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UMVU-ESTIMATORS BASED ON CONDITIONAL UMVU-ESTIMATORS

by

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Summary

Let $\hat{a}$ be an estimator of $a(\theta)$ and Y a statistic which is sufficient and complete for each fixed value of θ. Torgersen (1965) has stated that if $\hat{a}$ is unbiased, then it is conditionally unbiased, given Y, and if it is UMVU conditionally, given Y, then it is UMVU also unconditionally. This paper provides a rigorous proof after introducing (necessary?) additional conditions. A principle of conditional inference, due to Sandved, and the UMVU-criterion are elucidated by the theorem. A sufficient condition for the non-existence of an unbiased estimator is derived from the theorem. It is suggested that statistical methods should be judged after one has deleted from the sample space such events, which in a certain sense contain no information concerning the parameter (-component) in interest.

1. Introductory remarks on an earlier result by Torgersen and Sandved.

Sandved (1968) considers two observable random variates X and Y, possibly vectorvalued. The marginal distribution of Y depends on a parameter ν, and the conditional distribution of X, given Y, depends on a parameter θ. θ and ν may be vectorvalued, and they take on values independently of each other, i.e. (θ,ν) may be any point in a product space $\Theta \times N$. The parameter of interest is θ, while ν is nuisance. In this situation Sandved speaks of Y as <u>ancillary for θ</u>, and she states as a principle that inference concerning θ should be made conditionally, given the ancillary statistic Y. This principle can be motivated by considering the experiment as being performed in two steps: First we observe the value of Y. This part of the experiment contains no information concerning the value of θ, since the distribution of Y depends only on ν, and θ takes on values independently of ν. Next we observe X, given the value of Y, and from this additional experiment we can draw inference concerning the value of θ. Thus the value of Y contains in itself no information about θ, but it is decisive for the amount of information that X gives concerning θ.

We want a point estimate of a realvalued parameter $a(\theta)$. An estimator $\hat{a}$ is <u>unbiased</u> if $E_{\theta,\nu}\hat{a} = a(\theta)$ for all (θ,ν), and it is a <u>UMVU-estimator</u> (Uniformly Minimum Variance Unbiased estimator) if it is unbiased and any other unbiased estimator a^* satifies $\text{var}_{\theta,\nu}a^* \geq \text{var}_{\theta,\nu}\hat{a}$ - or equivalent

$E_{\theta,\nu}a^{*2} \geq E_{\theta,\nu}\hat{a}^2$ - for all (θ,ν) .
Sandved (1968) states that if the ancillary statistic Y is also complete, then

$$\hat{a} \text{ is unbiased} \Rightarrow \hat{a} \text{ is conditionally unbiased, given } Y=y \;, \tag{1.1}$$

and

$$\hat{a} \text{ is UMVU conditionally, given } Y \Rightarrow \hat{a} \text{ is UMVU.} \tag{1.2}$$

(Fraser (1956) points out that if there exists an unbiased estimator of the realvalued parameter $b(\nu)$, which is a function of Y , then it is essentially unique UMVU. (1.2) constitutes a supplement to Fraser's result.) We give an example where (1.2) is useful:

<u>Example 1</u>. The intensities λ_1 and λ_2 of two independent poisson processes are to be compared by estimating $\theta = \lambda_1/(\lambda_1+\lambda_2)$, the expected number of poisson events in process number 1 per time unit in fractions of the expected total number of poisson events per time unit. One decides to observe the two processes during a time period of length t_0 and, if necessary, to continue the observation beyond this period until a total number of at least m events are recorded. Let X_i be the number of recorded events in process no. i , $i=1,2$. The point of time when the experiment is finished is not recorded, so that (X_1,Y) is a sufficient statistic, where $Y = X_1+X_2$. The conditional distribution of X_1 , given $Y=y$, is binomial with parameters y and θ , i.e.

$$P_{\lambda_1,\lambda_2}(X_1=j\,|\,y) = \binom{y}{j}\theta^j(1-\theta)^{y-j} \;,\; j=0,1,\dots,y \;,\quad y=m,m+1,\dots$$

Let N be the (unobserved) number of claims in the first time period of length t_o. N has a poisson distribution with parameter $\nu = (\lambda_1+\lambda_2)t_o$. $Y = \max(N,m)$, and hence the distribution of Y depends on (λ_1,λ_2) only through ν. Suppose nothing is known a priori concerning the values of λ_1 and λ_2, except that they are strictly positive. Then $(\theta,\nu)\in(0,1)X(0,\infty)$, and hence Y is ancillary for θ according to Sandved's definition. Furthermore, since N is complete, Y is also complete, and by (1.2) the conditional UMVU-estimator $\hat{\theta}=X_1/Y$ is UMVU also unconditionally.

