degrees of freedom. Goodman consider in [2], ch.4 some hypotheses of the form \(H_1\) for comparison of \(2 \times 2\)-tables. The tests developed there are special cases of (12).

The LSD-test for differences \(d_i - d_j\) consists of two steps:

Step 1. \(H\) is tested by (12).

a) If \(H\) is accepted, the procedure stops.

b) If \(H\) is rejected, we go on to step 2.
Step 2. Compare the \( \hat{d}_i - \hat{d}_j \) pairwise and assert:

\[
d_i > d_j \text{ if } T^0_{ij} > x(\frac{\alpha}{2}) \quad \text{i.e.}
\]

\[
\hat{d}_i - \hat{d}_j > x(\frac{\alpha}{2}) \left( \frac{S^2_{d_i}}{n_i} + \frac{S^2_{d_j}}{n_j} \right)^{\frac{1}{2}}
\]

(13)

**Lemma 5**

a) \( \lim_n P \) (at least one false statement: \( d_i > d_j \mid H \)) \( \leq \alpha \).

b) \( \lim_n P \) (state \( d_i > d_j \mid d_i < d_j \)) = 0.

c) \( \lim_n P \) (state \( d_i > d_j \mid d_i = d_j \)) \( \leq \frac{\alpha}{2} \).

**Proof.**

a) \( \lim \) \( P \) (at least one false statement: \( d_i > d_j \mid H \)) \( \leq \alpha \).

\[
\lim_n P(U > z(K-1,\alpha)) = \alpha.
\]

b) The event \{state \( d_i > d_j \)\} implies \( \hat{d}_i - \hat{d}_j > 0 \) and b) follows from the fact that \( \hat{d}_i - \hat{d}_j \leq d_i - d_j \).

c) \( \lim_n P \) (state \( d_i > d_j \mid d_i = d_j \)) \( \leq \lim_n P \) (\( T^0_{ij} > x(\frac{\alpha}{2}) \)) = \( \frac{\alpha}{2} \).

Q.E.D.

Next, we show some results concerning \( \lim_n a(d_1, \ldots, d_K, \sigma_d) \) for the LSD-test.

**Theorem 2**

a) For \( (d_1, \ldots, d_K, \sigma_d) \in \omega(V_1, \ldots, V_t) \):

\[
\lim_n a(d_1, \ldots, d_K, \sigma_d) = \begin{cases} 
\lim_n P(U > z(K-1,\alpha)) \cap \max_{i,j} T^0_{ij} > x(\frac{\alpha}{2}), & \text{for } t = 1, \\
\lim_n P(\bigcup_{h=1}^t \max_{i,j \in V_h} T^0_{ij} > x(\frac{\alpha}{2})), & \text{for } t > 1.
\end{cases}
\]
b) \[ \lim_{n} \alpha(d_1, \ldots, d_K, \bar{g}_d) \leq \frac{(K+1-t)(K-t)}{2} \alpha \] for \( t \geq 2 \) (14)

Proof. From lemma 5 b) it follows directly that

\[ \lim P( \bigcup_{n} \bigcup_{i \in V} \bigcup_{j \in V} (false\ statement: d_i > d_j)) = 0. \]

Hence b) is proved for \( t = K \). We assume now \( t < K \).

a) The result for \( t = 1 \) (i.e. \( d_1 = d_2 = \ldots = d_K \)) follows directly.

Let \( t \geq 2 \).

\[ \lim \alpha(d_1, \ldots, d_K, \bar{g}_d) = \lim P( \bigcup_{n} \bigcup_{i \not\in V} \bigcup_{j \in V} (false\ statement: d_i > d_j)) \]

\[ = \lim P( \bigcup_{h=1}^{t} \bigcup_{i \not\in j} (state\ d_i > d_j)) \]

\[ = \lim P(U > z(K-1, \alpha)) \bigcup_{h=1}^{t} \bigcup_{i < j} |T_{ij}^o| > x(\frac{\alpha}{2}) \]

\[ = \lim P(U > z(K-1, \alpha)) \bigcup_{h=1}^{t} \max_{i,j \in V} T_{ij}^o > x(\frac{\alpha}{2}) . \]

Let \( V = U/n \). Since \( t \geq 2 \) we then have that \( \hat{d}_k \neq d_i \) for at least one pair. Then \( \hat{d} - d_i \sim \alpha \), where \( \alpha \) is a constant greater than zero. Hence \( \hat{d}_i - \hat{d} \sim \alpha \alpha > 0 \) and

\[ V \sim \alpha \frac{K \sum_{k=1}^{n} \frac{k}{\sigma_{\hat{d}, k}}}{\sigma_{\hat{d}, k}} = b > 0 . \]
Let now \( 0 < c < b \). Then
\[
\lim_{n} P(U > nc) = 1, \text{ which gives that the test } (12) \text{ is consistent,}
\]
i.e. \( \lim_{n} P(U > z(K-1,\alpha)) = 1 \). Hence
\[
\lim_{n} \alpha(d_1, \ldots, d_K; \sigma_d) = \lim_{n} P(\bigcup_{h=1}^{t} \max_{i,j \in V_h} T_{ij}^{0} > x(\frac{c}{\sigma_d}))
\]
\[
= 1 - \lim_{n} P(\bigcap_{h=1}^{t} \max_{i,j \in V_h} T_{ij}^{0} \leq x(\frac{c}{\sigma_d}))
\]
\[
= 1 - \prod_{h=1}^{t} \lim_{n} P(\max_{i,j \in V_h} T_{ij}^{0} \leq x(\frac{c}{\sigma_d})).
\]

b) From a) and (10):
\[
\lim_{n} \alpha(d_1, \sigma_d) \leq \sum_{h=1}^{t} \sum_{i < j} \lim_{n} P(|T_{ij}^{0}| > x(\frac{c}{\sigma_d})) = \alpha \sum_{h=1}^{t} \frac{\nu_{h}(\nu_{h}-1)}{2}
\]
\[
= \frac{\alpha}{2} \sum_{h=1}^{t} \nu_{h}^{2} - K \leq \frac{\alpha}{2} [(K-t+1)^2 + t - 1 - K] = \frac{\alpha}{2}(K-t+1)(K-t)
\]
Q.E.D.

