ON THE WAITING TIME DISTRIBUTION OF BULK QUEUES

by

John Dagsvik
Abstract

The waiting time process of the n-th arriving group is considered for the general bulk queueing model $GI^X/G^Y/1$.

A generalisation of Lindley's waiting time equation is established.

By a generalisation of Kingman's method [3], this equation is solved for the models $GI^X/E_k^Y/1$ and $E_k^X/G^Y/1$.

When the service time is Erlang distributed $E_k$, the results are applied to the case where the service- and the arrival groups are of constant size.

Key words: Wendel projection, Group Waiting time, Restbatch, Waiting time equation, Erlang distributions, Hyperexponential distribution, Stationary distributions.
Contents

1. Introduction 1
2. The algebraic formalism 1
3. Group waiting time 3
4. The model $\text{GI}^X/\text{E}_k^Y/1$ 12
5. Special cases 16
6. The model $\text{E}_k^X/\text{G}^Y/1$ 19
7. The stationary solution 21
1. Introduction

In the present paper we shall assume that customers arrive in groups $G_n$, $n = 0,1,2,\ldots$. The group size is a stochastic variable $X$, with probability distribution $f(.)$. The inter-arrival intervals $A_n$, $n = 0,1,\ldots$ are independent and have the same distribution $a(.)$. The service mechanism is described as follows: At the end of a service period the server accepts $Y$ customers from the waiting line, or a smaller number if the line is shorter. $Y$ is called the service group capacity. The length of the service time $B$, has the distribution $b(.)$. We shall assume the existence of two integers $m, 1$ such that $X \leq 1$, $Y \leq m$.

The most general works on bulk queues seems to be those of Keilson [2], Cohen [1], Le Gall [5], Lambotte and Teghem [4]. They obtain the distribution of the queue length from which the waiting time distribution is derived. However, there exists no such results for general distributions $a(.)$, $b(.)$, $f(.)$ and $g(.)$. Earlier works are restricted to the case where $a(.)$ or $b(.)$ are the exponential distribution. Even if $b(.)$ is exponential the analysis are only limited to bulk service models (Cohen [1], Le Gall [5]).

2. The algebraic formalism.

Let $\Omega_n$ denote the set of $n \times n$ matrices whose components are finite complex measures on the Borel subsets of the real line. According to Kingman [3] the product of two measures is defined as their convolution. An operator $T : \Omega_1 \to \Omega_1$ is defined by
(2.1) \((Tv)(E) = v(E \cap R^+) + v(-R^+)\varepsilon(E), \ v \in \Omega_1,\)

where \(\varepsilon\) is the measure

\[
\varepsilon(E) = \begin{cases} 
1 & \text{if } 0 \in E \\
0 & \text{if } 0 \notin E
\end{cases}
\]

and \(R^+ = (0, \infty)\).

This operator has the property that if \(X\) is a random variable with distribution

\[
v(E) = \Pr\{X \in E\},
\]

then

\[
(Tv)(E) = \Pr\{X^+ \in E\}.
\]

Kingman shows that \(\Omega_1\) is an commutative algebra over the complex field \(\mathbb{C}\) with identity \(\varepsilon\). Furthermore, he shows that the image \(\Omega_1^+\) and the kernel \(\Omega_1^-\) of \(T\) are both disjoint subalgebras of \(\Omega_1\). \(T\) is extended to \(\Omega_n\) by \(T[v_{ij}] = \{Tv_{ij}\}\), \(\{v_{ij}\} \in \Omega_n\) and multiplication in \(\Omega_n\) is defined in the obvious way. With \(I_n\varepsilon\) as identity it is easy to verify that \(\Omega_n\) has the same properties as \(\Omega_1\) except that \(\Omega_n\) is no longer commutative. The norm on \(\Omega_n\) is defined by

\[
\|v\| = \max_j \sum_i |d_{v_{ij}}|, \ v = \{v_{ij}\}.
\]

The set whose elements are Fourier-Stieltjes transform of elements from \(\Omega_1\) we denote \(\hat{\Omega}_1\), and extend the definition to \(\hat{\Omega}_n\) in the obvious manner.
3. Group waiting time.

We shall study the group waiting time process defined as follows:

Definition 3.1

By the group waiting time \( W_n, n = 0, 1, 2, \ldots \) we mean the waiting time (excluding service time) of the first customer from \( C_n \) who is taken into service.

It is convenient to work with the random variable \( Z_n \) which is defined as the time \( C_n \) spends in the queue until there is no one ahead, except the ones being served. If \( L_n \) is the time interval from the instant when \( Z_n \) becomes zero and until the last service group with customers from \( C_n \) starts, then the sequence \( |Z_n|, n = 0, 1, 2, \ldots \) satisfies

\[
Z_{n+1} = (Z_n + L_n - A_n)^+,
\]

with the initial condition \( Z_0 = z \). We recognize the expression above as an equation of the similar type as the waiting time equation found by Lindley [6]. However, there is an important difference: The random variables \( Z_n \) and \( L_n \) are no longer independent as in the model \( GI/G/1 \).

