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PROGRAMS ON HP-25 DESK CALCULATOR OF MATHEMATICAL FUNCTIONS FOR USE IN STATISTICS

by

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Corrections and changes to:

"PROGRAMS ON HP-25 DESK CALCULATOR OF MATHEMATICAL FUNCTIONS FOR USE IN STATISTICS." Statistical Research Report No.7, 1976.

None of the errors affect the programs or instructions for the use of the program, except perhaps that on page 37.

Page 5, line 5 f.a. read: $F_{\mu,\nu}(\lambda) = \nu Z_{\mu}(\lambda)/\mu Z_{\nu}$

Page 28, line 12 f.a. In place of $\sum_{i=o}^{\infty}$ read $\sum_{i=1}^{\infty}$

Page 28, after last line, add:

g and G are the gaussian density and gaussian integral respectively (see page 22) and Γ^* is given by

$$\Gamma_{\nu}^{*}(z) = \sum_{j=o}^{\frac{\nu}{2}-1} (\tfrac{z}{2})^{j} \, e^{-\frac{z}{2}}/j!$$

Page 37, line 10 f.b. In place of 41 STO + 3, read 41 STO + 2 .

Page 41: Replace the 5 first lines f.a. by:

STO	0	$b_n(j,p)$	STO	4	1-p,i
	1	$\frac{\mu+\nu}{2}$ +i-j		5	b
	2	$\frac{\mu}{2}$ -1+i, $B_n(j,p)$		6	$\lambda/2$
	3	f μ/ν		7	j , $\Sigma(1-K)b$

For comment on the double use of STO 2, STO 4 and STO 7 , see the last lines .....

CONTENTS

PREFACE

Revolutionary as the small desk electronic calculators, like the Hewlett Packard 25, are in the manual calculations, the first impression the present author got was that programming capacity was very limited. The programs presented by Hewlett Packard in its handbook seem to support that conclusion. To begin with the present author, therefore, concentrated on finding computing routines for important statistical functions which were mixtures of stages of running programs and manual treatments.

It therefore came as a distinct surprise when it was realized that functions like the hypergeometric distribution (p.12-13) eccentric or central; or the eccentric Fisher and Student distributions (p.37,29) could be handled by almost completely automized programs squeezed into the 49 program lines. Considering the extensive but insufficient tables and nomograms existing e.g. for the hypergeometric distribution, this is indeed important.

The running time for most of the programs presented here will be from seconds to a couple of minutes for the usual combinations of parameter values. When the programs are running the attention of the computer will usually not be needed. In some cases the running time will be longer. For the hypergeometric distribution with a population size of $N = 800$ the running time will be about 15 minutes to obtain one value, the other values will then be run off quickly. 2 - 8 minutes will usually be the running time for the eccentric Fisher distribution, but with both degrees of freedom = 200 and eccentricity = 120 it will take 2½ hours ! A drawback by the program for the Fisher distribution is that they assume one of the degrees of freedom to be even. Hence an interpolation is needed if both are odd.

In the hypergeometric distribution, with

$$h(x) = \frac{\binom{a}{x}\binom{N-a}{n-x}}{\binom{N}{n}} \qquad H(x) = \sum_{j=o}^{x} h(j) \quad ,$$

h(0) is first found by summing logarithms, keeping characteristic and mantissa seperated in different stores to obtain high capacity. Then $h(x)$ is found by recursion and $H(x)$ by summing. The eccentric hypergeometric distribution with odds ratio λ is found from

$$H(x;\lambda) = K(x;\lambda)/K(n,\lambda) \text{ , where } K(x;\lambda) = \sum_{j=o}^{x} h(j)\,\lambda^{j} \text{ .}$$

Only a slight correction of the program for the central distribution is needed.

The program for the eccentric Fisher distribution is based on the fact that the cumulative probability function can be written

$$E\ K_{\mu+2H,\nu}(f\ \tfrac{\mu}{\nu})$$

where $K_{\mu,\nu}(g)$ is the central cumulative distribution of the ratio between chi-squares with μ and ν degrees of freedom and H is Poisson distributed with $E\ H$ = eccentricity/2 .

