UNIFORMLY MINIMUM VARIANCE UNBIASED (UMVU) ESTIMATORS BASED ON SAMPLES FROM RIGHT TRUNCATED AND RIGHT ACCUMULATED EXPONENTIAL DISTRIBUTIONS

by

Erik N. Torgersen
ABSTRACT

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BASED ON SAMPLES FROM RIGHT TRUNCATED AND RIGHT ACCU-MULATED EXPONENTIAL DISTRIBUTIONS

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Consider a sample; $X_1, \ldots, X_n$; of size $n$ from the distribution $F$ on $[0,1]$ given by:

$$F(0,x] = 1 - e^{-\lambda x}; x \in [0,A[$$

and

$$F([1]) = e^{-\lambda}$$

Here $\lambda > 0$ is an unknown parameter.

It is shown that this experiment does not admit a boundedly complete and sufficient statistic when $n \geq 2$. We provide answers to the following problems:

Which functions of $(X_1, \ldots, X_n)$ are UMVU estimators of their expectations?

Which functions of $\lambda$ has UMVU estimators?

Suppose a function of $\lambda$ has an UMVU estimator. How do we find it?

How must $n$ be chosen so that a given function of $\lambda$ does have an UMVU estimator based on $n$ observations?
Introduction

Consider a sample; \(X_1, \ldots, X_n\); of size \(n\) from the distribution \(F\) on \([0,1]\) given by:

\[
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and

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Here \(\lambda > 0\) is an unknown parameter.

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Which functions of \((X_1, \ldots, X_n)\) are UMVU estimators of their expectations?

Which functions of \(\lambda\) has UMVU estimators?

Suppose a function of \(\lambda\) has an UMVU estimator. How do we find it?

How must \(n\) be chosen so that a given function of \(\lambda\) does have an UMVU estimator based on \(n\) observations?

The basic results on UMVU estimation in Lehmann and Scheffé [1950; Completeness, similar regions and unbiased estimation. Sankhyā 10, 305-340] and in Bahadur [1957; On unbiased estimates of uniformly minimum variance. Sankhyā 18, 211-224] are used constantly and without explicit references. Our use of Laplace transforms appears to be similar to the use of Laplace transforms in Linnik [1966, Statistical problems with nuisance parameters. Translations Monographs Volume 20, 1968, American mathematical society].
UMVU estimators based on samples from right truncated and right accumulated exponential distributions.

Suppose the probability of death within the infinitesimal time interval \((x, x + dx)\) is \(\lambda e^{\lambda x} dx\). Inference on \(\lambda\) based on, either the observed lifespans of \(n\) randomly chosen individuals, or on the time of occurrence of the \(n\)-th death may - in both cases - be based on a complete and sufficient statistic. If, however, our experiment is obtained by observing the times of death within a given period for a fixed sample of at least two individuals, then, as we shall see, no complete and sufficient statistic is available*. UMVU estimation must then be based on first principles; and it is this analysis which is the subject of this section. We may, without loss of generality, assume that the period chosen is of unit length.

Our experiment, \(G^n\), is then obtained by making \(n\) independent observations \(X_1, \ldots, X_n\) of a random variable \(X\) whose distribution is given by:

\[
P(0 < X < x) = 1 - e^{-\lambda x}; \quad x \in [0, 1]
\]

and

\[
P(X = 1) = e^{-\lambda}
\]

We will assume that \(\lambda > 0\) is totally unknown.

Most of the analysis, however, carries over to the case where the parameter set is a specified sub set of \([0, \infty[\) having at least one point of accumulation.

If \(n = 1\), then our experiment is a sub experiment of a complete experiment and, consequently, is itself a complete experiment.

*) The problem of completeness was brought to the authors attention by E. Sverdrup.
In order to describe the joint distribution of $X_1, \ldots, X_n$ in a convenient way let us introduce the functions:

\[ d(x) = 1 \text{ if } 0 \leq x < 1 \]

\[ = 0 \text{ if } x = 1 \]

\[ t(x) = xd(x) ; x \in [0, 1] \]

\[ d_n(x_1, \ldots, x_n) = \sum_i d(x_i) ; x_1, \ldots, x_n \in [0, 1] \]

and

\[ t_n(x_1, \ldots, x_n) = \sum_i t(x_i) ; x_1, \ldots, x_n \in [0, 1] \]

Let $U$ and $\delta_1$ denote, respectively, the rectangular distribution on $[0, 1]$ and the one point distribution in 1. Then

\[ d_n(x) = \left( \frac{1 - e^{-\lambda}}{\lambda} \right)^d e^{-\lambda(1-d)} ; d = 0, 1, \ldots, n \]

is a version of $\frac{dP}{d\lambda}(dU + \delta_1)^n$.

It follows that $(d_n(X), t_n(X))$ constitutes a minimal sufficient statistic and that the family of distributions of this statistic is of the Darmois, Koopman type.

The joint distribution of $D_n = d_n(X)$ and $T_n = t_n(X)$ is given by:

(i) $P_\lambda(D_n = d) = \binom{n}{d}(1-e^{-\lambda})^d e^{-(n-d)\lambda} ; d = 0, 1, \ldots, n$.

ie $D_n$ is binomially distributed with success parameter $1-e^{-\lambda}$.

(ii) The conditional distribution of $T_n$ given $D_n = d$ has, for each $d = 0, 1, \ldots, n$, density:

\[ \frac{\lambda^d}{(1-e^{-\lambda})^d} e^{-\lambda t} ; t \in [0, d] \]

w.r.t $U^d$ where $U^d$ is the $d$ fold convolution of $U$.