II.3. Simultaneous confidence intervals for general linear functions and power-products of measures of association from \( K \) contingency tables

II.3 (i). General linear functions \( \sum_{i=1}^{K} c_i d_i \)
A procedure for comparison of all linear functions in \( d_1, \ldots, d_K \) can be constructed from simultaneous confidence intervals. The construction of joint confidence intervals for \( \sum_{i=1}^{K} c_i d_i \) is similar to the method applied by Spjøtvoll in [9].

From lemma 3 we have an asymptotic \((1-\alpha)\) confidence region for \( d = (d_1, \ldots, d_K) \) given by
\[ A(\hat{\sigma}_d^2, S_d^2) = \{ \hat{\sigma} : \sum_{i=1}^{K} \frac{(\hat{\sigma}_i - \sigma_i)^2}{S_d^2} n_i \leq z(K, \alpha) \} \]  

(15)

where \( \hat{\sigma} = (\hat{\sigma}_1, \ldots, \hat{\sigma}_K) \) and \( S_d^2 = (S_{d,1}^2, \ldots, S_{d,K}^2) \).

**THEOREM 3.** The simultaneous confidence intervals for all linear functions \( \sum c_i \hat{\sigma}_i \) based on the region \( A \) is of the form

\[
(a) \quad \sum c_i \hat{\sigma}_i - \sqrt{c_i S_d^2} \leq \sum c_i \hat{\sigma}_i \leq \sum c_i \hat{\sigma}_i + \sqrt{c_i S_d^2} 
\]

Asymptotically the probability is equal to \((1-\alpha)\) for \((a)\) to be true for all linear functions, i.e.

\[
\lim_{n \to \infty} P\left( \sum c_i \hat{\sigma}_i - \sqrt{c_i S_d^2} \leq \sum c_i \hat{\sigma}_i \leq \sum c_i \hat{\sigma}_i + \sqrt{c_i S_d^2} \right) = 1 - \alpha .
\]

**Proof.**

Let \( E_d = \{ (\hat{\sigma}_d^2, S_d^2) : \sum_{i=1}^{K} \frac{(\hat{\sigma}_i - \sigma_i)^2}{S_d^2} n_i \leq z(K, \alpha) \} \)

and \( F_d = \{ (\hat{\sigma}_d^2, S_d^2) : \sum_{i=1}^{K} c_i \hat{\sigma}_i - \sqrt{c_i S_d^2} \leq \sum_{i=1}^{K} c_i \hat{\sigma}_i \leq \sum_{i=1}^{K} c_i \hat{\sigma}_i + \sqrt{c_i S_d^2} \} \) for all \((c_1, \ldots, c_K)\).

We are going to show that \( E_d = F_d \) (for all \( n \)). The theorem then follows.
Let for a given vector $c = (c_1, \ldots, c_K)$: $a_1(c, S_d^2) = \min_{d \in A} (\sum_{i=1}^{K} c_i d_i)$ and $a_2(c, S_d^2) = \max_{d \in A} (\sum_{i=1}^{K} c_i d_i)$, then $E_d \subset \mathbb{R}^* = \{(\hat{d}, s_d^2) : a_1^c \leq \sum_{i=1}^{K} c_i d_i \leq a_2^c\}$ for all vectors $c$.

Let us find $a_1^c$ and $a_2^c$. The maximum and minimum of $\sum c_i d_i$ are obtained at some point $d^*$ satisfying

$$\frac{K}{\sum_{i=1}^{K} S_{d,i}^2} n_i = z(K, a). \quad (17)$$

Assume for example that $a_2^c = \sum c_i d_i^*$ where $\frac{K}{\sum_{i=1}^{K} S_{d,i}^2} n_i < z(K, a)$. Let $d_1^** = d_1^* + c_i/T$ for some constant $T$. Then

$$\sum_{i=1}^{K} c_i d_i^* = \sum_{i=1}^{K} c_i d_i^* + \frac{1}{T} \sum_{i=1}^{K} c_i^2 > \sum_{i=1}^{K} c_i d_i^*$$

Since $\hat{d}^*$ is an inner point of $A$, there exists $\varepsilon$ such that

$$|d^* - d| = \sqrt{\frac{K}{\sum_{i=1}^{K} (\hat{d}_i^*-d_i)^2}} < \varepsilon \Rightarrow \{d \in A\}$$

and

$$|d^*-d^**|^2 = \frac{1}{T^2} \sum_{i=1}^{K} c_i^2 < \varepsilon^2 \text{ for } T^2 > \frac{1}{\varepsilon^2} \sum_{i=1}^{K} c_i^2.$$ 

That is, if $T$ is sufficiently large, then $d^** \in A$, and we get a contradiction. In a similar way it can be shown that minimum also is obtained for a point $d^0$ satisfying (17).

The extreme values are found by the Lagrange method under the condition (17). Let

$$g(d) = \sum_{i=1}^{K} c_i d_i - \lambda \left( \sum_{i=1}^{K} \frac{(\hat{d}_i^*-d_i)^2}{S_{d,i}^2} n_i - z(K, a) \right)$$
\[ \frac{\partial g}{\partial d_k} = 0 : c_k + \lambda \frac{d_k - d_k^*}{S_{d,k}^2} n_k = 0 \quad \text{for} \quad k = 1, \ldots, K \]

\[ \downarrow \]

\[ d_k - d_k^* = \frac{1}{\lambda} \frac{S_{d,k}^2}{n_k} c_k \quad \iff \quad \frac{(d_k - d_k^*) \sqrt{n_k}}{S_{d,k}} = \frac{1}{\lambda} \frac{S_{d,k}}{\sqrt{n_k}} c_k. \]

This gives:

\[ \sum_{k=1}^{K} \frac{(\hat{d}_k - d_k)^2}{\frac{S_{d,k}^2}{n_k}} = \frac{1}{\lambda^2} \sum_{k=1}^{K} \frac{c_k^2 S_{d,k}^2}{n_k}. \]