A different method is used in the case of the eccentric Student distribution. Numerical integration is used on the integral expression involving the chi-square and the gaussian distribution.

Besides the above mentioned programs, which the present author found non-trivial to construct, other useful programs are presented, e.g. for the normal distribution, the binomial distribution, the eccentric chi-square distribution etc.

The program for the normal distribution is different from the one presented by Hewlett Packard in its handbook, and has the advantage of requiring no fixed storage in the register, leaving the stores free to be used for manual computations. On the other hand the running time for the handbook program is shorter. However, no program is really needed in this case. Even without the many good tables, the normal distribution is easily found by a simple routine as suggested below.

I. NOTATIONS

A. Binomial, Poisson and hypergeometric distributions.

X_{np} and X_λ are respectively binomially and Poisson distributed

$$Pr(X_{np}=x) = b_n(x;p) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$Pr(X_\lambda = x) = b(x;\lambda) = \frac{\lambda^x}{x!} e^{-\lambda}$$

$$B_n(x;p) = \sum_{j=o}^{x} b_n(j;p) \quad , \quad B(x;\lambda) = \sum_{j}^{x} b(j;\lambda) \ .$$

In two independent Bernoulli trial series of n and $N-n$ trials respectively, let p_1 be the probability of an event A in a trial in the first series and p_2 be the probability of H in a trial in the second series. Then the conditional probability of A occuring x times in the first series given that A occurs a times among all N trials, is

$$h(x;\lambda) = \binom{a}{x}\binom{N-a}{n-x} \lambda^x / \sum_{j} \binom{a}{j}\binom{N-a}{n-j} \lambda^j$$

$$= h(x) \lambda^x / \sum_{j=1}^{a} h(j)\lambda^j$$

where

$$h(x) = h(x;1) = \binom{a}{x}\binom{N-a}{n-x} / \binom{N}{n}$$

$\lambda =$ odds ratio $= \frac{p_1}{1-p_1} / \frac{p_2}{1-p_2} =$ eccentricity.

(The first Bernoulli series may be thought of as the sample of size n from the lot of size N. The bias of the sampling in favour of A is measured by λ.) We also introduce

$$H(x,\lambda) = \sum_{j=o}^{x} h(j;\lambda)$$

$$K(x;\lambda) = \sum_{j=o}^{x} h(j)\lambda^j \quad ; \quad H(x;\lambda) = K(x;\lambda)/K(a;\lambda) \ .$$

$H(x;\lambda)$ defines the eccentric hypergeometric distribution, central when $\lambda = 1$.

B. Normal, Student- chi-square and Fisher distribution.

If Y is normal $(0,1)$, then

$$Pr(Y \leq y) = G(y) \quad , \quad g(y) = G'(y) \ .$$

If $Z_\mu(\lambda)$ is chi-square distributed with μ degrees of freedom and eccentricity $\lambda = E\,Z - \mu$, then

$$Pr(Z_\mu \leq z) = \Gamma_\mu(z;\lambda) \quad , \quad \gamma_\mu(z;\lambda) = \frac{d}{dz}\Gamma_\mu(z;\lambda) \ .$$

If $\lambda = 0$, then we write for the central chi-square distribution,

$$Z_\mu(0) = Z_\mu \quad , \quad \Gamma_\mu(z;0) = \Gamma_\mu(z) \quad , \quad \gamma_\mu(z;0) = \gamma_\mu(z) \ .$$