The result (1.2) is of some fundamental interest, by characterizing a (nonvoid) class of inference problems where Sandved's conditioning principle is supported by the "classical" UMVU-criterion.

Torgersen presented the results (1.1) and (1.2) as early as in 1965 in a discussion at the Nordic Conference on Mathematical Statistics. Instead of assuming that Y is ancillary for θ and complete, he assumed only that Y is sufficient and complete for ν for each fixed value of θ.

Example 1 (continued). Suppose that $\lambda_2 > 0$ and $\lambda_1 \geq a$ is known a priori. Then $\theta\in(0,1)$ and $\nu \geq t_o a/\theta$, and Y is no longer ancillary for θ. But still Y is sufficient for each fixed value of θ, and Y is also complete for each fixed value θ_o of θ, since the distribution of N belongs to a regular Darmois-Koopman exponential family of distributions when $\nu \geq t_o a/\theta_o$. Hence, according to Torgersen's version of (1.2) $\hat{\theta}$ is still a UMVU-estimator of θ .

This example lays open an important consequence of Torgersen's result, namely the existense of estimation problems where the UMVU-criterion leads to conditional inference even if ancillarity considerations do not apply.

However, the UMVU-criterion should by no means be considered as canonical, and the above example could as well be turned to criticism of this criterion: Y is an estimator of ν , and therefore Y contains information about θ when $\nu \geq t_0 a/\theta$. Conditioning with respect to Y amounts to neglecting this information.

It is interesting to note that the UMVU-property of $\hat{\theta}$ is no longer implied by (1.2) when ν is known, since Y is not complete then. Thus, even if $\hat{\theta}$ in example 1 is optimal (in the UMVU-sense) when ν is unknown, it might happen that knowledge of ν can be utilized so as to construct another unbiased estimator which has smaller variance than $\hat{\theta}$ for some θ .

Torgersen gave the following proof of (1.1) and (1.2):

$$(1.1):\quad E_{\theta,\nu}\hat{a} \equiv a(\theta) \Rightarrow E_{\theta,\nu}E_{\theta}(\hat{a}|Y) \equiv a(\theta)$$

$$\Rightarrow \forall\theta[E_{\theta}(\hat{a}|Y) = a(\theta) \text{ a.s.}] .$$

The last implication is due to the completeness of Y for each fixed θ .

$$(1.2):\quad E_{\theta,\nu}a^* \equiv a(\theta) \Rightarrow \forall\theta[E_\theta(a^*|Y) = a(\theta) \text{ a.s.}]$$

$$\Rightarrow \forall\theta[E_\theta(a^{*2}|Y) \geq E_\theta(\hat{a}^2|Y) \text{ a.s.}]$$

$$\Rightarrow \forall(\theta,\nu)[E_{\theta,\nu}a^{*2} \geq E_{\theta,\nu}\hat{a}^2] .$$

A careful examination of the statements (1.1) and (1.2) reveals that the concepts "conditionally unbiased" and "conditional UMVU-estimator" must be defined precisely and that further assumptions are needed to make the above proof rigorous. General versions of (1.1) and (1.2) are stated and proved in section 3.