The density of $T_n$ given $D_n = d$ w.r.t Lebesgue measure on $[0, d]$ is thus:

\[ \frac{\lambda}{(1-e^{-\lambda})} e^{-\lambda t} f_d(t) ; t \in [0, d] \]
where \( f_d(t) = \frac{1}{(d-1)!} \left[ x^{d-1} - \binom{d}{1} (t-1)^{d-1} + \cdots + \binom{d}{d-1} (t-[t])^{d-1} \right] \); 
\( t \in [0, d] \), is a version of the density of \( U^d \) w.r.t. Lebesgue measure on \([0, d]\).

Before proceeding let us note the fact that \( \int h(x) U^n (dx) < \infty \)

if and only if \( \int h(x) x^{n-1} (n-x)^{n-1} dx < \infty \). It follows that a statistic \( \delta( D_n, T_n ) \) is integrable if and only if:

\[
\int_0^d \delta(d, t) t^{d-1} (d-t)^{d-1} dt < \infty; d = 1, ..., n
\]

We shall use the fact that any integrable statistic may be represented by a sequence \( \nu_0, \nu_1, \nu_2, \ldots, \nu_n \) of finite measures so that: \( \nu_i \ll U^n \); \( i=0, 1, \ldots, n \) and

\[
\frac{d\nu_{i-1}}{dU^i} = \left( \begin{array}{c} n \\ i \end{array} \right) e^{-(n-i)} e^{-t} \delta(i, t); t \in [0, i]; i = 0, 1, \ldots, n
\]

This representation is 1-1 onto and linear. The expectation of an integrable statistic may be expressed in terms of these measures as follows:

**Proposition E.3.**

Let \( \kappa_0 \) be the one point distribution in 0 and \( \kappa_i, i = 1, 2, \ldots \) the probability distribution on \([i, \infty[\) whose density w.r.t. Lebesgue measure on \([i, \infty[\) may be specified as:

\[
\Gamma(i)^{-1} (t-i)^{i-1} e^{-(t-i)}; t \geq i
\]

Then the expectation of an integrable statistic \( \delta( D_n, T_n ) \) may be written:

\[
E \delta(D_n, T_n) = \chi^n \int e^{(1-\lambda) t} \prod_{d=0}^{n} \kappa_{n-d} \nu_d (dt).
\]
Proof:

\[ E \delta(D, t) = \sum_{d=0}^{n} \lambda^d e^{-(n-d)\lambda} \int_0^d e^{-(1-\lambda)t} v_\delta, d(dt) \]

\[ = \sum_{d=0}^{n} \lambda e^{(1-\lambda)(n-d)\lambda} \int_0^\infty e^{-(1-\lambda)t} v_\delta, d(dt) \]

\[ = \lambda \sum_{d=0}^{n-1} \int_0^\infty e^{-(1-\lambda)t} [t^{n-d} v_\delta, d](dt) \]

\[ = \lambda \int_0^\infty e^{-(1-\lambda)t} \left[ \sum_{d=0}^{n-1} t^{n-d} v_\delta, d \right](dt) . \]

Corollary E. 2

\( \delta(D_n, T_n) \) is an unbiased estimator of zero if and only if it is integrable and

\( \sum_{d=0}^{n} \int_0^\infty e^{-(1-\lambda)t} \nu^a_{\delta, d} = 0 \)

The corollary tells us - in principle - how to construct the most general unbiased estimator of zero. To see this rewrite (§) as:

\( \nu^a_{\delta, n} = -\sum_{d=0}^{n-1} \kappa_{n-d} v_\delta, d \)

The procedure is therefore: Choose \( \delta(d, \cdot) ; d=0, 1, \ldots, n-1 \) so that \( \sum_{d=0}^{n-1} \kappa_{n-d} v_\delta, d \) has no mass on \( [n, \infty[ \). Finally, \( \nu^a_{\delta, n} \) and thus \( \delta(n, \cdot) \) is obtained by (§§).

Proposition E. 3

Let \( \delta(d, t) ; t\in[0, d] \): \( d=0, 1, \ldots, n-1 \) be given functions. Then there is a function \( \delta(n, t) \) so that \( \delta(D_n, T_n) \) is an unbiased estimator of zero if and only if:
(i) \[ \int_0^d |\delta(d,t)| t^{d-1}(d-t)^{d-1} dt < \infty ; \quad d=1, \ldots, n-1 \]

(ii) \[ \delta(o,o) = 0 \]
and
\[ \sum_{d=1}^i \frac{e^{n-d} (n-d-1)}{\Gamma(n-d)} \int_0^d e^{x(d-x)i-d \nu_\delta, d} (dx) = 0 ; \quad i=1, \ldots, n-1 \]

Proof:

We must show that (ii) is - assuming (i) is satisfied - a necessary and sufficient condition. The measures \( \kappa_{n-d}^{*\nu_\delta, d} \); \( d=o, 1, \ldots, n-1 \) are absolutely continuous with, respectively, densities:

\[ \int_0^d \Lambda(t-n+d) \frac{n-1}{\Gamma(n-d)} (t-x-n+d)^{n-d-1} e^{-(t-x-n+d) \nu_\delta, d} (dx) ; \quad t \geq n-d \]

It follows that a version of the density of

\[ \sum_{d=0}^{n-1} \kappa_{n-d}^{*\nu_\delta, d} \]

is:

\[ \Gamma(n)^{-1} (t-n)^{n-1} e^{-(t-n) \nu_\delta, o} \{0\} I_n, \infty [ \]

\[ + \sum_{d=1}^{n-1} \int_0^d \Gamma(n-d)^{-1} (t-x-n+d)^{n-d-1} e^{-(t-x-n+d) \nu_\delta, d} (dx) ; \quad t \geq 1 \]

