OPTIMAL MULTIPLE TESTS FOR SOME DISCRETE DISTRIBUTIONS

by

Tore Skrøppa
ABSTRACT

For comparison of pairs of proportions in one multinomial distribution and in several binomial distributions multiple tests are found which maximize the minimum power of the individual tests over certain alternatives. The individual tests are found to be wellknown conditional tests at equal levels. When optimality is defined relative to the conditional power the levels of the individual tests differ as shown by some examples.
1.1 Tests maximizing minimum power and minimum average power.

Let $X$ be a random variable with a probability distribution depending upon a parameter $\theta$, $\theta \in \Omega$. Consider a family of hypothesis testings problems

$$(1) \quad H_t : \theta \in \Omega_{0t} \text{ against } K_t : \theta \in \Omega_{1t}, \; t \in T,$$

where $\Omega_{it} \subset \Omega$, $i = 0, 1$, and $T$ is an index set with a finite number $N$ of elements. A test of the problem (1) is defined to be the vector $(\varphi_1, \ldots, \varphi_N)$, where $\varphi_t$ is an ordinary test function of the hypothesis $H_t$ against the alternative $K_t$, $t \in T$. The power function of $(\varphi_1, \ldots, \varphi_N)$ is defined to be the vector $(E_{\theta} \varphi_1(X), \ldots, E_{\theta} \varphi_N(X))$, where $E_{\theta} \varphi_t(X)$ is the power function of $\varphi_t$, $t \in T$. Let $S(\gamma)$ be the set of all tests $(\varphi_1, \ldots, \varphi_N)$ such that

$$\sum_{t=1}^{N} E_{\theta} \varphi_t(X) \leq \gamma \text{ when } \theta \in \Omega_0,$$

where $\Omega_0 = \bigcap_{t=1}^{N} \Omega_{0t}$.

Spjøtvoll (1972) has formulated two optimality criteria of tests of the problem (1). Let $\omega_t \subset \Omega_{1t}, \; t \in T$.

(I) A test $(\varphi_1, \ldots, \varphi_N) \in S(\gamma)$ will be said to maximize the minimum average power over $\omega_t$, $t \in T$, if it maximizes

$$\sum_{t=1}^{N} \inf_{\omega_t} E_{\theta} \varphi_t(X)$$

among tests $(\psi_1, \ldots, \psi_N) \in S(\gamma)$. 

A measurable space \((\mathcal{X}, \mathcal{F})\) is given, and let \(f_0, \ldots, f_N, f_1, \ldots, f_N\) be integrable functions with respect to a \(\sigma\)-finite measure \(\mu\) on \((\mathcal{X}, \mathcal{F})\). Let \(S(\gamma)\) be the set of all tests \((\psi_1, \ldots, \psi_N)\) satisfying

\[
\sum_{t=1}^{N} \int \psi_t(x) f_{0t}(x) d\mu(x) \leq \gamma
\]

and \(S'(\gamma)\) the corresponding set when we have equality in (5).

The following two theorems and Lemma 1 are due to Spjøtvoll (1972) and show how we can find tests maximizing (3) and (4).

**Theorem 1.** Suppose there exists a test \((\varphi_1, \ldots, \varphi_N) \in S'(\gamma)\) defined by

\[
\varphi_t(x) = \begin{cases} 
1 & \text{when } f_t(x) > cf_{0t}(x) \\
0 & \text{when } f_t(x) < cf_{0t}(x)
\end{cases}
\]

Then \((\varphi_1, \ldots, \varphi_N)\) maximizes

\[
\sum_{t=1}^{N} \int \psi_t(x) f_t(x) d\mu(x)
\]

among all tests \((\psi_1, \ldots, \psi_N) \in S'(\gamma)\).
Theorem 2. Suppose there exists a test \((\varphi_1, \ldots, \varphi_N) \in S'(\gamma)\) defined by

\[
\begin{align*}
1 & \quad \text{when } c_t f_t(x) > f_{0t}(x) \\
\varphi_t(x) & = a_t \quad \text{when } c_t f_t(x) = f_{0t}(x) \\
0 & \quad \text{when } c_t f_t(x) < f_{0t}(x)
\end{align*}
\]

where \(a_1, \ldots, a_N, c_1, \ldots, c_N\) are such that

\[
\int \varphi_t(x) f_t(x) d\mu(x) = \inf_{T} \int \varphi_t(x) f_t(x) d\mu(x), \quad t \in T,
\]

and \(c_t \geq 0, \quad t \in T, \quad \sum_{t=1}^{N} c_t > 0\). Then \((\varphi_1, \ldots, \varphi_N)\) maximizes

\[
\inf_{T} \int \psi_t(x) f_t(x) d\mu(x)
\]

among all tests \((\psi_1, \ldots, \psi_N) \in S'(\gamma)\).

It follows from the proof of Theorem 2 (see [2], pp. 400-401) that the test defined by (8) and (9) also maximizes (10) among all tests in \(S(\gamma)\). If \(f_{0t}\) and \(f_{1t}\), \(t \in T\), are probability densities this also holds for the test defined by (6).

Lemma 1. Let \(\omega_t\) be subsets of \(\Omega_{1t}, \quad t \in T\). Suppose there exists a test \((\varphi_1, \ldots, \varphi_N)\) such that

(i) there exist points \(\theta_t^* \in \omega_t, \quad t \in T\), such that

\((\varphi_1, \ldots, \varphi_N)\) maximizes

\[
\inf_{T} \sum_{t=1}^{N} E_{\theta_t^*} \psi_t(X)
\]

among tests \((\psi_1, \ldots, \psi_N) \in S(\gamma)\).
(ii) \( \inf_{\theta \in \Theta_t} E_{\theta} \phi_t(x) = E_{\Theta_t} \phi_t(x), \ t \in T. \)

Then \((\phi_1, \ldots, \phi_N)\) maximizes \((4)\) \((3)\) among tests in \(S(\gamma)\).

In many problems an optimal test of each hypothesis \(H_t\) against the alternative \(K_t, t \in T\), is known, and a vector consisting of such tests can in some situations be shown to maximize \((4)\).

