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ON THE REVERSIBILITY OF THE INPUT
AND OUTPUT PROCESSES FOR A GENERAL
BIRTH-AND-DEATH QUEUEING MODEL

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ABSTRACT

In this paper we consider the general birth-and-death queueing model of Natvig (1975). Define the input and output processes by the steady-state behaviour of respectively successive input and output intervals. Ignoring balking customers, two cases are considered. In the first case we treat a lost customer neither as an input nor as an output, then secondly as both. For both cases we show the input and output processes to be reverse processes. One mistake and two erroneous comments in Natvig (1975) are also corrected.

GENERAL BIRTH-AND-DEATH QUEUEING MODEL ; STATE-DEPENDENT ;
BALKING ; DEFECTIVE CUSTOMERS ; LOSS ; STEADY-STATE ;
INPUT PROCESS ; OUTPUT PROCESS ; REVERSIBILITY

1. INTRODUCTION

Define the input and output processes by the steady-state behaviour of respectively successive input and output intervals, bearing in mind that the precise interpretation of "input" and "output" varies throughout the paper. Let us denote the stationary distributions of the number of customers in the system, just before the commencement of an input interval and just after the completion of an output interval by the letters π and q respectively. Talking about steady-state in connection with the input and output processes it is tacitly understood that the π - and q -distributions exist.

These processes are considered for the following birth-and-death queueing model. There are N waiting positions ($0 \leq N \leq \infty$), s servers ($1 \leq s \leq \infty$) and an arbitrary queueing discipline. Let an index n indicate that the quantity in question depends on the number of customers in the system (the number being served plus the number queueing), but not on time t . The instantaneous arrival rate is λ , the probability of balking (i.e. not trying to obtain service) being ξ_n and hence $\lambda_n = \lambda(1-\xi_n)$ is the arrival rate of customers trying to obtain service. The instantaneous service rate of each server is ν_n , and the defection rate of customers from the system before service completion is γ_n . Hence the instantaneous departure rate, μ_n , of customers having joined the system, is given by

$$\mu_n = \begin{cases} n\nu_n + \gamma_n & , \quad 1 \leq n \leq s \\ s\nu_n + \gamma_n & , \quad s < n \leq s + N . \end{cases}$$

In our paper Natvig (1975) three cases were considered giving the following results:

1. We started by ignoring both balking and lost customers. For $s + N > 1$ it was shown that the input and output processes are identical iff $s + N = \infty$, $\lambda_n = \lambda_0 n \geq 1$ in which case they are Poisson (λ_0). In the case $s = 1$, $N = 0$ the input and output processes are identical to a renewal process which is not Poisson. The steady-state joint distribution of the input and output numbers during an interval is infinitely divisible iff $s + N = \infty$, $\lambda_n = \lambda_0$ and $\mu_n = n\mu$, $n \geq 1$ corresponding to the M/M/ ∞ model. The sufficiency part of this statement is due to Milne (1971).

2. Next, with again no balking and registering losses both as inputs and outputs, it was shown for $s + N > 1$ that the input and output processes are identical iff $\lambda_n = \lambda_0$, $1 \leq n \leq s + N$, and again Poisson (λ_0). In the case $s = 1$, $N = 0$ the input and output processes are identical to a renewal process which in general is not Poisson.

3. Finally, by registering balking and lost customers both as inputs and outputs, it was shown that the input and output processes are Poisson (λ), thus generalizing Boes (1969).

In the first two cases we showed the input and output processes to be different if they are non-renewal. This was done by simply stating that a single input and output interval are differently distributed, which is in fact wrong according to Conolly and Chan (1976) treating the specialization of the model above with $(N, s) = (\infty, 1)$. However, our conclusion is nevertheless correct since we can show the input and output processes to be reverse processes in both cases. The deduction of this result which is

the main contribution of this paper, will be given in Section 2. For the third case the input and output processes are identical and hence obviously reverse processes.