On \([n, \infty [\) this reduces to

\[ \Gamma(n)^{-1} (t-n)^{n-1} e^{-(t-n) \nu_\delta, o} \{0\} \]

\[ + \sum_{d=1}^{n-1} \int_0^d \Gamma(n-d)^{-1} (t-x-n+d)^{n-d-1} e^{-(t-x-n+d) \nu_\delta, d} (dx) \]

It follows - by continuity - that \( \sum_{d=0}^{n-1} \kappa_{n-d}^{*\nu_\delta, d} \) is concentrated on \([o, n]\) if and only if:

\[ \Gamma(n)^{-1} (t-n)^{n-1} \nu_\delta, o \{0\} \]

\[ + \sum_{d=1}^{n-1} \int_0^d \Gamma(n-d)^{-1} (t-x-n+d)^{n-d-1} e^{-(t-x-n+d) \nu_\delta, d} (dx) = 0, \quad t > n \]

i.e. \( \Gamma(n)^{-1} t^{n-1} \nu_\delta, o \{0\} \)

\[ + \sum_{d=1}^{n-1} \int_0^d \Gamma(n-d)^{-1} (t-x+d)^{n-d-1} e^{-(t-x+d) \nu_\delta, d} (dx) = 0, \quad t > 0 \]

*) If \( a \) and \( b \) are numbers then \( a \wedge b \equiv \min \{a, b\} \).
The left hand side of this identity is - in any case - a polynomial of degree at most $n-1$. The proof is now completed by checking that the equations in (ii) just states that the coefficients of this polynomial are all zero. \[\Box\]

(ii) may be rewritten as:

(ii') $\delta(o,o) = 0$

and

$$\sum_{d=1}^{n} \frac{e^{n-d}}{\Gamma(n-d)} \int_0^{n-j-1} e^{x} \nu_{j+1}^{(n-j-1)}(dx) = \sum_{d=1}^{n} \frac{e^{n-d}}{\Gamma(n-d)} \int_0^{n-j-2} e^{x} \nu_{j+1}^{(n-j-2)}(dx);$$

Suppose now that $\delta(d,\cdot)$ are constructed for $d=0,1,\ldots,i \leq n-2$

so that $\int_0^d |\delta(d,t)| t^{d-1}(d-t)^{d-t} < \infty$ and (ii) holds i.e:

$$\sum_{d=1}^{n} \frac{e^{n-d}}{\Gamma(n-d)} (n-d-2) \int_0^{n-d-2} e^{x} \nu_{j+1}^{(n-d)-2}(dx) = 0; \quad i'=1,2,\ldots,i$$

and $\delta(o,o) = 0$.

By proposition E.3 there are functions $\delta(d,\cdot); i < d \leq n$ so that $\delta(D_n, T_n)$ is an unbiased estimator of zero.

Consider now any UMVU estimator $\varphi$. By proposition E.3 there is no restriction on $\varphi(o,o)$. Let $\delta$ be an UMVU estimator of zero. By

$$\int_0^1 e^x \nu_{j+1}(dx) = 0 \text{ i.e: } \int_0^1 \delta(1,x)dx = 0.$$ Hence

$$\int_0^1 \varphi(1,x) \delta(x)dx = 0$$ for any square integrable function $\delta(x), x \in [0,1]$ such that $\int_0^1 \delta(x)dx = 0$. This imply that there is a constant $b$ so that $\varphi(1,\cdot) = b$ a.e Lebesgue on $[0,1]$.

We shall now demonstrate that $\varphi(d,\cdot) = b$ a.e Lebesgue on $[0,d]$ for $d = 1,2,\ldots,n-1$. Suppose we have shown that $\varphi(d,\cdot) = b$ a.e. Lebesgue on $[0,d]$ for $d = 1,2,\ldots,j < n-1$. 
Let $\delta$ be a square integrable unbiased estimator of zero.

Then $\varphi \delta$ is an unbiased estimator of zero so that:

$$\frac{e^{n-j-1}}{\Gamma(n-j-1)} \int_0^{j+1} e^x \varphi(x) \eta_\delta, j+1(dx) = -\sum_{d=1}^{j} \frac{e^{n-d}}{\Gamma(n-d)} \left( \int_0^{n-d-1} d e^x (d-x)^{j+1-d} \varphi_d, d(dx) \right)$$

Hence

$$\frac{e^{n-j-1}}{\Gamma(n-j-1)} \int_0^{j+1} e^x \varphi(x) \eta_\delta, j+1(dx) = -\sum_{d=1}^{j} \frac{e^{n-d}}{\Gamma(n-d)} \left( \int_0^{n-d-1} d e^x (d-x)^{j+1-d} \varphi_d, d(dx) \right)$$

It follows that:

$$\int_0^{j+1} \varphi(x) \eta_\delta, j+1(dx) = \int_0^{j+1} \varphi(x) \eta_\delta, j+1(dx)$$

for any bounded function $\delta$ on $[0, j+1]$ satisfying

$$\int_0^{j+1} \delta(x) \eta_\delta, j+1(dx) = 0.\text{ Hence } \varphi(j+1, \cdot) = b \text{ a.e.} \text{ Lebesgue. By induction: } \varphi(d, \cdot) = b \text{ a.e.} \text{ Lebesgue on } [0, d]; d = 1, \ldots, n-1.$$  