**Theorem 3.** Suppose there exists a test \((\phi_1, \ldots, \phi_N)\) where \(\phi_t\) is most powerful (MP) for testing \(f_{ot}\) against \(f_t\) at level \(\alpha_t\), when \(f_{ot}\) and \(f_t\) are probability densities, \(t \in T\), and \(\alpha_1, \ldots, \alpha_N\) are such that

\[
\sum_{t=1}^{N} \alpha_t = \gamma
\]

\[
\int \phi_1(x)f_1(x)d\mu(x) = \ldots = \int \phi_N(x)f_N(x)d\mu(x).
\]

Then \((\phi_1, \ldots, \phi_N)\) maximizes

\[
\inf_{T} \int \psi_t(x)f_t(x)d\mu(x)
\]

among all tests \((\psi_1, \ldots, \psi_N) \in S(\gamma)\).

**Proof.** \(\phi_t\) is MP for testing \(f_{ot}\) against \(f_t\) and can be written

\[
\phi_t(x) = \begin{cases} 
1 & \text{when } f_t(x) > c_t f_{ot}(x) \\
\alpha_t & \text{when } f_t(x) = c_t f_{ot}(x) \\
0 & \text{when } f_t(x) < c_t f_{ot}(x)
\end{cases}
\]

for some constant \(c_t > 0, t \in T\).
Suppose \( c_t' > 0, \ t \in T \). Then \( (\varphi_1, \ldots, \varphi_N) \) is as in Theorem 2 with \( c_t = c_t'^{-1}, \ t \in T \).

Next, suppose \( c_t' = 0 \) for at least one \( t \in T \), let \( c_s' = 0 \).

We then have

\[
1 \quad \text{when } f_s(x) > 0 \\
\varphi_s(x) = a_s \quad \text{when } f_s(x) = 0 \\
0 \quad \text{when } f_s(x) < 0
\]

and

\[
\int \varphi_s(x)f_s(x)\,d\mu(x) = \int f_s(x)\,d\mu(x) = 1. \quad (12) \implies \int \varphi_1(x)f_1(x)\,d\mu(x) = \ldots = \int \varphi_N(x)f_N(x)\,d\mu(x) = 1.
\]

But then

\[
\inf_{T} \int \varphi_t(x)f_t(x)\,d\mu(x) = 1 \geq \inf_{T} \int \psi_t(x)f_t(x)\,d\mu(x)
\]

for any other test \( (\psi_1, \ldots, \psi_N) \in S(\gamma) \), and the theorem is proved.

In some problems there exist uniformly most powerful (UMP) tests of each hypothesis \( H_t \) against the alternative \( K_t \), \( t \in T \). According to Theorem 3 we can choose \( \varphi_t \) as such a UMP level \( \alpha_t \) test, \( t \in T \), and determine \( \alpha_1, \ldots, \alpha_N \) such that

\[
\sum_{t=1}^{N} \alpha_t = \gamma \quad \text{and} \quad \inf_{t} \inf_{\vartheta_t} E_{\vartheta_t} \varphi_t(X) = \text{constant}, \ t \in T.
\]

\( (\varphi_1, \ldots, \varphi_N) \) then maximizes (4).

In problems where no UMP test exists of some hypothesis, we can choose \( \varphi_t \) as the MP level \( \alpha_t \) test for testing \( f_{0t} \) against \( f_{0*}, \) where \( \vartheta_* \in \Omega_{0t}, \ t \in T \). Suppose that the levels \( \alpha_1, \ldots, \alpha_N \) can be determined such that (11) and (12) hold with \( f_t = f_{0*}, \ t \in T \). Then the test \( (\varphi_1, \ldots, \varphi_N) \)
maximizes (4) for any \( \omega_t \subset \Omega_1 t, t \in T, \) such that
\[
\inf_{\omega_t} E_\theta \varphi_t(X) = \int \varphi_t(x) f_{\theta t}^*(x) d\mu(x), \quad t \in T.
\]

1.2 Unbiased tests.

A test \((\varphi_1, \ldots, \varphi_N)\) is defined to be unbiased if
\[
\sup_{\Omega_{0 t}} E_\theta \varphi_t(X) \leq \inf_{\Omega_{1 t}} E_\theta \varphi_t(X), \quad t \in T.
\]

Suppose that the power function \( E_\theta \varphi_t(X) \) is a continuous function of \( \theta \) and let \( \Omega_t \) be the common boundary of \( \Omega_{0 t} \) and \( \Omega_{1 t}, t \in T. \) Then (13) implies
\[
E_\theta \varphi_t(T) = \text{constant} = \gamma_t \quad \text{when } \theta \in \Omega_t, \quad t \in T.
\]

Furthermore, if \( Z_t \) is a sufficient and complete statistic relative to \( \Omega_t, \) then (14) is equivalent to
\[
E_\theta [\varphi_t(X) | Z_t] = \gamma_t \quad \text{when } \theta \in \Omega_t, \quad t \in T.
\]

Hence \( \varphi_t \) is a conditional test given \( Z_t, \) and attention can be restricted to conditional tests.

Lemma 2. Let \((\psi_1, \ldots, \psi_N)\) be an unbiased test for the problem (1) and let \( \alpha_t \) be the level of \( \psi_t, t \in T, \sum_{t=1}^N \alpha_t = \gamma. \)

Suppose there exists a UMP unbiased level \( \alpha^* \) test \( \varphi_t \) of the hypothesis \( H_t \) against the alternative \( K_t, t \in T. \) Then
\[
\inf_{T} \inf_{\omega_t} E_\theta \varphi_t(X) \geq \inf_{T} \inf_{\omega_t} E_\theta \psi_t(X)
\]
Proof. The test \( \varphi_t \) is UMP unbiased. Then \( E_\theta \varphi_t(X) \geq E_\theta \psi_t(X) \), \( \theta \in \Omega_{1t} \) and \( \inf_{\omega_t} E_\theta \varphi_t(X) \geq \inf_{\omega_t} E_\theta \psi_t(X) \). This holds for all \( t \in T \), and consequently (16) and (17) must be true.

Lemma 2 shows that for any unbiased test a vector consisting of UMP unbiased tests, if it exists, will have equal or greater minimum power and minimum average power.

The following theorem will be helpful when determining tests maximizing minimum power among unbiased tests.

**Theorem 4.** Suppose that \( \varphi_t \) is UMP unbiased level \( \alpha_t \) test for testing \( H_t \) against \( K_t \), \( t \in T \), where

\[
(18) \quad \frac{1}{N} \sum_{t=1}^{N} \alpha_t = \gamma
\]

\[
(19) \quad E_{\theta^*} \varphi_1(X) = \ldots = E_{\theta^*} \varphi_N(X),
\]

\( \theta^* \in \Omega_{1t}, t \in T \). Then \( (\varphi_1, \ldots, \varphi_N) \) maximizes

\[
(20) \quad \inf_{T} \frac{1}{T} E_{\theta^*} \psi_t(X)
\]

among unbiased tests \( (\psi_1, \ldots, \psi_N) \in S(\gamma) \).