If Y is normal $(\delta,1)$ and independent of Z_μ, then, $T_\mu(\delta) = Y\sqrt{\mu} / \sqrt{Z_\mu}$ is eccentric Student distributed with μ degrees of freedom and eccentricity δ, and

$$Pr(T_\mu(\delta) \leq t) = G_\mu(t,\delta) \ .$$

If $\delta = 0$ we write for the (central) Student distribution

$$G_\mu(t;0) = G_\mu(t) .$$

If $Z_\mu(\lambda)$ and Z_ν are chi-square distributed (as explained above) and independent, then

$$F_{\mu,\nu}(\lambda) = \sqrt{\nu}\, Z_\mu(\lambda) / \sqrt{\mu}\, Z_\nu$$

is eccentric Fisher-distributed with μ and ν degrees of freedom and eccentricity λ and we write

$$\Pr(F_{\mu,\nu}(\lambda) \leq f) = G_{\mu,\nu}(f;\lambda) .$$

If $\lambda = 0$ we have the (central) Fisher-distribution with μ and ν degrees of freedom and we write

$$F_{\mu,\nu}(0) = F_{\mu,\nu} \quad , \quad G_{\mu,\nu}(f;\lambda) = G_{\mu,\nu}(f) .$$

We shall also need

$$R_{\mu,\nu}(\lambda) = Z_\mu(\lambda) / Z_\nu$$

which is "eccentric chi-square-ratio-distributed", and we write

$$\Pr(R_{\mu,\nu}(\lambda) \leq r) = K_{\mu,\nu}(r;\lambda) .$$

We obviously have $G_{\mu,\nu}(f;\lambda) = K_{\mu,\nu}(f\frac{\mu}{\nu},\lambda)$. The reason for needing both K and G is that we shall use $K_{\mu',\nu}(f\frac{\mu}{\nu},\lambda)$ with $\mu' \neq \mu$.

II. THE BINOMIAL DISTRIBUTION

The programs are based upon

$$b_n(0,p) = (1-p)^n \quad , \quad b_n(j,p) = \frac{n-j+1}{j} \circ \frac{p}{1-p} \circ b_n(j-1,p)$$

$$B_n(j,p) = \sum_{i=0}^{j} b_n(i,p) \; .$$

A. Program for tabulation of B_n .

1	STO 3	11	g 1/x	21	RCL 0
2	≷	12	1	22	STO +7
3	STO 2	13	-	23	RCL 7
4	f y^x	14	STO 4	24	R/S
5	STO 0	15	RCL 2	25	1
6	STO 7	16	×	16	STO +1
7	R/S	17	RCL 1	27	STO -2
8	1	18	f pause	28	RCL 4
9	STO 1	19	÷	29	GTO 15
10	RCL 3	20	STO × 0		

Run the program by setting

n , enter , 1-p , R/S.

First stop displays $B_n(0,p)$.
Setting R/S repeatedly displays j when pausing and $B_n(j,p)$ when stopping in turn for $j=1,2,\ldots,n$.

If $b_n(j,p)$ is wanted, press RCL 0 after a stop where j has been displayed in the pause. The process can afterwards be continued by pressing R/S .

B. Program for single values of B_n .

Use the program above with the following alterations and additions:

18	g NOP
23	g NOP
24	GTO 30
30	RCL 5
31	RCL 1
32	f x $\geq$ y
33	GTO 35
34	GTO 25
35	RCL 7
36	GTO 00

Run the program by setting

x , STO 5 , n , enter , 1-p , R/S .

$B_n(0,p)$ is first displayed, pressing R/S again displays $B_n(x,p)$. Press RCL 0 to find $b_n(x,p)$.

Examples. (with "f fix 5"). For tabulation of $B_n(x,p)$ for n=100 , p=0.2 , use program A above and set 100, enter 0.8 , R/S. Repeat R/S.

The following values are displayed. (Not all displays are written down.)

x	$B_n(x,p)$	x	$B_n(x,p)$
0	$2.0370360 \cdot 10^{-10}$	15	0.12851
1	$5.2962936 \cdot 10^{-9}$	20	0.55946
2	$6.8317095 \cdot 10^{-8}$	25	0.91252
5	0.00002	30	0.99394
10	0.00570	35	0.99985

To find $B_{100}(20,0.2)$ use program B above and set 20, STO 5, 100, enter, 0.8, R/S . Repeat R/S after first stop to display 0.55946.

