Indifference pricing of weather futures based on electricity futures

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1. INTRODUCTION

Increasing the share of renewable energies within the energy supply of European countries and the USA poses various challenges to energy markets, regulatory bodies and capital markets. In particular, with the emergence of renewable power sources such as wind and photovoltaic, weather factors now also have a substantial impact on the supply side in the power market. Previously, temperature has played a crucial role in determining the demand side of power, as households require heating in the winter and cooling in the summer. As the risk in operating in power markets has a strong relation to weather factors, it is of crucial importance to understand the interplay between weather and power prices and how these risk factors can be hedged.

The Chicago Mercantile Exchange (CME) provides a wide range of weather-linked futures contracts that can be used for hedging weather risk in power markets. At the CME, futures contracts are written on weather indices measured in locations world-wide. The most important futures are written on temperatures recorded in several cities in the US, Europe, Canada and Asia.

As the underlying of derivative contracts on weather is typically not tradeable we are facing an incomplete market situation in which we need to use an appropriate pricing approach. In this paper, we propose a framework for pricing temperature futures based on the indifference pricing approach. As temperature drives the demand of power, we suggest to use power futures as a correlated asset in our pricing approach and derive "fair prices" of temperature futures (as a side effect this increases also the liquidity of the contracts used for pricing). The indifference approach sets up two stochastic control problems of an agent to compare maximizing the expected utility of trading in power futures to first issuing a weather derivative and then trading optimally in power futures. The indifference price will be the one that makes the agent indifferent between the two. We derive both a seller and a buyer price based on this approach.

We model temperature using a continuous-time autoregressive model, which has turned out to explain the stochastic evolution of temperatures very well (see Benth et al. (2007) and Härdle and Cabrera (2012)). Power spot prices are modelled by a seasonally varying Ornstein-Uhlenbeck process, from which we can compute explicit power forward prices. We derive, using an exponential utility function, analytic temperature futures prices by solving the corresponding Hamilton-Jacobi-Bellman equations arising from the stochastic control problems, introducing a factorization of the solution.

We apply our results to pricing temperature futures written on the city of Essen, Germany. We use power spot and forward prices collected from the German power exchange EEX, and analyse our results with a view towards the actually observed futures prices at CME for this German city.

The indifference pricing approach is a well-established technique for derivatives pricing in incomplete markets (see Henderson and Hobson (2009) and further articles collected in Carmona (2009) and references therein). It has been applied to price rainfall derivatives by Carmona and Diko (see Carmona and Diko (2005)). The novelty of this paper is that we apply the approach to temperature derivatives, and can test the conclusions on actually traded weather derivatives in the market. Our theoretical solution also requires very different models and some new results regarding forwards and optimal control.

Furthermore, due to the explicit solution of the CAT futures price process, we will be able to calculate the price process for the whole trading period of a CAT contract and compare these results to the real data observed at the CME. Based on the results of the comparison it will be possible to develop strategies for the investor/hedger whether he should be active at the exchange or rather go to the OTC market. From our point of view this is the first time that the results of a mathematical model can be compared to real CME data for different contracts and that strategies for investors can be deduced. Additionally, from the indifference pricing approach, we will be able to analyze the risk preferences of the market participants as well as any implied risk premiums.

We present our results as follows: In §2 we give an overview of the market for temperature derivatives and the most important derivative structures. §3 is devoted to introduce our modelling approach. We then use the indifference approach to price the relevant futures contracts. In the following section §4 we perform an empirical analysis of the relevant data with a focus on the correlation of temperature and power prices. We then use in §5 our approach to calculate prices for various derivatives and compare them with market prices. §6 then gives a sensitivity analysis of important model parameters. We conclude in §7.

2. THE TEMPERATURE DERIVATIVE MARKET

Weather related derivatives present a relatively newly developed class of derivatives compared to derivatives based on other financial assets (e. g. currency swaps etc.). The first weather derivatives were traded in 1997 probably due to the impact of the weather phenomenon El Niño on some industries. The first weather derivatives which are mentioned in the literature, were traded by Aquila Energy (a weather option embedded in a power contract (Considine, 2000)) and Enron Corp. which structured a weather related bond with Koch Industries Inc. In this contract Enron agreed to pay $10,000 for every degree which was below the normal temperature index for Milwaukee, US for the winter period 1997-1998.

The CME in Chicago is currently the only exchange which offers weather derivatives. The largest part of weather derivatives which are traded at the CME are related to the temperature indices HDD, CDD and CAT. The temperature indices differ in the way the daily average temperature is accumulated over the observation period:

- Heating-degree-day

  The degree-day approach originates from the heating industry in the US which found out that the demand for heating and cooling in households depends strongly on the deviation of the daily average temperature from 65 degrees Fahrenheit (°F) or 18 degrees Celsius (°C). The HDD of day i measures the degrees of daily average temperatures which are below 18°C on day i or mathematically

$$HDD_i = \max\{18 - T_i, 0\}$$

  where T_i is the average temperature on day i. The HDD of a given observation period with N days is then defined as the sum of the $HDD_i's$ of the N days or

$$HDD = \sum_{i=1}^{N} HDD_i.$$

  In the following we will synonymously use the term HDD for either the HDD of one day or of an observation period.

- Cooling-degree-day

  Similar to the definition of the HDD, the *cooling-degree-day* (CDD) describes the days on which the daily average temperature exceeds 18°C which are typically the days where a higher load consumption of US households is expected due to the use of air conditioning. Consequently, the CDD of the day i is defined as

$$CDD_i = \max\{T_i - 18, 0\}$$

  and the CDD of a period of N days is given by

$$CDD = \sum_{i=1}^{N} CDD_i.$$

- Cumulative average temperature

  For Europe the CAT index is used instead of the CDD index in the summer months which is the sum of the daily average temperatures. The cumulative average temperature index (CAT-index) for the period of N days is defined as

$$CAT = \sum_{i=1}^{N} T_i.$$

For the Asia-Pacific region a slight modification of the CAT temperature index, the PRim (*Pacific Rim index*) index, is used for both time periods, winter and summer.

As seen in the example of Enron and Aquila Energy, the main purpose of weather derivatives is to hedge risks which a company is facing and which are correlated to the influence of weather. It is estimated that about 80% of the global economy is directly or indirectly affected by the weather (Auer, 2003).

Currently the CME offers temperature futures for 24 US cities, six cities in Canada, ten cities in Europe, and three cities in Australia and the Pacific Rim region (see (CME Group, 2009)). For most cities monthly and seasonal contracts as well as options of European style are available. For the US cities, HDD and CDD futures whereas for Europe HDD and CAT contracts are offered. All US futures are settled with a tick size of $20 whereas the

European futures are settled with €20 or £20 for London. The HDD contracts are tradable for the winter months November-March whereas both the CDD and CAT contracts are available for May-September. For the months October and April both CDD/CAT and HDD contracts are available (CME Group, 2009). All contracts are financially settled and the last trading day is the first business day which is at least five days after the end of the contract month.

3. INDIFFERENCE PRICING OF TEMPERATURE FUTURES

As mentioned in the introduction we focus on deriving an explicit and closed form expression for a weather derivative based on the indifference pricing approach. In the following we introduce the pricing framework and the assumed dynamics for the temperature process as well as the process of the correlated asset, the electricity futures contract. The advantage of the electricity futures contract traded for example at the EPEX are that there are contracts available with the same delivery period as weather derivatives (e.g. monthly contracts).

Let $T < \infty$ be a fixed time in the future denoting the end of the planning horizon which covers all times of interest and $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ be a probability space with an augmented filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ (satisfying the usual conditions) generated by the two Brownian motions W^{temp} and W^{elect}. Furthermore, let $t \in [0, T]$ denote the current time and T_1, T_2 be the beginning and the end of the measurement period of the CAT futures contract, i. e. $t \leq T_1 < T_2 \leq T$. Hence, we require a real measurement period instead of a one time event. Additionally, the focus lies on the price process of the CAT futures before the start of the measurement period.

For the dynamics of the temperature process we follow Benth et al. (2007) who propose to use an continuous-autoregressive process (CAR) model. The advantage of the CAR models lies in the implementation procedure. CAR models are continuous time models for which closed form expressions for various derivatives can be derived using standard techniques from financial mathematics. In order to estimate the parameters of the model, the close relation to the analogous AR model is used. In a first step, the parameters of the AR model are estimated which is convenient as temperature data are usually available on a daily basis and a broad range of estimate procedures for time series data exist.

Our analysis of daily temperatures in Germany concludes that the model of Benth et al. (2007) which was also applied by Härdle and Cabrera (2012) fits best for temperatures in Germany. The model has the following structure. Let $T(t)$ be the temperature at time t which is driven by the dynamics of a CAR(p) process of the following form:

$$T(t) = \Lambda^{temp}(t) + \mathbf{X}_1(t) \tag{3.1}$$

where $\Lambda^{temp}(t)$ is a deterministic and seasonal function and $\mathbf{X}_1(t)$ be the first coordinate of a state space vector which is driven by a mean-reverting Ornstein-Uhlenbeck process with dynamics:

$$d\mathbf{X}(t) = A\mathbf{X}(t)dt + e_p\eta(t)\,dW^{temp}(t) \tag{3.2}$$

where e_k is the k^{th}-unit vector of $\mathbb{R}^p$, $p \in \mathbb{N}_+$, $\eta(t)$ a positive and square-integrable function such that the Itô integral is well defined and A and $\mathbf{X}(t)$

given by

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\alpha_p & -\alpha_{p-1} & -\alpha_{p-2} & \dots & -\alpha_1 \end{pmatrix}; \qquad \mathbf{X}(t) = \begin{pmatrix} X(t) \\ X^{(1)}(t) \\ \vdots \\ X^{(p-2)}(t) \\ X^{(p-1)}(t) \end{pmatrix} \tag{3.3}$$

with $\alpha_i > 0$ for $i = 1 \dots p$ and $X^{(k)}(t)$ denoting the k-th derivative of $X(t)$. For the special case $p = 1$ the matrix A reduces to the constant $-\alpha_1$.

For a constant volatility function $\eta(t) \equiv \eta$ the process $\mathbf{X}(t)$ is stationary if and only if the eigenvalues of A have all strictly negative real parts and $\mathbf{X}(0)$ is Gaussian distributed with variance $\eta^2 \int_0^\infty e^{At} e_p^\top e_p e^{A^\top t} dt$ where $A_p^\top$ denotes the transpose of the matrix A (for a proof see (Brockwell and Hyndman (1992); Brockwell (2009)). For the setting above the requirements are slightly more restrictive (for the proof of the following proposition we refer to Benth et al. (2007)).

Proposition 3.1 *The solution of the process $\mathbf{X}(t)$ is stationary if all eigenvalues of A have strictly negative real parts and the volatility function $\eta(t)$ is* *bounded which ensures that the variance matrix converges,*

$$\lim_{t\to\infty}\int_0^t \eta^2(t-u)e^{Au}e_p^\top e_p e^{A^\top u}du \ < \infty.$$

with $A^\top$ denotes the transpose of the matrix A.

In order to derive a closed form expression for the price process we assume a rather simple spot price process which is based on the arithmetic model of Benth et al. (2008b) and is similar to the model of Lucia and Schwartz (2002). We assume that the dynamics of the spot price are mainly driven by an mean reverting Ornstein-Uhlenbeck process with time dependent and strictly positive volatility (i.e. $\sigma(t) \geq \delta$ for a constant $\delta > 0$):

$$\begin{aligned} S(t) &= \Lambda^{arith}(t) + Z(t) \qquad \text{with} \\ dZ(t) &= -\kappa Z(t)dt + \sigma(t)\, dW^{elect}(t). \end{aligned} \tag{3.4}$$

If instead of the spot price, the log spot price is used, this is often called a geometric spot price model. Before we can start with the indifference price framework we need to derive the dynamics of electricity futures based on the given spot price models. In order to obtain a closed form expression for the indifference price we concentrate on the *arithmetic model* (3.4) ($n = 1$, $m = 0$). For this part we simplify the notation by suppressing the superscript *elect* in W^{elect} as we deal only with the Brownian motion driving the spot price process here.

Following Benth et al. (2008b) the futures price at time t with delivery period $[T_1, T_2]$ is given as

$$F(t, T_1, T_2) = \mathbb{E}_{\mathbb{Q}}[\int_{T_1}^{T_2} \frac{1}{T_2 - T_1} S(u) du | \mathcal{F}_t]$$

assuming a constant risk free interest rate r, a pricing measure $\mathbb{Q}$ and financial settlement at the end of the delivery period.