Proof. By Lemma 2 it follows that a test maximizing (20) must be a vector of UMP unbiased tests. Let \( (\psi_1, \ldots, \psi_N) \) be such a vector with levels \( (\varepsilon_1, \ldots, \varepsilon_N) \), \( \frac{1}{N} \sum_{t=1}^{N} \varepsilon_t = \gamma \), and let
If \( e_q > e_q \) and \( e_t > e_t \), \( t \in T \), which is against the assumption
\[
\sum_{t=1}^{N} e_t = \sum_{t=1}^{N} e_t = \gamma.
\]
Hence \( E_{q^T}(X) \leq E_{q^T}(X) \) and
\[
\inf_T E_{q^T}(X) = E_{q^T}(X) = \inf_T E_{q^T}(X).
\]
This proves the theorem, since \((\psi_1, \ldots, \psi_N)\) was chosen arbitrary among the vectors of UMP unbiased tests in \( S(\gamma) \).

The following result follows directly from Theorem 4 and Lemma 1.

**Corollary 1.** Let \((\varphi_1, \ldots, \varphi_N)\) be as in Theorem 4. Then

\((\varphi_1, \ldots, \varphi_N)\) maximizes \( \inf_T \inf_{\omega_t} E_{q^T}(X) \) among unbiased tests in \( S(\gamma) \) for any \( \omega_t \subset \Omega_{1t} \), \( t \in T \), such that
\[
\inf_{\omega_t} E_{q^T}(X) = E_{q^T}(X) = \text{constant and } \theta^* \in \omega_t, t \in T.
\]

In many multiple problems the individual hypotheses against their alternatives are symmetrical, and it seems reasonable to choose symmetrical subsets \( \omega_t \subset \Omega_{1t} \), \( t \in T \).

(18) and (19) are then equivalent to \( \alpha_1 = \ldots = \alpha_N = \frac{\gamma}{N} \), and a vector of UMP unbiased tests at equal levels \( \frac{\gamma}{N} \) maximizes the minimum power.

To find tests maximizing the minimum average power we have to maximize \( \sum_{t=1}^{N} E_{q^T}(X) \) under the condition \( \sum_{t=1}^{N} \alpha_t = \gamma \), where \( \psi_t \) is a UMP unbiased level \( \alpha_t \) test of \( H_t \) against \( K_t \), \( t \in T \). It is difficult to give a general solution of this problem, and in the following we will concentrate on finding tests maximizing the minimum power.
2.1 Paired comparisons in a multinomial distribution.

Let \( X_1, \ldots, X_k \) be multinomially distributed with probability distribution

\[
P[X_1 = x_1, \ldots, X_k = x_k] = \frac{n!}{x_1! \cdots x_k!} p_1^{x_1} \cdots p_k^{x_k},
\]

\[
\sum_{s=1}^{k} x_s = n, \quad \sum_{s=1}^{k} p_s = 1, \quad p_s > 0, \quad s = 1, \ldots, k.
\]

Consider the problem

(21) \( H_{ij} : p_i = p_j \) against \( K_{ij} : p_i > p_j, \ i \neq j, \ (i, j) \in T, \)

where \( T \) is an index set consisting of \( N \) pairs \( (i, j) \), \( N \leq k(k-1) \).

Let \( (i, j) \in T \) be fixed. The joint distribution of \( X_1, \ldots, X_k \) is given by

(22) \[
dP = k^{n} p_{ij}^{n} e^{\sum_{s \neq i, j} x_s \log \frac{p_s}{p_j}} + x_i \log \frac{p_i}{p_j} \frac{dP_0}{dP}
\]

\[
= k^{n} p_{ij}^{n} e^{\sum_{s \neq i, j} x_s \log \frac{p_s}{p_j}} + x_i \log \rho_{ij} \frac{dP_0}{dP}
\]

where \( p_j = 1 - \sum_{s \neq j} p_s, \ \rho_{ij} = \log \frac{p_i}{p_j} \) and \( P_0 \) is the distribution when \( p_1 = \ldots = p_k = \frac{1}{k} \). The distribution (22) defines an exponential family, and we will use the results of chapter 4.4 in Lehmann (1959) to find UMP unbiased tests.

For testing the hypothesis

\( H'_{ij} : \rho_{ij} = 0 \) against \( K'_{ij} : \rho_{ij} > 0 \)

there exists a UMP unbiased level \( \phi_{ij} \) test \( \phi_{ij} \) defined by
According to Lemma 2 a test maximizing minimum power among unbiased tests has to be a vector with component tests defined by (23) and (24) for some $a_{ij}$. We will now study these tests in detail.

First we need the conditional distribution of $X_i$ given $Z_{ij} = z_{ij}$. We easily find

$$
P[X_i=x_i | Z_{ij}=z_{ij}] = \frac{(n-\sum x_s)! \prod_{s \neq j} p_s^{(1-\sum p_s)} p_j^{n-\sum x_s}}{x_i!(n-\sum x_s)! (1-\sum p_s)^n} = \frac{v_{ij}}{x_i!} \left(\frac{p_i}{p_i+p_j}\right)^{x_i} \left(\frac{p_j}{p_i+p_j}\right)^{v_{ij}-x_i}, \quad x_i = 0, 1, \ldots, v_{ij},$$

where $V_{ij} = X_i + X_j$. Hence conditionally $X_i$ is binomially distributed with

$$(v_{ij}, \frac{p_i}{p_i+p_j}),$$

and (24) becomes

$$\mathbb{E}[\varphi_{ij}(X_i, Z_{ij}) | Z_{ij}] = a_{ij} \text{ when } p_i = p_j,$$

where we have written $v = v_{ij}$ for convenience.
Let $\Delta_{ij} = \frac{p_i}{p_j}$. Then the power of the conditional test $\varphi_{ij}$ given $V_{ij} = v$ equals

$$
(25) \quad a_{ij}(c(v)) \left(\frac{\Delta_{ij}}{1+\Delta_{ij}}\right)^{c(v)} \left(1 - \frac{1}{1+\Delta_{ij}}\right)^{v-c(v)} \frac{\Gamma(v)}{\Delta_{ij}^v} \frac{1}{\Gamma(1+\Delta_{ij})} \left(1+\Delta_{ij}\right)^{-v-x} \quad x > c(v) \left(\frac{\Delta_{ij}}{1+\Delta_{ij}}\right)^x \left(1+\Delta_{ij}\right)^{-v-x}
$$

This is the power of the optimum test for testing $\frac{p_i}{p_i+p_j} = \frac{1}{2}$ against $\frac{p_i}{p_i+p_j} > \frac{1}{2}$ based on $v$ binomial trials and is an increasing function both of $v$ and of $\frac{p_i}{p_i+p_j}$ and consequently of the ratio $\Delta_{ij} = \frac{p_i}{p_j}$.