The register in the above programs is used as follows:

STO 0	$b_n(j,p)$	STO 4	$p/(1-p)$
STO 1	j	STO 5	x
STO 2	$n-j+1$	STO 6	not used
STO 3	$1-p$	STO 7	$B_n(j,p)$.

III. THE POISSON DISTRIBUTION

The programs are based on

$$b(j;\lambda) = \frac{\lambda}{j}\, b(j-1,\lambda) \quad , \quad b(0,\lambda) = e^{-\lambda} \quad , \quad B(j,\lambda) = \sum_{i=0}^{j} b(i,\lambda) \, .$$

A. Tabulation of B.

1	STO 2	9	STO + 1
2	CHS	10	RCL 1
3	g e^x	11	f pause
4	STO 0	12	STO ÷ 0
5	STO + 7	13	RCL 2
6	RCL 7	14	STO × 0
7	R/S	15	RCL 0
8	1	16	GTO 05

Run the program by setting

λ , R/S .

First stop displays $B(0,\lambda)$. Pressing R/S repeatedly displays,

j when pausing, and $B(j,\lambda)$ when stopping in turn for $j=1,2,\ldots$.

If $b(j,\lambda)$ is wanted, press RCL 0 after a stop where j has been displayed in the pause. Continue the process afterwards by pressing R/S .

B. Program for single values of B .

1	STO	3	8	RCL	3	15	STO ÷	0
2	R ↓		9	RCL	1	16	RCL	2
3	STO	2	10	f $x \geq y$		17	STO ×	0
4	CHS		11	GTO	20	18	RCL	0
5	g e^x		12	1		19	GTO	07
6	STO	0	13	STO	+1	20	RCL	7
7	STO	+7	14	RCL	1	21	GTO	00

Run the program by setting

λ , enter, x , R/S .

$B(x,\lambda)$ is then displayed. Press RCL 0 to find $b(x,\lambda)$.

Examples: (with f fix 5).

For tabulation of $B(j,\lambda)$ for $\lambda = 4.68$ use program A above and set 4.68 , R/S . Repeat R/S . The following values are displayed

0	0.00928	6	0.80732
1	0.05270	7	0.89785
2	0.15432	8	0.95081
3	0.31284	9	0.97835
4	0.49831	10	0.99124
5	0.67191	11	0.99672
		12	0.99886

To find B(5, 4.68) directly use program B and set 4.68 , enter , 5 , R/S .

B = 0.67191 is displayed. RCL 0 gives b = 0.17360 .

The registers in the above programs are used as follows:

STO 0	$b(j,\lambda)$	STO 3	x
STO 1	j	STO 4-6	not used
STO 2	λ	STO 7	$B(j,\lambda)$

IV THE (ECCENTRIC) HYPERGEOMETRIC DISTRIBUTION

The programs for

$$H(x) = \sum_{j=o}^{x} h(j) \quad , \quad \text{where} \quad h(x) = \binom{a}{x}\binom{N-a}{n-x} / \binom{N}{n}$$

and

$$H(x;\lambda) = \sum_{j=o}^{x} \binom{a}{j}\binom{N-a}{n-j} \lambda^j \Big/ \sum_{j=o}^{n} \binom{a}{j}\binom{N-a}{n-j} \lambda^j$$

are based on

$$h(0) = \frac{(N-a)\dots(N-a-n+1)}{N(N-1)\dots(N-n+1)} = 10^{\sum_{o}^{n-1} \log(1-\frac{a}{n-j})} ,$$

$$h(j+1) = \frac{(a-j)(n-j)}{(j+1)(N-a-n+j+1)} h(j) ;$$

$$H(x;\lambda) = K(x,\lambda) / K(n;\lambda)$$

where

$$K(x,\lambda) = \sum_{j=o}^{x} h(j) \lambda^j .$$

(By using the logarithmic form of $h(0)$, we obtain almost "unlimited" capacity.)