Our finding is in agreement with Reich (1957) where it is proved that a stationary birth-and-death process is reversible, i.e.,

$$p_i P_{ij}(t) = p_j P_{ji}(t) \quad i, j = 0, 1, \dots$$

Here the p_i 's are from the stationary distribution of the number of customers in the system at an arbitrary point of time, henceforth denoted by the letter p and the P_{ij} 's are transition probabilities. Note, however, that he does not define the input and output processes in terms of the π - and q -distributions. According to Natvig (1975) this difference can be decisive since for $s + N = \infty$ there are cases where the p -distribution exists and not the π - and q -distributions which in fact are identical, and vica versa.

The author has recently received a paper by Venter and Swanepoel (1971) overlapping some of the results in Natvig (1975) for the special case $N = \infty$. They also show the input and output processes, as defined by this author, to be identical. However, we will in Section 3 make an attempt to correct an apparent mistake in their argument, leading to a conclusion being in opposition to ours. In this section we will also correct two erroneous comments in Natvig (1975) including one on a reversibility argument by Daley (1975).

Finally in Section 4 we arrive at the common expectation of the input and output intervals for the two first non-trivial cases above. This is also done in Conolly and Chan (1976) for their specialization using a somewhat intuitive approach based on the

p-distribution rather than the π - and q-distributions.

2. ON THE REVERSIBILITY OF THE INPUT AND OUTPUT PROCESSES

We start by treating the case where both balking and lost customers are ignored. Denote the output interval separated by the departure of the n^{th} and $(n+1)^{\text{th}}$ customer by D_n and let Z_n be the number of customers in the system just after the departure of the n^{th} customer. Let $m = s + N$ and introduce $(i=0, \dots, m-1)$

$$g_n(x, y) dx dy = P(x \leq D_n \leq x+dx, y \leq D_{n+1} \leq y+dy)$$

$$g_n(x, y | i) dx dy = P(x \leq D_n \leq x+dx, y \leq D_{n+1} \leq y+dy | Z_n = i) .$$

Let $g_n^*(z, w | i)$ be the Laplace Transform (L.T.) of $g_n(x, y | i)$. Extending the argument leading to (2.16)* (a "*" on the number of an equation refers henceforth to the paper Natvig (1975)) one realizes

$$g_n^*(z, w | i) = \sum_{j=\max(0, i-1)}^{m-1} \sum_{k=\max(0, j-1)}^{m-1} \gamma_i^{(j+1-i)^*}(z) \gamma_j^{(k+1-j)^*}(w),$$

$$0 \leq i \leq m-1 .$$

Here $\gamma_k^{(j)^*}(\cdot)$ is the L.T. of the density function associated with the following degenerate r.v.

$X_k^{(j)}$ = the length of an output interval starting with k customers in the system during which there are j arrivals (which are not lost)

$$(k=0, 1 \leq j \leq m; 1 \leq k \leq m-1, 0 \leq j \leq m-k) .$$

The latter L.T. is given by (2.15)*.

Denote the L.T. of the steady-state version of $g_n(x, y)$ by $g^*(z, w)$. Then

$$g^*(z,w) = \sum_{i=0}^{m-1} \sum_{j=\max(0,i-1)}^{m-1} \sum_{k=\max(0,j-1)}^{m-1} \pi_i \gamma_i^{(j+1-i)*}(z) \gamma_j^{(k+1-j)*}(w), \quad (2.1)$$

where the π_i 's are from the π -distribution given by (2.7)*. This follows since the π - and q -distributions are identical.