Define a subset of the alternative by

$$
(26) \quad w_{ij} = \{(p_1, \ldots, p_k); \Delta_{ij} \geq \Delta_{ij}^* > 1, \Delta_{ij}^* \text{ fixed}, \sum_{s=1}^{k} p_s = 1\}.
$$

The conditional power attains its minimum value over $w_{ij}$ for $\Delta_{ij} = \Delta_{ij}^*$.

The power of interest, however, is the unconditional power. Using that the distribution of $V_{ij}$ is given by

$$
P[V_{ij} = v] = \binom{n}{v} (p_i + p_j)^v (1 - (p_i + p_j))^{n-v} = \binom{n}{v} (1 + \Delta_{ij} p_j)^v (1 - (1 + \Delta_{ij} p_j) p_j)^{n-v},
$$

$v = 0, 1, \ldots, n$, we find

$$
(27) \quad \beta_{ij}(p_j, p_i) = \beta_{ij}(p_j, \Delta_{ij} p_j) = \mathbb{E}[\varphi_{ij}(X)|V_{ij}]
$$

$$
= \sum_{v=0}^{n} a_{ij}(c(v)) \left(\frac{\Delta_{ij}}{1+\Delta_{ij}}\right)^{c(v)} \left(1 - \frac{1}{1+\Delta_{ij}}\right)^{v-c(v)} \frac{\Gamma(v)}{\Delta_{ij}^v} \frac{1}{\Gamma(1+\Delta_{ij})} \left(1+\Delta_{ij}\right)^{-v-x} \quad x > c(v) \left(\frac{\Delta_{ij}}{1+\Delta_{ij}}\right)^x \left(1+\Delta_{ij}\right)^{-v-x}.
$$
Let $\omega_{ij}$ be defined by (26) and let $\Delta_{ij}^* = \Delta^*$ for all $(i,j) \in T$. Furthermore, let $(\varphi_{1,2}, \ldots)$ be the vector test with components defined by (23) and (24) with $a_{ij} = \frac{\gamma}{N}$, $(i,j) \in T$. Then the power function of each test $\varphi_{ij}$ tends to $\frac{\gamma}{N}$ as $p_j \to 0$ on any line $p_i = \Delta_{ij} p_j$, $(i,j) \in T$, hence \[ \inf E_p \varphi_{ij}(X) = \frac{\gamma}{N}, (i,j) \in T. \] It follows that any unbiased test will maximize the minimum power over the subsets $\omega_{ij}$, $(i,j) \in T$. Let us therefore restrict the alternatives more.

Consider the subsets

(28) $\omega_{ij}^* = \{(p_1, \ldots, p_k); \Delta_{ij} > \Delta_{ij}^* > 1, p_j > \Delta_{ij}^* < 0.5, \sum_{s=1}^{k} p_s = 1\}, (i,j) \in T$,

where $\Delta_1^*$, $\Delta_2^*$ are fixed. See Figure 2.

![Figure 2](image-url)
Let the test \( (\varphi_{12}, \ldots) \) be as before. We will show that \( \inf_{p_i} \mathbb{E}_p \varphi_{ij}(X) \) is attained for \( p_j = \Delta^* \), \( p_i = \Delta^* \Delta^* \), \((i,j) \in T\).

Previously we have seen that \( \beta_{ij} \) for fixed \( \Delta_{ij} \) is a non-decreasing function of \( p_j \), and in a similar way we will show that \( \beta_{ij} \) for fixed \( p_j \) is a non-decreasing function of \( \Delta_{ij} \).

To do this we need the following extension of Lemma 2 of Chapter 3 in Lehmann (1959).

**Lemma 3.** Let \( p_\theta(x) \) be a family of densities on the real line with monotone likelihood ratio in \( x \). If \( g_\theta \) is a non-decreasing function of \( \theta \) for any fixed \( x \) and a non-decreasing function of \( x \) for any fixed \( \theta \), then \( \mathbb{E}_\theta g_\theta(X) \) is a non-decreasing function of \( \theta \).

The proof follows the proof on page 74 in Lehmann (1959).

Let \((i,j) \in T\) be fixed and put \( p_j = p^* \), \( 0 < p^* < 0.5 \). According to (27) we have

\[
\beta_{ij}(p^*, \Delta_{ij}, p^*) = \sum_{v=0}^{\infty} \{a_{ij}(c(v))(\frac{\Delta_{ij}}{1+\Delta_{ij}})^{c(v)}(\frac{1}{1+\Delta_{ij}})^{v-c(v)} + \\
\sum_{x>c(v)} (x)(\frac{\Delta_{ij}}{1+\Delta_{ij}})^x(\frac{1}{1+\Delta_{ij}})^{x-x} \}^n((1+\Delta_{ij})p^*)^v(1-(1+\Delta_{ij})p^*)^{n-v}.
\]

It is obvious that the family of distributions of \( V_{ij} \) for varying \( \Delta_{ij} \) has monotone likelihood ratio in \( v \). Consider the function \( g_{\Delta_{ij}}(v) = \mathbb{E}_{\Delta_{ij}}[\varphi_{ij}(X)|V_{ij}=v] \). We have previously seen that \( g_{\Delta_{ij}}(v) \) is an increasing function both of \( v \) and of \( \Delta_{ij} \). Using Lemma 3 we find that

\[
\beta_{ij}(p^*, \Delta_{ij}, p^*) = \mathbb{E}_{\Delta_{ij}} g_{\Delta_{ij}}(V_{ij}) \text{ is a non-decreasing function of } \Delta_{ij}.
\]

But then
\[
\inf_{\omega_{ij}} E_{\omega_{ij}(x)} = \inf_{\omega_{ij}} \beta_{ij}(p_j, p_i) = \beta_{ij}(\Delta_2^*, \Delta_1^*).$
\]

Furthermore, \(\inf_{\omega_{ij}} E_{\omega_{ij}(x)}\) is constant, \((i, j) \in T\), and the conditions (18) and (19) of Theorem 4 are fulfilled. According to Corollary 1 the test \((\varphi_{12}, \ldots)\) maximizes the minimum power over the subsets \(\omega_{ij}\), \((i, j) \in T\), among unbiased tests in \(S(\gamma)\). This result holds uniformly in \(\Delta_1^*\) and \(\Delta_2^*\).