A. Tabulation of the central hypergeometric distribution.

1	+	13	f x=y	25	g 10^x	37	÷
2	STO 5	14	GTO 18	26	STO 6	38	STO × 6
3	÷	15	RCL 2	27	STO +7	39	RCL 5
4	CHS	16	≷	28	RCL 7	40	f pause
5	1	17	GTO 03	29	f pause	41	1
6	+	18	RCL 2	30	R/S	42	STO -2
7	f log	19	STO -4	31	RCL 2	43	STO -3
8	STO +0	20	1	32	RCL 3	44	STO +5
9	1	21	STO +4	33	×	45	STO +4
10	STO -5	22	STO 5	34	RCL 5	46	RCL 6
11	RCL 4	23	RCL 0	35	RCL 4	47	GTO 27
12	RCL 5	24	g frac	36	×	48	GTO 00

1. Be sure that $N \geq a+n$. If not, then change notations; let $a \to N-a$ if $a \geq n$ and $n \to N-n$ if $n > a$.

2. Run the program by setting:

a , STO 2 , n , STO 3 , N-n , STO 4 , R/S .

3. $H(0) \circ 10^p$ is first displayed. Pressing RCL 0 displays the integer part as $-p$. Hence $H(0)$ is found.

4. Setting R/S repeatedly displays:

j when pausing, and $H(j) \circ 10^p$ when stopping , in turn for j=1,2,... . Hence j and H(j) are given for j=0,1,2,.. . Finally H(j) = 1.00 .... for large j .

Example 1 (with "f fix 4").

N = 8 , a = 3 , n = 5 . Set 3, STO 2, 5 , STO 3 , 3 , STO 4 , R/S , displays $0.1786 = H(0) \circ 10^p$. RCL 0 gives $-p = -1$, hence H(0) = 0.01786. Setting R/S repeatedly gives :

j	$H(j)10^1$		j	H(j)
1	2.8571		0	0.01786
2	8.2143	, hence	1	0.28527
3	10.0000		2	0.82143
			3	1.00000

B. Tabulation of the eccentric hypergeometric distribution.

Use the same program as above, but with the following alterations and additions:

46	RCL	1
47	STO ×	6
48	RCL	6
49	GTO	27

(Of course this program could have been used also under A , setting $\lambda = 1$.)

Use the same procedure as above 1 - 4 , but under 2 set

λ , STO 1, a, STO 2, n, STO 3, N-n, STO 4, R/S .

$K(j,\lambda)\circ 10^p$ is now displayed (p is not needed). Continue until $K(j,\lambda)\circ 10^p$ is stabilized. It can then be set equal to $K(n,\lambda)\circ 10^p$. Finally we find

$$H(j,\lambda) = K(j,\lambda)\circ 10^p / K(n,\lambda)\circ 10^p \ .$$

Example 2 (with f fix 4). N = 8, a = 3, n = 5, (as above) and $\lambda = 2$.

j	$K\circ 10^p$	H
0	0.1786	0.0043
1	5.5357	0.1342
2	26.9643	0.6537
3	41.2500	1.0000
4	41.2500	

C. Significance limits for testing $\lambda = 1$.

For significance testing $H(x)$ from program A , where x is the observed value; is used and checked against $1-\frac{\epsilon}{2}$, $\frac{\epsilon}{2}$ (ϵ=level).

However, for the purpose of finding the power critical values of the observed x may be needed.

Hence to find the smallest x such that $H(x) > \alpha$.

Use the program A above with the following alterations and changes:

24	g NOP	40	g NOP
29	RCL 1	48	f x≥y
30	GTO 48	49	GTO 31

Use the same procedure as above 1 - 3, but under 2 set

α STO 1, a, STO 2, n, STO 3, N-m, STO 4, R/S .

When the machine stops, α is displayed. Then press RCL 5 and

x = RCL 5 - 1 .

Example 3. $N = 26$, $a = 14$, $n = 10$, $\alpha = 0.95$.

Set 0.95, STO 1, 14, STO 2, 10, STO 3, 16, STO 4, R/S.

Then RCL 5 = 8. Hence $x = 7$.

If $\alpha = 0.05$ in STO 1 we get RCL 5 = 4, hence $x = 3$. This means that the critical values with level 0.90 are those < 2 or > 7 .