Consider now the servicing of the \( n \)-th arrival group \( C_n \). Service may be performed in one or several groups. The last service group which has customers from \( C_n \) will be called the \( n \)-th rest batch. The rest batch may be filled with customers from \( C_n \) or there may be places available for customers from \( C_{n+1} \).
We define two random variables $T_n$ and $J_n$, as follows:

If the $n$-th restbatch can accept $s$ customers and contains only $t \leq s$ customers from $C_n$, then $|J_n| = s-t$. If the $n$-th restbatch contains customers from $C_{n-1}$; $J_n = -|J_n|$. Otherwise $J_n = |J_n|$. $T_n$ is the number of customers which can be accepted from $C_{n+1}$ in the first service group with customers from $C_{n+1}$. We define $J_{-1} = T_{-1}$; the capacity of the initial service group.

If $S(T_{n-1})$ denotes service time of $C_n$, excluding the service time of the $n$-th rest batch, we realize that $Z_n > 0$ implies that

$$(3.2) \quad I_n = \begin{cases} 
S(T_{n-1}) + B0, T_{n-1} = S'(T_{n-1}) & \text{when } J_n \geq 0 \\
0 & \text{when } J_n < 0,
\end{cases}$$

and $T_{n-1} = J_{n-1}$,

since $C_n$ will have to wait an extra service period when $J_{n-1} = 0$, $J_n \geq 0$. When $Z_n = 0$ the first service group from $C_n$ has capacity $Y$ and $C_n$ must wait for the time $W_n$ before service starts, hence

$$(3.3) \quad I_n = \begin{cases} 
S(T_{n-1}) + W_n & \text{when } J_n \geq 0 \\
0 & \text{when } J_n < 0,
\end{cases}$$

and $T_{n-1} = Y$.

We define matrices of distribution functions

$U_n(t) = \{U_n^i(t) \delta(j,0)\}$, $V_n(t) = \{V_n^i(t) \delta(j,0)\}$, $K(t) = \{K_{ij}\}$ and $H(t) = \{H_{ij}\}$

for $i,j = -(m-2), -(m-1), \ldots, m-1$, 


by

\[ U_n^i(t) = \Pr\{Z_n \leq t, J_{n-1} = i\}, \]

\[ V_n^i(t) = \Pr\{W_n \leq t, J_{n-1} = i\}, \]

\[
K_{ij} = \begin{cases} 
\Pr\{S'(T_{n-1}) \leq t, J_n = i | J_{n-1} = j, Z_n > 0\} & \text{when } i \geq 0 \\
\Pr\{0 \leq t, J_n = i | J_{n-1} = j\} & \text{when } i < 0 , \end{cases}
\]

\[
H_{ij} = \begin{cases} 
\Pr\{S'(T_{n-1}) \leq t, J_n = i | J_{n-1} = j, Z_n = 0\} & \text{when } i \geq 0 \\
0 & \text{when } i < 0 . \end{cases}
\]

If \( \mu \) is the probability distribution of a random variable \( X \), \( \mu^* \) denote the distribution of \( (-X) \).

Let

\[ M_n = Z_n + S'(J_{n-1}) - A_n, \quad N_n = W_n + S(Y) - A_n, \quad D_{ij} = \{J_n = i, J_{n-1} = j\} . \]

Considering the possible events between the \( n \)-th and the \( n+1 \)-th arrival we find

\[
(3.8 \ a) \quad U_{n+1}^i(t) = \sum_j \Pr\{(Z_n - A_n)_+ \leq t, Z_n > 0, D_{ij}\} \quad \text{when } i < 0 ,
\]

\[
(3.8 \ b) \quad U_{n+1}^i(t) = \sum_j \left[ \Pr\{M_n^+ \leq t, Z_n > 0, D_{ij}\} + \Pr\{N_n^+ \leq t, Z_n = 0, D_{ij}\} \right] \quad \text{when } i \geq 0 ,
\]

\[
(3.9 \ a) \quad V_{n+1}^i(t) = \sum_j \left[ \Pr\{(Z_n + 5(0, J_n) - A_n)_+ \leq t, Z_n - A_n > 0, D_{ij}\} ight.
\]

\[
+ \Pr\{(Z_n + A_n)_+ \leq t, Z_n - A_n \leq 0, Z_n > 0, D_{ij}\} \right]
\]

when \( i < 0 \). (3.9 a) can be written
\[ V_{n+1}^i(t) = \sum_j \Pr\{ (Z_n - A_n)^+ + \delta(0, J_n) B \leq t, D_{ij} \} \]

\[-\Pr\{ \delta(0, J_n) B \leq t, (Z_n - A_n)^+ = 0, D_{ij} \} \]

\[+\Pr\{ (Z_n - A_n)^+ B \leq t, Z_n > 0, D_{ij} \} \]

\[-\Pr\{ B \leq t, (Z_n - A_n)^+ = 0, Z_n > 0, D_{ij} \} \]

\[= ((b^\delta(0, i) - b)(u_{n+1}^i - \varepsilon u_{n+1}^i(0))(t) + \sum_j \Pr\{ (Z_n - A_n + B)^+ \leq t, Z_n > 0, D_{ij} \}. \]

When \( i \geq 0 \) the expression is more complicated:

\[ (3.9 \text{ b}) \quad V_{n+1}^i(t) = \sum_j \Pr\{ M_n + B \delta(0, J_n) \leq t, M_n > 0, Z_n > 0, D_{ij} \} \]

\[+\Pr\{ (M_n + B)^+ \leq t, M_n \leq 0, Z_n > 0, D_{ij} \} \]

\[+\Pr\{ N_n + B \delta(0, J_n) \leq t, N_n > 0, Z_n = 0, D_{ij} \} \]

\[+\Pr\{ (N_n + B)^+ \leq t, N_n \leq 0, Z_n = 0, D_{ij} \}. \]