Finally consider a third subset of the alternatives. Let
\[
\omega_{ij}' = \left\{ (p_1, \ldots, p_k); p_1 \geq p_j + \Delta_2^*, p_j \geq \Delta_1^*, 0 < \Delta_2^* \leq 0.5, 0 < \Delta_1^* \leq 1, \sum_{s=1}^{k} p_s = 1 \right\},
\]

\((i, j) \in T\). See Figure 3.

Each component test has the same minimum power over \(\omega_{ij}\), \((i, j) \in T\), and the power has its minimum for \(p_j = \Delta_2^*, p_i = \Delta_2^* + \Delta_1^*\). The test \((\varphi_{12}, \ldots)\) maximizes the minimum power over \(\omega_{ij}\), \((i, j) \in T\), as well.
2.2 Paired comparisons for the Poisson distribution.

Let \( X_1, \ldots, X_k \) be independent with Poisson distributions
\[
P[X_S = x_S] = \frac{\lambda_s^{x_S} e^{-\lambda_s}}{x_S!}, \quad x_S = 0, 1, \ldots, \lambda_S > 0, \quad s = 1, \ldots, k.
\]

Consider the problem
\[
H_{ij} : \lambda_i = \lambda_j \text{ against } K_{ij} : \lambda_i > \lambda_j, \quad i \neq j, \ (i, j) \in T.
\]
\( T \) is an index set with \( N \) elements.

Using the same arguments as in 2.1 we find that the test
\( (\varphi_{i2}, \ldots) \) where
\[
\varphi_{ij}(x_i, v_{ij}) = \begin{cases} 1 & \text{when } x_i > c(v_{ij}) \\ a(v_{ij}) & \text{when } x_i = c(v_{ij}) \\ 0 & \text{when } x_i < c(v_{ij}) \end{cases}
\]
\[
\mathbb{E}[\varphi_{ij}(X_i, V_{ij})|V_{ij}] = \frac{\lambda_j}{N} \quad \text{when } \lambda_i = \lambda_j, \quad \text{and } V_{ij} = X_i + X_j,
\]
maximizes the minimum power over the subsets
\[
\omega_{ij} = \{(\lambda_1, \ldots, \lambda_k); \frac{\lambda_i}{\lambda_j} \geq \Delta^*_1 > 1, \lambda_j \geq \Delta^*_2 > 0, \lambda_s > 0 \ s = 1, \ldots, k\},
\]
\( (i, j) \in T \). The power function of \( \varphi_{ij} \) has analogous monotonicity properties and attains its minimum value over \( \omega_{ij} \) when
\[
\lambda_j = \Delta^*_2, \quad \lambda_i = \Delta^*_1 \Delta^*_2, \quad (i, j) \in T.
\]

2.3 Paired comparisons for the binomial distribution.

Let \( X_1, \ldots, X_r \) be independent with binomial distributions
\[
P[X_S = x_S] = \binom{n_S}{x_S} p_S^{x_S} (1-p_S)^{n_S-x_S}, \quad x_S = 0, 1, \ldots, n_S, \quad s = 1, \ldots, r.
\]

Consider the problem
\(H_{ij}: p_i = p_j\) against \(K_{ij}: p_i > p_j\), \(i \neq j\), \((i,j) \in T\), where \(T\) is an index set with \(N \leq r(r-1)\) elements.

Denote the joint distribution of \(X_1, \ldots, X_r\) by \(P_0\) when \(p_1 = \cdots = p_r = 0.5\). Keep \((i,j) \in T\) fixed. Then

\[
(29) \quad dP = 2^{s=1} n_s \prod_{s=1}^r (1-p_s) n_s \prod_{s=1}^r x_s \log \frac{p_s}{1-p_s} dP_0
\]

The family of distributions defined by (29) is exponential, and we can find a UMP unbiased test for the hypothesis

\(H_{ij}: \rho_{ij}\) against the alternative \(K_{ij}: \rho_{ij} > 0\), where

\[
\rho_{ij} = \log \frac{p_i(1-p_j)}{p_j(1-p_i)}.
\]

Testing of \(H_{ij}\) against \(K_{ij}\) is equivalent to testing of \(H_{ij}'\) against \(K_{ij}'\), and we are lead to a conditional test given \(V_{ij} = X_i + X_j\).

The UMP unbiased level \(\alpha_{ij}\) test \(\varphi_{ij}\) is

\[
(30) \quad \varphi_{ij}(x_i, V_{ij}) = a(V_{ij}) \quad \text{when} \quad x_i = c(V_{ij}),
\]

\[
0 \quad \text{when} \quad x_i < c(V_{ij}),
\]

where \(a(V_{ij})\) and \(c(V_{ij})\) are determined such that

\[
(31) \quad \mathbb{E}[\varphi_{ij}(X_i, V_{ij})|V_{ij}] = \alpha_{ij} \quad \text{when} \quad p_i = p_j.
\]
The conditional distribution of $X_i$ given $V_{ij} = v$ is

$$P[X_i = x_i | V_{ij} = v] = \frac{\binom{n_i}{x_i} \binom{n_j}{x_j} x_i}{\min(n_i, v)} \sum_{s=0}^{\min(n_i, v)} \binom{n_i}{s} \binom{n_j}{v-s} x_i^s$$

where $x_{ij} = \frac{p_i(1-p_j)}{p_j(1-p_j)}$. When $p_i = p_j$ this is a hypergeometric distribution. The inequality $p_i > p_j$ is equivalent to $x_{ij} > 1$.