Let θ be constant, then by the Girsanov theorem (see Bingham and Kiesel (2004), Karatzas and Shreve (1997)) the process

$$W^{\theta}(t) = W(t) - \theta t \quad \text{for } 0 \leq t \leq T$$

is a Brownian motion and the measure $\mathbb{Q}$ is equivalent to $\mathbb{P}$ with *Radon-Nikodym* derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = \exp\{-\theta W(t) - \frac{1}{2}\theta^2 t\}.$$

Moreover, the explicit solution of the spot price dynamic under the pricing measure $\mathbb{Q}$ is given by

$$Z(t) = e^{-\kappa(t-u)}Z(u) + \int_u^t e^{-\kappa(t-s)}\sigma(s)dW^\theta(s) + \int_u^t \theta e^{-\kappa(t-s)}\sigma(s)ds$$

for $0 \leq u \leq t \leq T$.

After the introduction of a pricing measure we can proceed with the valuation of electricity futures

Theorem 3.2 *Let the spot price dynamics given by* (3.4). *Furthermore, assume that the pricing measure* $\mathbb{Q}$ *is given by the Radon-Nykodym derivative* $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = \exp\{-\theta W(t) - \frac{1}{2}\theta^2 t\}$ *where* θ *is a constant measuring the market price of risk. Then the price of a futures contract at time* t *with delivery period* $t \leq T_1 \leq T_2 \leq T$, *constant interest rate* r *and financial settlement at the end of the delivery period is given by*

$$\begin{aligned} F(t, T_1, T_2) = \frac{1}{T_2 - T_1}\{&\int_{T_1}^{T_2} \Lambda(u)du + Z(t)\int_{T_1}^{T_2} e^{-\kappa(u-t)}du \\ &+ \theta \int_t^{T_2}\int_{\max\{T_1,s\}}^{T_2} e^{-\kappa(u-s)}\sigma(s)du\,ds\}. \end{aligned}$$

A detailed proof of the theorem can be found in Benth et al. (2008b). As a consequence we obtain the following two propositions.

Proposition 3.3 *The $\mathbb{Q}$-Dynamics for the futures price $F(t, T_1, T_2)$ is given by*

$$dF(t, T_1, T_2) = \tilde{\sigma}(t, T_1, T_2)\, dW^{\theta}(t) \tag{3.5}$$

with

$$\tilde{\sigma}(t, T_1, T_2) = \frac{\sigma(t)}{T_2 - T_1} \int_{T_1}^{T_2} e^{-\kappa(u-t)} du$$

Proposition 3.4 *The $\mathbb{P}$-dynamics of the futures price is given by*

$$dF(t, T_1, T_2) = \tilde{\theta}(t, T_1, T_2) dt + \tilde{\sigma}(t, T_1, T_2) dW(t) \tag{3.6}$$

with

$$\tilde{\sigma}(t, T_1, T_2) = \frac{\sigma(t)}{T_2 - T_1} \int_{T_1}^{T_2} e^{-\kappa(u-t)} du \quad \textit{and} \tag{3.7}$$

$$\tilde{\theta}(t, T_1, T_2) = -\theta \tilde{\sigma}(t, T_1, T_2) \tag{3.8}$$

Note that $\tilde{\sigma}(t, T_1, T_2)$ is a deterministic, strictly positive and bounded function due to the characteristic of $\sigma(t)$ and the fact that $T_1 < T_2$.

Furthermore, we assume that the two Brownian motions W^{temp} and W^{elect} are correlated with coefficient ρ which is a valid assumption for Germany as we will show empirically later. For the investor we assume an exponential utility function with risk aversion coefficient $\gamma > 0$,

$$U(x) = 1 - e^{-\gamma x}. \tag{3.9}$$

Moreover, the interest rate $r \geq 0$ is assumed to be constant and hence the bank account dynamics $R(t)$ are given by

$$dR(t) = rR(t)dt, \tag{3.10}$$

with $R(0) = 1$.

3.2. OPTIMAL FUTURES INVESTMENT

Let us consider the investment into electricity futures contracts with financial settlement. Suppose the investor is long $\xi(t) \in \mathbb{R}$ futures, all with the same delivery period, at time t and has $\zeta(t)$ monetary units in the bank account. Hence we allow for the purchase or sell of parts of futures. Let us now look at the value of the portfolio between time t and $t+dt$. Assume that the investor closes the futures position by going short at time $t + dt$, then the investor gains/loses $\xi(t)dF(t)$. On the other hand the value of the bank account increases by $\zeta(t)\, r\, R(t)dt$ (the earnings from the interest rate). Furthermore, assume that the investor can close the futures position at every time and transfers the gain/loss to/from the bank account (and if necessary borrowing money from the bank unconstrained). This means that every change in the futures price is continuously transferred to the accounts of both parties. The dynamics of the wealth process $Y(t)$ of the portfolio at time t, with $t_0 < t < T_1$ is then given by

$$dY(t) = \xi(t)dF(t) + \zeta(t)rR(t)dt \qquad \text{with} \quad Y(t_0) = y_{t_0} \tag{3.11}$$

where y_{t_0} denotes the initial investment.

Due to the fact that it is costless to enter a futures contract and the marking-to market of the futures position, the wealth of the portfolio at every time t with $t_0 \leq t \leq T_1$ can be written as:

$$Y(t) = \zeta(t)R(t). \tag{3.12}$$

Note that the dynamics of the wealth process implies that the futures position is finally closed at the start of the delivery period T_1 since we get the proceeds from the futures position by canceling out the position.

Let us assume that the number of futures contracts in the portfolio is limited in order to avoid an infinite investment in futures. Hence, the set of admissible trading strategies can be defined as

Definition 3.5 *An investment strategy process $\pi \in \{\pi(t)|0 \leq t \leq T_1\}$ is called admissible and we write $\pi \in \mathcal{A}$ if $\pi(t)$ is progressively measurable and $|\pi(t)| < K$ a.s. for $K \in \mathbb{R}_+$.*

Note that the boundedness of the investment strategy ensures that the wealth process has a unique and strong solution. We can simplify the dynamics of the wealth process by inserting (3.12) into (3.11) and using the futures dynamics in order to derive for an investment strategy π

$$\begin{aligned} dY^\pi(t) &= \pi(t)dF(t) + \frac{Y^\pi(t)}{R(t)} rR(t)dt \\ &= \big(\pi(t)\tilde{\theta}(t) + rY^\pi(t)\big)dt + \pi(t)\tilde{\sigma}(t)\, dW^{elect}(t), \end{aligned} \tag{3.13}$$

where we used for simplification $\tilde{\theta}(t)$ and $\tilde{\sigma}(t)$ instead of $\tilde{\theta}(t, T_1, T_2)$ and $\tilde{\sigma}(t, T_1, T_2)$ resp. For an admissible control $\pi(t)$ and $\tilde{\sigma}$ as defined in (3.7) the

Itô integral is well defined.

Based on the definition the wealth process is also depending on the portfolio strategy π. Due to the boundedness of π, the wealth process $Y^{\pi}(t)$ has a unique t-continuous solution (see for details Ebbeler (2012)).

In order to maximize the expected utility of the investor based on the chosen (exponential) utility function, the stochastic control problem has to be solved

$$\sup_{\pi(t)\in\mathcal{A}} \mathbb{E}[U(Y^{\pi}(T_1))|Y(t)=y] = \sup_{\pi(t)\in\mathcal{A}} \mathbb{E}[1-\exp(-\gamma Y^{\pi}(T_1))|Y(t)=y]$$

which is analog to

$$\Phi(t,y) := \sup_{\pi(t)\in\mathcal{A}} \mathbb{E}[-\exp(-\gamma Y^{\pi}(T_1))|Y(t)=y] = \sup_{\pi(t)\in\mathcal{A}} \mathbb{E}[\tilde{U}(Y^{\pi}(T_1))|Y(t)=y] \tag{3.14}$$

with the reduced utility function $\tilde{U} : y \mapsto -\exp\{-\gamma y\}$.

3.2.1. Hamilton-Jacobi-Bellman Equations and the Verification Theorem

We use the Hamilton-Jacobi-Bellman equations (HJB) (see Øksendal (1998, Ch.11)) to solve the stochastic control problem (3.14). For all functions

$\phi(t, y) \in C^{2,2}([0, T] \times \mathbb{R})$ and $\pi \in \mathcal{A}$ we introduce the functional operator

$$(L^\pi \phi)(t, y) = (\pi(t)\tilde{\theta}(t) + ry)\phi_y(t, y) + \frac{1}{2}\pi^2(t)\tilde{\sigma}^2(t)\phi_{yy}(t, y).$$

Then the HJB equation for the wealth process $Y^\pi(t)$ is given by

$$\Phi_t(t, y) + \sup_\pi\{(L^\pi \Phi)(t, y)\} = 0 \tag{3.15}$$

for all $t \in [0, T_1]$ and $y \in \mathbb{R}$, with terminal condition

$$\Phi(T_1, y) = -\exp(-\gamma y) = \tilde{U}(y) \text{ for all } y \in \mathbb{R}. \tag{3.16}$$

where $\phi_t, \phi_y, \phi_{yy}$ denote the partial derivatives of the function ϕ with respect to t, resp. y.

In order to ensure that the obtained solutions are the solutions to the optimal control problem in (3.14) we state the following *HJB Verification theorem* (see (Øksendal, 1998, Ch. 11) and Benth et al. (2003)).

Theorem 3.6 (Verification Theorem I) *Let $\phi(t, y) \in C^2([0, T_1] \times \mathbb{R})$ be a solution of the HJB equation* (3.15) *with terminal condition* (3.16). *Assume* *that*

$$\int_0^{T_1} \mathbb{E}[\pi^2(s)\tilde{\sigma}^2(s)\phi_y^2(s, Y(s)]ds < \infty.$$

for all admissible controls $\pi(t) \in \mathcal{A}$.

Then $\phi(t, y) \geq \Phi(t, y)$ *for all* $(t, y) \in [0, T_1] \times \mathbb{R}$.

Moreover, if π^* *is a maximizer of the HJB equation* (3.15) *and* π^* *is an admissible trading strategy, then*

$$\phi(t, y) = \Phi(t, y) \text{ for all } (t, y) \in [0, T_1] \times \mathbb{R}$$

and π^* *is an optimal trading strategy.*

Proof: Using Itô's formula for ϕ and the fact that ϕ is a solution of the HJB equation (3.15) proves that $\phi(t, y) \geq \Phi(t, y)$. If π^* is a maximizer of (3.15) then $\phi \leq \Phi$, which shows the equality.

3.2.2. Reduction of HJB equation

Let us assume that $\Phi_{yy} < 0$. Then the mapping $\pi \mapsto (\pi\tilde{\theta}(t) + ry)\Phi_y(t, y) + \frac{1}{2}\pi^2\tilde{\sigma}^2(t)\Phi_{yy}(t, y)$ is concave and so $\pi^* \in (-K, K)$. ($K > 0$ can always be chosen such that the function lies inside the interval.) The first-order condition for an optimal control policy is found by differentiating the HJB equation with respect to π which leads to

$$\Phi_y(t,y)\tilde{\theta}(t) + \pi\tilde{\sigma}^2(t)\Phi_{yy}(t,y) = 0$$

which implies that the maximizer is given by

$$\pi^*(t,y) = -\frac{\tilde{\theta}(t)\Phi_y(t,y)}{\tilde{\sigma}^2(t)\Phi_{yy}(t,y)}.$$

Inserting the optimal control π^* into (3.15) gives the nonlinear partial differential equation

$$\Phi_t(t,y) + ry\Phi_y(t,y) - \frac{1}{2}\frac{\tilde{\theta}^2(t)\Phi_y^2(t,y)}{\tilde{\sigma}^2(t)\Phi_{yy}(t,y)} = 0 \tag{3.17}$$

We try to find a solution of the value function of the form

$$\Phi(t,y) = h(t)\exp\{-\gamma g(t)y\}$$

with suitable functions h, g (i.e. we assume $h(t) \neq 0$). Inserting into the partial differential equations of Φ solutions for g and h are given by

$$g(t) = \exp\{r(T_1 - t)\} \text{ and } \qquad h(t) = -\exp\{-\int_t^{T_1} \frac{1}{2}\frac{\tilde{\theta}^2(s)}{\tilde{\sigma}^2(s)} ds\}$$

Summing up all the results we obtain that the value function Φ is given by

$$\Phi(t, y) = -\exp\{-\int_{T_1}^{T_2} \frac{1}{2}\frac{\tilde{\theta}^2(s)}{\tilde{\sigma}^2(s)}\, ds - \gamma e^{r(T_1 - t)} y\}$$

and the optimal trading strategy π^*:

$$\pi^* = \frac{\tilde{\theta}(t) e^{-r(T_1 - t)}}{\tilde{\sigma}^2(t)\gamma}.$$

Clearly, it is possible to chose K such that the optimal strategy lies within the interval $(-K, K)$.

Before we state the final theorem which proves the existence and uniqueness of the solution to the HJB equation and the optimal control strategy we state an important characteristic of the value function which is used in the proof of the *Existence and Uniqueness Theorem.*

Proposition 3.7 *Let $Y^\pi(t)$ be the wealth process as described in* (3.13) *and*

π be an admissible and deterministic trading strategy i. e. the strategy only independents on the time t.