Hence:

\[ \frac{1}{\lambda^2} \sum_{k=1}^{K} \frac{c_k^2 S_{d,k}^2}{n_k} = z(K, \alpha) \quad \text{and:} \]

\[ \frac{1}{\lambda} = \sqrt[2]{\frac{z(K, \alpha)}{\sum_{k=1}^{K} \frac{c_k S_{d,k}^2}{n_k}}}. \]

It now follows that

\[ \sum_{i=1}^{K} c_i d_i = \sum_{i=1}^{K} c_i \hat{d}_i + \frac{1}{\lambda} \sum_{i=1}^{K} \frac{S_{d,i}^2}{n_i} c_i = \sum_{i=1}^{K} c_i d_i + \sqrt{z(K, \alpha)} \sqrt{\sum_{i=1}^{K} \frac{c_i^2 S_{d,i}^2}{n_i}}. \]

I.e.

\[ a_1^c = \sum_{i=1}^{K} c_i \hat{d}_i - \sqrt{z(K, \alpha)} \sqrt{\sum_{i=1}^{K} \frac{c_i^2 S_{d,i}^2}{n_i}} \]

\[ a_2^c = \sum_{i=1}^{K} c_i \hat{d}_i + \sqrt{z(K, \alpha)} \sqrt{\sum_{i=1}^{K} \frac{c_i^2 S_{d,i}^2}{n_i}}. \]

This implies that

\[ F_{d}^* = F_{d} \]
and hence

\[ E_d \subseteq F_d. \]

We have left to show \( F_d \subseteq E_d \). Assume \((\hat{d}, S_d^2) \in \bar{F}_d\), i.e.

\[ a_1^c \leq \Sigma c_i d_i \leq a_2^c \quad \text{for all } c. \]

Particular for the following \( c \) given by

\[ c_i = \frac{I d_i - I d_i}{S_{d,i}^2} n_i \frac{1}{\sqrt{z(K, a)}} \]

we have

\[ \sqrt{K} \frac{1}{\sqrt{z(K, a)}} \left( \Sigma c_i (I d_i - I d_i) \right) \leq \sqrt{z(K, a)} \sqrt{K \frac{1}{i=1} n_i} \]

From this:

\[ \frac{1}{\sqrt{z(K, a)}} \frac{K (I d_i - I d_i)^2}{\Sigma \frac{1}{S_{d,i}^2} n_i} \leq \sqrt{z(K, a)} \sqrt{K (I d_i - I d_i)^2} \frac{1}{\Sigma \frac{1}{S_{d,i}^2} n_i} \]

\[ \iff \frac{K (I d_i - I d_i)^2}{\Sigma \frac{1}{S_{d,i}^2} n_i} \leq z(K, a) \iff (\hat{d}, S_d^2) \in E_d. \]

It is hereby shown that \( E_d = F_d \) for all \( n \) such that

\[ \lim_{n} P(F_d) = \lim_{n} P(E_d) = 1 - \alpha. \]

Q.E.D.

For differences \( d_i - d_j \) the intervals in theorem 3 are of the form:

\[ \hat{d}_i - \hat{d}_j - \sqrt{z(K, a)} \sqrt{\frac{S_{d,i}^2}{n_i} + \frac{S_{d,j}^2}{n_j}} \leq d_i - d_j \leq \hat{d}_i - \hat{d}_j + \sqrt{z(K, a)} \sqrt{\frac{S_{d,i}^2}{n_i} + \frac{S_{d,j}^2}{n_j}} \]

(18)

The test-procedure for linear functions in \( d_i \)'s consists now in
stating $\Sigma c_i d_i \neq 0$ if the corresponding interval does not cover 0, i.e. if

$$\left| \sum_{i=1}^{K} c_i \hat{d}_i \right| > \sqrt{\frac{z(K, \alpha)}{\sqrt{\sum_{i=1}^{K} \frac{c_i^2 S_i^2}{n_i}}}}. \quad (19)$$

For differences the test is: State $d_i \neq d_j$ if $|T_{ij}^0| > \sqrt{z(K, \alpha)}$

which leads to:

State $d_i > d_j$ if $T_{ij}^0 > \sqrt{z(K, \alpha)} \quad (20)$

Comparison of this test and multiple N-tests is done later. One property of the test (19) is given in the next result.

**Lemma 6.**

$$\lim_{n \to \infty} P_{d} (\text{at least one false statement: } \Sigma c_i d_i \neq 0) = \begin{cases} a & \text{if } d = 0 \\ <a & \text{if } d \neq 0 \end{cases}$$

**Proof.**

Assume first $d = 0$. Then $\Sigma c_i d_i = 0$ for all $c' = (c_1, \ldots, c_k)$.

Hence $P (\text{asserting at least one erroneous } \Sigma c_i d_i \neq 0) = P (\text{asserting at least one } \Sigma c_i d_i \neq 0) = 1 - P (\text{all intervals (16) cover } 0) = 1 - P (\text{all intervals (16) cover } \Sigma c_i d_i \to \infty) = - (1 - a) = a$.

Let now $d \neq 0$.

$P_d (\text{at least one false statement: } \Sigma c_i d_i \neq 0) = 1 - P_d (\text{no false statement})$

$= 1 - P_d (\text{the intervals (16) cover } 0 \text{ for those } c \text{ with } \Sigma c_i d_i = 0) \leq 1 - P_d (\text{all intervals (16) cover } \Sigma c_i d_i \to \infty) = (1 - a) = a$.

Q.E.D.

If $\Sigma c_i \hat{d}_i > 0$ and (19) holds, it is natural to assert $\Sigma c_i d_i > 0$.

Then the following result holds:
Lemma 7.

\[ \lim P \left( \text{stating } \sum_{i=1}^{K} c_i d_i > 0 \mid \sum_{i=1}^{K} c_i d_i < 0 \right) = 0 \]

Proof.

\[ \sum_{i=1}^{K} c_i \hat{d}_i \leq \sum_{i=1}^{K} c_i d_i. \] This implies: \[ \sum_{i=1}^{K} c_i \hat{d}_i \leq \sum_{i=1}^{K} c_i d_i \]

Then

\[ \lim P \left( \text{asserting } \sum_{i=1}^{K} c_i d_i > 0 \mid \sum_{i=1}^{K} c_i d_i < 0 \right) \leq \lim P \left( \sum_{i=1}^{K} c_i \hat{d}_i > 0 \mid \sum_{i=1}^{K} c_i \hat{d}_i < 0 \right) = 0. \]

Q.E.D.