The distribution of $V_{ij}$ can be written

$$P[V_{ij} = v] = \left( \sum_{s=0}^{\min(n_i, v)} \binom{n_i}{s} \binom{n_j}{v-s} p_j^{v-s} (1-p_j)^v \right) \left( 1 + (n_i - 1)p_j \right)^{-n_i}$$

$v = 0, 1, \ldots, n_i + n_j$, and using this we find the power function of $\varphi_{ij}$

$$\beta_{ij}(p_j, p_1) = \beta_{ij}(p_j, \frac{x_{ij} p_j}{1 + (x_{ij} - 1)p_j}) =$$

$$\sum_{v=0}^{n_i + n_j} \left\{ a(v) \binom{n_i}{v} \binom{n_j}{v} x_{ij} \right\} + \sum_{x>c(v)}^{n_i + n_j} \frac{n_i}{v-c(v)} \binom{n_j}{v-x} x_{ij}^x \right\}.$$
Arguing as in 2.1 we can show that $\beta_{ij}$ attains its minimum value over $w_{ij}$ when $\kappa_{ij} = \Delta^*_1$, $p_j = \Delta^*_2$.

Consider the test $(\varphi_{12}, \ldots)$ where $\varphi_{ij}$ is defined by (30) and (31) with $\alpha_{ij} = \frac{\gamma}{N}$, $(i,j) \in T$. In the case $n_1 = \ldots = n_r = n$ the minimum powers over the subsets $w_{ij}$ are all equal, and by Theorem 4 and Corollary 1 we find that this test maximizes the minimum power over $w_{ij}$, $(i,j) \in T$, among unbiased tests in $S(y)$.

If the $n_i$s are different, the levels $\alpha_{ij}$, $(i,j) \in T$, have to be determined such that $\beta_{ij}(p_j, p_i) = \text{constant}$ when $p_j = \Delta^*_2$, $p_i = \frac{\Delta^*_1 \Delta^*_2}{1 + (\Delta^*_1 - 1) \Delta^*_2}$, $(i,j) \in T$. If such a test can be found it maximizes the minimum power over the subsets $w_{ij}$, $(i,j) \in T$.

![Figure 4](image-url)
2.4 Paired comparisons against two-sided alternatives.

The alternatives considered so far have been one-sided. In some situations we might be interested in two-sided alternatives and we will now find tests maximizing minimum power for this problem.

Consider the multinomial situation in 2.1. Instead of (21) we have the problem

\[ H_{ij} : p_i = p_j \text{ against } K_{ij} : p_i \neq p_j, \quad i \neq j, \quad (i, j) \in T. \]

The number of hypotheses is \( N \).

Using Theorem 3 of Chapter 4 in Lehmann (1959) we find a UMP unbiased level \( \alpha_{ij} \) test of \( H_{ij} \) against \( K_{ij} \)

\[
\varphi_{ij}(x_i, v_{ij}) = \begin{cases} 
1 & \text{when } x_i < c_1(v_{ij}) \text{ or } x_i > c_2(v_{ij}), \\
0 & \text{when } c_1(v_{ij}) < x_i < c_2(v_{ij}),
\end{cases}
\]

where \( c_1(v_{ij}), c_2(v_{ij}), a_1(v_{ij}), a_2(v_{ij}) \) are determined such that

\[
E[\varphi_{ij}(X_i, V_{ij})|V_{ij}] = \alpha_{ij} \quad \text{when } p_i = p_j
\]

\[
E[X_i\varphi_{ij}(X_i, V_{ij})|V_{ij}] = \alpha_{ij}E[X_i|V_{ij}] \quad \text{when } p_i = p_i, \quad (i, j) \in T.
\]

Let \( \alpha_{ij} = \frac{\gamma}{N} \), \( (i, j) \in T \), and consider the subsets

\[
\omega_{ij} = \left\{ (p_1, \ldots, p_k); \frac{p_i}{p_j} < \Delta^* < 1, \frac{p_i}{p_j} > \Delta^* > 1, p_i > \Delta^*, p_j > \Delta^*, \sum_{s=1}^{k} p_s = 1 \right\},
\]

\( (i, j) \in T \).

Then \( \inf_{i,j} \varphi_{ij}(X) = \text{constant}, \ (i,j) \in T \), and the test \( (\varphi_{12}, \ldots) \)
maximizes the minimum power over \( \omega_{ij} \), \( (i,j) \in T \), among unbiased tests in \( S(\gamma) \).
2.5 Tests maximizing the conditional power.

So far we have considered optimality relative to the unconditional power of the tests. This power is found by averaging the conditional power with respect to all values of the conditioning variable, and it may be felt inappropriate taking into consideration values not occurred in the problem at hand. We will now find tests maximizing minimum conditional power and minimum average conditional power for the problem of paired comparisons in one multinomial distribution. This power has the advantage that it does not depend on nuisance parameters besides the one of interest.

For testing $H_{ij} : p_i = p_j$ against $K_{ij} : p_i > p_j$, $(i,j) \in T$, we shall use a conditional test given $V_{ij} = v_{ij}$. Let $f_{ij}^0$ and $f_{ij}$ be the conditional distribution of $X_i$ given $V_{ij} = v_{ij}$ when $p_i = p_j$ and $\frac{p_i}{p_j} = \Delta^* > 1$ respectively. Then

$$f_{ij}^0(x_i|v_{ij}) = \left(\frac{v_{ij}}{x_i}\right)^{\frac{1}{2}}$$

$$f_{ij}(x_i|v_{ij}) = \left(\frac{v_{ij}}{x_i}\right)^{\frac{\Delta^*}{1+\Delta^*}} X_i^1 \left(\frac{1}{1+\Delta^*}\right)^{v_{ij} - x_i}$$

Optimal tests will be found by using Theorem 1 and Theorem 2.

$$c_{ij} f_{ij} > f_{ij}^0$$

is equivalent to

$$x_i > c_{ij} + \Delta v_{ij} = c(v_{ij})$$

where $c_{ij} = -\frac{\log c_{ij}}{\log \Delta^*}$, $\Delta = \frac{\log \frac{1+\Delta^*}{2}}{\log \Delta^*}$. 