What can we say about $\varphi(n, \cdot)$? Let $\delta$ be any unbiased estimator of zero. The density (strictly speaking; a version of) $\eta_\delta, n$ is $t \mapsto e^{-\delta(n, t)} f_n(t)$ on $[1, n]$. By the identity: $\eta_\delta, n = \sum_{d=0}^{n-1} \eta_{n-d} \ast \eta_\delta, d$, this density may also be written:

$$-\sum_{d=0}^{n-1} \int_0^{d\wedge(t-n+d)} \Gamma(n-d-1) (t-x-n+d)^{d-1} e^{-(t-x-n+d)} \eta_\delta, d(dx)$$

It follows – provided $\delta(D_n, T_n)^2$ is integrable – that:

$$e^{-\delta(n, t)} \varphi(n, t) f_n(t) = -\sum_{d=1}^{n-1} \int_0^{d\wedge(t-n+d)} \Gamma(n-d-1) (t-x-n+d)^{n-d-1} e^{-(t-x-n+d)} \eta_\delta, d(dx)$$

$$= b e^{-\delta(n, t)} f_n(t) ; \text{ almost all } t \epsilon [1, n]$$
Hence \([\varphi(n, t) - b] \delta(n, t) f_n(t) = 0\); almost all \(t \in [1, n]\)

Consider now the space of unbiased estimators \(\delta\) of zero such that: \(\delta(0, \cdot) = 0, \delta(1, \cdot) = 0, \ldots, \delta(n-2, \cdot) = 0\).

By proposition E.3:
\[
\int_0^{n-1} e^{-x \delta(n-1, x)} f_{n-1}(x) dx = 0
\]

i.e.
\[
\int_0^{n-1} \delta(n-1, x) f_{n-1}(x) dx = 0, \text{ and this is the only requirement on } \delta(n-1, \cdot).
\]

By the density considerations above:
\[
e^{-t \delta(n, t)} f_n(t) = -\int_0^{t-1} \Gamma(1)^{-1} (t-x-1)^{n-2} e^{-(t-x-n-n-1)} \delta(n-1, x) f_{n-1}(x) dx
\]

\(= -n e^{-t} \int_0^{t-1} \delta(n-1, x) f_{n-1}(x) dx; \text{ almost all } t \in [1, n]\)

It follows that
\[
[\varphi(n, t) - b] \int_0^{t-1} \delta(n-1, x) f_{n-1}(x) dx = 0, \text{ on almost all } t \in [1, n], \text{ provided } \delta(D_n, T_n) \text{ is square integrable.}
\]

It follows that
\[
[\varphi(n, t) - b] \int_0^{t-1} \gamma(x) dx = 0; \text{ for almost all } t \in [1, n].
\]

provided
\[
\int_0^{n-1} \gamma(x) f_{n-1}(x) dx = 0, \int_0^{n-1} \gamma(x)^2 f_{n-1}(x) dx < \infty
\]

and
\[
\int_0^{1} \left[ 1/f_n(t) \right] \left[ \int_0^{t-1} \gamma(x) f_{n-1}(x) dx \right]^2 dt < \infty.
\]

These conditions are satisfied with \(\gamma = \gamma_\varepsilon\) where

\[
\gamma_\varepsilon(x) = \begin{cases} 0 & \text{when } 0 < x < \varepsilon \\ -1/f_{n-1}(x) & \varepsilon \leq x \leq n-1 - \varepsilon \\ 1/f_{n-1}(x) & n-1 - \varepsilon < x \leq n-1 - \varepsilon \\ 0 & x > n-1 - \varepsilon \end{cases}
\]

and \(\varepsilon \in [0, n-1]\) is a constant. Letting \(\varepsilon \to 0\) we get:
\[
[\varphi(n, t) - b] \int_0^{t-1} \gamma(x) dx = 0; \text{ for almost all } t \in [1, n]
\]

where \(\gamma(x) = -1\) or \(+1\) as \(x < n-1/2\) or \(x > n-1/2\).
Hence

\[ \varphi(n, \cdot) = b \text{ a.e. on } [1, n] \]

If \( \delta \) is an unbiased estimator of zero then, by the identity:

\[ \nu_{\delta, n} = -\sum_{d=1}^{n-1} k_{n-d} \nu_{\delta, d} \],

\( \nu_{\delta, n} \) is concentrated on \([1, n]\). It follows that \( \delta(n, \cdot) = o \text{ a.e on } [0, 1]\). This implies that there is no restrictions on \( \varphi(n, \cdot) \) on \([0, 1]\), except for the condition of square integrability. We have proved the "only if" part of

**Theorem E.5**

\( \varphi(D_n, T_n) \) is an UMVU estimator of its expectation if and only if:

(i) \( \varphi \) is a.e. a constant on

\[ \{(d, t): 1 \leq d \leq n-1\} \cup \{(n, t): 1 \leq t \leq n\} \]

and

(ii) \( \int_0^1 \varphi(n, t)^2 t^{n-1} dt < \infty \)

**Proof:** Suppose \( \varphi \) satisfies (i) and (ii) and let \( \delta \) be any square integrable estimator of zero. We must show that \( \varphi \delta \) is an unbiased estimator of zero. (It is easily seen that \( \varphi \) is square integrable). Let \( b \) denote the constant in(i).

Then

\[ \sum_{d=0}^{n-1} k_{n-d} \nu_{\delta, d} = b \sum_{d=0}^{n-1} k_{n-d} \nu_{\delta, d} = -b \nu_{\delta, n} = -\nu_{\delta, n} = -\nu_{\delta, n} \].