Consider the following stochastic process $Z(t) = \exp\{-\gamma e^{r(T_1-t)} Y^\pi(t)\}$. Then (cp. Benth et al. (2003))

$$\mathbb{E}[|Z(t)|^m] < \infty \qquad \forall t \in [0, T_1] \qquad m \geq 2.$$

Proof: For the proof see Ebbeler (2012).

Prop. 3.7 shows that due to the Lipschitz and bounding conditions of volatility and drift term the exponential of the wealth process possesses finite moments. This fact will be used in the following theorem which shows that the value function obtained is unique and optimal.

Theorem 3.8 (Existence and Uniqueness) *Let the wealth process $Y^\pi(t)$ be given by* (3.11) *and let* (3.15) *be the corresponding HJB equation with terminal condition* (3.16)*. Then the optimal control is given by*

$$\pi^*(t) = \frac{\tilde{\theta}(t)}{\gamma \tilde{\sigma}^2(t)} e^{-r(T_1-t)}. \tag{3.18}$$

Moreover the solution of the HJB equation is given by

$$\Phi(t,y) = -\exp\{-\frac{1}{2}\int_t^{T_1} \frac{\tilde{\theta}(s)}{\tilde{\sigma}(s)} ds - \gamma e^{r(T_1-t)} y\}. \tag{3.19}$$

Proof: The control π^* is independent of y and deterministic. Additionally, for $K \in \mathbb{R}_+$ large we have $\pi^* \in [-K, K]$ and so we obtain that $\pi^* \in \mathcal{A}$. Obviously, $\Phi \in C^2([0,T_1] \times \mathbb{R})$ and a solution to the HJB equation. Furthermore, the process $Y^\pi(t)$ has a unique and continuous solution with $\mathbb{E}[\int_0^{T_1} |Y^\pi(t)|^2 dt] < \infty$ for all $\pi \in \mathcal{A}$. Additionally, Prop. 3.7 shows that $Z(t) = \exp\{-\gamma e^{r(T_1-t)} Y^\pi(t)\}$ possesses finite moments i. e. $\mathbb{E}[\int_0^{T_1} |Z(t)|^2 dt] < \infty$. As $\tilde{\theta}$ and $\tilde{\sigma}$ are bounded functions, $\int_0^{T_1} (\int_t^{T_1} \frac{\tilde{\theta}(s)}{\tilde{\sigma}(s)} ds)^2 \, dt < \infty$ and therefore

$$\Phi_y(t, Y(t)) = \gamma e^{r(T_1-t)} \exp\{-\frac{1}{2}\int_t^{T_1} \frac{\tilde{\theta}(s)}{\tilde{\sigma}(s)} ds\} \, Z(t)$$

is square-integrable. Consequently

$$\int_0^{T_1} \mathbb{E}[\pi^2(t)\tilde{\sigma}^2(t)\Phi_y^2(t, Y(t))] dt < \infty.$$

Hence the requirements of the *Verification Theorem* 3.6 are fulfilled which concludes the proof.

Remark 3.9 *The optimal portfolio strategy π^* is independent of the wealth process but changes over time. Plugging in the formula for $\tilde{\theta}(t) = -\theta\tilde{\sigma}(t)$ leads to*

$$\pi^*(t) = \frac{-\theta}{\gamma\tilde{\sigma}(t)} e^{-r(T_1-t)},$$

which presents the optimal control as a linear function in the market price of risk θ.

3.3. FUTURES PORTFOLIO OPTIMIZATION EXTENDED BY CAT FUTURES

Let us suppose that the futures portfolio is extended with a long position in a CAT futures, and $G(t)$ denotes the CAT futures price with time period $[T_1, T_2]$. From the temperature model in (3.1), the index amount $I(T_1, T_2)$ the buyer receives is given by

$$I(T_1, T_2) = \int_{T_1}^{T_2} T(s)ds = \int_{T_1}^{T_2} \Lambda(s) + e_1^\top \mathbf{X}(s)ds.$$

Considering (3.13) the following value function is optimized over the set of admissible trading strategies $\pi \in \mathcal{A}$.

$$\begin{aligned}\Gamma_p(t) &= \sup_{\pi\in\mathcal{A}} \mathbb{E}\big[-\exp\{-\gamma(Y^\pi(T_1) - e^{-r(T_2-T_1)}(G(t) - \int_{T_1}^{T_2} \Lambda(s) + e_1^\top \mathbf{X}(s)ds))\}|\mathcal{F}_t\big] \\ &= \exp\big\{-\gamma e^{-r(T_2-T_1)} \int_{T_1}^{T_2} \Lambda(s)ds\big\}\,\tilde{\Gamma}^p(t)\end{aligned}$$

with

$$\tilde{\Gamma}^p(t) = \sup_{\pi\in\mathcal{A}} \mathbb{E}\big[-\exp\{-\gamma(Y^\pi(T_1) - e^{-r(T_2-T_1)}(G(t) - \int_{T_1}^{T_2} e_1^\top \mathbf{X}(s)ds))\}|\mathcal{F}_t\big]$$

where we use the superscript p in order to highlight that we are dealing with a p-dimensional case. Using the solution of $\mathbf{X}(t)$ for a fixed T_1 we obtain

$$\int_{T_1}^{T_2} e_1^\top \mathbf{X}(s)ds = \bar{A}(T_2-T_1)\mathbf{X}(T_1) + \eta \int_{T_1}^{T_2} \bar{A}(T_2-u)e_p\, dW^{temp}(u)$$

where $\bar{A} : \mathbb{R} \mapsto \mathbb{R}^{1\times p}$ is defined as $\bar{A}(u) = e_1^\top\big[A^{-1}(\exp\{Au\} - I_p)\big]$ and I_p denotes the identity matrix of $\mathbb{R}^{p\times p}$. Using double conditioning and the fact that the stochastic integral is independent of $\mathcal{F}_{T_1}$ the value function can be rewritten as

$$\begin{aligned}\tilde{\Gamma}^p(t) &= \sup_{\pi\in\mathcal{A}} \mathbb{E}\Big[\mathbb{E}\big[-\exp\{-\gamma(Y^\pi(T_1) - e^{-r(T_2-T_1)}(G(t) - \bar{A}(T_2-T_1)\mathbf{X}(T_1) \\ &\qquad - \eta\int_{T_1}^{T_2} \bar{A}(T_2-u)e_p\, dW^{temp}(u)))\}|\mathcal{F}_{T_1}\big]|\mathcal{F}_t\Big] \\ &= \exp\{\frac{1}{2}\gamma^2 e^{-2r(T_2-T_1)}\eta^2 \int_{T_1}^{T_2} (\bar{A}(T_2-s)e_p)^2 ds\}\, \bar{\Gamma}^p(t)\end{aligned}$$

with

$$\bar{\Gamma}^p(t) = \sup_{\pi\in\mathcal{A}} \mathbb{E}\Big[-\exp\{-\gamma(Y^\pi(T_1) - e^{-r(T_2-T_1)}(G(t) - \bar{A}(T_2-T_1)\mathbf{X}(T_1)))\}|\mathcal{F}_t\Big].$$

The CAT futures price $G(t)$ is based on information up to time t and therefore adapted to $\mathcal{F}_t$. Consequently, the term $G(t)$ can be factored out of the expectation and hence

$$\bar{\Gamma}^p(t) = \exp\{\gamma e^{-r(T_2-T_1)}G(t)\}\, \Psi^p(t,x,y)$$

where

$$\Psi^p(t,x,y) = \sup_{\pi\in\mathcal{A}} \mathbb{E}\Big[-\exp\{-\gamma(Y^\pi(T_1) + e^{r(T_2-T_1)}\bar{A}(T_2-T_1)\mathbf{X}(T_1))\}|\mathbf{X}(t)=x, Y(t)=y\Big]. \tag{3.20}$$

Note that the last part of the value function $\bar{A}(T_2-T_1)\mathbf{X}(T_1)$ can be see as a sum of the components of $\mathbf{X}(t)$ (i. e. $\bar{A}(T_2-T_1)\mathbf{X}(T_1) = \sum_{k=1}^{p} \bar{A}_i \mathbf{X}_i(T_1)$ where the i-th subscripts denote the i-th component of the vector).

It is sufficient to optimize the value function Ψ^p (3.20).

3.4. REDUCTION OF THE HJB EQUATION

Again the optimal trading strategy π^* is bounded (i. e. $\pi^* \in [-K,K]$) in order to avoid an optimization which goes to infinity. Note that the vector $y \in \mathbb{R}$ whereas x is p-dimensional with components $x_1, x_2, \ldots, x_p$. The gradient of the function Ψ^p with respect to the vector x is denoted by $\nabla_x \Psi = (\frac{\partial\Psi}{\partial x_1}, \ldots, \frac{\partial\Psi}{\partial x_p})^\top$ and Ψ_{x_i} denotes the partial derivative of Ψ with respect to x_i. Let the value function Ψ^p be given as above, then the HJB equation is defined as

$$\Psi_t^p + \sup_{\pi \in \mathcal{A}} \{ (\pi \tilde{\theta}(t) + ry) \Psi_y^p + \frac{1}{2} \pi^2 \tilde{\sigma}^2(t) \Psi_{yy}^p + \pi \tilde{\sigma}(t) \eta \rho \Psi_{yx_p}^p \}$$
$$+ \sum_{i=1}^{p-1} x_{i+1} \Psi_{x_i}^p + \sum_{i=1}^{p} -a_i x_{p-i+1} \Psi_{x_p} + \frac{1}{2} \eta^2 \Psi_{x_p x_p}^p = 0$$

or using the vector notation

$$\Psi_t^p + \sup_{\pi \in \mathcal{A}} \{ (\pi \tilde{\theta}(t) + ry) \Psi_y^p + \frac{1}{2} \pi^2 \tilde{\sigma}^2(t) \Psi_{yy}^p + \pi \tilde{\sigma}(t) \eta \rho \Psi_{yx_p}^p \}$$
$$+ (Ax)^\top \nabla_x \Psi^p + \frac{1}{2} \eta^2 \Psi_{x_p x_p}^p = 0. \tag{3.21}$$

The terminal condition for $t = T_1$ is given by

$$\Psi^p(T_1, x, y) = -\exp\{-\gamma(y + e^{-r(T_2 - T_1)} \bar{A}(T_2 - T_1) x)\}. \tag{3.22}$$

Theorem 3.10 (Verification Theorem II) *Let* $\psi^p(t, x, y) \in C^2([0, T_1] \times \mathbb{R}^p \times \mathbb{R})$ *be a solution of the HJB equation* (3.21) *with terminal condition*

(3.22). *Assume that*

$$\int_0^{T_1} \mathbb{E}[\pi^2(s)\tilde{\sigma}^2(s)\big(\psi_y^p(s,\mathbf{X}(s),Y^\pi(s))\big)^2]ds < \infty$$
$$\int_0^{T_1} \mathbb{E}[\eta^2\big(\psi_{x_p}^p(s,\mathbf{X}(s),Y^\pi(s))\big)^2]ds < \infty$$

for all admissible controls $\pi(t) \in \mathcal{A}$.

Then $\psi^p(t,x,y) \geq \Psi^p(t,x,y)$ *for all* $(t,x,y) \in [0,T_1] \times \mathbb{R}^p \times \mathbb{R}$.

Moreover, if π^* *is a maximizer of the HJB equation 3.21 and* π^* *is an admissible trading strategy, then*

$$\psi^p(t,x,y) = \Psi^p(t,x,y) \text{ for all } (t,x,y) \in [0,T_1] \times \mathbb{R}^p \times \mathbb{R}^2$$

and π^* *is an optimal trading strategy.*

Proof: The proof is analogous to the proof of 3.6.