2. Basic assumptions and notations.

Let X be an observable random variate, whose distribution belongs to a parametrized family $\mathcal{P} = \{P_\omega\}_{\omega\in\Omega}$ of probability distributions over the Borel class $\mathcal{A}$ in a Euclidean space $\mathcal{X}$. Expectation and variance with respect to P_ω is denoted by E_ω and var_ω respectively. The parameter of interest is the function $\psi : \Omega \to \Theta$, where Θ may be taken as the image under ψ of Ω. Let $Y = Y(X)$ be a statistic which takes on values in the measurable space $(\mathcal{Y},\mathcal{B})$. The sampling distribution of Y is denoted by P_ω^Y. We introduce the notation $\mathcal{P}^Y = \{P_\omega^Y\}_{\omega\in\Omega}$, and for each $\theta \in \Theta$ we define $\mathcal{P}_\theta = \{P_\omega\}_{\omega\in\psi^{-1}(\{\theta\})}$ and $\mathcal{P}_\theta^Y = \{P_\omega^Y\}_{\omega\in\psi^{-1}(\{\theta\})}$.

If Y is sufficient for $\mathcal{P}_\theta$, then by definition there exists for each $A \in \mathcal{A}$ a joint version of the conditional probability of A given Y for all P_ω in $\mathcal{P}_\theta$, say

$\bar{P}_\theta(A|Y)$. Since $\mathcal{X}$ is Euclidean, we can choose the functions $\bar{P}_\theta(A|\cdot)$ such that for each fixed y , $\bar{P}_\theta(\cdot|y)$ is a probability measure over $\mathcal{A}$, viz. the conditional distribution of X , given $Y = y$, relative to any P_ω in $\mathcal{P}_\theta$. We denote expectation and variance with respect to $\bar{P}_\theta(\cdot|y)$ by $\bar{E}_\theta(\cdot|y)$ and $\overline{\text{var}}_\theta(\cdot|y)$ respectively. If Y is sufficient for $\mathcal{P}_\theta$ for each θ in Θ , we define for each y the class $\bar{\mathcal{P}}_y = \{\bar{P}_\theta(\cdot|y)\}_{\theta\in\Theta}$.

3. UMVU-estimators based on conditional UMVU-estimators .

Let a be a realvalued function defined on Θ . We will now prove general versions of (1.1) and (1.2), and hence we assume that

Y is sufficient and complete for $\mathcal{P}_\theta$ for each θ in Θ .(3.1)

Then $\bar{P}_\theta(\cdot|y)$ and $\bar{\mathcal{P}}_y$ can be defined as in section 2. To make the proofs rigorous we shall also need one of the following additional (sets of) assumptions:

(i) $\mathcal{P}^Y$ is homogeneous, (which means that any two measures in $\mathcal{P}^Y$ are absolutely continuous with respect to each other), and (ii) $\mathcal{Y}$ is countable, (3.2)

or

(i) $\mathcal{P}^Y$ is homogeneous, (ii) a is continuous,
(iii) Θ is Euclidean and hence posesses a countable dense subset Θ' , (iv) (the conditional distributions $\bar{P}_\theta(\cdot|y)$ can be chosen such that) for each y the class

$\bar{\mathcal{P}}_y$ is dominated by a σ-finite measure $\varkappa_y$, and the density $f_\theta(x|y) = d\bar{P}_\theta(x|y)/d\varkappa_y$ is a continuous function of θ, and (v) for each y and θ_o we can find a constant $c > 0$ and a finite set $\{\theta_1,\ldots,\theta_k\}$ in Θ' such that $f_\theta(x|y) \leq c \sum_{i=1}^{k} f_{\theta_i}(x|y)$ for all θ in some neighbourhood (in Θ) of θ_o. (c and $\{\theta_1,\ldots,\theta_k\}$ may depend on y and θ_o.) (3.3)

Lemma 3.4 Assume that (3.1) and (at least) one of the conditions (3.2) and (3.3) are true. Then

$E_\omega \hat{a} \underset{\omega}{\equiv} a[\psi(\omega)] \Rightarrow \bar{E}_\theta(\hat{a}|y) \underset{\theta}{\equiv} a(\theta)$ for $y \notin N$, where N is a $\mathcal{P}^Y$ - nullset.