The test (20) for differences has similar properties as the multiple N-tests:

Theorem 4.

For \((d, \sigma_d) \in \omega(V_1, \ldots, V_t)\)

\[ a) \lim_{n} \alpha(d_1, \ldots, d_K, \sigma_d) = \lim_{n} P \left( \max_{n} T_{ij}^0 < \sqrt{z(K, \alpha)} \right) = 1 - \frac{1}{n} \lim_{n} P \left( \max_{i,j \in V_h} T_{ij}^0 < \sqrt{z(K, \alpha)} \right). \]

\[ b) \lim_{n} \alpha(d_1, \ldots, d_K, \sigma_d) \leq \alpha. \]

Proof.

a) Completely analogous to the proof for theorem 1a) by substituting

\[ x(\alpha/K(K-1)) \text{ with } \sqrt{z(K, \alpha)}. \]

b) From a): \( \lim_{n} \alpha(d, \sigma) = 1 - \lim_{n} P(\text{the intervals (16) cover 0 for all } c \text{ such that } c_i = 1 \text{ and } c_j = -1, c_k = 0 \text{ for } k \neq i, j \text{ and } d_i = d_j) \)
\[ 1 - \lim \frac{P}{n} \text{(the intervals (16) cover } d_i - d_j \text{ for those } i, j \text{ with } d_i = d_j) \]
\[ \leq 1 - \lim \frac{P}{n} \text{(the intervals (16) cover } \sum c_i d_i \text{ for all } c) = \alpha. \]

Q.E.D.

Analogous with theorem 1b one sees that (from (10)):
\[ \lim_{n} \alpha(d_1, \ldots, d_K, c_d) \leq 2[1 - \frac{1}{2}(\sqrt{z(K, \alpha)})\sum_{h=1}^{t} \frac{v_h(v_h-1)}{2} \leq [1 - \frac{1}{2}(\sqrt{z(K, \alpha)})] (K-1-t)(K-t). \]

In II.5 \( x(a/K(K-1)) \) and \( \sqrt{z(K, \alpha)} \) are compared, and it shows that for all usual choice of \( a \) \( x(a/K(K-1)) < \sqrt{z(K, \alpha)} \). In that case
\[ 1 - \frac{1}{2}(\sqrt{z(K, \alpha)}) \leq \frac{\alpha}{K(K-1)}, \text{ and hence: } \lim_{n} \alpha(d_1, \ldots, d_K, c_d) \leq (1 - \frac{t-1}{K})(1 - \frac{t-1}{K-1}) \alpha. \]

II.3 (ii) General power-products:
\[ \prod_{k=1}^{K} d_{ik}^{c_k} \]

It is now assumed that the measure of association \( d \) is positive, i.e. \( d_k > 0 \) for \( k = 1, \ldots, K \). If we are interested in comparing the degree of association in the tables this will always be the case. Let us as an example suppose that we have chosen the ordinal measure \( \gamma \). We know from [1] that \( \gamma \in [-1, 1] \). We then use simply \( d = \gamma^2 \) to compare the strength of association in the tables.

Let us consider linear functions of \( \ln d_k \) instead of \( d_k \). Then
\[ \frac{\partial \ln d_k}{\partial p_{ijk}} = \frac{1}{d_k} d_{ij,k} \text{ and } S_{\ln d,k}^2 = \left( \frac{1}{d_k} \right)^2 S_{d,k}^2 \]

Then from theorem 3:
\[
\lim_{n \to \infty} P \left( \sum_{k=1}^{K} c_k \frac{\ln \hat{d}_k}{\sqrt{z(K, \alpha)}} \leq \sqrt{\sum_{k=1}^{K} \frac{c_k^2 d_k^2}{n_k \hat{d}_k^2}} \right) \leq \frac{K}{n} \frac{c_k \ln \hat{d}_k}{\sqrt{z(K, \alpha)}} \leq \frac{K}{n} \frac{c_k \ln \hat{d}_k}{\sqrt{z(K, \alpha)}} \text{ for all } c = 1 - \alpha.
\]

Now, the inequality inside the parenthesis can be expressed as follows:

\[
\ln \prod_{k=1}^{K} \frac{c_k \hat{d}_k}{n_k \hat{d}_k^2} - \frac{z(K, \alpha)}{\sum_{k=1}^{K} \frac{c_k^2 d_k^2}{n_k \hat{d}_k^2}} \leq \ln \prod_{k=1}^{K} \hat{d}_k \leq \ln \prod_{k=1}^{K} \frac{c_k \hat{d}_k}{n_k \hat{d}_k^2}
\]

for all \( c \)

Hence the following result has been proven.

**THEOREM 5.** Simultaneous confidence intervals for all power-products based on the region

\[
A(n, \hat{d}, S_d^2) = \left\{ \hat{d} : \frac{\sum_{i=1}^{K} (\ln \hat{d}_i - \ln d_i)^2}{S_d^2} \leq z(K, \alpha) \right\}
\]

is of the form

\[
-\left[ \frac{z(K, \alpha)}{\sum_{i=1}^{K} \frac{c_i^2 d_i^2}{n_i \hat{d}_i^2}} \right]^{1/2} \leq \prod_{k=1}^{K} \frac{c_k \hat{d}_k}{n_k \hat{d}_k^2} \leq \left[ \frac{z(K, \alpha)}{\sum_{i=1}^{K} \frac{c_i^2 d_i^2}{n_i \hat{d}_i^2}} \right]^{1/2}
\]

(a) \( \prod_{k=1}^{K} \frac{c_k \hat{d}_k}{n_k \hat{d}_k^2} \leq \prod_{k=1}^{K} \frac{c_k \hat{d}_k}{n_k \hat{d}_k^2} \)

\[
(21)
\]
The probability of (a) to hold for all \((c_1, \ldots, c_K)\) is asymptotically equal to \(1 - \alpha\).