Let \((\psi_{12}, \ldots)\) be a test for the problem (21). By Theorem 1 the test maximizes
\[
\sum_{(i,j) \in \mathcal{T}} \int \psi_{ij}(x_i, v_{ij}) f_{ij}(x_i | v_{ij}) d\mu(x_i)
\]
among tests \((\psi_{12}, \ldots) \in S(\gamma)\) if

\[
1 \quad \text{when } x_i > c + v_{ij}\Delta
\]

\[
(36) \quad \varphi_{ij}(x_i, v_{ij}) = a(v_{ij}) \quad \text{when } x_i = c + v_{ij}\Delta
\]

\[
0 \quad \text{when } x_i < c + v_{ij}\Delta
\]

where \(c\) and \(a(v_{ij}), (i,j) \in \mathcal{T}\), are constant such that

\[
(37) \quad \sum_{(i,j) \in \mathcal{T}} \{a(v_{ij})(c + v_{ij}\Delta) \frac{1}{2} v_{ij} + \sum_{x_i > c + v_{ij}\Delta} (v_{ij})(\frac{1}{2}) v_{ij}\} = \gamma.
\]

By Theorem 2 \((\psi_{12}, \ldots)\) maximizes
\[
\inf_{(i,j) \in \mathcal{T}} \int \psi_{ij}(x_i, v_{ij}) f_{ij}(x_i | v_{ij}) d\mu(x_i)
\]
among tests \((\psi_{12}, \ldots) \in S(\gamma)\) if

\[
1 \quad \text{when } x_i > c(v_{ij})
\]

\[
(38) \quad \varphi_{ij}(x_i, v_{ij}) = a(v_{ij}) \quad \text{when } x_i = c(v_{ij})
\]

\[
0 \quad \text{when } x_i < c(v_{ij})
\]

where \(a(v_{ij})\) and \(c(v_{ij}), (i,j) \in \mathcal{T}\), are such that

\[
(39) \quad \sum_{(i,j) \in \mathcal{T}} \{a(v_{ij})(c(v_{ij})\Delta) \frac{1}{2} v_{ij} + \sum_{x_i > c(v_{ij})} (v_{ij})(\frac{1}{2}) v_{ij}\} = \gamma
\]

and

\[
(40) \quad a(v_{ij})(c(v_{ij})\Delta) + \sum_{x_i > c(v_{ij})} (v_{ij})(\frac{1}{2}) v_{ij} = \text{constant}, (i,j) \in \mathcal{T}.
\]
Consider the subsets

\[ \omega_{ij} = \{(p_1, \ldots, p_k); \frac{p_i}{p_j} \geq \Delta^* > 1, \sum_{s=1}^{k} p_s = 1\}, (i,j) \in T. \]

By using Lemma 1 the following conclusion is obtained.

(i) A test \( \varphi_{12, \ldots} \) where \( \varphi_{ij} \) is defined by (36) and (37), \( (i,j) \in T \), maximizes the minimum average conditional power over \( \omega_{ij}, (i,j) \in T \), among tests in \( S(\gamma) \).

(ii) A test \( \varphi_{12, \ldots} \) where \( \varphi_{ij} \) is defined by (38), (39) and (40), \( (i,j) \in T \), maximizes the minimum conditional power over \( \omega_{ij}, (i,j) \in T \), among tests in \( S(\gamma) \).

Each component test \( \varphi_{ij} \) depends on \( \Delta^* \) in addition to \( v_{ij} \), and the levels of the test are not in general all equal. The conditional power is an increasing function of the conditioning variable, and if \( v_{ij} > v_{kl} \) the level of \( \varphi_{ij} \) has to be chosen less than that of \( \varphi_{kl} \) in order to get the same minimum power. In the case \( v_{ij} = v \), \( (i,j) \in T \), which is unlikely to occur, equal levels are equivalent to constant minimum power, and we get the same tests as in 2.1.

Usually we are interested in non-randomized tests and then optimal tests can not generally be found. We have used the normal approximation to the binomial distribution, and have found tests which are approximately optimal in a few constructed examples.

The constant \( c \) in (36) is determined such that

\[ \sum_{(i,j) \in T} \left( 1 - \frac{c + \Delta v_{ij} - 0.5 v_{ij}}{0.5 v_{ij}} \right) = \gamma \]
and \( c(v_{ij}) \) in (38), \((i,j) \in T\), are determined such that

\[
\sum_{(i,j) \in T} \left( 1 - \frac{c(v_{ij}) - 0.5v_{ij}}{0.5v_{ij}} \right) = \gamma
\]

and

\[
1 - \phi\left( \frac{c(v_{ij}) - p*v_{ij}}{\sqrt{p*(1-p*)v_{ij}}} \right) = \text{constant}, \quad (i,j) \in T,
\]

where \( p^* = \frac{\Delta^*}{1 + \Delta^*} \).

All paired comparisons are done, hence \( N = k(k-1) \). We have chosen \( \gamma = 0.05\), \( \Delta^* = 1.5\), \( \Delta^* = 2.5\), \( k = 3\) and \( k = 4\).

The examples show the levels of the component tests \( \psi_{ij} \), \( i < j \), in a vector test which a) approximately maximizes the minimum average conditional power and b) approximately maximizes the minimum conditional power.

**Example 1.** \( k = 3\), \( X_1 = 10\), \( X_2 = 20\), \( X_3 = 20\).

<table>
<thead>
<tr>
<th>( (i,j) )</th>
<th>( V_{ij} )</th>
<th>( \Delta^* = 1.5 )</th>
<th>( \Delta^* = 2.5 )</th>
<th>( \Delta^* = 1.5 )</th>
<th>( \Delta^* = 2.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (1,2) )</td>
<td>30</td>
<td>0.0069</td>
<td>0.0087</td>
<td>0.0094</td>
<td>0.0106</td>
</tr>
<tr>
<td>( (1,3) )</td>
<td>30</td>
<td>0.0069</td>
<td>0.0087</td>
<td>0.0094</td>
<td>0.0106</td>
</tr>
<tr>
<td>( (2,3) )</td>
<td>40</td>
<td>0.0109</td>
<td>0.0080</td>
<td>0.0059</td>
<td>0.0038</td>
</tr>
</tbody>
</table>
Example 2. \( k = 3, x_1 = 10, x_2 = 30, x_3 = 30 \)

<table>
<thead>
<tr>
<th>((i,j))</th>
<th>(V_{ij})</th>
<th>(\Delta^*=1.5)</th>
<th>(\Delta^*=2.5)</th>
<th>(\Delta^*=1.5)</th>
<th>(\Delta^*=2.5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,2)</td>
<td>40</td>
<td>0.0066</td>
<td>0.0093</td>
<td>0.0103</td>
<td>0.0113</td>
</tr>
<tr>
<td>(1,3)</td>
<td>40</td>
<td>0.0066</td>
<td>0.0093</td>
<td>0.0103</td>
<td>0.0113</td>
</tr>
<tr>
<td>(2,3)</td>
<td>60</td>
<td>0.0112</td>
<td>0.0063</td>
<td>0.0046</td>
<td>0.0019</td>
</tr>
</tbody>
</table>