Now denote the input interval separated by the arrival of the n^{th} and $(n+1)^{\text{th}}$ customer by T_n and let

$$f_n(x,y)dx dy = P(x \leq T_n \leq x+dx, y \leq T_{n+1} \leq y+dy).$$

Denote the L.T. of the steady-state version of $f_n(x,y)$ by $f^*(z,w)$. By an argument completely parallel to the one leading to (2.1), we get

$$f^*(z,w) = \sum_{i=0}^{m-1} \sum_{j=0}^{\min(i+1,m-1)} \sum_{k=0}^{\min(j+1,m-1)} \pi_i \gamma_{i+1}^{(i+1-j)*}(z) \gamma_{j+1}^{(j+1-k)*}(w), \quad (2.2)$$

where the π_i 's are still from the π -distribution given by (2.7)* whereas $\gamma_k^{(j)*}(\cdot)$ is now given by (2.2)*. We will in the following establish the relation

$$g^*(z,w) = f^*(w,z) \quad (2.3)$$

By interchanging the order of summation in (2.1) we get

$$g^*(z,w) = \sum_{k=0}^{m-1} \sum_{j=0}^{\min(k+1,m-1)} \sum_{i=0}^{\min(j+1,m-1)} \pi_i \gamma_i^{(j+1-i)*}(z) \gamma_j^{(k+1-j)*}(w).$$

Hence what remains to be shown, is that

$$\pi_i \gamma_i^{(j+1-i)*}(z) \gamma_j^{(k+1-j)*}(w), \quad (2.4)$$

where $\gamma_k^{(j)*}(\cdot)$ is given by (2.15)*, is identical to

$$\pi_k \gamma_{k+1}^{(k+1-j)*}(w) \gamma_{j+1}^{(j+1-i)*}(z), \quad (2.5)$$

with $\gamma_k^{(j)*}(\cdot)$ from (2.2)* for $0 \leq k \leq m-1$, $0 \leq j \leq \min(k+1, m-1)$, $0 \leq i \leq \min(j+1, m-1)$. By restricting to the case with $j > 0$, $i > 0$, (2.5) is identical to

$$\begin{aligned} & \prod_{v=1}^k \left(\frac{\lambda_v}{\mu_v} \right) \pi_0 \frac{\lambda_j}{\mu_j} \prod_{v=j}^{k+1} \frac{\mu_v}{\lambda_v + \mu_v + w} \frac{\lambda_i}{\mu_i} \prod_{v=i}^{j+1} \frac{\mu_v}{\lambda_v + \mu_v + z} = \\ & = \prod_{v=1}^i \left(\frac{\lambda_v}{\mu_v} \right) \pi_0 \frac{\mu_{k+1}}{\lambda_{k+1} + \mu_{k+1} + w} \prod_{v=j}^k \frac{\lambda_v}{\lambda_v + \mu_v + w} \frac{\mu_{j+1}}{\lambda_{j+1} + \mu_{j+1} + z} \prod_{v=i}^j \frac{\lambda_v}{\lambda_v + \mu_v + z}, \end{aligned}$$

which is precisely (2.4). Note here and in the following that $\lambda_m = 0$. Next consider $j > 0$, $i = 0$ for which (2.5) equals

$$\begin{aligned} & \prod_{v=1}^k \left(\frac{\lambda_v}{\mu_v} \right) \pi_0 \frac{\lambda_j}{\mu_j} \prod_{v=j}^{k+1} \frac{\mu_v}{\lambda_v + \mu_v + w} \frac{\lambda_0}{\lambda_0 + z} \prod_{v=1}^{j+1} \frac{\mu_v}{\lambda_v + \mu_v + z} = \\ & = \pi_0 \frac{\mu_{k+1}}{\lambda_{k+1} + \mu_{k+1} + w} \prod_{v=j}^k \frac{\lambda_v}{\lambda_v + \mu_v + w} \frac{\lambda_0}{\lambda_0 + z} \frac{\mu_{j+1}}{\lambda_{j+1} + \mu_{j+1} + z} \prod_{v=1}^j \frac{\lambda_v}{\lambda_v + \mu_v + z}, \end{aligned}$$

again being equal to (2.4). Now let $j = 0$, $i = 1$ leading to the following expression for (2.5)

$$\prod_{v=1}^k \left(\frac{\lambda_v}{\mu_v} \right) \pi_0 \frac{\lambda_0}{\lambda_0 + w} \prod_{v=1}^{k+1} \frac{\mu_v}{\lambda_v + \mu_v + w} \frac{\lambda_1}{\lambda_1 + \mu_1 + z},$$