It can be of special interest to consider \(\frac{d_i}{d_j}\).

By putting \(c_i = 1\), \(c_j = -1\) and \(c_k = 0\) for \(k \neq (i,j)\), theorem 5 gives us the following intervals for \(\frac{d_i}{d_j}\):

\[
\frac{\hat{d}_i}{\hat{d}_j} \leq \frac{d_i}{d_j} \leq \frac{\hat{d}_i}{\hat{d}_j}
\]

(22)

From theorem 5: \(\lim P((22) \text{ is true for all } i,j) \geq 1 - \alpha\).

Multiple N-tests for differences \(\ln d_i - \ln d_j\) can also be applied to construct joint intervals for \(\frac{d_i}{d_j}\). (2) leads to the following intervals:

\[
\frac{\hat{d}_i}{\hat{d}_j} \leq \frac{d_i}{d_j} \leq \frac{\hat{d}_i}{\hat{d}_j}
\]

(23)

The probability of (23) for all \(i,j\) is asymptotically \(\geq 1 - \alpha\).

As mentioned earlier, \(x(\alpha/K(K-1)) < \sqrt{z(K,\alpha)}\) for all the usual choices of \(\alpha\), so that the intervals (23) are shorter than the intervals (22).

The test-procedure for power-products:

Assert \(\prod_{k=1}^{c_k} d_k \neq 1\) if the confidence interval (21) lies outside of 1. i.e. if

\[
\prod_{k=1}^{c_k} \hat{d}_k > e^{\left[ z(K,\alpha) \sum_{i=1}^{K} \frac{K c_i^2 d_i^2}{i n_i d_i^2} \right]^\frac{1}{2}} \quad \text{or} \quad \prod_{k=1}^{c_k} \hat{d}_k < e^{-\left[ z(K,\alpha) \sum_{i=1}^{K} \frac{K c_i^2 d_i^2}{i n_i d_i^2} \right]^\frac{1}{2}}
\]

(24)
**Lemma 8.**

\[
\lim_{n \to \infty} P(\text{at least one false statement: } \Pi d_k^c \neq 1) \leq \alpha.
\]

**Proof.**

\[P(\text{at least one false statement}) = 1 - P(\text{no false statement})\]

\[= 1 - P(\text{the intervals (21) cover 1 for those } c \text{ such that } \Pi d_k^c = 1)\]

\[\leq 1 - P(\text{the intervals (21) cover } \Pi d_k^c \text{ for all } c) \to 1 - (1 - \alpha) = \alpha.\]

Q.E.D.

By applying theorem 3 and 5 we can construct simultaneous confidence intervals for all linear functions and all power-products at the same time. The result follows from a simple relation for two events \( R_1, R_2 \):

\[P(R_1 \cap R_2) \geq 1 - P(\bar{R}_1) - P(\bar{R}_2) = P(R_1) + P(R_2) - 1\]

**Lemma 9.**

\[
\lim_{n \to \infty} P \left( \sum_{k=1}^{K} c_k \hat{d}_k \left[ z(K, a_i^2) \sum_{i=1}^{K} \frac{c_i^2 d_i^2}{n_i} \right]^{1/2} \leq \sum_{k=1}^{K} c_k \hat{d}_k \right) \\
\geq \left[ z(K, a_i^2) \sum_{i=1}^{K} \frac{b_i^2 d_i^2}{n_i} \right]^{1/2}
\]

for all \( c = (c_1, \ldots, c_k) \)

\[
\cap \prod_{k=1}^{K} \hat{d}_k \leq \prod_{k=1}^{K} b_k \hat{d}_k \\
\geq \left[ z(K, a_i^2) \sum_{i=1}^{K} \frac{b_i^2 d_i^2}{n_i} \right]^{1/2}
\]

for all \( b = (b_1, \ldots, b_k) \) \( \geq 1 - \alpha.\)

As an example of application of multiple inference methods we shall consider \( K \ 2 \times 2 \)-tables.
II. 4. Comparison of 2x2-tables

From [1] we know that the natural measure of association is essentially the cross-product ratio. Let \( \Delta_k = \frac{p_{11k} p_{22k}}{p_{12k} p_{21k}} \) for \( k = 1, \ldots, K \). \( \Delta_k \) is the cross-product ratio in table \( k \).

Let further \( \hat{\Delta}_k = \frac{q_{11k} q_{22k}}{q_{12k} q_{21k}} \), and \( s^2_k = x_{11k}^{-1} + x_{22k}^{-1} + x_{12k}^{-1} + x_{21k}^{-1} \), for \( k = 1, \ldots, K \).

From [1], lemma 19:

\[
S^2_{\Delta,k} = n_k \Delta_k^2 s^2_k, \quad \text{for} \quad k = 1, \ldots, K.
\]

Let \( \rho_k = \ln \Delta_k \) and \( \hat{\rho}_k = \ln \hat{\Delta}_k \). Then

\[
S^2_{\rho,k} = n_k s^2_k
\]

From II.1 (2) we can construct the following confidence intervals for \( \rho_i - \rho_j \):

\[
\hat{\rho}_i - \hat{\rho}_j - x(\frac{a}{K(K-1)})\sqrt{S^2_i + S^2_j} \leq \rho_i - \rho_j \leq \hat{\rho}_i + \hat{\rho}_j + x(\frac{a}{K(K-1)})\sqrt{S^2_i + S^2_j}
\]

Multiple N-tests for \( \rho_i - \rho_j \) are now to assert \( \rho_i \neq \rho_j \) if

\[
|\hat{\rho}_i - \hat{\rho}_j| > x(\frac{a}{K(K-1)})\sqrt{S^2_i + S^2_j}
\]

For \( K = 2 \) the results (25) and (26) are the same as those in [3], (p. 97, (49) and (52)).