If $\Psi_{yy}^p < 0$ the mapping $\pi \mapsto (\pi\tilde{\theta}(t) + ry)\Psi_y^p + \frac{1}{2}\pi^2\tilde{\sigma}^2(t)\Psi_{yy}^p + \pi\tilde{\sigma}(t)\eta\rho\Psi_{yx_p}^p$ is concave and hence the optimal trading strategy π^* lies in the open interval $(-K,K)$. The first order condition for the optimality is given by

$$\pi^*(t,x,y) = -\frac{\tilde{\theta}(t)\Psi_y^p + \tilde{\sigma}(t)\eta\rho\Psi_{yx_p}^p}{\tilde{\sigma}^2(t)\Psi_{yy}^p}.$$

Inserting the optimal portfolio strategy π^* into the HJB equation (3.21) and simplifying the expression leads to the following equation

$$\Psi^p_t - \frac{\tilde{\theta}^2(t)(\Psi^p_y)^2}{\tilde{\sigma}^2(t)\Psi^p_{yy}} - \frac{\tilde{\theta}(t)\eta\rho\Psi^p_{yx_p}\Psi^p_y}{\tilde{\sigma}(t)\Psi^p_{yy}} + ry\Psi^p_y + \frac{1}{2}\frac{(\tilde{\theta}(t)\Psi^p_y + \tilde{\sigma}(t)\rho\eta\Psi^p_{yx_p})^2}{\tilde{\sigma}^2(t)\Psi^p_{yy}} - \frac{\tilde{\theta}(t)\eta\rho}{\tilde{\sigma}(t)}\frac{\Psi^p_y\Psi^p_{yx_p}}{\Psi^p_{yy}}$$
$$- \frac{\eta^2\rho^2(\Psi^p_{yx_p})^2}{\Psi^p_{yy}} + (Ax)^\top \nabla_x \Psi^p + \frac{1}{2}\eta^2\Psi^p_{x_p x_p} = 0. \tag{3.23}$$

Let $a(t), c(t)$ be suitable functions from $\mathbb{R} \mapsto \mathbb{R}$ and $b : \mathbb{R} \mapsto \mathbb{R}^p$ a p-dimensional function with components $b_i(t)$ $1 \leq i \leq p$. Then we try to identify a solution of the value function of the form

$$\begin{aligned} \Psi^p(t,x,y) &= -\exp\{a(t) + c(t)y + (b(t))^\top x\} \\ &= -\exp\{a(t) + c(t)y + \sum_{i=1}^{p} b_i(t)x_i\} \end{aligned} \tag{3.24}$$

where $(b(t))^\top$ denotes the transpose of the function $b(t)$. If $b_p(t) = e_p^\top b(t)$ the p-th component of $b(t)$ exists and is integrable then $a(t)$ and $c(t)$ possess unique solutions of the form

$$c(t) = -\gamma \exp\{r(T_1 - t)\} \tag{3.25}$$

$$a(t) = -\frac{1}{2}\int_t^{T_1} \Big(\frac{\tilde{\theta}^2(s)}{\tilde{\sigma}^2(s)} - \eta^2 b_p^2(s)(1-\rho^2) + 2\frac{\tilde{\theta}(s)}{\tilde{\sigma}(s)}\eta\rho b_p(s)\Big)ds. \tag{3.26}$$

Since the differentiation is with respect to the time t, the unique solution of $b(t)$ is given by

$$
\begin{aligned}
b(t) &= e^{-(t-T_1)A^\top} b(T_1) \\
&= -\gamma e^{-r(T_2-T_1)} (\bar{A}(T_2-T_1)e^{-(t-T_1)A})^\top. \qquad (3.27)
\end{aligned}
$$

Summing up all the results above we can conclude that the function Ψ^p of the form

$$
\begin{aligned}
\Psi^p(t,x,y) &= -\exp\{-\gamma e^{r(T_1-t)}y - \gamma e^{-r(T_2-T_1)}\bar{A}(T_2-T_1)e^{-(t-T_1)A}x + a(t)\} \\
&= \Phi(t,y)\exp\Big\{ -\gamma e^{-r(T_2-T_1)}\bar{A}(T_2-T_1)e^{-(t-T_1)A}x \\
&\quad + \frac{\eta^2(1-\rho^2)}{2}\int_t^{T_1} b_p^2(s)ds - \eta\rho\int_t^{T_1}\frac{\tilde{\theta}(s)}{\tilde{\sigma}(s)}b_p(s)ds\Big\}
\end{aligned}
$$

with $a(t)$ as defined above, solves the HJB equation (3.21). The optimal trading strategy is given by

$$
\begin{aligned}
\pi^*(t,x,y) &= \frac{\tilde{\theta}(t)}{\tilde{\sigma}^2(t)\gamma}e^{-r(T_1-t)} + \frac{\eta\rho}{\tilde{\sigma}(t)\gamma}e^{-r(T_1-t)}b_p(t) \\
&= \pi_0^*(t) + \frac{\eta\rho}{\tilde{\sigma}(t)\gamma}e^{-r(T_1-t)}b_p(t)
\end{aligned}
$$

Theorem 3.11 *Let the process $Y^\pi(t)$ and $\mathbf{X}(t)$ be defined as in (3.13) resp.*

(3.1), and let (3.21) *be the corresponding HJB equation with terminal condition* (3.22). *Then the optimal control is given by*

$$\pi^*(t) = \frac{\tilde{\theta}(t)}{\tilde{\sigma}^2(t)\gamma} e^{-r(T_1-t)} + \frac{\eta\rho}{\tilde{\sigma}(t)\gamma} e^{-r(T_1-t)} b_p(t)$$

with $b_p(t)$ *as defined in* (3.27). *Moreover, the solution of the HJB equation is given by*

$$\Psi^p(t, \mathbf{X}(t), Y^{\pi^*}(t)) = -\exp\{-\gamma e^{r(T_1-t)} Y^{\pi^*}(t) - \gamma e^{-r(T_2-T_1)} \bar{A}(T_2-T_1) e^{-(t-T_1)A} \mathbf{X}(t) + a(t)\}$$

with $a(t)$ *as defined in (3.26).*

Proof: The control π^* is independent of y and deterministic. Additionally, for $K \in \mathbb{R}_+$ large we have $\pi^* \in [-K, K]$ and so we obtain that $\pi^* \in \mathcal{A}$. Obviously, $\Psi \in C^2([0, T_1] \times \mathbb{R}^{p+1})$ and a solution to the HJB equation. In order to apply the *Verification Theorem* we need to show

$$(i) \int_0^{T_1} \mathbb{E}[\pi^2(s)\tilde{\sigma}^2(s)\big(\psi_y^p(s, \mathbf{X}(s), Y^{\pi^*}(s))\big)^2]ds < \infty$$

$$(ii) \int_0^{T_1} \mathbb{E}[\eta^2\big(\psi_{x_p}^p(s, \mathbf{X}(s), Y^{\pi^*}(s))\big)^2]ds < \infty$$

Observe that $\mathbf{X}_t = e^{A(t-T_1)}\mathbf{X}(T_1) + \int_{T_1}^{t} e^{A(t-u)} e_p \eta \, dW^{temp}(u)$ is normally distributed due to the stochastic integral and hence $\exp\{X(t)\}$ possesses finite moments. The rest of the proof is similar to the proof of Thm. 3.8.

The indifference price of a CAT futures can be derived by setting $\Phi(t, y) = \Gamma_p(t, x, y)$ and solving for the price $G(t)$. Observe that the optimized wealth process for the portfolio of electricity futures and CAT futures is

$$\Gamma_p(t,x,y) = \exp\Big\{ -\gamma e^{-r(T_2-T_1)} \int_{T_1}^{T_2} \Lambda(s)ds + \frac{1}{2}\gamma^2\eta^2 e^{-2r(T_2-T_1)} \int_{T_1}^{T_2} (\bar{A}(T_2-s)e_p)^2 ds \Big\} \\ \times \exp\{\gamma e^{-r(T_2-T_1)} G(t)\} \Psi^p(t,x,y).$$

Hence, the CAT price is given by

$$G(t) = \int_{T_1}^{T_2} \Lambda(s)ds - \frac{1}{2}\gamma e^{-r(T_2-T_1)}\eta^2 \int_{T_1}^{T_2} (\bar{A}(T_2-s)e_p)^2 ds + \frac{1}{\gamma} e^{r(T_2-T_1)} \ln\Big(\frac{\Phi(t,y)}{\Psi^p(t,x,y)}\Big).$$

Recall the fact that the function Ψ^p is a multiple of the function Φ

$$\Psi^p(t,x,y) = \Phi(t,y) \exp\Big\{ -\gamma e^{-r(T_2-T_1)} \bar{A}(T_2-T_1) e^{-(t-T_1)A} x + \frac{\eta^2(1-\rho^2)}{2} \int_t^{T_1} b_p^2(s)ds \\ - \eta\rho \int_t^{T_1} \frac{\tilde{\theta}(s)}{\tilde{\sigma}(s)} b_p(s)ds \Big\}.$$

Using $b_p(t) = e_p^\top b(t) = -\gamma e^{-r(T_2-T_1)} e_p^\top (e^{-(t-T_1)A})^\top \bar{A}(T_2-T_1)^\top$ the CAT price can be simplified further to

$$\begin{aligned} G(t) = &\int_{T_1}^{T_2} \Lambda(s)ds + e_1^\top [A^{-1}(e^{A(T_2-t)} - e^{A(T_1-t)})]\mathbf{X}(t) \\ &- \eta\rho \int_t^{T_1} \frac{\tilde{\theta}(s)}{\tilde{\sigma}(s)} e_p^\top e^{-A^\top(s-T_1)} ds \bar{A}(T_2-T_1)^\top \\ &- \frac{1}{2}\gamma e^{-r(T_2-T_1)}\eta^2 \Big(\int_{T_1}^{T_2} (\bar{A}(T_2-s)e_p)^2 ds \\ &\quad + (1-\rho^2) \int_t^{T_1} (e_p^\top e^{-A^\top(s-T_1)} \bar{A}(T_2-T_1)^\top)^2 ds \Big). \end{aligned} \tag{3.28}$$

In view of (3.28) the price of the CAT temperature futures at time t consists of 4 parts, the integral of the seasonal function over the measurement period, the stochastic price process of the deseasonalized temperature process $\mathbf{X}$ at time t and two adjustment factors which are driven by the market price of risk of the electricity market and the volatility factor of the temperature dynamics respectively. If we analyze the first two components of the CAT price more precisely and consider the results of Benth et al. (2007), we obtain that these two components describe the expected CAT temperature futures price. Therefore, (3.28) can be rewritten as

$$G(t) = \mathbb{E}\big[\int_{T_1}^{T_2} T(s)\,ds|\mathcal{F}_t\big] - \rho R_p^{el}(t,T_1,T_2) - \gamma R_p^{temp}(t,T_1,T_2)$$

where the risk premium for hedging in electricity futures R^{el} is defined as

$$R_p^{el}(t,T_1,T_2) = \eta \int_t^{T_1} \frac{\tilde{\theta}(s)}{\tilde{\sigma}(s)} e_p^\top e^{-A^\top(s-T_1)} ds \bar{A}(T_2-T_1)^\top \tag{3.29}$$

and the risk premium from the risk aversion towards the trading of CAT futures R^{temp} is given by

$$R_p^{temp}(t,T_1,T_2) = \frac{1}{2}\eta^2 e^{-r(T_2-T_1)}\Big(\int_{T_1}^{T_2}(\bar{A}(T_2-s)e_p)^2 ds \tag{3.30}$$

$$+ (1-\rho^2)\int_t^{T_1}(e_p^\top e^{-A^\top(s-T_1)}\bar{A}(T_2-T_1)^\top)^2 ds\Big). \tag{3.31}$$

Moreover, using (3.8) the risk premium R^{el} can be simplified to

$$R_p^{el}(x,y,z) = \eta\theta e_p^\top[(A^\top)^{-1}(e^{-A^\top(t-T_1)} - I)]\bar{A}(T_2-T_1)^\top$$

which shows that the risk premium is proportional to the market price of risk. Furthermore, the premium is scaled by the "mean reversion coefficient"

A of the temperature process, the time-to-measurement as well as the length of the measurement period. The sign of R_p^{el} mainly depends on the sign of θ as η is naturally positive. If θ is positive R_p^{el} is negative and vice versa. If the time-to-measurement converges to zero, i. e. t goes to T_1, the risk premiums will converge to zero which is what we would expect as the influence of the trade in electricity should vanish close to the start of the measurement period. The risk premium R^{temp} depends on the volatility of the temperature and some averaging of the speed of mean reversion, discounted with the interest rate back to the start of the measurement period. R^{temp} is always positive due to the definition of $\bar{A}$ and the fact that $\rho \leq 1$ and consequently the risk premium contributes negatively to the CAT price. Furthermore, for t converging to T_1 the risk premium does not converge to zero but rather

$$\lim_{t \to T_1} R^{temp} = \frac{1}{2}\eta^2 e^{-r(T_2-T_1)} \int_{T_1}^{T_2} \bar{A}^2(T_2-u)du.$$

This shows that driven by the utility function, the price the investor is willing to pay for a CAT contract is below the risk neutral price of the CAT at the start of the measurement period (t close to T_1). Hence, the investor wants to have a discount for bearing the risk of the CAT futures.

Combining the two risk premia from above and define the overall risk premium R as

$$R(t, T_1, T_2) = -\rho R^{el}(t, T_1, T_2) - \gamma R^{temp}(t, T_1, T_2) \tag{3.32}$$

which describes the difference between the CAT price and the predicted payments of a long position in the CAT futures.

If ρ and θ are positive the first component R_p^{el} of R is negative and it will be a matter of relative size of the two terms if R is positive or not. This will depend on the length of the measurement period as well as the time-to-measurement. For $t = T_1$, i. e. at the start of the measurement period, the overall risk premium is negative as the first component vanishes.