Proof. By the rule of "double expectation" the left side of the implication is $E_\omega \bar{E}_{\psi(\omega)}(\hat{a}|Y) \underset{\omega}{\equiv} a[\psi(\omega)]$, or equivalent $E_\omega \bar{E}_\theta(\hat{a}|Y) = a(\theta)$ for all $\omega \in \psi^{-1}(\{\theta\})$ and all θ. Hence, by the assumed completeness of Y for each θ, it follows that

$$\text{for each } \theta \text{ in } \Theta,\ \bar{E}_\theta(\hat{a}|y) = a(\theta) \text{ for all } y \text{ except (possibly) those in a } \mathcal{P}^Y_\theta \text{ - nullset } N_\theta. \tag{3.5}$$

The trouble now arizes that Θ is usually uncountable, and therefore the union of all exeptional sets N_θ need not be contained in a $\mathcal{P}^Y$-nullset. Thus we can not conclude directly from (3.5) that $\hat{a}$ is conditionally unbiased in the sense of the lemma. To solve this difficulty, assume first that (3.2) is satisfied. Then, by the homogeneity of $\mathcal{P}^Y$, each N_θ is a $\mathcal{P}^Y$-nullset. Moreover, the countability of $\mathcal{Y}$ implies that $\mathcal{B}$ is countable, and hence there is at most a countable number of distinct

sets N_θ . It follows that the union $N = \bigcup_{\theta\in\Theta} N_\theta$ of the sets N_θ in (3.5) is a $\mathcal{P}^Y$-nullset, and hence the right side of the implication in the lemma is true. Assume instead that (3.3) is satisfied. By the assumed homogeneity of $\mathcal{P}^Y$ all N_θ are $\mathcal{P}^Y$-nullsets, and hence also $N = \bigcup_{\theta\in\Theta'} N_\theta$ is a $\mathcal{P}^Y$-nullset. We now fix a $y \notin N$ and a $\theta_o \in \Theta$ and let c and $\{\theta_1,\dots,\theta_k\}$ be as in (v) . Let $\{\theta_n'\}$ be a sequence in Θ' which converges to θ_o . By (v) we have
$|\hat{a}(x)| f_{\theta_n'}(x|y) \leq c \sum_{i=1}^{k} |\hat{a}(x)| f_{\theta_i}(x|y)$ for n large enough.
Considered as a function of x the right side of this inequality is $\varkappa_y$-integrable, since $\bar{E}_{\theta_i}(\hat{a}|y)$ exists (and equals $a(\theta_i)$) for $i = 1,\dots,k$. By Lebesgue's theorem of dominated convergence and the assumed continuity in θ of $f_\theta(x|y)$ and $a(\theta)$ it then follows that

$$a(\theta_o) = \lim_{n\to\infty} a(\theta_n') = \lim_{n\to\infty} \int \hat{a}(x) f_{\theta_n'}(x|y)d\varkappa_y(x)$$

$= \int \hat{a}(x) f_{\theta_o}(x|y)d\varkappa_y = \bar{E}_{\theta_o}(\hat{a}|y)$. Since θ_o was an arbitrary point in Θ , the right side of the implication in the lemma is true. $\square$

<u>Theorem 3.6</u> Assume that (3.1) and (at least) one of the assumptions (3.2) and (3.3) are true. Then, $\hat{a}$ is a UMVU-estimator of $a(\theta)$ with respect to $\bar{\mathcal{P}}_y$ for all y , and $E_\omega \hat{a}^2 < \infty$ for all ω => $\hat{a}$ is a UMVU-estimator of $a[\psi(\omega)]$.

<u>Proof.</u> $\hat{a}$ is unbiased, since $E_\omega \hat{a} = E_\omega \bar{E}_{\psi(\omega)}(\hat{a}|Y) = E_\omega a[\psi(\omega)] = a[\psi(\omega)]$ for all ω . Let a^* be any other unbiased estimator of $a[\psi(\omega)]$. Then, by lemma 3.4, a^* is an unbiased estimator of $a(\theta)$ with respect to $\bar{\mathcal{P}}_y$ for all y except possibly

those in a $\mathcal{P}^Y$-nullset N. By the conditional UMVU-property of $\hat{a}$ we have $\bar{E}_\theta(\hat{a}^2|y) \leq \bar{E}_\theta(a^{*2}|y)$ for $y \notin N$. Integrating with respect to P_θ^Y we get $E_{\psi(\omega)}\hat{a}^2 \leq E_{\psi(\omega)}a^{*2}$ for all ω such that $E_\omega a^{*2} < \infty$. (If $E_\omega a^{*2} = \infty$, this inequality is trivially satisfied.) This proves that $\hat{a}$ is UMVU. □

It is easily verified that the conditions (3.1) and (3.2) are satisfied in example 1. Hence the conclusions in that example are valid.