**Corollary E.5**

\( \varphi(D_n, T_n) \) is an UMVU estimator of its expectation if and only if it is square integrable and measurable w.r.t. the \( \sigma \)-algebra \( \mathcal{F} \) of sets generated by the sets:

\[ [D_n = P_n = 0], \]

\[ [1 \leq D_n \leq n-1] \cup [D_n = n, T_n \geq 1] \]

and

\[ [D_n = n] \land [T_n \leq t]; t \in [0, 1]. \]
Let \( \varphi(D_n, T_n) \) be \( \mathcal{S}_\lambda \)-measurable and integrable. Put
\[
a = \varphi(o, o),
b = \varphi(0, T_n) \; ; \text{a.s. on the set } [1 \leq D_n \leq n-1] \cup [D_n = n, T_n \geq 1]\
\]
and write \( c(t) = \varphi(n, t) \) when \( t \in [0, 1] \). Then
\[
E\varphi(D_n, T_n) = aP(D_n = T_n = o) + b [1 - P(D_n = T_n = o) - P(D_n = n, P_n < 1)] + P(D_n = n) \int_0^1 c(t) P(T_n \in dt | D_n = n) = a e^{-n\lambda} + b (1 - e^{-n\lambda}) \int_0^{n-1} e^{-\lambda t} \frac{t^{n-1}}{(n-1)!} dt
\]
\[
+ \lambda \int_0^{n-1} c(t) e^{-\lambda t} \frac{t^{n-1}}{(n-1)!} dt
\]
\[
= A + Be^{-n\lambda} \int_0^{n-1} C(t) e^{-\lambda t} dt
\]
where \( A = b, B = a - b \) and \( C(t) = \frac{t^{n-1}}{(n-1)!} (c(t) - b) \)
P-integrability of \( \varphi \) is equivalent with Lebesgue integrability of \( C \). We have almost proved:

**Proposition E.6**

\( \mathcal{S}_\lambda \) is complete and a function \( g \) of \( \lambda \) has a \( \mathcal{S}_\lambda \)-measurable unbiased estimator if and only if \( g \) is of the form:

\[
g(\lambda) = A + Be^{-n\lambda} + \lambda \int_0^{n-1} C(t) e^{-\lambda t} dt; \; \lambda > 0
\]
where \( A \) and \( B \) are constants and \( C \) is Lebesgue integrable on \([0, 1]\). If \( g \) is of this form then its \( \mathcal{S}_\lambda \) measurable unbiased estimator \( \varphi \) is given by:

\[
\varphi(o, o) = A + B
\]
\[
\varphi(d, t) = A ; \; 1 < d < n \; \text{or} \; (d = n \; \text{and} \; t > 1)
\]
and
\[
\varphi(n, t) = A + \frac{(n-1)!}{t^{n-1}} C(t) ; \; t \in [0, 1]
\]

**Completion of the proof of the proposition:**

Consider any function
\[
g(\lambda) = A + Be^{-n\lambda} + \lambda \int_0^{n-1} C(t) d^{-\lambda t} dt; \; \lambda > 0
\]
where $A$ and $B$ are constants and $C$ is integrable (Lebesgue) on $[0,1]$. Define $\varphi$ as in the proposition. By the calculations immediately before the proposition, $\varphi$ is an unbiased estimator of $g$. Let $\varphi$ be a measurable unbiased estimator of zero. Write $a = \varphi(o,o)$, $b = \varphi(d,t)$ when $1 \leq d < n$ or $(d = n$ and $t \geq 1)$ and $C(t) = \varphi(n,t); te[0,1]$. Then - by the same calculations:

$$g(\lambda) = A + B e^{-\lambda n + \lambda} \int_0^1 C(t) e^{-\lambda t} dt = 0$$

where $A = b$, $B = a - b$ and $C(t) = \frac{t^{n-1}}{(n-1)!} (C(t) - b)$.

We may - without loss of generality - assume $n \geq 2$. Then $o = \lim_{\lambda \to 0} g(\lambda) = A + B$ and $o = \lim_{\lambda \to 0} g'(\lambda) = -nB$ so that $A = B = 0$. Hence $C = 0$ so that $\varphi = o$.

**Corollary E.7**

A function $g$ of $\lambda$ has an UMVU estimator if and only if it is of the form:

$$g(\lambda) = A + B e^{n \lambda + \lambda} \int_0^1 C(t) e^{-\lambda t} dt; \lambda > 0$$

where $A$ and $B$ are constants and $C$ is a function on $[0,1]$ such that $\int_0^1 \frac{C(t)^2}{t^{n-1}} dt < \infty$.

If $g$ is of this form then its UMVU estimator is the function $\varphi$ defined in proposition E.6.

Let $F$ denote the probability distribution on $[0,\infty[$ whose density (w.r.t. Lebesgue Measure) is $x \mapsto e^{-x}$

Then $\int e^{(1-\lambda)t} F(dt) = \lambda^{-1}; \lambda > 0$

Let $g$ be of the form $(\S)$. Then:

$$g(\lambda) = \int_0^\infty e^{(1-\lambda)t} [A \delta_n + Be^{-\lambda \delta_n} + C(t)] dt; \lambda > 0$$

where $\delta_n$ is the one point distribution in $n$ and $\kappa$ is the distribution on $[0,1]$ whose density w.r.t. Lebesgue measure is $t \mapsto e^{-t} \kappa(t)$.
It follows that a function \( g(\lambda) ; \lambda > 0 \) has a measurable* unbiased estimator if and only if it is of the form:

\[
(\S \S) \quad g(\lambda) = \lambda^n \int_0^\infty e^{(1-\lambda)t} [A \eta^n + B \eta^m G_n^{*} \delta_{n+\lambda}] \, dt ; \lambda > 0
\]

where \( A \) and \( B \) are constants and \( \xi \) is an absolutely continuous finite measure on \([0, 1]\). This representation is unique (ie \( A, B \) and \( \xi \) are determined by \( g \)) and a \( g \) of the form \((\S \S)\) has an UMVU estimator if and only if

\[
\int_0^1 \left[ \frac{d\xi}{dt} \right]^2 / n^{-1} \, dt < \infty
\]

or equivalently that:

\[
\int_0^1 c(t)^2 \, dt < \infty
\]

where \( \kappa(C) = \kappa \).