The register in the above program is used as follows:

STO 0	log h(0)	STO 4	N-n , N-n-a+j+1
STO 1	λ	STO 5	N, j
STO 2	a - j	STO 6	h(j)
STO 3	n - j	STO 7	H(j)

Further examples.

Example 4. (Program A.) $N = 400$, $a = 50$, $n = 100$, $\lambda = 1$.
The machine ran for 2 minutes to obtain

$$H(0)\cdot 10^{p} = 0.182716045 \;,\; -p = -6 \text{ (RCL 0)} .$$

Repeatedly pressing R/S gives:

j	$H(j)\cdot 10^{p}$	H(j)
0	0.18272	
1	3.82248	0.000004
2	38.85519	0.000039
3	255.97489	0.000256
4	1230.23547	0.001230
5	4604.61563	0.004605
6	13996.20	0.013996
7	35587.97	0.035588
8	77422.03	0.077422
9	146768.58	0.146769
10	246280.87	0.246281
15	852302.02	0.852302
20	996382.26	0.996382
25	999990.55	0.999991
30	1000000.00	1.000000
31	Rep.	

After H(10) had been displayed 30 R/S was changed to 30 f pause (by setting PRGM, BST, BST, f pause, SST, RUN), in order to save time. Then running was stopped at first pause after 10, 15, etc., had been displayed.

Example 5. $N = 800$, $a = 300$, $n = 400$. First display, after 9 minutes, was:

$$H(0)\cdot 10^{p} = 0.10858 \;,\; -p = -131 .$$

In order not to exhaust exponential capacity $H(0)\cdot 10^{131}$ must now be divided by some power of 10 , say 10^{90} and then stored in STO 6 and 7 . (Set EEX, 90, CHS, X, STO 6, STO 7.)
Hence we now tabulate $H(j)\cdot 10^{p'}$ where $p' = p-90 = 41$.

(Unless this is done the machine will be OF with $H(72)\cdot 10^p = 7.75\cdot 10^{99}$.)

Repeatedly pressing R/S we get:

j	H(j)	j	H(j)
0	$1.0858 \cdot 10^{-132}$	141	0.10719
1	$1.2911 \cdot 10^{-129}$	142	0.13665
2	$7.5570 \cdot 10^{-127}$	145	0.25552
123	$5.2706 \cdot 10^{-5}$	150	0.52910
124	$9.5400 \cdot 10^{-5}$	155	0.78910
125	$1.6914 \cdot 10^{-4}$	160	0.93747
130	$2.1802 \cdot 10^{-3}$	165	0.98825
134	0.011752	170	0.99864
140	0.082609	180	0.999996
		190	0.99999995 is repeated.

Example 6. (Program B.) Two Bernoulli sequences I and II containing 16 and 80 trials respectively, are compared with respect to the occurance of an event A in each trial. The result is:

	A	$\bar{A}$	Sum	% of A	Odds
Sequence I	x=6	10	n=16	37.5	0.6
Sequence II	16	64	80	20	0.25

a = 22

Confidence interval for odds ratio = λ is wanted. (Let p_1 and p_2 be the probabilities of A by a trial in sequence I and II respectively.

Then $\lambda = \frac{p_1}{1-p_1} / \frac{p_2}{1-p_2}$.) A point estimate of λ is $0.6/0.25 = 2.4$. We want to find an upper limit λ_2 such that $H(x,\lambda_2) \geq 0.95$ and a lower limit λ such that $H(x,\lambda_2) \leq 0.05$. We use program B and try out different values of λ .

λ	K(x,λ)	K(n,λ)	H(x,λ)
1	9.6271	10.0000	0.96271
1.1	13.5914	14.3391	0.9479
1.08	12.7011	13.3550	0.9510
<u>1.087</u>	13.0068	13.6923	0.94994
4.0	5821.50	21638.64	0.2690
7.0	129896.	2391983.	0.0543
7.5	192287.		0.0422
7.2	152429.		0.0491
7.18	150039.		0.0496
7.17	148855.		0.0498
<u>7.16</u>	147680.	2949665.	0.05007

Hence $1.087 < \lambda < 7.16$.