immediately reducing to (2.4). Finally let $j = 0$, $i = 0$ for which (2.5) is equal to

$$\prod_{v=1}^k \left(\frac{\lambda_v}{\mu_v} \right) \pi_0 \frac{\lambda_0}{\lambda_0 + w} \prod_{v=1}^{k+1} \frac{\mu_v}{\lambda_v + \mu_v + w} \frac{\lambda_0}{\lambda_0 + z} \frac{\mu_1}{\lambda_1 + \mu_1 + z},$$

again reducing to (2.4) completing the proof of (2.3).

It is just a matter of patience to generalize this argument to more than two successive intervals. Hence we have proved that the input and output processes are reverse processes. From the generalized version of (2.3) it now immediately follows that the input and output processes are identical if one of them is renewal. If on the other hand none of them are, (2.3) implies that they can not be identical. In particular we have the input and output processes to be identical iff the input process is renewal, a necessary and sufficient condition for which was established in Natvig (1975). Hence the argument on the output process of this section makes the one in Natvig (1975) superfluous. By setting $w = 0$ in (2.3) we realize that a single input and output interval is identically distributed being in agreement with Conolly and Chan (1976). The associated L.T. is given by (2.14)*.

We next treat the case where balking customers are ignored registering losses both as inputs and outputs. In order to prove relation (2.3) it easily follows that we must show a modified (2.4), with $\gamma_k^{(j)*}(\cdot)$ from (2.15)* where $\mu_m/(\mu_m+z)$ is replaced by 1 and m by $m+1$, being identical to

$$\pi_k \gamma_{\min(k+1,m)}^{(\min(k+1,m)-j)*}(w) \gamma_{\min(j+1,m)}^{(\min(j+1,m)-i)*}(z), \quad (2.6)$$

with $\gamma_k^{(j)*}(\cdot)$ from (2.2)* with $\lambda_m > 0$. This identity must hold for $0 \leq k \leq m$, $0 \leq j \leq \min(k+1,m)$, $0 \leq i \leq \min(j+1,m)$.

Note that both in the modified (2.4) and (2.6) the π -distribution is given by (2.7)* with m instead of $m-1$.

Considering $k < m$, $j < m$, we treat the same four cases as earlier in this section the argument being completely parallel. Next let $k = m$, $j < m$ and start with the case $j > 0$, $i > 0$.

Then (2.6) is identical to

$$\begin{aligned} & \prod_{v=1}^m \left(\frac{\lambda_v}{\mu_v}\right) \pi_0 \frac{\lambda_j}{\mu_j} \prod_{v=j}^m \frac{\mu_v}{\lambda_v + \mu_v + w} \frac{\lambda_i}{\mu_i} \prod_{v=i}^{j+1} \frac{\mu_v}{\lambda_v + \mu_v + z} = \\ & = \prod_{v=1}^i \left(\frac{\lambda_v}{\mu_v}\right) \pi_0 \prod_{v=j}^m \frac{\lambda_v}{\lambda_v + \mu_v + w} \frac{\mu_{j+1}}{\lambda_{j+1}} \prod_{v=i}^{j+1} \frac{\lambda_v}{\lambda_v + \mu_v + z} , \end{aligned}$$

which is precisely the modified version of (2.4). This reduction is easily modified for the three other cases ($j > 0$, $i = 0$), ($j = 0$, $i = 1$) and ($j = 0$, $i = 0$). Now consider $k = m-1$, $j = m$ and start with the case $i > 0$.

Then (2.6) equals

$$\begin{aligned} & \prod_{v=1}^{m-1} \left(\frac{\lambda_v}{\mu_v}\right) \pi_0 \frac{\lambda_m}{\lambda_m + \mu_m + w} \frac{\lambda_i}{\mu_i} \prod_{v=i}^m \frac{\mu_v}{\lambda_v + \mu_v + z} = \\ & = \prod_{v=1}^i \left(\frac{\lambda_v}{\mu_v}\right) \pi_0 \frac{\mu_m}{\lambda_m + \mu_m + w} \prod_{v=i}^m \frac{\lambda_v}{\lambda_v + \mu_v + z} , \end{aligned}$$

again being the modified version of (2.4). For the other case $i = 0$, the argument is easily modified. Finally, consider $k = m$, $j = m$. This situation is very parallel to the one above and is left to the reader.