Consider the situation of no correlation between temperature and spot price, the CAT price $G(t)$ is different from the expected discounted payments of the futures. The risk premium reduces mainly to R_p^{temp} which is due to the exponential nature of the utility function and nature of the aggregated temperatures over the measurement period.

For the case $p = 1$, the temperature process is driven by an OU process, the CAT price reduces to

$$
\begin{aligned}
G(t) = &\int_{T_1}^{T_2} \Lambda^{temp}(s)ds + \frac{1}{\alpha}(e^{-\alpha(T_1-t)} - e^{-\alpha(T_2-t)})X(t) \\
&- \rho\eta\bar{\alpha}(T_2 - T_1)\int_t^{T_1} \frac{\tilde{\theta}(s)}{\tilde{\sigma}(s)} e^{-\alpha(T_1-s)}ds \\
&- \frac{1}{2}\gamma\eta^2 e^{-r(T_2-T_1)}\big(\frac{1}{2}\bar{\alpha}^2(T_2 - T_1)(1-\rho^2)\bar{\alpha}(2(T_1 - t)) + \int_{T_1}^{T_2} \bar{\alpha}^2(T_2 - u)du\big),
\end{aligned}
\tag{3.33}
$$

with the notation $\bar{\alpha}(u) = \frac{1}{\alpha}(1 - e^{-\alpha u})$.

3.5. INDIFFERENCE SELLING PRICE OF CAT FUTURES

In a similar way a CAT futures price for the selling side can be derived. Analogous to the previous setting the optimization problem is extended by a short position in a CAT futures and given by

$$
\begin{aligned}
\Gamma_p^s(t) &= \sup_{\pi\in\mathcal{A}} \mathbb{E}\big[-\exp\{-\gamma(Y^\pi(T_1) + e^{-r(T_2-T_1)}(G^s(t) - I(T_1, T_2)))\}|\mathcal{F}_t\big] \\
&= \sup_{\pi\in\mathcal{A}} \mathbb{E}\big[-\exp\{-\gamma(Y^\pi(T_1) + e^{-r(T_2-T_1)}(G^s(t) - \int_{T_1}^{T_2} \Lambda(s) + e_1^\top \mathbf{X}(s)ds))\}|\mathcal{F}_t\big].
\end{aligned}
$$

With similar calculation as before the selling price of the CAT futures is given by

$$
\begin{aligned}
G^s(t) = &\int_{T_1}^{T_2} \Lambda(s)ds + e_1^\top [A^{-1}(e^{A(T_2-t)} - e^{A(T_1-t)})]\mathbf{X}(t) \\
&- \eta\rho \int_t^{T_1} \frac{\tilde{\theta}(s)}{\tilde{\sigma}(s)} e_p^\top e^{-A^\top(s-T_1)} ds \bar{A}(T_2 - T_1)^\top \\
&+ \frac{1}{2}\gamma e^{-r(T_2-T_1)}\eta^2 \Big(\int_{T_1}^{T_2} (\bar{A}(T_2 - s)e_p)^2 ds \\
&\quad + (1-\rho^2) \int_t^{T_1} (e_p^\top e^{-A^\top(s-T_1)} \bar{A}(T_2 - T_1)^\top)^2 ds \Big) \\
= &\mathbb{E}[\int_{T_1}^{T_2} T(s)ds|\mathcal{F}_t] - \rho R_p^{el} + \gamma R_p^{temp}.
\end{aligned}
$$

4. PARAMETER ESTIMATION AND CORRELATION ANALYSIS

4.1. PARAMETER ESTIMATION TEMPERATURE MODEL

For the empirical analysis of temepratures we select eight cities in Germany (Munich, Stuttgart, Frankfurt, Essen, Leipzig, Berlin, Hannover, Hamburg) which represent a comprehensive grid of the temperature landscape in Germany. Additionally to the eight time series of temperature data, we also calculated an artificial *Germany Average Temperature* which is the (unweighted) average temperature of the eight cities. For the following we will denote this time series as Germany-Average or Germany.

All temperature data are obtained from the *Deutsche Wetterdienst* (DWD)[2]. For each weather station we use temperature data from January 1, 1993 to June 30, 2010. In order to have equally sized years we removed all data points of February 29 which leads to a total of 6386 data points except for Munich for which we observe one missing value on January 20,1999.

The best fit for the seasonal function Λ^{temp} to the data is obtained by the following form:

$$\Lambda^{temp}(t) = b_1 + b_2 t + b_3 \cos\Big(\frac{2\pi}{365}(t - b_4)\Big). \tag{4.1}$$

The parameters are obtained using the least-square estimation and are presented in 1 (for more details see Ebbeler (2012)). Conducting the ADF

	Munich	Stuttgart	Frankfurt	Essen	Leipzig	Berlin	Hannover	Hamburg	Germany
b_1	8.827 (< 0.001)	9.724 (< 0.001)	10.730 (< 0.001)	10.395 (< 0.001)	9.478 (< 0.001)	9.746 (< 0.001)	9.428 (< 0.001)	9.255 (< 0.001)	9.698 (< 0.001)
b_2	6.00×10^{-05} (0.021)	1.95×10^{-05} (0.439)	5.35×10^{-05} (0.030)	2.58×10^{-05} (0.300)	7.52×10^{-05} (0.004)	1.06×10^{-04} (< 0.001)	1.14×10^{-04} (< 0.001)	8.86×10^{-05} (< 0.001)	6.77×10^{-05} (0.004)
b_3	9.867 (< 0.001)	9.269 (< 0.001)	9.333 (< 0.001)	8.163 (< 0.001)	9.513 (< 0.001)	9.758 (< 0.001)	8.592 (< 0.001)	8.523 (< 0.001)	9.124 (< 0.001)
b_4	−167.598 (< 0.001)	−166.810 (< 0.001)	−167.420 (< 0.001)	−164.275 (< 0.001)	−165.523 (< 0.001)	−166.084 (< 0.001)	−163.893 (< 0.001)	−163.277 (< 0.001)	−165.691 (< 0.001)

Table 1: Parameter estimation for the seasonal temperature function Λ^{temp} with the corresponding p-values in parentheses below.
Source: Own calculations

and KPSS test we observe that the remaining deseasonalized temperature

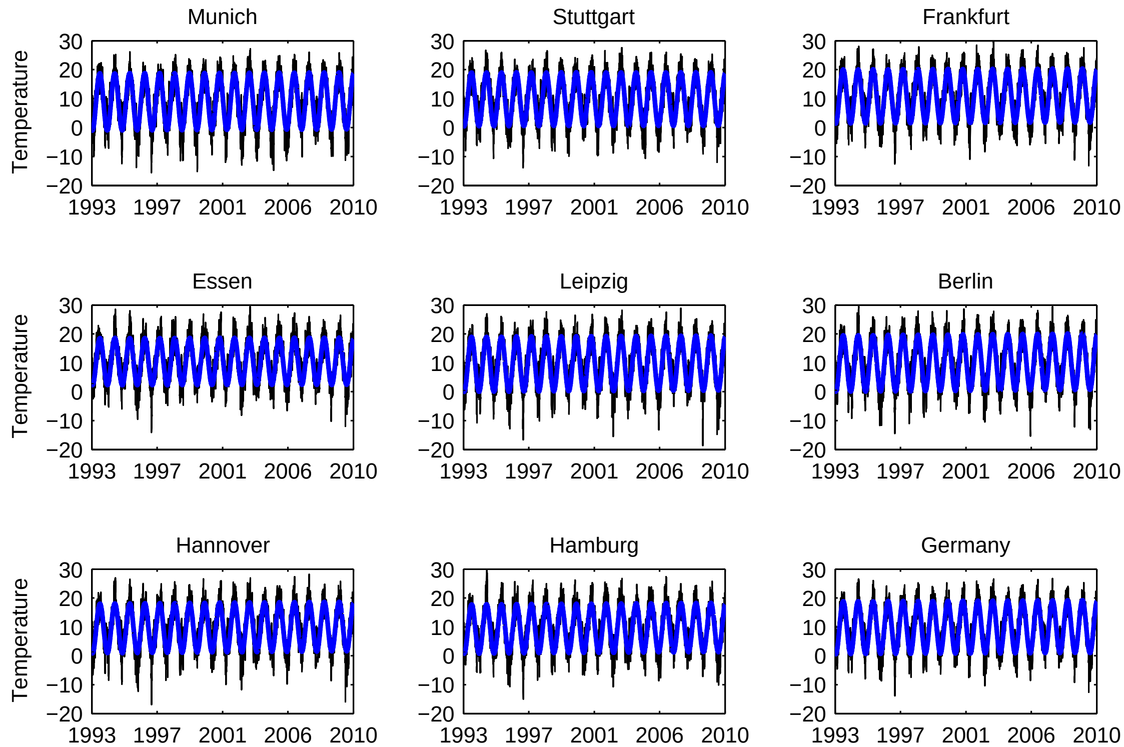


Figure 1: Daily average temperature (black solid line) and the estimated deterministic, seasonal function Λ^{temp} (blue line).
Source: Own calculations based on DWD data

data are stationary. Analyzing the autocorrelation function (ACF) and the partial autocorrelation function (PACF) of the deseasonalized temperatures conclude that an AR(p) process should be used. Using the Schwarz information criteria and the plots of the PACF an AR(3) model ist most suitable for the data which is in line with the results of Härdle and Cabrera (2012). The results of the parameters and the related CAR parameters $(\alpha_1, \alpha_2, \alpha_3)$

can be found in Table 2.

	Munich	Stuttgart	Frankfurt	Essen	Leipzig	Berlin	Hannover	Hamburg
$\mathbf{a_1}$	0.952 (< 0.001)	0.969 (< 0.001)	0.927 (< 0.001)	0.993 (< 0.001)	0.959 (< 0.001)	0.957 (< 0.001)	0.939 (< 0.001)	0.892 (< 0.001)
$\mathbf{a_2}$	-0.222 (< 0.001)	-0.241 (< 0.001)	-0.200 (< 0.001)	-0.297 (< 0.001)	-0.246 (< 0.001)	-0.230 (< 0.001)	-0.226 (< 0.001)	-0.183 (< 0.001)
$\mathbf{a_3}$	0.056 (< 0.001)	0.063 (< 0.001)	0.060 (< 0.001)	0.110 (< 0.001)	0.095 (< 0.001)	0.081 (< 0.001)	0.091 (< 0.001)	0.088 (< 0.001)
α_1	2.048	2.031	2.073	2.007	2.041	2.043	2.061	2.108
α_2	1.318	1.303	1.346	1.311	1.327	1.316	1.348	1.400
α_3	0.214	0.209	0.213	0.194	0.191	0.192	0.196	0.204
λ_1	-0.244	-0.239	-0.232	-0.207	-0.199	-0.205	-0.201	-0.200
$\lambda_{2,3}$	$-0.902 \pm 0.254i$	$-0.896 \pm 0.268i$	$-0.920 \pm 0.267i$	$-0.900 \pm 0.358i$	$-0.921 \pm 0.335i$	$-0.919 \pm 0.309i$	$-0.930 \pm 0.330i$	$-0.954 \pm 0.328i$

Table 2: The parameter estimation for the autoregression coefficients a_1, a_2, a_3 with the corresponding p-values in parentheses below; CAR parameter estimates $\alpha_1, \alpha_2, \alpha_3$ and the corresponding eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of the coefficient matrix A.
Source: Own calculations

The remaining residuals are best modeled by a volatility function $\eta^2(t)$ which is based on a truncated Fourier Series of the form

$$\eta^2(t) = c_1 + \sum_{k=1}^{K} c_{2k} \cos(\frac{2k\pi t}{365}) + c_{2k+1} \sin(\frac{2k\pi t}{365}). \tag{4.2}$$

For the different cities in Germany we accept that the number of Fourier terms is different as the residuals show significant differences in the plots. In order to determine the appropriate length K of the Fourier Series the parameters are estimated with different length $K = 1, 2, \ldots, T$. K is then chosen to be the maximum value for which all parameters c_{2k} and c_{2k+1} are significant to the 5% significance level, i. e. $K = \max\{J = 1, 2, \ldots | c_{2j}, c_{2j+1}$ are significant $\forall\, j \in$

$\mathbb{N}$ and $j \leq J\}$. In the case, $K = 0$, the variance function is modeled by a constant $\eta^2(t) \equiv \eta^2$ which is given by the variance of the white noise term derived by the AR regression. This procedure is conservative regarding the number of parameters in the model as the model is reduced to the smallest number of Fourier Series terms which contribute significantly to the variance function. The results are shown in Table 3 (for details see Ebbeler (2012)).