If (3.9 b) is rewritten in the same way as (3.9 a) we obtain

\[ V_{n+1}^i(t) = ((b^\delta(0, i) - b)(u_{n+1}^i - \varepsilon u_{n+1}^i(0))(t) + \sum_j \Pr\{ (M_n + B)^+ \leq t, Z_n > 0, D_{ij} \} \]

\[+ \Pr\{ (N_n + B)^+ \leq t, Z_n = 0, D_{ij} \}, i \geq 0. \]
Since $Z_n > 0$ implies $W_n = Z_n + B\delta(0, J_{n-1})$, the probability of 
\{(N_n + B)^+ \leq t, Z_n = 0\} can be written

$$
\Pr\{N_n + B)^+ \leq t, Z_n = 0\} = \Pr\{(N_n + B)^+ \leq t\}
$$

$$
-\Pr\{(Z_n + S(1) + (1 + \delta(0, J_{n-1}))B - A_n)^+ \leq t, Z_n > 0\},
$$

whence

\[ (3.10 \ a) \quad U_{n+1}^i = \sum_j T(a \cdot K_{ij}(U_n^j - \epsilon U_n^j(0))) \text{ when } i < 0 , \]

\[ (3.10 \ b) \quad U_{n+1}^i = \sum_j [T(a \cdot (K_{ij} - b \delta(0, j) H_{ij})(U_n^j - \epsilon U_n^j(0))) + T(a \cdot H_{ij}V_n^j)] \text{ when } i \geq 0 \]

and

\[ (3.10 \ c) \quad V_{n+1}^i = (b \delta(0, i) - b)(U_{n+1}^i - \epsilon U_{n+1}^i(0)) + \sum_j T(a \cdot b K_{ij}(U_n^j - \epsilon U_n^j(0))) \text{ when } i < 0 , \]

\[ (3.10 \ d) \quad V_{n+1}^i = (b \delta(0, i) - b)(U_{n+1}^i - \epsilon U_{n+1}^i(0)) + \sum_j [T(a \cdot b (K_{ij} - b \delta(0, j) H_{ij})(U_n^j - \epsilon U_n^j(0))) + T(a \cdot b H_{ij}V_n^j)] \text{ when } i \geq 0 . \]

Hence we have established a set of equation for $U_n$ and $V_n$. 
Lemma 3.1

(i) \( K_{ij} = \begin{cases} f(|j|+i), & \text{when } i < 0 \\ \sum_{r \geq 1} b^r \sum_{p > 0} (f(g^*)^{r-1})(p+|j|)g(i+p)+b(0,i) f(|j|) e^r & \text{when } i \geq 0 , \end{cases} \)

(ii) \( H_{ij} = \begin{cases} \sum_{r \geq 0} (f(g^*))^r(p)g(i+p), & \text{when } i \geq 0 \\ 0 & \text{when } i < 0 , \end{cases} \)

(iii) \( bH_{ij} = K_{i0} \text{ when } i \geq 0 . \)

Proof:

Let \( R(J_{n-1})+1 \) denote the number of service groups with customers from \( C_n \). When \( J_n \geq 0 \), \( J_{n-1} = j \neq 0 \), \( Z_n > 0 \), \( R(J_{n-1}) \) must satisfy

\[(3.11 \ a) \quad J_n + X = \sum_{s=1}^{R(j)} Y_s + |j|, \quad X > \sum_{s=1}^{R(j)-1} Y_s + |j| \]

because

\[ \sum_{s=1}^{R(j)} Y_s + |j| \]

customers are served in \( R(j)+1 \) groups and the rest of \( C_n \) is served in the \( n \)-th restbatch. When \( J_{n-1} = 0 \), the restbatch is complete and \( T_{n-1} = Y \), so that

\[(3.11 \ b) \quad J_n + X = \sum_{s=1}^{R(0)} Y_s + Y, \quad X > \sum_{s=1}^{R(0)-1} Y_s + Y. \]
By an elementary argument

\[ \Pr[R(J_{n-1})=r, J_n=i|J_{n-1}=j] = \sum_{p>0} (f(g*)^r)(p+j)g(i+p), \]

\( i \geq 0, j \neq 0, r \geq 1, \)

\[ \Pr[R(J_{n-1})=r, J_n=i|J_{n-1}=0] = \sum_{p>0} (f(g*)^r)(p)g(i+p), i \geq 0, r \geq 0, \]

\[ \Pr[S'(J_{n-1}) \leq t|R(J_{n-1})=r, J_n=i|J_{n-1}=j, Z_n > 0] = b^r\delta(0,j)(t). \]

When \( J_n < 0, Z_n > 0, \) the relation

\[ X + |J_n| = |J_{n-1}| \]

must be valid. Furthermore \( S(X) \) is seen to have the same distribution as \( S(0) \). The theorem now follows easily.

It is convenient to introduce some further matrix notations.