We shall use this representation to study the set of positive integers \( n \) such that a given \( g \) has an UMVU estimator based on \( n \) observations.

**Proposition E.8**

Let \( m < n \) be positive integers, \( A_m, B_m, A_n, B_n \) constants; and \( C_m \) and \( C_n \) Lebesgue integrable functions on \([0, 1]\). Consider first the case \( m > 1 \). Then

\[
A_m^* B_m^* e^{-m \lambda^* \lambda^m} \int_0^1 e^{-\lambda^* t} C_m(t) \, dt =
\]

\[
A_n^* B_n^* e^{-n \lambda^* \lambda^n} \int_0^1 e^{-\lambda^* t} C_n(t) \, dt
\]

if and only if:

(i) \( A_m = A_n \)

(ii) \( B_m = B_n = 0 \) and

(iii) \( \kappa(C_n) = \kappa(C_m)^* F(n-m)^* \)

*) From here on we write \( \mathcal{S}_n^* \) instead of \( \mathcal{S}_n \).
Let \( n > 1 \). Then
\[
A_1 + B_1 e^{-\lambda t} + \lambda \int_0^1 e^{-\lambda t} c(t) dt =
\]
\[
A_n + B_n e^{-\lambda t} + \lambda \int_0^1 e^{-\lambda t} c_n(t) dt
\]
if and only if

(i') \( A_1 + B_1 = A_n \)

(ii') \( B_n = 0 \)

and

(iii') \( \kappa(C_n) = \kappa(C_1)^{F(n-1)*} + B_1 e^{-\lambda t} \delta_1 = B_0 \]

Proof:

The identity may - in both cases - be written:

\[
\lambda^m \int_0^\infty (1-\lambda) t [A_m e^{-m F m} + B_m e^{-m F m} \delta_m + \kappa(C_m)] dt =
\]

\[
\lambda^n \int_0^\infty (1-\lambda) t [A_n e^{-n F n} + B_n e^{-n F n} \delta_n + \kappa(C_n)] dt
\]

Dividing by \( \lambda^n \) on both sides and using the identity
\[
\int_0^\infty (1-\lambda) t F(dt) = \lambda^{-1} ; \lambda > 0 \]
we may write this:

\[
A_m F^* + B_m e^{-m F^*} \delta_m + \kappa(C_m) F(n-m)^*
\]

(\( \dagger \))

\[
= A_n F^* + B_n e^{-n F^*} \delta_n + \kappa(C_n) .
\]

Further more - by letting \( \lambda \to 0 \) in the identity as it is written in the proposition - we obtain \( A_m F^* + B_m = A_n F^* + B_n \).

On \([m,n]\) (\( \dagger \)) may be written:

\[
A_m F^* + B_m e^{-m F^*} \delta_m + \kappa(C_m) F(n-m)^*
\]

= \( \dagger \) .

Or - equivalently:

\[
(A_n - A_m) F^* - B_m e^{-m F^*} \delta_m = \kappa(C_m) F(n-m)^*
\]

Writing out densities we get:

\[
(A_n - A_m) \frac{x^{n-1}}{\Gamma(n)} e^{-x} - B_m e^{-m \frac{(x-m)^{n-1}}{\Gamma(n)}} e^{-(x-m)}
\]

\[
= \int_0^1 (x-s)^{n-1} e^{-(x-s)} s C_m(s) ds
\]
i.e.:

\[(A_n - A_m) \frac{x^{n-1}}{\Gamma(n)} - B_m \frac{(x-m)^{n-1}}{\Gamma(n)} = \int_0^1 \frac{(x-s)^{n-m-1}}{\Gamma(n-m)} C_m(s) ds\]

for almost (Lebesgue) all \(x\) on \([m,n]\). Hence - since both sides of this equation are polynomials - this extends to all real numbers \(x\). The right hand side is of degree \(\leq n-m-1 \leq n-2\). It follows that the coefficient of \(x^{n-1}\) on the left hand side must vanish i.e.:

\[A_n - A_m = B_m.\]  Hence

\[B_n = A_n + B_n - A_n = A_m + B_m - A_n = 0\]

Let us now distinguish the two cases.

1°. \(m > 1\). Then the coefficient of \(x^{n-2}\) on the left is zero i.e.:

\[B_m = 0\]

It follows that \(A_m = A_n\) and that \(B_m = B_n = 0\) (iii) follows now by inserting this in (†). By (†) again (i), (ii) and (iii) are sufficient.

2°. \(m = 1\). We have shown that the identity imply (i') and (ii'). It follows that (†) may be written as (iii'):

Using proposition E.8 we will show that the set of positive integers \(n\) such that a given \(g\) has an UMVU estimator based on \(n\) observations is an interval.