	Munich	**Stuttgart**	**Frankfurt**	**Essen**	**Leipzig**	**Berlin**	**Hannover**	**Hamburg**
$\mathbf{c_1}$	4.955 (< 0.001)	4.543 (< 0.001)	4.561 (< 0.001)	4.246 (< 0.001)	4.779 (< 0.001)	4.633 (< 0.001)	4.691 (< 0.001)	4.606 (< 0.001)
$\mathbf{c_2}$	1.465 (< 0.001)	1.041 (< 0.001)	0.830 (0.030)		0.530 (< 0.001)	0.383 (0.002)	0.764 (< 0.001)	0.491 (< 0.001)
$\mathbf{c_3}$	0.498 (< 0.001)	0.322 (0.011)	0.361 (0.003)		0.566 (< 0.001)	0.700 (< 0.001)	0.401 (0.001)	0.561 (< 0.001)
$\mathbf{c_4}$						0.572 (< 0.001)		0.721 (< 0.001)
$\mathbf{c_5}$						−0.304 (0.015)		−0.236 (0.036)
K	1	1	1	0	1	2	1	2

Table 3: The parameter estimation of the variance function $\eta^2(t)$ with the corresponding p-values in parentheses below. The number of truncated Fourier Series terms K is given in the last line.
Source: Own calculations

Analysing the EPEX spot prices we obtain the best fit for the seasonal function Λ as

$$\Lambda(t) = \beta_0 + \beta_1 \mathbf{1}_{\{t=Sat,Sun\}} + \beta_2 t + \beta_3 \cos\big(\frac{2\pi}{365}(t+\beta_4)\big) + \beta_5 \cos\big(\frac{6\pi}{365}(t+\beta_6)\big). \tag{4.3}$$

In this function t is measured in days and $\mathbf{1}$ denotes the indicator function which is equal to one if t is Saturday or Sunday. This model accounts for the weekly pattern by using the dummy variable and hence creates different price levels for the weekend and the week. Furthermore, the two cosine functions account for the annual seasonality and the semi-annual peaks in the data.

In order to receive stable parameter estimates outliers are excluded from the parameter estimation. Data points which are above or below four standard deviations from the mean are defined as outliers. This leads to an exclusion of 16 data points which is less than 0.5% of the data sample. The parameters are estimated using the least-square regression[3]. The results of the parameters are shown in table 4.

All parameters are significant to the 1% level except of the parameter β_6

β_0	β_1	β_2	β_3	β_4	β_5	β_6
21.386 (< 0.001)	-12.985 (< 0.001)	0.010 (< 0.001)	3.474 (< 0.001)	41.118 (< 0.001)	-2.116 (< 0.001)	6.068 (0.042)

Table 4: Results of the parameter estimation of the seasonal function for arithmetic and geometric spot price model and the corresponding p-values in parentheses below.
Source: Own calculations

which is significant to the 10% level (see Ebbeler (2012) for more details). We also tested a seasonal function with a 4π cosine term which is used in Benth et al. (2008a) but we obtained a better fit with the chosen function (4.3).

The ADF test and the Phillips-Perron test show that the deseasonalized temperatures are stationary time series.

For simplification we further assume that the volatility is constant. Considering the unique solution of (3.4) the parameters can be estimated using the procedures for AR processes.

κ	σ
0.323	13.127

Table 5: Results of the parameter estimation for the mean reversion and volatility of the spot price model.
Source: Own calculations

In order to estimate the market-price-of-risk parameter θ we use the approach proposed by Cartea and Figueroa (2005) based on monthly electricity contracts (EEX Baseload contracts) for the time period 2006-2009 under the assumption of a constant volatility σ. The resulting market-price-of-risk for the indifference pricing approach is calculated as the average of each monthly θ_i (see Ebbeler (2012)). For the observed time period we obtain a market-price-of-risk of $\theta = 0.233$ which we will use later in the indifference pricing valuation.

4.3. CORRELATION ANALYSIS BETWEEN TEMPERATURE AND ELECTRICITY SPOT PRICE

Many papers have analyzed the relationship between temperatures and the electricity demand in different countries.

Peirson and Henley (1994) analyze the effect of temperature on load in Great Britain. They observe that the influence of temperature on load is statistical significant irrespective if the model considers autocorrelation or not.

Similarly Pardo et al. (2002) analyze the influence of temperature and seasonality on the load in Spain and observe that both parameters are significant.

Weron (2006) uses a slightly different approach to analyze the dependency between temperature and electricity load in California. In his study, he calculates the Pearson correlation coefficient between the load and the temperature in California for the years 1999-2002. The results show that the correlation between temperature and load is significant and the correlation coefficient is even higher if the weekly pattern in the load series is considered.

While all the papers mentioned above analyze the relation between air temperature and system load explaining the relation to the demand side of the electricity price evaluation, a different approach is used by Lucia and Torró (2005). They analyze among other weather variables, the influence of air temperature to the electricity spot price at Nord Pool. This approach differs in the way that going from the system load to the price traded at an exchange can change the dependency structure since known demand patterns might be considered in the capacity forecast and hence have no effect on the spot price. They calculate the correlation between the average of the 168 hourly spot system prices and the maximum of the difference between the *Heating degree of the Week* and its historical average and 0. The results show a correlation of about 40%, stating that significant abnormal cold weather waves (over at least one week or more) have an impact on the average spot price in Norway.

Similar results are also observed by Weron and Misiorek (2008) for four 5-weeks periods between the hourly log-price at Nord Pool and the hourly air temperature.

The approach which is chosen here differs significantly from all other approaches. For the correlation analysis between temperature and spot price we define for each time series an appropriate stochastic model and analyze the correlation between the stochastic components of both time series and not between the price and the temperature itself. In this way, we can exclude that the correlation is affected by the deterministic and a-priori known temperature and spot price movements within a year. The fact that the air temperature is lower in Germany in the winter months compared to summer is obvious and therefore already priced-in the spot price which generally has higher prices in winter compared to summer. Consequently, analyzing the temperature and the spot price directly without removing the deterministic part leads to a correlation effect which is only caused by the general effect of colder temperature and higher spot prices in winter. From our point of view, such an correlation effect is not adequate to decide if electricity contracts can be used to price weather derivatives. Moreover, utility companies usually set up a capacity plan for their production facilities in advance which should cover the expected electricity demand. This plan is especially important for the nuclear and older coal plants. Short term changes in the demand are mostly covered by gas turbine plants or similar plants which are more flexible in the capacity loads. The drawback of this flexibility lies in the higher running costs of these plants. As a consequence we could expect that deviations from the expected level for example caused by unexpected temperatures, lead to price movements at the spot price (away from the expected spot price). Due to these reasons we estimate the deterministic components of the temperature time series and the spot price time series. After removing the seasonal component from the data, the obtained time series show the deviations of the temperature and the spot price from the expected values. Based on these data, the correlation between the stochastic processes driving the time series can be analyzed. In the given situation this means that the effect of deviations of the daily average temperatures in Germany on the spot price traded at the EPEX is analyzed.

The following correlation analysis is split into two parts. The first part deals with the correlation between the deseasonalized daily average temperatures and the deseasonalized spot prices in winter whereas the second part analyzes the correlation effect in summer months.

In order to avoid that negative correlation effects of the winter months are counterbalanced by positive correlation effects in the summer, the correlation is calculated separately for winter and summer.
Additionally, the analysis of the correlation is conducted for each year separately in order to gain further insides about the development of the correlation over time.

4.3.1. Correlation Analysis Winter Months

As the primary reason for additional energy consumption in winter months is heating, the analysis concentrates on the heating period in Germany. For most parts of Germany, the heating period starts in November and goes until end of February or mid of March. Consequently, the considered time period in the winter is from November 1^{st} until February 28^{th} of the following year. If we would include also March in the analysis, the results change only marginally. The Pearson and the Spearman correlation coefficient are calculated for each year separately, i. e. we calculate the correlation for the time period 2009/11/01-2010/02/28 and 2008/11/01-2009/02/28 etc. separately. As during the Christmas break (December 24^{th} - January 1^{st}), many industry companies are shutting down their production, the energy demand differs significantly from the usual demand level. This often leads to very unusual spot prices at the EPEX. In order to ensure that the measured correlation effects are not driven by these effects we excluded the Christmas break from our data set for each year. As a consequence, the sample for each year consists of 111 data points, which is an adequate sample size for the calculations and to obtain reliable results.

For the calculations the deseasonalized German average temperature is used which is the average of the daily average temperatures of all 8 cities. On the other side, we used the deseasonalized arithmetic spot price (3.4) as well as the log-price (the geometric model). In the correlation analysis the temperature of a specific day is compared to the EPEX Spot Price of the same day. In general the temperature forecasts for Germany are very precise for the next 24 hours with only an error rate of less than a few percent. Therefore the comparison of the real temperatures measured on day t and the spot price on the same day does not possess a strong bias although the spot price for day t is already determined on the day before.

The results of the correlation analysis for the winter periods can be seen in Table 6. We obtain correlation coefficients for both models which are all significant to the 2%-level. Most of them are even significant to 0%-level. Concentrating on the arithmetic case the Pearson correlations mainly are between 31% and 57% with a very high correlation in the period 2004/2005 and 2009/2010. On the other hand, the lowest correlation coefficient is calculated for the period 2008/2009 which is probably due to the financial crisis and the uncertainty in the commodity markets. The deviations between the Spearman and the Pearson coefficient can in large be explained by two effects. Firstly (esp. in 2001/2002 and 2002/2003) the dependence structure between temperature and spot price is not linear but rather quadratic, secondly in 2007/2008 the introduction of the CO_2 certificates led to some price movements in first days of January 2008 which weakens the correlation coefficient.

The picture in the geometric case is slightly different. In general, both correlations are slightly higher compared to the arithmetic case for most years. Additionally, all coefficients are significant to the 1%-level.

4.3.2. Correlation Analysis Summer Months

Similar to the correlation analysis of the winter period, the correlation for the summer period is calculated for each year separately. The time period

Timeframe	Arithmetic Model		Geometric Model	
	Pearson r	Spearman ρ	Pearson r	Spearman ρ
2009/11/01-2010/02/28	-73.46% (0.00)	-73.67% (0.00)	-69.80% (0.00)	-69.38% (0.00)
2008/11/01-2009/02/28	-25.89% (0.01)	-22.76% (0.02)	-28.30% (0.00)	-23.35% (0.01)
2007/11/01-2008/02/28	-35.43% (0.00)	-28.84% (0.00)	-30.36% (0.00)	-30.64% (0.00)
2006/11/01-2007/02/28	-33.67% (0.00)	-39.75% (0.00)	-40.28% (0.00)	-41.76% (0.00)
2005/11/01-2006/02/28	-39.51% (0.00)	-38.40% (0.00)	-49.64% (0.00)	-47.13% (0.00)
2004/11/01-2005/02/28	-56.60% (0.00)	-57.04% (0.00)	-58.01% (0.00)	-59.33% (0.00)
2003/11/01-2004/02/28	-33.18% (0.00)	-24.78% (0.01)	-36.86% (0.00)	-30.02% (0.00)
2002/11/01-2003/02/28	-43.07% (0.00)	-51.84% (0.00)	-47.83% (0.00)	-48.27% (0.00)
2001/11/01-2002/02/28	-31.54% (0.00)	-68.94% (0.00)	-57.17% (0.00)	-73.29% (0.00)

Table 6: Results of the correlation analysis for the winter period 11/01-02/28 excluding the Christmas break. The results of the Pearson and Spearman coefficient for the arithmetic model are presented in columns 2 and 3 respectively with the p-values in parentheses below. The correlation results for the geometric model are stated in columns 4 and 5.
Source: Own calculations

which is analyzed each year starts on June 1st and ends at the last day of August. Hence the sample of each year consists of 92 data points. We expect that on warmer days, the electricity demand increases which would lead to a positive correlation between temperature and spot price.

The Pearson correlation for the summer varies between 25% to 46% for the arithmetic case and between 25% and 58% in the geometric case (see Table 7). The years 2008 and 2009 show significantly lower correlation results for both models which could be based on the financial crisis. Especially the 2008 results differ from the other results which coincides with the peak of the financial crisis in summer 2008 and therefore is probably caused by the high insecurity in the commodity markets. Excluding the years 2008 and 2009, all estimated Pearson correlation results are significant to the 1% level confirming our assumption of the dependence structure. Moreover, the results of the Spearman correlation coefficients are similar to the values of the Pearson correlation clearly indicating that the dependence structure exists and is not driven by one-time effects or special effects in the different years.

The results of the correlation analysis for the winter and the summer period demonstrates that a correlation structure exists between the deseasonalized temperatures and the deseasonalized spot price in Germany. This effect is significant for most of the years (only excluding 2008). Additionally, the effect is independent of the chosen model for the electricity spot price.

The correlation values indicate that the use of electricity based derivatives are appropriate as an hedging instrument for weather derivatives. In comparison to the winter period, the correlation values are slightly smaller supporting the hypothesis that the 'air-conditioning-effect' in Germany has a smaller impact on the spot price compared to the 'heating-effect' in the winter.