Example 3. \( k = 3, x_1 = 10, x_2 = 40, x_3 = 40 \)

<table>
<thead>
<tr>
<th>((i,j))</th>
<th>(V_{ij})</th>
<th>(\Delta^*=1.5)</th>
<th>(\Delta^*=2.5)</th>
<th>(\Delta^*=1.5)</th>
<th>(\Delta^*=2.5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,2)</td>
<td>50</td>
<td>0.0070</td>
<td>0.0102</td>
<td>0.0107</td>
<td>0.0121</td>
</tr>
<tr>
<td>(1,3)</td>
<td>50</td>
<td>0.0070</td>
<td>0.0102</td>
<td>0.0107</td>
<td>0.0121</td>
</tr>
<tr>
<td>(2,3)</td>
<td>80</td>
<td>0.0112</td>
<td>0.0050</td>
<td>0.0037</td>
<td>0.0011</td>
</tr>
</tbody>
</table>

Example 4. \( k = 3, x_1 = 10, x_2 = 60, x_3 = 60 \)

<table>
<thead>
<tr>
<th>((i,j))</th>
<th>(V_{ij})</th>
<th>(\Delta^*=1.5)</th>
<th>(\Delta^*=2.5)</th>
<th>(\Delta^*=1.5)</th>
<th>(\Delta^*=2.5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,2)</td>
<td>70</td>
<td>0.0075</td>
<td>0.0111</td>
<td>0.0112</td>
<td>0.0124</td>
</tr>
<tr>
<td>(1,3)</td>
<td>70</td>
<td>0.0075</td>
<td>0.0111</td>
<td>0.0112</td>
<td>0.0124</td>
</tr>
<tr>
<td>(2,3)</td>
<td>120</td>
<td>0.0103</td>
<td>0.0029</td>
<td>0.0025</td>
<td>0.0004</td>
</tr>
</tbody>
</table>
Example 5. \( k = 4 \), \( x_1 = 10 \), \( x_2 = 20 \), \( x_3 = 30 \), \( x_4 = 40 \)

<table>
<thead>
<tr>
<th>((i,j))</th>
<th>(v_{ij})</th>
<th>(\Delta^* = 1.5)</th>
<th>(\Delta^* = 2.5)</th>
<th>(\Delta^* = 1.5)</th>
<th>(\Delta^* = 2.5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,2)</td>
<td>30</td>
<td>0.0012</td>
<td>0.0049</td>
<td>0.0087</td>
<td>0.0142</td>
</tr>
<tr>
<td>(1,3)</td>
<td>40</td>
<td>0.0027</td>
<td>0.0048</td>
<td>0.0054</td>
<td>0.0053</td>
</tr>
<tr>
<td>(1,4)</td>
<td>50</td>
<td>0.0042</td>
<td>0.0043</td>
<td>0.0035</td>
<td>0.0020</td>
</tr>
<tr>
<td>(2,3)</td>
<td>50</td>
<td>0.0042</td>
<td>0.0043</td>
<td>0.0035</td>
<td>0.0020</td>
</tr>
<tr>
<td>(2,4)</td>
<td>60</td>
<td>0.0057</td>
<td>0.0036</td>
<td>0.0023</td>
<td>0.0008</td>
</tr>
<tr>
<td>(3,4)</td>
<td>70</td>
<td>0.0069</td>
<td>0.0030</td>
<td>0.0016</td>
<td>0.0003</td>
</tr>
</tbody>
</table>

Example 6. \( k = 4 \), \( x_1 = 20 \), \( x_2 = 40 \), \( x_3 = 60 \), \( x_4 = 80 \)

<table>
<thead>
<tr>
<th>((i,j))</th>
<th>(v_{ij})</th>
<th>(\Delta^* = 1.5)</th>
<th>(\Delta^* = 2.5)</th>
<th>(\Delta^* = 1.5)</th>
<th>(\Delta^* = 2.5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,2)</td>
<td>60</td>
<td>0.0021</td>
<td>0.0092</td>
<td>0.0107</td>
<td>0.0182</td>
</tr>
<tr>
<td>(1,3)</td>
<td>80</td>
<td>0.0035</td>
<td>0.0056</td>
<td>0.0055</td>
<td>0.0046</td>
</tr>
<tr>
<td>(1,4)</td>
<td>100</td>
<td>0.0044</td>
<td>0.0034</td>
<td>0.0030</td>
<td>0.0011</td>
</tr>
<tr>
<td>(2,3)</td>
<td>100</td>
<td>0.0044</td>
<td>0.0034</td>
<td>0.0030</td>
<td>0.0011</td>
</tr>
<tr>
<td>(2,4)</td>
<td>120</td>
<td>0.0050</td>
<td>0.0020</td>
<td>0.0016</td>
<td>0.0003</td>
</tr>
<tr>
<td>(3,4)</td>
<td>140</td>
<td>0.0054</td>
<td>0.0012</td>
<td>0.0009</td>
<td>0.0006</td>
</tr>
</tbody>
</table>

In a) the results are a bit confusing. When \( \Delta^* = 1.5 \) the levels are small for low \( v_{ij} \)s and increase as the \( v_{ij} \)s increase. When \( \Delta^* = 2.5 \) the situation is changed. We get the lowest levels for high \( v_{ij} \)s, and the difference between the levels increases as the difference between the \( v_{ij} \)s decreases.
In b) the examples show that the levels are high for low \( v_{ij} \)'s, and the difference between the levels becomes greater as the difference between the \( v_{ij} \)'s increases. The reason is that the conditional power is an increasing function of \( v_{ij} \), and to get the same minimum power the levels have to be high for small values and low for high values under the condition that their sum equals \( \gamma \).

We get different tests when maximizing the conditional power compared to the unconditional power. In practical applications the varying of the levels of the component tests does not seem to be of any great importance. Comparing our examples with the tests we would get by using equal levels, we only find great differences in Example 4 and Example 6 when \( \Delta^* = 2.5 \).

Situations where it may be useful to vary the levels occur when comparing both small proportions and high proportions in the same multiple problem. Then a choice of higher levels for the smaller proportions may be appropriate.
REFERENCES