Having established (2.3), the rest of the argument from the first case of this section still applies. In particular we have the L.T. of the p.d.f. of the input and output interval given by (3.5)*.

3. AN ATTEMPT TO CORRECT AN APPARENT MISTAKE IN VENTER AND SWANEPOEL (1971) AND CORRECTIONS OF TWO ERRONEOUS COMMENTS IN NATVIG (1975)

Venter and Swanepoel (1971) start by considering a discrete time birth-and-death process $\{X(t)\} t = 0, 1, 2, \dots$, i.e., a

discrete time Markov chain on $\{0,1,2,\dots\}$ with one step stationary transition matrix

$$\begin{bmatrix} r_0 & p_0 & 0 & 0 & \dots & \dots \\ q_1 & r_1 & p_1 & 0 & \dots & \dots \\ 0 & q_2 & r_2 & p_2 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

It is assumed that $p_i > 0, i \geq 0, q_i > 0, i \geq 1$. Let M_1^B, M_2^B, \dots denote the moments of births and M_1^D, M_2^D, \dots the moments of deaths. Assuming $\{X(t)\}$ stationary, where the steady-state distribution is found given by

$$\hat{\pi}_n = \frac{\prod_{i=1}^n (p_{i-1}/q_i)}{(1 + \sum_{n=1}^{\infty} \prod_{i=1}^n (p_{i-1}/q_i))} = \prod_{i=1}^n (p_{i-1}/q_i) \hat{\pi}_0, \quad (3.1)$$

they claim the processes $\{M_n^B\}$ and $\{M_n^D\}$ to be identically distributed. This is a cornerstone in showing the input and output processes to be identical also in the continuous time case.

However, their claim seems to be wrong since we in the following will show

$$\begin{aligned} P(M_1^B = n, M_2^B = n+m) &= P(M_1^D = m, M_2^D = n+m), \\ m \geq 0, n \geq 0, \end{aligned} \quad (3.2)$$

indicating the reversibility of the two processes. For simplicity assume $r_i = 0, i \geq 0$ and $p_0 = 1$. We start with the case $m \geq 2, n \geq 2$. Then using (3.1)

$$\begin{aligned}
 & P(M_1^D = m, M_2^D = n+m) = \\
 & \sum_{k=1}^{\infty} \prod_k q_k \prod_{v=k-1}^{k+m-3} p_v q_{k+m-2} \prod_{v=k+m-3}^{k+m+n-5} p_v q_{k+m+n-4} = \\
 & = \sum_{i=m+n-4}^{\infty} \prod_0^{i+4-m-n} (p_v/q_v) \prod_{v=i+4-m-n}^{i+2-n} p_v q_{i+3-n} \prod_{v=i+2-n}^i p_v q_{i+1} = \\
 & = \prod_0^{m-2} p_v q_{m-1} \prod_{v=m-2}^{m+n-4} p_v q_{m+n-3} + \\
 & + \sum_{i=m+n-3}^{\infty} \prod_i p_i \prod_{v=i+5-m-n}^i q_v/p_v \prod_{v=i+4-m-n}^i p_v p_{i+2-n} q_{i+3-n} q_{i+1} = \\
 & = \prod_{m+n-4}^{m+n-3} p_{m+n-4} \prod_{v=m-1}^{m+n-3} q_v p_{m-2} \prod_{v=1}^{m-1} q_v + \\
 & + \sum_{i=m+n-3}^{\infty} \prod_i p_i \prod_{v=i+3-n}^{i+1} q_v p_{i+2-n} \prod_{v=i+5-m-n}^{i+3-n} q_v p_{i+4-m-n} = \\
 & = P(M_1^B = n, M_2^B = n+m) .
 \end{aligned}$$