Let \( F = \{ F_{ij} \}, G = \{ G_{ij} \}, E = \{ E_{ij} \}, K = \{ K'_{ij} \} \) and \( \nu_r = \{ \nu^r_{ij} \} \) be the matrices with entries

\[ F_{ij} = f(j-i), K'_{ij} = \sum_{p \geq 0} b^p \sum_{r \geq 0} (f(g*)^r)(p+j)g(i+p), \]

for \( i,j = 0,1,2,\ldots m-1, \)

and \( F_{ij} = K'_{ij} = 0 \) for \( i,j = m, \ldots, m+(l-n)^+/1 \).

\[ G_{ij} = g(i-j), E_{ij} = \sum_{p>0} g(i+p)f(p+j), \text{ for } i,j = 0,1,\ldots m+(l-m)^+/1, \]

\[ \nu^r_{ij} = \delta(i,0) \text{ for } i,j = 0,1,\ldots r-1. \]
Lemma 3.2

(i) \( K' = E (e I_m - b G)^{-1} I_m \)

(ii) \( K_{ij} = \begin{cases} e F_{-ij}, & \text{when } i < 0, j \geq 0 \\ b K_{ij} + b(0,i) F_{ij}^e, & \text{when } i \geq 0, j \geq 0 \end{cases} \)

Proof:

(ii) is seen immediately.

(i): Since \( EG^r = \{ (EG^r)_{ij} \} = \{ \sum_{p>0} (f(g^*)^r)(p+j)g(i+p) \} \),

\( i,j = 0,1,\ldots,m-1 \), the theorem follows. Observe that \( G^r = 0 \)

when \( r > m+(1-m)^+ \), hence there is no convergence problem.

Remark:

Even if the matrices involved are defined \( m \) dimensional they
are understood to be \( m+(n-m)^+ \) dimensional with the undefined
entries equal to zero.

By the substitutions \( Q_n = \{ Q_{ik} \}, P_n = \{ P_{ik} \} \), where

\[ (3.12 \ a) \quad Q_{ik}^n = (V_{ik}^n + (b-b(0,i))(U_{ik}^n - e U_{ik}^n(0))) \delta(k,0) \]

and

\[ (3.12 \ b) \quad P_{ik}^n = (U_{ik}^n + U_{ik}^n - \delta(0,i) \sum_r (U_{ik}^n-r+U_{ik}^n)) \delta(k,0) \]

\( i, k = 0,1,\ldots,m-1 \),

it is possible to write \( (3.10) \) on the form
(3.13 a) \[ U_{n+1} = T(a^K \begin{bmatrix} 0 \\ bK' \\ -v_{2m-1} \end{bmatrix} Q_n) + T(a^K \begin{bmatrix} F \\ 0 \\ bK' \end{bmatrix} (P_n - e P_n(0))) , \]

(3.13 b) \[ Q_{n+1} = T(a^b \begin{bmatrix} 0 \\ 0 \\ v_{2m-1} \end{bmatrix} Q_n) + T(a^b \begin{bmatrix} F \\ 0 \\ bK' \end{bmatrix} (P_n - e P_n(0))) , \]

whence theorem 3.3 follows.

**Theorem 3.3**

Let

\[ \Phi_n = \begin{bmatrix} P_n \\ v_{2m-1} Q_n \end{bmatrix} \]

and assume that \( U_0(t) = I_{2m-1} e(z+t) \), \( V_0(t) = I_{2m-1} e(w+t) \).

Then \( U_n \) and \( V_n \) are uniquely determined by (3.13) and

(3.14) \[ \Phi_{n+1} = T(a^K_1 (\Phi_n - e \Phi_n(0))) + T(a^K_2 \Phi_n(0)) , \quad n = 0, 1, \ldots , \]

where

\[ K_1 = \begin{bmatrix} (I_m - v_m) (F + b K') \\ v_m (F + b K') \end{bmatrix} , \quad K_2 = \begin{bmatrix} 0, (I_m - v_m) K' \\ v_m K' \end{bmatrix} . \]

Equation (3.14) is called the waiting time equation.

**Corollary 3.4**

When the traffic intensity \( \rho = E(X)E(B)/E(Y)E(A) \) is less than unity the stationary distribution \( \Phi = \lim_{n \to \infty} \Phi_n \) exists and is determined by

(3.15) \[ \Phi = T(a^K_1 (\Phi - e \Phi(0))) + T(a^K_2 \Phi(0)) . \]
A proof of the existence of the stationary distribution is given in [7] and [2].

We will assume that the customers in an arrival group are ordered and the queuing discipline is first come first served.

Let $W_n^p$ be the waiting time of customer nr. $p$ in $C_n$ given that $C_n$ contains at least $p$ customers. Let

$$\Gamma_n^p = \Pr \{ W_n^p \leq t \}.$$  

The problem of finding $\Gamma_n^p$ can be solved by the following argument:

If $C_n$ contains exactly $p$ customer then obviously

$$\Gamma_n^p = \sum_{i} U_{n+1}^i .$$

Hence we must have

$$\Gamma_n^p = T(a^{*}v_{mK'}p^{*}v_{2m-1Q_{n}}) + T(a^{*}v_{2m-1Kp}(P_n - eP_n(0))) ,$$

where $K'p$ and $K_p$ are $K'$ and $K$ respectively, when $f(\cdot)$ is replaced by $\delta(p, \cdot)$.