By applying theorem 3 simultaneous confidence intervals for all linear functions in \( \Delta_k 's \) can be obtained:

\[
\frac{K}{\Sigma c_i k \hat{\Delta}_k} - \sqrt{z(K,a)} \sqrt{\frac{K}{\Sigma c_i^2 \hat{\Delta}^2 s^2_i}} \leq \frac{K}{\Sigma c_i k \Delta_k} \leq \frac{K}{\Sigma c_i k \hat{\Delta}_k} + \sqrt{z(K,a)} \sqrt{\frac{K}{\Sigma c_i^2 \hat{\Delta}^2 s^2_i}}
\]

If one is only interested in differences \( \Delta_i - \Delta_j \), one obtains as
mentioned earlier usually shorter intervals by using (2), which in this case gives:

\[
\hat{\Delta}_i - \hat{\Delta}_j - x(\frac{\alpha}{K(K-1)}) \sqrt{\frac{\hat{\sigma}^2 + \hat{\Delta}_j^2 S_j^2}{\hat{\Delta}_i^2 S_i^2 + \hat{\Delta}_j^2 S_j^2}} \leq \Delta_i - \Delta_j \leq \hat{\Delta}_i - \hat{\Delta}_j + x(\frac{\alpha}{K(K-1)}) \sqrt{\frac{\hat{\Delta}_i^2 S_i^2 + \hat{\Delta}_j^2 S_j^2}{\hat{\Delta}_i^2 S_i^2 + \hat{\Delta}_j^2 S_j^2}}
\]

(28)

Simultaneous confidence intervals for power-products is of the form:

\[
\prod_{k=1}^{K} \sum_{i=1}^{2} \frac{z(K, \alpha) \Sigma c_i^2 S_i^2}{c_k^2} \leq \prod_{k=1}^{K} \hat{\Delta}_k \leq \prod_{k=1}^{K} \sum_{i=1}^{2} \frac{z(K, \alpha) \Sigma c_i^2 S_i^2}{c_k^2}
\]

(29)

Joint intervals for \( \frac{\Delta_i}{\Delta_j} \) will usually be shorter by applying (23):

\[
\hat{\Delta}_i - x(\frac{\alpha}{K(K-1)}) \sqrt{\frac{S_i^2 + S_j^2}{\hat{\Delta}_i^2 S_i^2 + \hat{\Delta}_j^2 S_j^2}} \leq \frac{\Delta_i}{\Delta_j} \leq \hat{\Delta}_i - x(\frac{\alpha}{K(K-1)}) \sqrt{\frac{S_i^2 + S_j^2}{\hat{\Delta}_i^2 S_i^2 + \hat{\Delta}_j^2 S_j^2}}
\]

II. 5. A discussion of the proposed test methods for differences

Three test procedures have been proposed for differences \( d_i - d_j \):

A: Multiple N-tests.

State \( d_i > d_j \) if \( T_{ij}^0 > x(\frac{\alpha}{K(K-1)}) \).

B: The LSD-test.

State \( d_i > d_j \) if \( U > z(K, \alpha) \) and \( T_{ij}^0 > x(\frac{\alpha}{2}) \).

C: Derived from simultaneous confidence intervals for linear functions.

State \( d_i > d_j \) if \( T_{ij}^0 > (\frac{1}{2}) \).

The probability of at least one false statement is \( \leq \alpha \) (asymptotically) for the tests A and C. For test B this is generally true only when \( d_1 = \ldots = d_k \). From theorem 2 a) we see that for \( t > 2 \lim_{n \to \infty} a(\delta, \sigma^2) \) is equal to or greater than \( \alpha \). The tests A and C should therefore be preferred to test B. Of the tests A and C we
choose the one with highest probability of "stating $d_i > d_j$" when $d_i > d_j$. I.e. $A$ is better than $C$ when

$$x^2\left(\frac{a}{K(K-1)}\right) < z(K,a),$$

that is, when the intervals (2) are shorter than the intervals (18).

Let now $Y$ be a random variable with distribution $N(0,1)$.

Then: $1 - \alpha = P(-x(\frac{\alpha}{2}) \leq Y \leq x(\frac{\alpha}{2})) = P(Y^2 < x^2(\frac{\alpha}{2}))$. Hence, since $Y^2$ is chi-square distributed with one degree of freedom:

$$x^2(\frac{\alpha}{2}) = z(1, \alpha).$$

From this:

$$x^2\left(\frac{a}{K(K-1)}\right) = z(1, \frac{2\alpha}{K(K-1)}).$$

For $K = 2$: $x^2\left(\frac{a}{K(K-1)}\right) = z(1, \alpha) < z(2, \alpha)$.

(Generally: $p < s \Rightarrow z(p, \alpha) < z(s, \alpha)$).

I.e. for $K = 2$ the multiple $N$-tests should be preferred for comparing $d_1$ and $d_2$. For $K \geq 3$ the situation is not so simple, but it looks as if $z(1, \frac{2\alpha}{K(K-1)}) < z(K, \alpha)$ for all usual choice of $\alpha(\leq 0.25)$. If, however, $\alpha$ is greater than 0.75, the opposite inequality will often hold. A table of these two fractiles for special values of $\alpha$ and $K$ will illustrate this.

**Table 1a:**

<table>
<thead>
<tr>
<th>K</th>
<th>$z(3, \alpha)$</th>
<th>$z(4, \alpha)$</th>
<th>$x^2(\frac{\alpha}{6}) = z(1, \frac{\alpha}{3})$</th>
<th>$x^2(\frac{\alpha}{12}) = z(1, \frac{\alpha}{6})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>11.35</td>
<td>13.28</td>
<td>8.61</td>
<td>9.92</td>
</tr>
<tr>
<td>4</td>
<td>7.81</td>
<td>9.49</td>
<td>5.73</td>
<td>6.96</td>
</tr>
<tr>
<td>5</td>
<td>6.25</td>
<td>7.78</td>
<td>4.53</td>
<td>5.73</td>
</tr>
<tr>
<td>6</td>
<td>4.11</td>
<td>5.39</td>
<td>3.00</td>
<td>4.15</td>
</tr>
<tr>
<td>7</td>
<td>2.37</td>
<td>3.36</td>
<td>1.91</td>
<td>3.00</td>
</tr>
<tr>
<td>8</td>
<td>1.21</td>
<td>1.92</td>
<td>1.32</td>
<td>2.35</td>
</tr>
<tr>
<td>9</td>
<td>0.58</td>
<td>1.06</td>
<td>1.08</td>
<td>2.07</td>
</tr>
<tr>
<td>10</td>
<td>0.35</td>
<td>0.71</td>
<td>1.00</td>
<td>1.99</td>
</tr>
<tr>
<td>11</td>
<td>0.12</td>
<td>0.30</td>
<td>0.95</td>
<td>1.93</td>
</tr>
</tbody>
</table>
From table 1a) and 1b) it seems obvious that the multiple N-tests are much better than test C for multiple comparison of differences in the d_i's. This is what we had to expect since test C is a special case of a multiple test for comparison of all linear functions in the d_i's. Reasonably it will then be a poorer test for differences than a test constructed to examine only differences.