With $m = 1, n = 2$ we get

$$\begin{aligned}
 & P(M_1^D = 1, M_2^D = 3) = \sum_{k=2}^{\infty} \prod_k q_k q_{k-1} p_{k-2} q_{k-1} = \\
 & = \sum_{i=0}^{\infty} \prod_i p_i q_{i+1} p_i p_{i+1} = P(M_1^B = 2, M_2^B = 3) ,
 \end{aligned}$$

the cases $m = 2, n = 1$ and $m = n = 1$ being completely parallel.

We finally correct two erroneous comments in Natvig (1975) (p.589) and start with apologizing for one on a reversibility argument by Daley (1975).

After finishing the work on Natvig (1975) we became aware that Daley (1975) has indicated how the result of Boes (1969) can be obtained by a reversibility argument. The process is represented

as $(Q(t), I(t))$ where $Q(t)$ is the system state at time t and $I(t)$ is a $\{0,1\}$ - valued flip-flop process that changes whenever an arrival balks (or is lost). Then it can be checked that $(Q(t), I(t))$ is a reversible Markov chain, from which the result of Boes (1969) immediately follows since we already know the input process to be Poisson (λ) . It is not necessary, as we claimed, to at least verify that the two-dimensional p -, π - and q -distributions are identical. His argument could easily be used in the more general third case mentioned in the introduction. Our argument seems, however, no longer, not much shorter as commented.

For the second case mentioned in the introduction we stated that the joint distribution of the input and output numbers during an interval is obviously not infinitely divisible, since lost customers are considered both as inputs and outputs. The statement is correct, the argument being somewhat unprecise. This case, however, does only differ from the first one when $s + N < \infty$. Then an argument of Shanbhag (1973) leads us immediately to our conclusion.

4. THE COMMON EXPECTATION OF THE INPUT AND OUTPUT INTERVAL

Before giving our deduction it should be stated that Conolly and Chan (1976) are also able to obtain higher order moments by recursive techniques for their specialization. It should also be admitted that their approach seems extendable to the model of this paper, though perhaps then involving far more algebra than in their special case where there are no losses.

We start by treating the case where both balking and lost customers are ignored knowing (2.14)* to give the L.T., $f^*(z)$, associated with a single input and output interval

$$f^*(z) = \sum_{i=0}^{m-1} \pi_i \left[\frac{\lambda_0}{\lambda_0+z} \prod_{v=1}^{i+1} \frac{\mu_v}{\lambda_v+\mu_v+z} + \sum_{j=0}^i \rho_{i+1-j} \prod_{v=i+1-j}^{i+1} \frac{\mu_v}{\lambda_v+\mu_v+z} \right] \quad (4.1)$$

Here the π_i 's are from the π -distribution given by (2.7)* and $\rho_i = \lambda_i/\mu_i$, $1 \leq i \leq m$. Remember that $\lambda_m = 0$. The expectation of interest, ET, is then found by differentiation

$$\begin{aligned} ET &= - f^{*'}(z) \Big|_{z=0} = \\ &= \sum_{i=0}^{m-1} \pi_i \prod_{v=1}^{i+1} \frac{\mu_v}{\lambda_v+\mu_v} \left[\frac{1}{\lambda_0} + \sum_{s=1}^{i+1} \frac{1}{\lambda_s+\mu_s} + \right. \\ &+ \left. \sum_{j=0}^i \rho_{i+1-j} \left(\prod_{v=1}^{i-j} \frac{\lambda_v+\mu_v}{\mu_v} \right) \sum_{s=i+1-j}^{i+1} \frac{1}{\lambda_s+\mu_s} \right] = \\ &= \sum_{i=0}^{m-1} \pi_i \prod_{v=1}^{i+1} \frac{1}{1+\rho_v} \left[\frac{1}{\lambda_0} + \sum_{s=1}^{i+1} \frac{1}{\lambda_s+\mu_s} (1+ \right. \\ &\left. \sum_{j=i+1-s}^i \left(\prod_{v=1}^{i+1-j} (1+\rho_v) - \prod_{v=1}^{i-j} (1+\rho_v) \right) \right] . \end{aligned}$$