4. The model $GI^X/Y_{k}/1$.

Within this model it is possible to obtain solutions of the waiting time equation (by using the approach suggested in [3] p.p. 312-313). Let $e_{\theta}; \theta \in \mathbb{C}$, denote the complex exponential measure defined by

$$e_{\theta}(E) = \int_{E \cap \mathbb{R}^+} e^{x} \exp(-\theta x) dx , \text{ if } \Re \theta > 0 ,$$

$$e_{\theta}(E) = \int_{E \cap \mathbb{R}^-} e^{x} \exp(-\theta x) dx , \text{ if } \Re \theta < 0 ,$$
The following lemma is proved in the same way as the analogous results in [3] p.p. 312-313.

Lemma 4.1

(i) \[ e_\theta^n e_\mu^n = \beta^n e_\theta^n + (1-\beta) \sum_{k=1}^{n} \beta^{n-k} e_\mu^k , \]

where

\[ \beta = \frac{\mu}{\mu-\theta} . \]

(ii) \[ T(a^m e_\mu^n) = \sum_{k=1}^{n} (e_\mu^k e)^n_{n-k+e} , \]

where

\[ \gamma_m = \int_0^\infty \frac{e^{-\mu Y} (\mu Y)^m}{m!} a(dy) . \]

By the introduction of the generating functions

\[ \psi_x(t) = \sum_{n \geq 0} P_n(t)x^n , \quad \varphi_x(t) = \sum_{n \geq 0} \nu_{2m-1} Q_n(t)x^n . \]

\[ \tilde{\psi}_x = \begin{pmatrix} \psi_x \\ \varphi_x \end{pmatrix} = \sum_{n \geq 0} P_n x^n . \]

(3.14) is transformed to the equivalent equation

(4.1) \[ \tilde{\psi}_x = xT(a^K_1(\tilde{\psi}_x-e\tilde{\psi}_x(0))) + xT(a^K_2)\tilde{\psi}_x(0) + \tilde{\phi}_0 . \]
By the transformation

$$\omega_x = (eI_m - bG)^{-1}(b(\psi_x - \psi_x(0)e) + \varphi_x)$$

we get

(4.2 i) \[ \psi_x = x(I_m - \nu_m)T(a*(F(\psi_x - e\psi_x(0)) + Ew_x)) + P_0, \]

(4.2 ii) \[ w_x - b(Gw_x + \psi_x - e\psi_x(0)) = x\nu_m T(a*b(F(\psi_x - e\psi_x(0)) + Ew_x)) + \nu_m Q_0. \]

By assumption \( b = e^{\frac{k}{\mu}} \). Assume that (4.2) has a solution of hyperexponential type; i.e,

(4.3) \[ \begin{pmatrix} \psi_x \\ \omega_x \end{pmatrix} = \begin{pmatrix} q_0^i \\ q_0^\nu \end{pmatrix} e + \sum_{j=1}^{\infty} \begin{pmatrix} q_j^i \\ q_j^\nu \end{pmatrix} e^{\theta_j}. \]

If \( U_0 = eU_0(0) \) and \( V_0 = eV_0(0) \), it follows from the definition of \( \tilde{P}_n \) that \( \tilde{P}_0 = \begin{pmatrix} (I_m - \nu_m)U_0 \\ \nu_m V_0 \end{pmatrix} \).

Inserting (4.3) in (4.2) gives

\[
\begin{align*}
\epsilon q_0^i + \sum_{j>0} \epsilon \theta_q q_0^i &= x(I_m - \nu_m) \sum_{j>0} T(a* e_{\theta_j} (Fq_j^i + Eq_j^\nu)) + x(I_m - \nu_m) Eq_0^\nu + \\
& \quad + (I_m - \nu_m)U_0, \\
(eI_m - bG)q_0^\nu + \sum_{j>0} (e \theta_q q_0^\nu - e \theta_q b(Gq_j^\nu + q_j^i)) \\
&= x\nu_m \sum_{j>0} T(a* e_{\theta_j} (Fq_j^i + Eq_j^\nu)) + x\nu_m T(a*b Eq_0^\nu) + \nu_m V_0.
\end{align*}
\]

\* From Lemma 4.1 follows
\( (4.4\ i) \quad q^i_j + \sum_{j > 0} q^j_j (I_m - \nu_m) x \sum_{j > 0} (Fq^j_j + Eq^j_j) (\alpha_j e_\theta_j + (1 - \alpha_j) e) \\
- x (I_m - \nu_m) Eq^j_j + (I_m - \nu_m) U_0 = 0 \),

\( (4.4\ ii) \quad \sum_{j > 0} \left( (e_\mu - \beta_j) q^j_j - \beta_j q^j_j \right) e_\theta_j - (e_\mu - \epsilon) G q^j_j - G q^j_j + q^j_j \epsilon \\
- \sum_{j > 0} \left( G q^j_j + q^j_j \right) (1 - \beta_j) \sum_{r=1}^k \beta_j^{k-r} (e_\mu - \epsilon) + (1 - \beta_j^k) e \\
- x v m \sum_{j > 0} (Fq^j_j + Eq^j_j) \beta_j^k (\alpha_j e_\theta_j + (1 - \alpha_j) e) \\
- x v m \sum_{j > 0} (Fq^j_j + Eq^j_j) (1 - \beta_j) \sum_{r=1}^k \beta_j^k \eta r_j (e_\mu - \epsilon) + (1 - \beta_j^k) e \\
- x v m Eq^j_j (\sum_{r=1}^k (e_\mu - \epsilon) \gamma_{k-r} \epsilon - v m V_0 = 0 \),

where
\[
\eta r_j = \sum_{s > r}^{k} \beta_j^{s-r} \gamma_{s-r}, \quad \alpha_j = \alpha(\theta_j).
\]