5. EMPIRICAL ANALYSIS OF INDIFFERENCE PRICES

5.1. CAT PRICE MODELS

The empirical analysis is conducted for two scenarios of the temperature process, the OU process (which is a CAR process with $p = 1$) and a CAR process with $p = 3$, which is the result of the temperature analysis in Germany. Based on the derived CAT prices 3.33 and 3.28 and based on the derived parameter estimates CAT prices for Essen. 2 - 4 show the results of the price models for the months May-September of 2010 and 2011. For each measurement month the CAT prices are calculated for at least the last three months before the start of the measurement period. The prices are compared to the CAT price of the corresponding month traded at the CME in Chicago (solid red line)[4]. In addition to the CME price which shows the

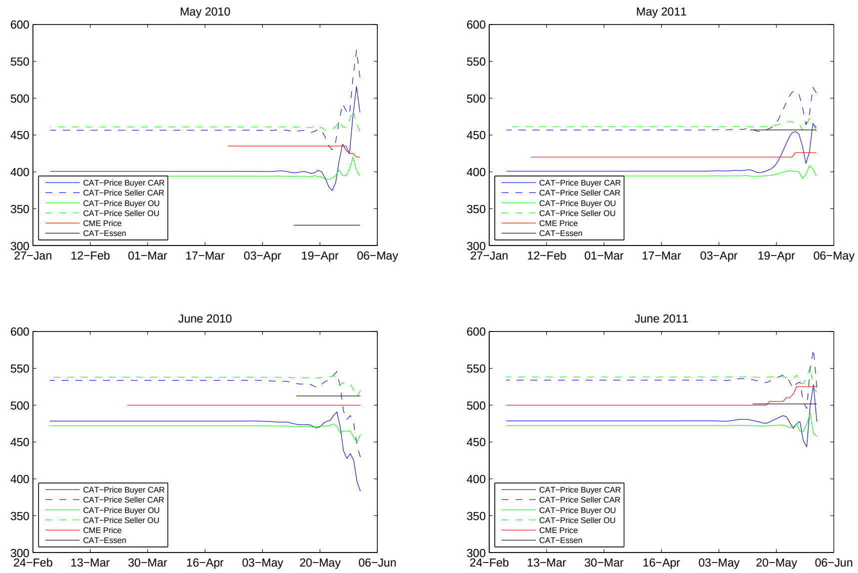


Figure 2: Comparison of the calculated CAT Prices of the two analyzed models (OU-model and CAR-model) compared to the CAT prices traded at the CME and the observed CAT for Essen (a-posterior) for May 2010 (left top), May 2011 (right top), June 2010 (left bottom) and June 2011 (right bottom) with parameters: $r = 3\%, \rho = 30\%, \theta = 0.23$ and $\gamma = 5.0\%$.
Source: Own calculations and Reuters data

CAT of Essen for the corresponding measurement month and is calculated a-posteriori, i. e. the CAT is calculated at the end of the measurement period. This value represents the value which is chosen by the clearing house to balance the traded futures contracts.

The indifference CAT prices are very similar in all analyzed months. The prices are stable in the first part of the observation period. In this time

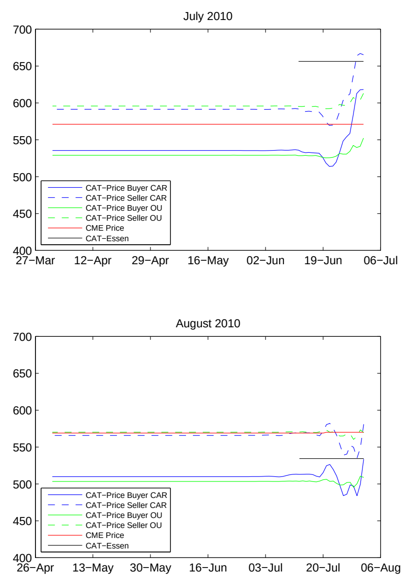


Figure 3: Comparison of the calculated CAT Prices of the two analyzed models (OU-model and CAR-model) compared to the CAT prices traded at the CME and the observed CAT for Essen (a-posterior) for July 2010 (left top), July 2011 (right top), August 2010 (left bottom) and August 2011 (right bottom) with parameters: $r = 3\%, \rho = 30\%, \theta = 0.23$ and $\gamma = 5.0\%$.
Source: Own calculations and Reuters data

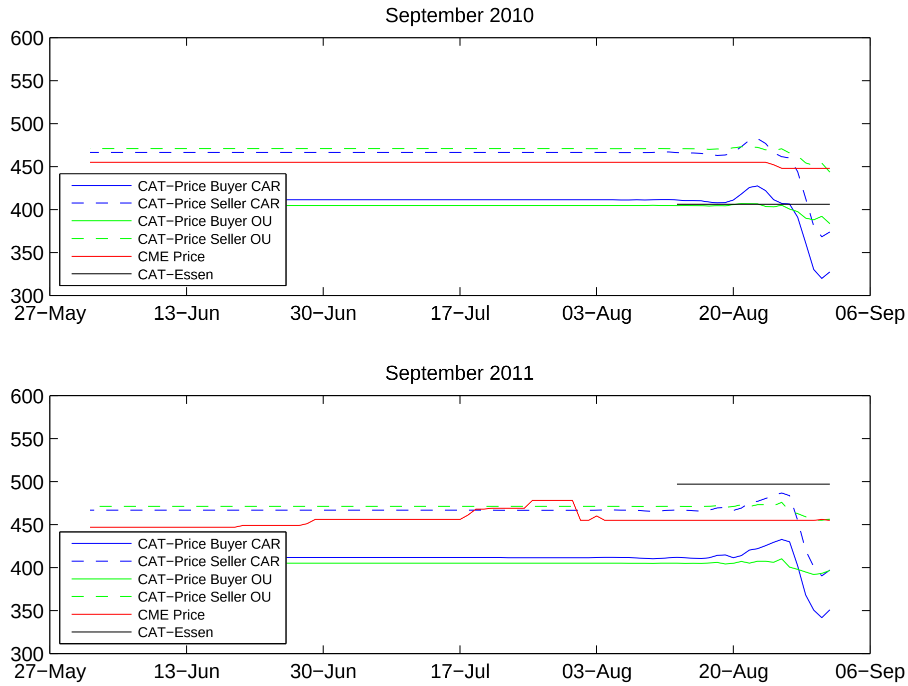


Figure 4: Comparison of the calculated CAT Prices of the two analyzed models (OU-model and CAR-model), the CAT prices traded at the CME and the observed CAT for Essen (a-posterior) for September 2010 and 2011 with parameters: $r = 3\%, \rho = 30\%, \theta = 0.23$ and $\gamma = 5.0\%$.
Source: Own calculations and Reuters data

period the prices are dominated by the seasonal function component and the risk premium. Since the seasonal component is independent of t, this is a constant value throughout the observation period. The risk premium on the other hand is dependent in both models on t but due to the exponential function, variations are only detectable shortly before the measurement period. In both models we obtain that the selling price is higher compared to the buying price due to the different sign of R^{temp} which is as expected due to the *bid-ask-spread*. The differences in the price processes are based on the different values of the risk premiums which is higher (in absolute values) in the OU-model. In the last two weeks before the measurement, the behavior of the indifference prices changes from almost constant to more volatile behavior. This behavior is driven by two effects. Firstly, the influence of the second component of the price equation which depends on $X(t)$ increases. Depending on the values of the deseasonalized temperatures on the current day (OU-model) and additionally the last two days in the CAR-model ($p = 3$), this component adds a positive or negative shift to the price process. Secondly, the risk premium changes close to the measurement period as the R^{el} component converges to zero reducing the risk premium to the negative component R^{temp}. Comparing the two price models it is observable that the CAR-model is more volatile. The reason for this lies in the fact that the deseasonalized temperatures of three days are considered and taken into account as a weighted sum whereas in the other model only the temperature of one day is considered. Consequently, the CAR-model possesses larger price movements in the last days before the start of the measurement period. Hence, in the OU-model the derived prices are closer to the value of the seasonal component (the expected CAT based on the seasonal function). Consequently, this model can be seen as more conservative.

Analyzing the 8 months under consideration, we observe that the indifference price models are adequate to price weather derivatives for the buyer and seller side. The derived mathematical price models possess a clear structure consisting of the expected CAT value (incl. a seasonal component) and a risk premium which only varies in the sign of the R^{temp} between the seller and buyer price. Due to this clear structure, all price models can be implemented without the use of Monte-Carlo simulations or other approximations. Furthermore and more importantly, the obtained prices for Essen are in a similar price range compared to the traded CME prices as well as the real Essen CAT. This shows the adequacy of the derived models.

Considering all analyzed months, it can be summarized that the buying prices for both models are mainly lower than the corresponding CME price. As a consequence investors would try to find a counterparty at the OTC market rather than buying at the CME.

A similar picture is observable for the selling prices which is higher compared to buying price due to the change in the sign of R^{temp}. For most of the observed months, the selling prices of both models (OU and CAR-model) are higher compared to the CME price which again leads to the situation where the investor sells the CAT OTC rather than at the CME.

A further characteristic of the presented price models can be seen, if the prices are compared to the real and a-posteriori calculated Essen CAT values. Considering the derived CAT prices for the buyer of a contract, it can be obtained that the price models show a better prediction of the Essen CAT compared to the CME. Especially, the CAR(3) model shows an appropriate fit to the Essen value. In cases where the CAR(3) price differs from the CAT value, the buying price is below the Essen CAT which implies a profit for the investor.

Based on the chosen approach it is also possible to derive an indication about the implied risk aversion of the current market participants. Using the proposed model, the best fit with the CME data is obtained with a small risk aversion coefficient γ ($< 1\%$), as a decrease in the risk aversion leads to an upward shift of the prices in the buying models. However, even with a very small γ the CME price is still above the buying CAT prices of the models for most months. Hence, the conclusions from above are still valid.

All price models show the expected characteristic of temperature futures with an almost constant price at the beginning of the trading period and an increase in volatility close to the measurement period, the so called *Samuelson effect* (see also Benth et al. (2007)). The selling prices of both models is clearly higher as the proposed buying prices which displays the typical bid-ask-spread. Moreover, the observed CME price is in most cases between the selling and buying price which would imply for investors using the indifference pricing approach and the assumed risk aversion that they would not trade at the exchange but go to the OTC market. Furthermore, the observed prices are appropriate for trading as the processes either predict the real CAT value quiet well or produce a positive payoff in case of deviations from the CAT value.

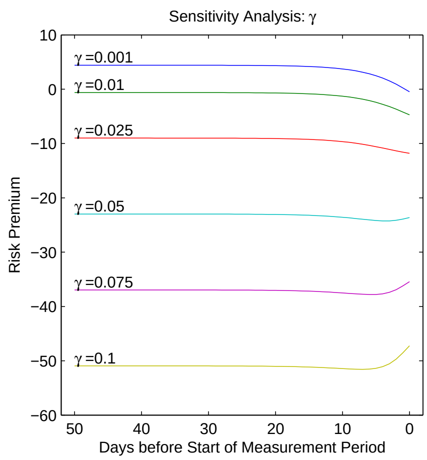
A detailed view towards the risk premium can be found in Ebbeler (2012).

6. SENSITIVITY ANALYSIS OF THE PRICING MODEL

In the previous sections the derived CAT price models were analyzed based on estimations from market data (EPEX spot market, EEX futures market, and weather data). In order to show the robustness of the CAT price model sensitivity analyses for the risk aversion factor γ, the market-price-of-risk θ and the correlation factor ρ are conducted.

The risk aversion coefficient γ describes the investors caution to enter a risky investment compared to a certain cash flow. The larger γ, the higher the aversion of the investor to buy a risky asset and to prefer a riskless and certain investment. A value for γ close to zero consequently implies that the investor is more willing to accept a fair game. In the CAT price model, the risk aversion coefficient is introduced through the utility function of the investor which is assumed to be negatively exponential in our setting. In the derived CAT price model γ is part of the risk premium R and the leading coefficient of R^{temp}. As discussed before the two components of the risk premium possess different signs and therefore the interaction between these two components determines the structure of R. Therefore, the sensitivity analysis primarily focuses on the impact of different values for γ on the risk premium. The obtained structure of R is then transferred unchanged to the CAT price due to the linear structure of R in $G(t)$.