Now applying (2.7)* we get

$$\begin{aligned} ET &= \pi_0 \sum_{i=0}^{m-1} \frac{1}{1+\rho_{i+1}} \prod_{v=1}^i \frac{\rho_v}{1+\rho_v} \left[\frac{1}{\lambda_0} + \sum_{s=1}^{i+1} \frac{1}{\lambda_s+\mu_s} \prod_{v=1}^s (1+\rho_v) \right] = \\ &= \pi_0 \left[\frac{1}{\lambda_0} \left(\sum_{i=0}^{m-2} \frac{1}{\rho_{i+1}} \prod_{v=1}^{i+1} \frac{\rho_v}{1+\rho_v} + \prod_{v=1}^{m-1} \frac{\rho_v}{1+\rho_v} \right) + \right. \\ &+ \left. \sum_{s=1}^m \frac{1}{\lambda_s+\mu_s} \prod_{v=1}^s (1+\rho_v) \left[\sum_{i=s-1}^{m-2} \frac{1}{\rho_{i+1}} \prod_{v=1}^{i+1} \frac{\rho_v}{1+\rho_v} + \prod_{v=1}^{m-1} \frac{\rho_v}{1+\rho_v} \right] \right] . \end{aligned}$$

Applying (2.6)* twice this reduces to

$$ET = \pi_0 \left[\frac{1}{\lambda_0} + \sum_{s=1}^m \frac{1}{\mu_s} \prod_{v=1}^{s-1} \rho_v \right] = \pi_0 / \lambda_0 p_0 ,$$

where p_0 is from the p -distribution given by (2.24)*. Using this once more along with (2.7)*, we finally get

$$ET = \left(\sum_{j=0}^{m-1} \lambda_j p_j^{-1} \right), \quad (4.2)$$

a result being in agreement with Conolly and Chan (1976) for $m = \infty$.

For the case where balking customers are ignored registering losses both as inputs and outputs, we apply (3.5)* rather than (2.14)*. The argument being completely parallel is left to the reader and gives (4.2) with $m - 1$ replaced by m .

For the case where $m = \infty$, we assume that λ_n and μ_n are such that the π -distribution exists, i.e., we claim the denominator of (2.7)* to be convergent. We further assume the existence of a $\delta > 0$ such that $f^*(z)$ given by (4.1) is uniformly convergent for $\text{Re } z > -\delta$. According to Theorem 3 (p.74) in Knopp (1945), repeated term by term differentiation of $f^*(z)$ is then allowed. In fact the new series are now uniformly convergent in an arbitrary closed subregion containing 0, thus implying the existence of moments of any order.

REFERENCES

- Boes, D.C. (1969). Note on the output of a queuing system.
J.Appl.Prob., 6, 459-461.
- Conolly, B.W. and Chan, J. (1976). Generalised birth and death
queueing processes: recent results. Private communication.
- Daley, D.J. (1975). Queueing output processes. Private
communication.
- Knopp, K. (1945). Theory of Functions, Part 1. Dover, New York.
- Milne, R.K. (1971). Stochastic analysis of multivariate point
processes. Ph.D. Thesis. Australian National University.
- Natvig, B. (1975). On the input and output processes for a general
birth-and-death queueing model. Adv. Appl. Prob., 7, 576-592.
- Reich, E. (1957). Waiting times when queues are in tandem.
Ann.Math.Statist., 28, 768-773.
- Shanbhag, D.N. (1973). Characterization for the queuing system
 $M/G/\infty$. Proc. Camb. Phil. Soc., 74, 141-143.
- Venter, J.H. and Swanepoel, J.W.H. (1971). Some properties of
birth and death processes. S. Afr. Statist. J., 5, 15-25.