**Proposition E.9**

Suppose \(g\) has an unbiased \(\Theta_n\) measurable estimator for \(n = n_1\) and \(n = n_2\). Then \(g\) has a \(\Theta_m\) measurable unbiased estimator for any integer \(m\) between \(n_1\) and \(n_2\). If, moreover, \(g\) has an UMVU estimator based on \(n\) observations for \(n = n_1\) and \(n = n_2\) then \(g\) has an UMVU estimator based on \(m\) observations for any \(m\) between \(n_1\) and \(n_2\).
Proof:

If suffices to prove the following statement:

Statement: Let \( n > m+1 \). Suppose \( g \) has an unbiased \( \mathbb{E}_m \) measurable estimator as well as an unbiased \( \mathbb{E}_n \) measurable estimator. Then \( g \) has an unbiased \( \mathbb{E}_{m+1} \) measurable estimator. If, moreover, \( g \) has an UMVU estimator based on \( m \) observations and an UMVU estimator based on \( n \) observations then \( g \) has an UMVU estimator based on \( m+1 \) observations.

Proof of the statement:

1°. \( m > 1 \). Suppose \( g \) has an unbiased \( \mathbb{E}_m \) measurable estimator and an unbiased \( \mathbb{E}_n \) measurable estimator. Then we may - by propositions E.6 and E.8 - write:

\[
g(\lambda) = A + \lambda^{-m} \int_0^1 e^{-\lambda t} C_m(t) dt
\]

\[
= A + \lambda^{-n} \int_0^1 e^{-\lambda t} C_n(t) dt
\]

We must show that there exists a \( C_{m+1} \) so that

\[
g(\lambda) = A + \lambda^{-m-1} \int_0^1 e^{-\lambda t} C_{m+1}(t) dt
\]

Or equivalently that:

\[
x(C_{m+1}) = x(C_m)^*F
\]

It suffices therefor to show that \( x(C_m)^*F \) is supported by \([0, 1]\).

The density of \( x(C_m)^*F \) on \([1, \infty[\) is:

\[
x \rightarrow \int_0^1 e^{-(x-s)} e^{-s} C_m(s) ds = e^{-x} \int_0^1 C_m(s) ds
\]

Hence we will be through if we can show that \( \int_0^1 C_m(s) ds = 0 \).

By proposition E.8 \( x(C_m)^*(n-m)^* \) is concentrated on \([0, 1]\) i.e.

\[
\int_0^1 (x-s)^{n-m-1} C_m(s) ds = 0 \text{ a.e., and hence - by continuity everywhere on } [1, \infty[. \text{ The left hand side is a polynomial in } x \text{ with } \int_0^1 \Gamma(n-m)^{-1} C_m(s) ds \text{ as coefficient of } x^{n-m-1}.
\]

It follows that \( \int_0^1 C_m(s) ds = 0 \). Suppose now that \( g \) has an
UMVU estimator based on $m$ observations, i.e:

$$\int_0^1 \frac{C_m(t)^2}{t^{m-1}} dt < \infty.$$  

It follows from the equation $\kappa(C_{m+1}) = \kappa(C_m)F$ that $e^{-x}C_{m+1}(x) = \int_0^x e^{-s}C_m(s)ds$ for almost all $x$ in $[0,1]$.

We may as well assume that:

$$C_{m+1}(x) = \int_0^x C_m(s)ds; x \in [0,1]$$

We get successively:

$$\int_0^1 \frac{C_{m+1}(x)^2}{x^m} dx = \int_0^1 \frac{\left(\int_0^x C_m(s)ds\right)^2}{x^m} dx \leq \int_0^1 \frac{1}{x^m} \int_0^x C_m(s)^2 ds dx = \int_0^1 \int_0^x \frac{C_m(s)^2}{x^m} ds dx = \int_0^1 C_m(s)^2 \int_0^1 \frac{1}{x^m} dx ds$$

If $m \geq 3$ then this may be written:

$$\int_0^1 C_m(s)^2 \frac{1}{s^{m-2}} \left(\frac{1}{s^{m-2}} - 1\right) ds$$

$$\leq \int_0^1 \frac{sC_m(s)^2}{s^{m-1}} ds = \int_0^1 \frac{C_m(s)^2}{s^{m-1}} ds < \infty.$$

If $m = 2$ then we get:

$$\int_0^1 C_m(s)^2 \left[\int_0^1 \frac{1}{x^m} dx\right] ds = \int_0^1 \frac{C_2(s)^2}{s} [-s \log s] ds < \infty.$$

$2^o. m = 1$. Suppose $g$ has an unbiased $B_1$ measurable estimator and an unbiased $B_n$ measurable estimator. Then we may - by propositions E.6 and E.8 - write:

$$g(\lambda) = A_1 + B_1 e^{-\lambda} + \lambda A_1 \int_0^1 e^{-\lambda t} C_1(t)dt$$

$$= A_1 + B_1 + \lambda B_1 \int_0^1 e^{-\lambda t} C_n(t)dt$$

where
\[ \kappa(C_n) = \kappa(C_1) F^{(n-1)} + B_1 F^{n\ast} (e^{-1} \delta_1 - \delta_0) \]

We must try to find a \( C_2 \) so that

\[ g(\lambda) = A_1 + B_1 + \lambda^2 \int_0^1 e^{-\lambda t} C_2(t) dt \]

i.e. \( C_2 \) must satisfy:

\[ \kappa(C_2) = \kappa(C_1) F + B_1 F^{2\ast} (e^{-1} \delta_1 - \delta_0) \]