Finally we now present a survey of the asymptotical significance levels, and rejecting levels \( x(\alpha/(K(K-1))) \) for multiple N-tests of each hypothesis \( H_{ij} : d_i = d_j \). The level at each comparison \( H_{ij} \) is given by

\[
\alpha_K = \lim_{n} \text{P (stating} \ d_i \neq d_j \ | \ d_i = d_j) = 2\alpha/K(K-1).
\]
Table 2.

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>20</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha=0.01$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha_K$</td>
<td>0.01</td>
<td>0.0033</td>
<td>0.0067</td>
<td>0.0100</td>
<td>0.0067</td>
<td>0.0036</td>
<td>0.0022</td>
<td>0.00053</td>
<td>0.00023</td>
</tr>
<tr>
<td>$x(\frac{K}{2})$</td>
<td>2.58</td>
<td>2.94</td>
<td>3.15</td>
<td>3.29</td>
<td>3.40</td>
<td>3.57</td>
<td>3.7</td>
<td>4.0</td>
<td>4.3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>20</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha=0.05$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha_K$</td>
<td>0.05</td>
<td>0.01667</td>
<td>0.00833</td>
<td>0.00500</td>
<td>0.00333</td>
<td>0.00179</td>
<td>0.00111</td>
<td>0.000263</td>
<td>0.000115</td>
</tr>
<tr>
<td>$x(\frac{K}{2})$</td>
<td>1.96</td>
<td>2.39</td>
<td>2.64</td>
<td>2.81</td>
<td>2.93</td>
<td>3.13</td>
<td>3.26</td>
<td>3.66</td>
<td>3.9</td>
</tr>
</tbody>
</table>

II.6. An example

We shall investigate the association between age and participation in the Storting elections in Norway in 1969 and in the advisory referendum on Norway's accession to the EC, for women and men separately. The interview surveys were undertaken by the Central Bureau of Statistics of Norway and are presented in [5] and [6].

In this case four tables are to be compared. For each table we present the portion of voters/non-voters within each age group.

Table 1. Age-participation in Storting elections 1969, females.

<table>
<thead>
<tr>
<th>Age</th>
<th>Voters</th>
<th>Non-voters</th>
<th>Number of respondents</th>
</tr>
</thead>
<tbody>
<tr>
<td>20-24 years</td>
<td>0.880</td>
<td>0.120</td>
<td>108</td>
</tr>
<tr>
<td>25-29</td>
<td>0.885</td>
<td>0.115</td>
<td>96</td>
</tr>
<tr>
<td>30-49</td>
<td>0.923</td>
<td>0.077</td>
<td>518</td>
</tr>
<tr>
<td>50-69</td>
<td>0.903</td>
<td>0.097</td>
<td>452</td>
</tr>
<tr>
<td>70-79</td>
<td>0.788</td>
<td>0.212</td>
<td>137</td>
</tr>
<tr>
<td>All ages</td>
<td>0.895</td>
<td>0.105</td>
<td>1311</td>
</tr>
</tbody>
</table>

Source: [5], table 3.
Table 2. Age-participation in Storting elections 1969, males.

<table>
<thead>
<tr>
<th>Age</th>
<th>Voters</th>
<th>Non-voters</th>
<th>Number of respondents</th>
</tr>
</thead>
<tbody>
<tr>
<td>20-24 years</td>
<td>0.822</td>
<td>0.178</td>
<td>152</td>
</tr>
<tr>
<td>25-29</td>
<td>0.872</td>
<td>0.128</td>
<td>109</td>
</tr>
<tr>
<td>30-49</td>
<td>0.918</td>
<td>0.082</td>
<td>563</td>
</tr>
<tr>
<td>50-69</td>
<td>0.930</td>
<td>0.070</td>
<td>459</td>
</tr>
<tr>
<td>70-79</td>
<td>0.935</td>
<td>0.065</td>
<td>108</td>
</tr>
<tr>
<td>All ages</td>
<td>0.909</td>
<td>0.091</td>
<td>1391</td>
</tr>
</tbody>
</table>

Source: [5], table 3.

Table 3. Age-participation in the EC referendum 1972, females.

<table>
<thead>
<tr>
<th>Age</th>
<th>Voters</th>
<th>Non-voters</th>
<th>Number of respondents</th>
</tr>
</thead>
<tbody>
<tr>
<td>20-24 years</td>
<td>0.673</td>
<td>0.327</td>
<td>52</td>
</tr>
<tr>
<td>25-29</td>
<td>0.821</td>
<td>0.179</td>
<td>56</td>
</tr>
<tr>
<td>30-49</td>
<td>0.803</td>
<td>0.197</td>
<td>208</td>
</tr>
<tr>
<td>50-69</td>
<td>0.859</td>
<td>0.141</td>
<td>205</td>
</tr>
<tr>
<td>70-79</td>
<td>0.745</td>
<td>0.255</td>
<td>47</td>
</tr>
<tr>
<td>All ages</td>
<td>0.808</td>
<td>0.192</td>
<td>568</td>
</tr>
</tbody>
</table>

Source: [6], table 3.2
Table 4. Age-participation in the EC referendum 1972, males.