Fig. 5 shows the risk premium for the values $\gamma = 0.1\%, 1\%, 2.5\%, 5\%, 7.5\%$ and $\gamma = 10\%$ for the last 50 days before the start of the measurement period (T_1) of the CAT contract. The case $\gamma = 0.1\%$ describes the case in which the investor is almost risk neutral. In this case, the risk premium is positive



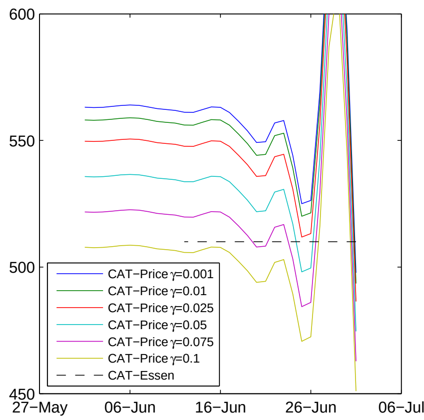


Figure 5: The results of the sensitivity analysis for the risk aversion coefficient parameter γ. The impact of different values for γ on the risk premium (R) (left) and the corresponding impact on the overall CAT price (right) exemplary for the CAT July 2011 contract.
Source: Own calculations

and almost constant for the first days and decreases sharply to zero close to T_1. The positivity of R is caused by the fact that for small γ the influence of R^{temp} in the risk premium is only marginal which contributes negatively to R. This described structure of R and especially the positive value of R implies that the investor is willing to pay an extra premium to enter the CAT futures contract. This willingness for the extra premium is mainly driven by the hedging effect of the electricity contracts considered in the model. Hence, for the investor, the advantages of the hedging effect predominates the own risk aversion. As in this setting we deal with an investor which is almost risk neutral, the additional hedging effect (additional information) through the futures contracts lead to a higher attractiveness for the CAT futures which is usually displayed in an higher bid price which coincides here with the observations in the sensitivity analysis and the CAT prices.

An increase in γ has mainly two effects. Firstly, the risk premium is vertically shifted in the direction to negative values of R (for the constant part) and secondly, the behavior close to T_1 changes. For values close to zero, the risk premium is decreasing in the last days whereas for larger values R firstly decreases slightly before it sharply increases in the last five days. The *turning* point for the last days can be observed for a value of $\gamma = 3.2\%$ in which R is almost constant. For $\gamma = 10\%$ the risk premium is around -50 which, in the context of the CAT contract, implies a reduction of more than 1.5 degrees Celsius per day in the measurement period in order to buy the contract. The increase in R in the last days before T_1 implies that the investor requires a smaller discount in order to buy the CAT. This is in line with the expectations as the aversion should reduce as more information about the temperatures in the observation period are available which is the case close to T_1. Hence, the structure of the risk premium R displays in an appropriate way the characteristic of both, a risk averse investor as well as a risk neutral investor.

Note, that this parameter represents the market-price-of-risk of the electricity market and not the market price of risk of the weather derivatives market. The empirical analysis showed that the market price of risk varies between the contracts and is mainly in the range between -0.5 and 0.5 which therefore also represents the analyzed value range in the following sensitivity analysis. 6 shows the structure of the risk premium R for different values of θ and

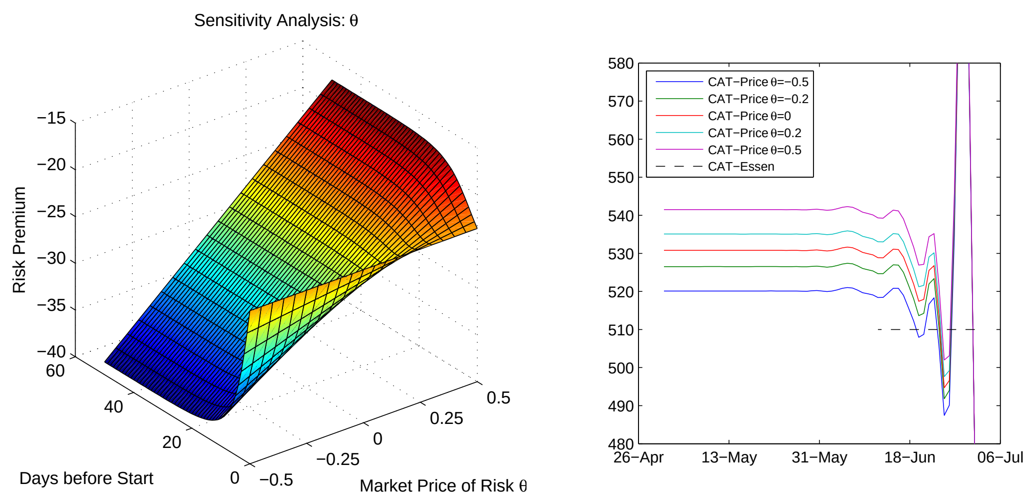


Figure 6: The results of the sensitivity analysis for the market-price-of-risk coefficient θ on the risk premium (R) (left) and the corresponding CAT price (right) exemplary for the CAT July 2011 contract.
Source: Own calculations

depending on the last days before the start of the measurement period at T_1. A number of characteristics are observable. For the time period 15 days and more before T_1, R is linear increasing in θ and flat in t due to the fact that the exponential terms of R are nearly constant over time and θ is a multiplicative factor in R^{el}. Depending on the value of θ the risk premium behaves differently on the last days before T_1. As $R^{el} \to 0$ for $t \to T_1$, the risk premium is independent of θ in T_1 and therefore, the shape of the risk premium in the last days before T_1 strongly depends on the chosen value for θ, since this determines if R is strongly decreasing (e. g. $\theta = 0.5$) or strongly increasing (e. g. $\theta = -0.5$). Consequently, we can conclude that the value of the market price of risk of the electricity market clearly influences the risk premium of the CAT price in the weeks before T_1. Nevertheless, this effect vanishes close to T_1. From an investor's point of view this means that the choice of the electricity model and the parameter estimation of the spot price model has to be made carefully especially in the time period of 10 days or more before T_1, because both effects influence the results of the estimated parameter θ and hence in the consequence affects the associated risk premium for the CAT price. However, close to T_1, this bias is negligible as the influence of the electricity contracts decreases and the risk premium is mainly driven by R^{temp}.

The correlation parameter ρ in the CAT price model describes the correlation between the electricity spot price and the temperature. As described before it seems natural to assume that the spot price and the temperature is positively correlated in the summer months. Consequently, the value range of $[0, 1]$ for ρ is considered in the analysis where $\rho = 1$ represents a perfect correlation between spot price and temperature whereas $\rho = 0$ induces that both times series are uncorrelated. In the obtained CAT price model, the parameter ρ influences both parts of the risk premium R. On the one hand side ρ is the leading coefficient of the electricity risk R^{el} and on the other hand side one summand of the temperature risk R^{temp}.

The results show that the risk premium R is smallest in absolute values for $\rho = 1$ for the time period 10 or more days before T_1 and increases (in absolute values) in the last days before T_1 sharply. This characteristic is in line with what we would expect from a market point of view for an indifference price model of weather derivatives. For $\rho = 1$ a strong correlation between spot price and temperature is assumed which should give the investor a perfect hedge of its weather derivative position by the electricity futures portfolio. Consequently, the investor requires only a small risk premium. Simply in

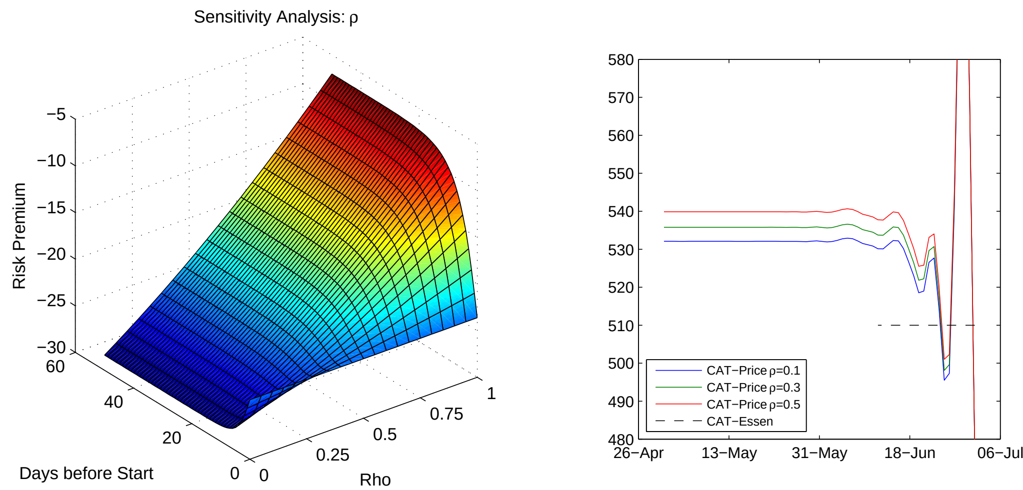


Figure 7: The results of the sensitivity analysis for the correlation ρ on the risk premium (R) (left) and the corresponding CAT price (right) exemplary for the CAT July 2011 contract.
Source: Own calculations

the last days before T_1 new and more precise information about the temperatures in $[T_1, T_2]$ are available and therefore the hedging effect decreases and hence the investor asks for a higher risk premium. In contrary to this, if it is assumed that spot price and temperature are uncorrelated ($\rho = 0$), no hedging effect is expected in the portfolio through the electricity futures position and hence the investor requires a high risk premium. This effect is well displayed in the risk premium of the weather derivative pricing model (see Fig. 7) which has the highest risk premium (in absolute values) for $\rho = 0$. In this situation the risk premium is almost six times higher compared to the case of $\rho = 1$. Another effect which was also observable in the sensitivity analysis of the market price of risk is visible here as well. The risk premium is independent of ρ in T_1 since $R^{el} \to 0$ for $t \to T_1$ as well as the summand of R^{temp} which possesses the factor $1 - \rho^2$. As a consequence, R is either increasing or decreasing sharply in the last days before T_1 depending on the chosen correlation. For the CAT price we observe that the prices are different for different values of ρ at the beginning but converges to the same CAT price in T_1 as the risk premia coincide.

7. CONCLUSION

In our analysis we studied the relation of electricity futures and CAT temperature futures and the applicability of the indifference pricing approach for CAT contracts using electricity futures. We conducted an empirical analysis of average daily temperatures which are best modeled by a deterministic function covering the seasonal effects, and a stochastic component which is driven by a CAR(3) process. After taking appropriately care of seasonalities, a significant correlation structure between the deseasonalized temperature and the deseasonalized spot price is observable for winter and summer months independently of the chosen spot price model. Furthermore, using the indifference pricing approach and electricity futures as the tradable asset a closed-form expression for the CAT price is derived which consists of the expected payments of the CAT and a risk premium. We show that the derived CAT prices for both models are adequate for pricing temperature derivatives and, conducting a thorough sensitivity analysis that CAT price models are robust under changes of key parameters.

Future lines of research could be an extension of the modelling approach by using different electricity price models, in particular models that include jumps as described in Benth et al. (2008b, Ch.4). Also, we could apply our approach to other temperature derivatives such as HDD and CDD contracts. The HDD contracts are traded for the winter period and hence are the counterpart of the CAT contracts for the cities in Europe. As the HDD of one day is defined by $HDD_t = \max\{18 - T(t), 0\}$ (with $T(t)$ is the daily average temperature), the independence properties used in the simplification of the HJB equation are not applicable. However, with some modifications in the presented approach it should be possible to derive also the price process of a HDD contract which could be at least solved numerically. Carmona and Diko (2005) solved a similar problem for precipitation based derivatives using a transformation formula for the payoff function. Another interesting extension would be the analysis of wind speed and rainfall derivatives (of which the latter is traded at CME, but former does not yet exist).

Notes

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[2]The DWD is part of the *Bundesministerium für Verkehr, Bau und Stadtentwicklung* and is running 181 weather stations in Germany (see `www.dwd.de` for details).

[3]Since the first day of the observation period is not January 1^{st} we adapt the parameter t in parameter estimation in the form that $t = 167$ for the first observed data on 16 June, 2000. In this way, every multiple of 365 for t denotes 1^{st} January.

[4]All CME data are obtained from Reuters.

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Timeframe	Arithmetic Model		Geometric Model	
	Pearson r	Spearman ρ	Pearson r	Spearman ρ
2009/06/01-2009/08/31	22.60% (0.03)	25.33% (0.02)	15.97% (0.13)	21.49% (0.04)
2008/06/01-2008/08/31	17.26% (0.10)	15.57% (0.14)	13.82% (0.19)	13.48% (0.20)
2007/06/01-2007/08/31	29.47% (0.00)	35.29% (0.00)	28.59% (0.01)	31.47% (0.00)
2006/06/01-2006/08/31	46.44% (0.00)	47.81% (0.00)	58.29% (0.00)	49.23% (0.00)
2005/06/01-2005/08/31	40.72% (0.00)	40.07% (0.00)	37.34% (0.00)	36.69% (0.00)
2004/06/01-2004/08/31	31.11% (0.00)	36.97% (0.00)	37.65% (0.00)	37.67% (0.00)
2003/06/01-2003/08/31	25.44% (0.01)	32.83% (0.00)	25.30% (0.01)	27.46% (0.01)
2003/06/01-2003/08/31	27.52% (0.01)	28.77% (0.01)	29.10% (0.00)	27.38% (0.01)

Table 7: Results of the correlation analysis for the summer period 06/01-08/31 separately for each year. The results of the Pearson and Spearman coefficient for the arithmetic model are presented in columns 2 and 3 respectively with the p-values in parentheses below. The correlation results for the geometric model are stated in columns 4 and 5.

Source: Own calculations