Example 2. The joint density of $X_1, X_2, \ldots, X_n, Y$ (with respect to the Lebesgue measure in $\mathbb{R}^{n+1}$) is

$$(2\pi)^{-\frac{n+1}{2}}(\sigma^2+y^2)^{-\frac{n}{2}}\exp\left\{-\frac{1}{2(\sigma^2+y^2)}\sum_{i=1}^{n}x_i^2 - \frac{1}{2}(y-\eta)^2\right\},$$

$$(x_1,\ldots,x_n,y) \in \mathbb{R}^{n+1}, \quad \sigma^2 > 0, \quad \eta \in \mathbb{R}.$$

By integration with respect to $x_1,\ldots,x_n$ we find that $Y \sim N(\eta,1)$. As conditional distribution of $X_1,\ldots,X_n$, given $Y=y$, we can take the n-variate normal distribution with density

$$f_{\sigma^2}(x_1,\ldots,x_n|y) = (2\pi)^{-\frac{n}{2}}(\sigma^2+y^2)^{-\frac{n}{2}}\exp\left\{-\frac{1}{2(\sigma^2+y^2)}\sum_{i=1}^{n}x_i^2\right\}.$$

A UMVU-estimator of σ^2+y^2 in this distribution is $\frac{1}{n}\sum_{i=1}^{n}X_i^2$ and hence $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}X_i^2 - Y^2$ is a conditional UMVU-estimator of σ^2, given Y. Y is sufficient and complete for each fixed value of σ^2. To establish that $\hat{\sigma}^2$ is a UMVU-estimator of σ^2 also unconditionally, it is sufficient to verify that condition (3.3) is satisfied. Only (v) deserves a comment:

For any fixed σ^2 and y and $\varepsilon > 0$ we have

$$f_{\sigma^2}(x_1,\ldots,x_n|y) \leq (2\pi)^{-\frac{n}{2}}(\sigma^2+y^2)^{-\frac{n}{2}}\exp\left\{-\frac{1}{2(\sigma^2+\varepsilon+y^2)}\sum_{i=1}^{n}x_i^2\right\}$$

$$= \left(\frac{\sigma^2+\varepsilon+y^2}{\sigma^2+y^2}\right)^{\frac{n}{2}} f_{\sigma^2+\varepsilon}(x_1,\ldots,x_n|y) .$$

Let Θ' be the set of rationals in $(0,\infty)$. By choosing ε such that $\sigma^2+\varepsilon \in \Theta'$, we see that condition (v) is satisfied with $c = \left(\frac{\sigma^2+\varepsilon+y^2}{\sigma^2+y^2}\right)^{\frac{n}{2}}$. Hence $\hat{\sigma}^2$ is a UMVU-estimator of σ^2 . This conclusion remains true if η depends on σ^2 . We only require that the set of possible η - values includes an open interval for each fixed value of σ^2 .

4. A result on the non-existence of an unbiased estimator.

We start by modifying example 1:

Example 3. Suppose in example 1 that one stops observing at time t_0 . Then $Y = N$ is poisson distributed with parameter $\nu = (\lambda_1+\lambda_2)t_0$. Lemma (3.4) states that any unbiased estimator of $\theta = \lambda_1/(\lambda_1+\lambda_2)$ is a.s. $(\mathcal{P}^Y)$ conditionally unbiased, given Y . But $Y = 0$ is a nonnull event, and the conditional distribution of (X_1,X_2) , given $Y = 0$, is the one-point distribution in $(0,0)$, which provides no (conditionally) unbiased estimator of θ . Hence, in this situation θ has no unbiased estimator, (and no UMVU-estimator).