V COMBINATORIAL FUNCTIONS

Program A for n! is based upon

$$n! = n(n-1)\dots 2.1$$

in that order of the factors. It can only be used for $n \leq 69$.

Program B for n! is based upon

$$n! = 10^{\sum_{j=1}^{n} \log(n-j+1)}$$

in that order of the sum. In order not to exhaust capacity A and p in $n! = A\ 10^p$ are stored separately.

<u>Program B for n! is recommended as all round useful for almost all practical purposes.</u>

Program C for n! is based upon

$$n! = 1.2\dots n$$

in that order of the factors. Whenever 1,2... i ; i=1,2,...

surpasses 10^{98j}, it is divided by 10^{98}. This guarantees high degree of accuracy and large capacity.

Program A for $\binom{n}{m}$ is based upon

$$\binom{n}{m} = \frac{n}{m} \circ \frac{n-1}{m-1} \circ \dots \circ \frac{n-m+1}{1}$$

and can be used for n up to 300 .

Program B for $\binom{n}{m}$ is based upon

$$\binom{n}{m} = 10^{\sum_{i=o}^{m-1} [\log(n-i)-\log(m-i)]}$$

In order not to exhaust capacity, A and p are stored separately.

n! program A .

1	STO 1	6	1
2	1	7	-
3	f x=y	8	STO ×1
4	GTO 10	9	GTO 02
5	≷	10	RCL 1

Assume $n \leq 69$. Set

n , R/S .

n! is displayed. Example: 10! = 3628800 takes a few seconds. $60! = 8.3209871 \circ 10^{81}$ takes 30 seconds.

n! program B.

1	f log	6	1	11	f int
2	g x=0	7	-	12	STO 1
3	GTO 09	8	GTO 01	13	-
4	STO +0	9	RCL 0	14	g 10^x
5	f last x	10	Enter		

Set

n , R/S .

In $n! = A \circ 10^p$, A is displayed and p obtained by pressing RCL 1 .

Examples: n = 5 . 1.20 is displayed , RCL 1 = 2 . Hence $5! = 1.20 \circ 10^2 = 120$. $10! = 3.6288 \circ 10^6 = 3628800$. $100! = 9.332622519 \circ 10^{157}$ after 1 min. and 10 seconds. See under program C for more accurate value of 100!

n! Program C .

1 STO 3	7 8	13 RCL 2	19 STO ÷1
2 1	8 f x <y	14 STO ×1	20 9
3 STO 1	9 GTO 19	15 f x <y	21 8
4 RCL 1	10 1	16 GTO 04	22 STO +0
5 EXX	11 STO +2	17 RCL 1	23 GTO 10
6 9	12 RCL 3	18 GTO 00	

(Instead of 20-23, one could use: 20 f log , 21 STO +0 , 22 GTO 10 .)

Set

n , R/S .

In $n! = A \circ 10^p$, A is displayed and p obtained by RCL 0 .

Examples: n = 100 . After 1 min. 10 seconds, $9.3326216 \circ 10^{59}$ is displayed. RCL 0 = 98. Hence $100! = 9.3326216 \circ 10^{157}$. Likewise $140! = 1.3462013 \circ 10^{45+196}$.

$\binom{n}{m}$ Program A.

1	Enter	7	R ↓	13	1
2	1	8	1	14	-
3	STO 0	9	f x=y	15	RCL 1
4	R ↓	10	GTO 19	16	1
5	STO ×0	11	R ↓	17	-
6	STO 1	12	STO ÷0	18	GTO 05
				19	RCL 0

Can be used for $n \leq 300$. (Let preferably $2m \leq n$ to save time.)

Set

m , enter, n , R/S .

$\binom{n}{m}$ is displayed.

Examples: $\binom{10}{3}$: Set 3 enter 10 , gives 120.
$\binom{52}{13} = 6.3501356 \cdot 10^{11}$ after a few seconds $\binom{300}{150} =$
$= 9.3759703 \cdot 10^{88}$ after 2 min. 40 seconds.