<table>
<thead>
<tr>
<th>Age</th>
<th>Voters</th>
<th>Non-voters</th>
<th>Number of respondents</th>
</tr>
</thead>
<tbody>
<tr>
<td>20-24 years</td>
<td>0.797</td>
<td>0.203</td>
<td>74</td>
</tr>
<tr>
<td>25-29</td>
<td>0.893</td>
<td>0.107</td>
<td>56</td>
</tr>
<tr>
<td>30-49</td>
<td>0.897</td>
<td>0.103</td>
<td>214</td>
</tr>
<tr>
<td>50-69</td>
<td>0.872</td>
<td>0.128</td>
<td>195</td>
</tr>
<tr>
<td>70-79</td>
<td>0.929</td>
<td>0.071</td>
<td>42</td>
</tr>
<tr>
<td>All ages</td>
<td>0.878</td>
<td>0.122</td>
<td>581</td>
</tr>
</tbody>
</table>

Source: [6], table 3.2

We will regard participation as an indicator of the interest for voting. Then it seems reasonable to regard "voters" and "non-voters" as ordered characteristics. We then have an ordered case, and from [1], part 1 we choose the ordinal measure \( \gamma \) as a measure of association for the four tables. Let then \( \gamma_i \) denote the measure \( \gamma \) in table \( i \), for \( i = 1, \ldots, 4 \). Notice that, roughly speaking, \( \gamma < 0 \) means that the probability of voting increases with increasing age. The estimator of \( \gamma_i \) is denoted by \( \hat{\gamma}_i \), and the estimator of the asymptotic variance of \( \sqrt{n_i} \hat{\gamma}_i \) is denoted by \( S_{\gamma,i}^2 \) (see [1], part 2).

The number of observations in the tables are:

\[ n_1 = 1311, n_2 = 1391, n_3 = 568, n_4 = 581. \]

The tables are obviously independent since they are from different samples.

We find the following results:

\[ \hat{\gamma}_1 = 0.1499, \quad S_{\gamma,1}^2 = 7.208 \]
\[ \hat{\gamma}_2 = -0.2517, \quad S_{\gamma,2}^2 = 7.022 \]
\[ \hat{Y}_3 = -0.1285 \quad S^2_{Y,3} = 3.994 \]
\[ \hat{Y}_4 = -0.1155 \quad S^2_{Y,4} = 5.619 \]

We will now try to order \( Y_1, Y_2, Y_3, Y_4 \) as far as possible by multiple \( N \)-tests.

We choose a level \( \alpha = 0.10 \) so that \( x(\frac{\alpha}{k(k-1)}) = 2.40 \).

From (6) we assert \( Y_i > Y_j \) if
\[ T_{ij}^0 = \frac{\hat{Y}_i - \hat{Y}_j}{\sqrt{\frac{S^2_{Y,i}}{n_i} + \frac{S^2_{Y,j}}{n_j}}} > 2.40 \]

Results:
\[ T_{12}^0 = 3.91 \quad T_{13}^0 = 2.49 \quad T_{14}^0 = 2.15 \quad T_{32}^0 = 1.12 \quad T_{42}^0 = 1.12 \quad T_{43}^0 = 0.10 \]

Conclusion: \( Y_1 > Y_2 \) and \( Y_1 > Y_3 \).

This means that in table 2 and 3 there is a larger tendency of more voting as age increases than in table 1.

It would be interesting to estimate the difference between average association in 1969 and in 1972, i.e. \( \frac{1}{2}(Y_1 + Y_2) - \frac{1}{2}(Y_3 + Y_4) \), and the difference between average association among females and among males, i.e. \( \frac{1}{2}(Y_1 + Y_3) - \frac{1}{2}(Y_2 + Y_4) \). If one is interested in such quantities in addition to comparing \( Y_i \) and \( Y_j \), one can instead of using multiple \( N \)-tests use the procedure in II.3i) with the simultaneous intervals (16). This method gives among others the following confidence intervals:

\[
\begin{align*}
0.1152 & \leq Y_1 - Y_2 \leq 0.6880 & -0.0338 & \leq Y_1 - Y_3 \leq 0.5906 \\
-0.0783 & \leq Y_1 - Y_4 \leq 0.6091 & -0.1834 & \leq Y_3 - Y_2 \leq 0.4298 \\
-0.2024 & \leq Y_4 - Y_2 \leq 0.4748 & -0.3477 & \leq Y_4 - Y_3 \leq 0.3737
\end{align*}
\]
Simultaneous confidence level for these intervals is approximately equal to 0.90.

Instead of comparing the $y_i$'s we could compare the strength of association in the four tables by considering $y^2$ as a measure of degree of association. If we do so by multiple N-tests we find no differences between the tables. Simultaneous confidence intervals for the same functions in $y^2_i$ as those in $y_i$ above can be obtained from (16), by noting that the asymptotic variance of $\sqrt{n_i} y^2_i$ is equal to $4 \hat{\sigma}_i^2 \hat{\mu}_i^2$ (see [1], part 2). This method gives with confidence level equal to 0.90 the following intervals:

\[
\begin{align*}
-0.1583 & \leq y_1^2 - y_2^2 \leq 0.0766 & -0.0804 & \leq y_1^2 - y_3^2 \leq 0.0923 \\
-0.0796 & \leq y_1^2 - y_4^2 \leq 0.0978 & -0.1633 & \leq y_3^2 - y_2^2 \leq 0.0696 \\
-0.1682 & \leq y_4^2 - y_2^2 \leq 0.0682 & -0.0905 & \leq y_4^2 - y_3^2 \leq 0.0842 \\
\end{align*}
\]

\[
\begin{align*}
-0.0452 & \leq \frac{1}{2}(y_1^2 + y_2^2) - \frac{1}{2}(y_3^2 + y_4^2) \leq 0.1020 \\
-0.0920 & \leq \frac{1}{2}(y_1^2 + y_3^2) - \frac{1}{2}(y_2^2 + y_4^2) \leq 0.0542 \\
0 & \leq \frac{1}{2}(y_1^2 + y_3^2) \leq 0.0627 & 0 & \leq \frac{1}{4}(y_1^2 + y_2^2 + y_3^2 + y_4^2) \leq 0.0655 \\
\end{align*}
\]
REFERENCES