Example 4. Let $(X_1,X_2,n-X_1-X_2)$ be trinomially distributed with probability distribution

$$P_{\pi_1,\pi_2}(X_1=x_1,X_2=x_2) = \frac{n!}{x_1!x_2!(n-x_1-x_2)!}\pi_1^{x_1}\pi_2^{x_2}(1-\pi_1-\pi_2)^{n-x_1-x_2}$$

for $x_1, x_2 = 0, 1, \ldots, n$, $x_1 + x_2 \leq n$, where $\pi_1, \pi_2 > 0$ and $\pi_1 + \pi_2 < 1$. $Y = X_1 + X_2$ is binomially distributed with parameters n and $\nu = \pi_1 + \pi_2$. Given $Y = y > 0$,the conditional distribution of X_1 is binomial with parameters y and $\theta = \frac{\pi_1}{\pi_1 + \pi_2}$, and X_1/Y is a conditional UMVU-estimator of θ . Conditions (3.1) and (3.2) are easily verified. However, given $Y = 0$, the conditional distribution of (X_1, X_2) is concentrated in $(0,0)$, and hence a conditionally unbiased estimator of θ does not exist for this value of Y . By the same argument as in example 3, we conclude that there exists no unbiased estimator and hence no UMVU-estimator of θ . (It has erronously been claimed that X_1/Y is UMVU in this model.)

The following theorem exposes the gist of the matter in the last two examples.

Theorem 4.1. Assume that (3.1) and at least one of the conditions (3.2) and (3.3) are true, and that the conditional distributions $\bar{P}_\theta(\cdot|y)$ can be chosen independently of θ on a set in $\mathcal{B}$ which is not a $\mathcal{P}^Y$-nullset. Then $a[\psi(\omega)]$ has no unbiased estimator, unless a is constant on Θ .

The proof is a trivial consequence of lemma 3.4.

5. Irrelevant events. A possible principle of conditioning.

For each $\omega \in \Omega$ and $A \in \mathcal{A}$ let $P_{\omega|A}$ be the measure P_ω restricted to the σ - field of subevents of A . A set in $\mathcal{A}$, which is not a $\mathcal{P}$ - nullset, will be called irrelevant for the parameterfunction $\psi : \Omega \to \Theta$, or ψ-irrelevant, if the family $\{P_{\omega|A}\}_{\omega\in\psi^{-1}(\{\theta\})}$ does not depend on θ. Suppose A is ψ - irrelevant. The event $X \in A$ is equally likely for all values of ψ in the sense that the class of probabilities $\{P_\omega(A)\}_{\omega\in\psi^{-1}(\{\theta\})}$ is the same for all θ in Θ. Thus the occurence of the event $X \in A$ gives in itself no information on ψ. Moreover it follows from the definition above that the class of conditional distributions $\{P_\omega(\cdot|X\in A)\}_{\omega\in\psi^{-1}(\{\theta\})}$ is independent of θ also, so that when X is known to be in A, no additional information on ψ can be obtained by specifying X further. Since in this sense the event $X\in A$ contains no information on ψ it seems reasonable to disregard A and judge any method for making inference about ψ after conditioning on the complementary event $X\in\mathcal{X}-A$. However the problem arizes that there might be more than one ψ-irrelevant set. In particular, if P_ω is absolutely continuous with respect to the Lebesgue measure and A is ψ-irrelevant, then $A\cup\{x\}$ is ψ-irrelevant for all $x \in\mathcal{X}$. The union of all ψ-irrelevant sets is therefore $\mathcal{X}$, which itself is not ψ-irrelevant. Examples are easily constructed which show that not even finite unions of ψ-irrelevant sets need to be ψ-irrelevant. Thus, when there are more than one ψ-irrelevant set, conclusions based on the suggested conditioning principle may depend on the arbitrary choice of ψ-irrelevant set. However, when there exists a ψ-irrelevant set A which contains all other ψ-irrelevant sets as subsets, then it seems reasonable to judge any statistical method for drawing inference about (only) ψ by studying its operating characteristic conditionally, given $X \in \mathcal{X} - A$. In example 3, when λ_1 and λ_2 are > 0, $Y = 0$ is the only event which is irrelevant for θ. When $Y > 0$, X_1/Y is a conditional UMVU-estimator of θ, given Y. Y is complete for each fixed value of θ also conditionally, given $Y > 0$. Hence, by theorem 3.6, X_1/Y is a UMVU-estimator of θ when attention is restricted to the relevant part of the sample space, $\{(X_1,X_2) | X_1+X_2 > 0\}$. Example 4 may be commented likewise.

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