It suffices therefor to show that

\[ \kappa(C_1) F + B_1 F^{2\ast} (e^{-1} \delta_1 - \delta_0) \]

is concentrated on \([0,1]\). The density on \([1, \infty]\) may be written:

\[ x \mapsto \int_0^1 e^{-(x-s)} e^{-s} C_1(s) ds + e^{-1} B_1 (x-1) e^{-(x-1)} - B_1 x e^{-x} \]

It follows that we will be through if we could show that:

\[ \int_0^1 C_1(s) ds + B_1 (x-1) - B_1 x = 0 \quad \text{i.e. that:} \]

\[ \int_0^1 C_1(s) ds = B_1 \]

By proposition E.8:

\[ \kappa(C_1) F^{(n-1)} + B_1 F^{n\ast} [e^{-1} \delta_1 - \delta_0] \]

is concentrated on \([0,1]\), i.e.

\[ \int_0^1 \frac{(x-s)^{n-2}}{\Gamma(n-1)} e^{-(x-s)} e^{-s} C_1(s) ds + B_1 e^{-1} \frac{(x-1)^{n-1}}{\Gamma(n)} e^{-(x-1)} - B_1 \frac{x^{n-1}}{\Gamma(n)} e^{-x} = 0 \]

or equivalently that:

\[ \int_0^1 \frac{(x-s)^{n-2}}{\Gamma(n-1)} C_1(s) ds + B_1 e^{-1} \frac{(x-1)^{n-1}}{\Gamma(n)} - B_1 \frac{x^{n-1}}{\Gamma(n)} = 0 \]

The coefficient of \( x^{n-2} \) on the left hand side is:

\[ \int_0^1 \frac{1}{\Gamma(n-1)} C_1(s) ds - B_1 \frac{(n-1)}{\Gamma(n)} \]

It follows that

\[ \int_0^1 C_1(s) ds = B_1 \].

Suppose finally that \( g \) has an UMVU estimator based on \( 1 \) observation, i.e. that

\[ \int_0^1 C_1(s)^2 ds < \infty \].

The density of \( \kappa(C_2) = \kappa(C_1) F + B_1 F^{2\ast} [e^{-1} \delta_1 - \delta_0] \) on \([0,1]\) is (almost everywhere)

\[ e^{-x} C_2(x) = \int_0^x e^{-(x-s)} e^{-s} C_1(s) ds \]

\[ + e^{-1} B_1 (x-1) e^{-(x-1)} - B_1 x e^{-x} \].
We may therefore as well assume that:

\[ C_2(x) = \int_0^x C_1(s) \, ds - B_1 = \int_1^x C_1(s) \, ds = -\int_0^x C_1(s) \, ds \]

Hence

\[ \int_0^1 C_2(x) \, dx = \int_0^1 \left[ \frac{\partial}{\partial x} C_1(s) \right] x \, dx \]

\[ = \int_0^1 x^2 \left[ \frac{\partial}{\partial x} C_1(s) \right] x \, dx \leq \int_0^1 x^2 C_1(s) \, dx \]

\[ = \int_0^1 \left[ \int_0^x C_1(s) \, ds \right] dx = \int_0^1 C_1(s) \, ds < \infty . \]

**Proposition E.8**

Consider an estimand \( g \) of the form:

\[ g(A) = A + B e^{-mA} + A e^{mA} \int_0^1 C_m(s) \, ds ; \lambda > 0 \text{ where } B_m = 0 \text{ when } m > 1. \]

Then \( g \) has a \( \mathcal{O}_n \) measurable unbiased estimator if and only if

\[ \int_0^1 (C_m(s) - B_m) \, ds = 0 ; i = 0, 1, \ldots, n-m-1. \]

It follows that the set of integers \( n \) such that a given non constant \( g \) has a \( \mathcal{O}_n \) measurable unbiased estimator is a finite interval. In particular the set of integers \( n \) so that a given non constant \( g \) has an UMVU estimator based on \( n \) observations is a finite interval.

**Proof:**

Suppose \( m > 1. \) By proposition E.8 \( g \) has a \( \mathcal{O}_n \) measurable unbiased estimator if and only if \( \kappa(C_m)*F(n-m)* \) is concentrated on \([0,1]\) i.e., if and only if:

\[ \int_0^1 \frac{(x-s)^{n-m-1}}{\Gamma(n-m)} e^{-(x-s) e^{-s} C_m(s) ds} = 0 \]

or equivalently that:

\[ \int_0^1 s^i C_m(s) ds = 0 ; i = 0, 1, \ldots, n-m-1. \]
If this holds for arbitrarily large \( n \) then
\[
C_m = o \ a.e. \ i.e. g = A.
\]
This proves the last two statements of the proposition.

It remains to consider the case \( m = 1 \).

Then
\[
g(\lambda) = A_1 + B_1 e^{-\lambda + \lambda^1} \int_0^1 e^{-\lambda t} C_1(s) ds
\]

By proposition E.8 \( g \) has an unbiased estimator based on \( n \) observations if and only if
\[
x(C_1) F(n-1) + B_1 F(n)^*(e^{-1} \delta_1 - 0)
\]
is concentrated on \([0,1] \ i.e. if and only if:
\[
\int_0^1 \left( \frac{x-s}{\Gamma(n-1)} C_1(s) ds + B_1 \frac{(x-1)^{n-1}}{\Gamma(n)} - B_1 \frac{x^{n-1}}{\Gamma(n)} \right) x = 0
\]
i.e. if and only if
\[
\int_0^1 (C_1(s) - B_1)s^k ds = 0; \ k = 0, 1, \ldots, n-2
\]