The equations above imply that the coefficients of 
\( e, (e_\mu - \epsilon), r = 1, 2, \ldots, k, \) and \( e_\theta_j, j = 1, 2, \ldots \) are zero. Hence, we are led to the equations

\( (4.5\ i) \quad q^i_j = \sum_{j > 0} \frac{1 - \alpha_j^i}{\alpha_j} q^j_j + x (I_m - \nu_m) Eq^j_j + (I_m - \nu_m) U_0 \),

\( (4.5\ ii) \quad q^j_j = \beta_j^k (I_m - \nu_m) (I_m - \beta_j^k g) q^j_j \).
(4.5iii) \((I_m - \mathbf{G} - x\nu_m \mathbf{E})q_o'' = \sum_{j>0} \frac{1-c}{\nu_m (I - \beta_j^k \mathbf{G}) - 1 + \beta_j^{-k}} q_j'' + \nu_m V_o\),

(4.5iv) \( (\delta(r,k)G + x\nu_k - r\nu_m \mathbf{E})q_o'' = \sum_{j>0} \left[(\beta_j - 1) \beta_j^{-r} + (\beta_j - 1)(\gamma_j a_j^{-1} - \beta_j^{-r}) \nu_m (I - \beta_j^k \mathbf{G})\right]q_j'', \quad r = 1, 2, \ldots, k, \)

(4.5v) \((I_n - x\alpha_j F)(I_m - \beta_j^k \mathbf{G}) - x\alpha_j \beta_j^k \mathbf{E})q_j'' = 0, \quad j = 1, 2, \ldots, \)

where \(\beta_j, j = 1, 2, \ldots, hk\), are the roots of

(4.5vi) \(\det [(I_n - x\alpha F)(I_m - \beta_j^k \mathbf{G}) - x\alpha \beta_j^k \mathbf{E}]\),

where \(h = m + (1-m)^+\).

Observe that (4.5 iv-v) constitutes a set of 2kh equations. Hence it is sufficient that (4.5 vi) has kh roots.

5. Special cases.

Unfortunately, the assumptions (4.3) are not always fully filled. For instance when \(Y = m > 1\). However, in this case we are able to slightly the assumptions (4.3). Observe that \(Y = m > 1\) implies \(\mathbf{G} = 0\). Let

\((5.1) \quad \xi_x = (F + b \mathbf{E})(\psi_x - c \psi_x(0)) + b \psi_x\)

whence

\((5.2a) \quad \psi_x = x(I_m - \nu_m)T(a \psi_x) + (I_m - \nu_m)U_o\),
and

\[(5.3) \quad \xi_x = x(F+bE)(I_m-\nu_m)(T(a*\xi_x)-\epsilon T(a*\xi_x)(0)) + x\nu_m T(a*\xi_x) + \nu_m V_0 . \]

We now suppose that (5.3) has a solution of the form

\[\xi_x = p_0 e^+ \sum_{j \geq 0} \theta_j p_j , \]

that is; \(p_0, p_j, j > 0\), must satisfy

\[p_0 e^+ \sum_{j \geq 0} \theta_j p_j = x(F+bE)(I_m-\nu_m) \sum_{j \geq 0} T(a*\theta_j) p_j + x\nu_m \sum_{j \geq 0} T(a*\theta_j) p_j \]

\[+ x\nu_m T(a*b)p_0 + \nu_m V_0 . \]

Exactly the same calculations as in the preceding section give

\[(5.4 \text{ i}) \quad p_j = x\alpha_j (F+b E)p_j , \quad j = 1, 2, \ldots , \]

\[(5.4 \text{ ii}) \quad \sum_{j \geq 0} (\beta_j - 1) \beta_j^k \alpha_j \delta (r,k) (1-\alpha_j) E(I_m-\nu_m) \]

\[+ (\beta_j - 1) \beta_j^k \eta_j \nu_m p_j = \gamma_{k-r} \nu_m p_0 , \quad r = 1, 2, \ldots , k , \]

\[(5.4 \text{ iii}) \quad (I_m-x\nu_m) p_0 = \sum_{j} (\alpha_j^{-1} - 1)(I_m-\nu_m) + x\nu_m + x\alpha_j E(I_m-\nu_m) \]

\[ - x\alpha_j \beta_j^k E \} p_j + \nu_m V_0 . \]
We shall now consider the case when both the arrival and the service group capacity are of constant size; i.e. \( X = 1 \), \( Y = m \). Assume first that \( m < 1 \). Then \( F = 0 \) and

\[
I_1 - \beta k G - \alpha \beta k E = I_1 - \beta k \begin{bmatrix}
0, x_m I_m \\
I_{1-m}, 0
\end{bmatrix}.
\]

It follows that

\[
(5.5) \quad \det(I_1 - \beta k G - \alpha \beta k E) = 1 - (x_\alpha)^m \beta_{kl}.
\]

From Lemma 1 in Takács [8] page 82 we conclude that (5.5) has \( k_l \) distinct roots \( \beta_j = \beta(\theta_j), j = 1, 2, \ldots, k_l \) for \( \text{Re} \theta \geq 0 \). Hence equations (4.5) have a solution. Assume that \( 1 \leq m \).

Then \( G = 0 \),

\[
I_m - x_\alpha (F + \beta k E) = I_m - x_\alpha \begin{bmatrix}
0, I_{m-1} \\
\beta k I_1, 0
\end{bmatrix},
\]

and

\[
(5.6) \quad \det(I_m - x_\alpha F - \alpha \beta k E) = 1 - (x_\alpha)^m \beta_{kl}.
\]

When \( p_0 \) is eliminated, (5.4ii) is a set of \( k_l \) equations because

\[
E = \begin{bmatrix}
0, 0 \\
I_1, 0
\end{bmatrix}.
\]

Accordingly, (5.4) has a solution since (5.6) has \( k_l \) distinct roots.

The stationary solution is obtained by multiplying (4.5) and (5.4) by \( 1-x \) and let \( x \) tend to \( 1- \).
6. The model $E_k^X/G^Y/1$.

The "key" to the solution is the fact that the assumption $a = e^k_\lambda$ enables us to express the operator $T$ in a special form.

**Lemma 6.1**

Suppose $\mu \in \Omega_m^+$. Then

$$T((e*)^k\mu) = (e*)^k\mu + (e-e*) \sum_{r=1}^{k} a_r(e*)^{k-r},$$

where

$$a_r = ((e^r\mu)(\sim \mathbb{R}^+)).$$

**Proof:**

For $k = 1$ the Lemma reduces to equation (72) in Kingman [3]. Assume (6.1) valid for $k = 1, 2, \ldots p$. Since $T(e*)^2 = e$ we get

$$T((e*)^{P+1}\mu) = T(e*T((e*)^P\mu)) = T(e*T((e*)^P\mu)) = T(e*T((e*)^P\mu)) = T(e*T((e*)^P\mu)) = T(e*T((e*)^P\mu)).$$

Now $\nu = T((e*)^P\mu) \in \Omega_m^+$ implies

$$T((e*)^{P+1}\mu) = T(e^\lambda \nu) = e^\lambda \nu + (e-e*)a',$$

where $a' = (e^\nu)(\sim \mathbb{R}^+)$. Obviously $T(e^\nu)(\sim \mathbb{R}^+) = (e^\nu)(\sim \mathbb{R}^+)$ which yields $a' = a_{p+1}$.

Hence, by the assumption

$$T(e\nu) = (e*)^{P+1}\mu + (e-e*) \sum_{r=1}^{P} a_r(e*)^{P+1-r} + (e-e*)a_{p+1}$$

and the Lemma follows by induction.
With \( \mu = K_1 \tilde{\psi}_x \) and \( e = e_\lambda \) application of (6.1) on (4.1) gives (with \( I = I_{2m-1} \))

\[
(6.2) \quad (eI-xa*K_1)\tilde{\psi}_x = x(e-e^*) \sum_{r=1}^{k} a_r (e^*)^{k-r} + xT(a^*(K_2-K_1))\tilde{\psi}_x(0) + \bar{F}_0.
\]

Let \( \Delta_x = eI-xa*K_1 \).

Inserting \( t = 0 \) gives an expression for \( a_k \), viz.,

\[
(6.3) \quad x\alpha_k = (eI-x(a^*(K_2-K_1))(0))\tilde{\psi}_x - \bar{F}_0(0).
\]

Since \( \|a*K_1\| \leq 1 \), \( \Delta_x^{-1} = \sum_{r \geq 0} (xa*K_1)^r \) exists when \( |x| < 1 \).

Thus (6.3) and (6.2) are equivalent to

\[
(6.4) \quad \tilde{\psi}_x - e\tilde{\psi}_x(0) = \Delta_x^{-1}(\bar{F}_0 - (e-e_\lambda^*)(\bar{F}_0(0))) + (e-e_\lambda^*)\Delta_x^{-1} \sum_{r=1}^{k-1} a_r (e_\lambda^*)^{k-r}
\]

\[
- e_\lambda^* \Delta_x^{-1}(Ie-x(e_\lambda^*)^{k-1}K_2)\tilde{\psi}_x(0).
\]

If \( k = 1 \), \( t = 0 \) determines \( \tilde{\psi}_x(0) \) by

\[
(6.5) \quad (e_\lambda^* \Delta_x^{-1}(eI-x(e_\lambda^*)^{k-1}K_2)(0)\tilde{\psi}_x(0) = (\Delta_x^{-1}(\bar{F}_0 - (e-e_\lambda^*)(\bar{F}_0(0))))(0).
\]

When \( k > 1 \) let

\[
c_r = (e-e_\lambda^*)(e_\lambda^*)^{k-r}, \quad r = 1,2,\ldots,k-1,
\]

\[
x = e_\lambda^*(eI-x(e_\lambda^*)^{k-1}K_2),
\]

\[
D = (\bar{F}_0 - (e-e_\lambda^*)(\bar{F}_0(0))),
\]

\[
E_j = (e_\lambda^*)^{j\Delta_x^{-1}}K_1.
\]
Thus, (6.4) can be written

\begin{equation}
\hat{\psi}_x = (e^{i\Delta_x^{-1}c_x})\hat{\psi}_x(0) + \sum_{r=1}^{k-1} c_r \Delta_x^{-1} c_r + \Delta_x^{-1} D .
\end{equation}

Furthermore \((E_j \hat{\psi}_x)(0) = \alpha_j\) leads to

\begin{equation}
\alpha_j = (E_j (\Delta_x^{-1}c_x))(0)\hat{\psi}_x(0) + \sum_{r=1}^{k-1} (E_j c_r)(0)\alpha_r + E_j D ,
\end{equation}

\(j = 1, 2, \ldots, k-1\),

\begin{equation}
(\Delta_x^{-1}c_x)(0)\hat{\psi}_x(0) = \sum_{r=1}^{k-1} c_r(0)\alpha_r + (E_j D)(0)
\end{equation}

which determines \(\alpha_j, j = 1, 2, \ldots k-1\), and \(\hat{\psi}_x(0)\).

7. The stationary solution

In this section we shall demonstrate how the stationary solution of (6.2) can be obtained by use of the Fourier-Stieltjes transform. In the stationary case we have

\begin{equation}
(e^{i-a*K_1})\hat{\theta} = (e-e_{\lambda\lambda}^*) \sum_{r=1}^{k} c_r(e_{\lambda\lambda}^*)^{k-r} + T(a*(K_2-K_1))\hat{\theta}(0) ,
\end{equation}

where now

\begin{equation}
\alpha_r = (e_{\lambda\lambda}^* R K_1 \hat{\theta})(0) .
\end{equation}

After the introduction of the Fourier transform an equation analogous to (6.6) is obtained

\begin{equation}
\hat{\Delta}_1(z)(\hat{\theta}(z) - \hat{\theta}(0)) = -\hat{c}_1(z)\hat{\theta}(0) + \sum_{r=1}^{k-1} \hat{c}_r(z)\alpha_r .
\end{equation}
Let \( \kappa \) be defined by
\[
\hat{\kappa} \hat{A}_1 = \text{det} \hat{A}_1 ,
\]
whence
\[
\text{det}(\Delta_1(z))(\hat{\Psi}(z) - \hat{\Psi}(0)) = -\hat{\kappa}(z) \hat{C}_1(z) \hat{\Psi}(0) + \sum_{r=1}^{k-1} \hat{c}_r(z) \hat{\kappa}(z) \alpha_r .
\]

Suppose that \( \text{det} \Delta_1(z) \) has \( k \) roots \( z_1, z_2, \ldots, z_k \). Then \( \Psi(0), \alpha_1, \ldots, \alpha_{k-1} \), are determined except for a constant by
\[
\text{det}(\Delta_1(z))(\hat{\Psi}(z) - \hat{\Psi}(0)) = \sum_{r=1}^{k-1} \hat{c}_r(z) \hat{\kappa}(z) \alpha_r , \quad i = 1, 2, \ldots, k .
\]

Premultiplicating (7.2) by \( u = \begin{bmatrix} v_m & 0 \\ 0 & v_m \end{bmatrix} \) gives an equation where both sides become zero when \( z = 0 \).

By l'Hôpital's rule,
\[
(a*\hat{K}_1)'(0)(\hat{\Psi} - \hat{\Psi}(0)) = u \hat{C}_1'(0) \hat{\Psi}(0) - i \sum_{r=1}^{k-1} \alpha_r u \alpha_r .
\]

Consider \( (a*\hat{K}_1)'(0) \hat{\Psi}(0) \).

The process \( \{J_n\} \) is recurrent and therefore \( J_n \) converges in distribution to \( J \), say. Since
\[
u K_1 = (b v_m K, v_m(K - F))
\]
and
\[ v_m k(t) = (k_0(t), k_1(t), \ldots, k_{m-1}(t)), \]

where
\[ k_j(t) = \sum_i k_{ij}(t), \]

it follows that
\[ (a*b v_m k)'(0) = i(E(S'(j))|J=j)-iE(A)+iE(B). \]

Thus
\[ (a*b v_m k)'(0) = v_m (a*b \hat{k})'(0)\hat{P}(0)+v_m (a*\hat{k})'(0)\nu_{2m-1}Q(0) \]
\[ = i \sum_j E(S'(j))|J=J \Pr(J=j) - iE(A)+iE(B). \]

By (3.11) we find
\[ E(S'(J)-A+B) = (p-1)E(A)+E(B) = k\lambda^{-1}(p-1)+E(B). \]

Equation (6.12) therefore reduces to
\[ (7.6) \ E(B)-k(1-p) = -i\lambda(a*b \hat{k})'(0)\hat{P}(0)+\nu_{2m-1}Q(0)-\sum_{r=1}^{k-1} r \nu_a r. \]

Together with (7.4) we have a set of \( k+1 \) matrix equations to determine the \( k+1 \) unknowns \( \hat{P}(0), a_1, \ldots, a_{k-1} \).

It is known that a probability distribution can be approximated by a linear combination of Erlang distributions. From Lemma 6.1 it is clear that the results in the last section can be generalized to the case when \( a(\cdot) \) is a linear combination of Erlang distributions. Accordingly, it is possible to obtain approximate solutions of the waiting time equation for general \( a(\cdot) \).
References