$\binom{n}{m}$ Program B.

1	STO 1	8	STO-0	15	GTO 01
2	f log	9	f last x	16	RCL 0
3	STO +0	10	1	17	Enter
4	R ↓	11	-	18	f int
5	f log	12	RCL 1	19	STO 1
6	g x=0	13	1	20	-
7	GTO 16	14	-	21	g 10^x

Can be used for very high n . (Let preferably $2m \leq n$ to save time.) Set

m , enter , n , R/S .

A in $\binom{n}{m} = A \cdot 10^p$ is displayed. RCL 1 = p .

Example: $\binom{52}{13}$: Set 13 , enter, 52. Then 6.35014 is displayed. RCL 1 = 11. Hence

$$\binom{52}{13} = 6.35014 \cdot 10^{11} .$$

VI NORMAL DISTRIBUTION

Manual routines or programs are not so much in need for the normal distribution, since convenient tables are easily available.

However, if you have the small calculator on your table anyhow, the manual routines with Simpson's formula or continuous fraction are easy to use (even on the old-fashioned slide ruler), and to memorize.

A complete program is also given below.
It has the advantage over the program given in the manual from the factory that no coefficients are stored permanently in the register. The register is at the disposal for manual calculations on the side.

The program is based on the continuous fraction formula (line 1-27) and Taylor's formula

$$G(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}(x - \frac{x^3}{6} + \frac{x^5}{40})$$

(line 28-43).

A. Manual routines for $G(x) = \int_{-\infty}^{x} g(v)dv$, where $g(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.

For $0 \leq x \leq 1.5$, Simpson's formula with three terms is used on $G(x) - 0.5 = \int_0^x g(v)dv$.

For $x \geq 1.5$ the continuous fraction formula

$$1 - G(x) = g(x) \cfrac{1}{x+\cfrac{1}{x+\cfrac{2}{x+\cdots\cfrac{n-1}{x+\frac{n}{x}}}}}$$

is used.

(i) $0 \leq x \leq 1.5$. Use

$$G(x) = (y^4+4y+1)\frac{x}{6\sqrt{2\pi}} + \frac{1}{2} \quad \text{where} \quad y = e^{-\frac{x^2}{8}} .$$

Compute first y , then $P = y^4 + 4y + 1$ and finally $G(x)$. Use register to store y and terms of P .

Gives almost 4 correct decimals.

(ii) $x \geq 1.5$. (Manual.) Use the continuous fraction formula with $n=5$. Write down the fraction and start from the bottom by dividing 5 by x (store and recall x). This amounts to using the recursion

$$Q_o = \frac{n}{x} \quad , \quad Q_j = \frac{n-j}{x+Q_{j-1}} \quad ; \; j=1,2,\ldots,n-1$$

$$1 - G(x) = Q_n \circ g(x) .$$

Gives 3 correct decimals with $n = 5$ and 4 correct decimals with $n = 10$.

[Under (i) five term Simpson formula

$$G(x) = (y^{16} + 4y^9 + 2y^4 + 4y + 1)\frac{x}{12\sqrt{2\pi}} + 0.5$$

$$y = e^{-\frac{x^2}{32}}$$

would give 4 correct decimals for $0 \leq x \leq 2.5$].

B. Program for $G(x) = \int_{-\infty}^{x} g(v)dv$, where $g(v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}$.

1	STO 1	12	GTO 15	23	2	34	÷
2	$\lessgtr$	13	RCL 0	24	×	35	6
3	STO 2	14	GTO 04	25	f $\sqrt{\ }$	36	g 1/x
4	÷	15	RCL 0	26	÷	37	-
5	STO 0	16	g 1/x	27	GTO 00	38	×
6	RCL 2	17	RCL 2	28	STO 0	39	1
7	STO +0	18	g x^2	29	g x^2	40	+
8	1	19	g e^x	30	Enter	41	RCL 0
9	STO -1	20	f $\sqrt{\ }$	31	Enter	42	×
10	RCL 1	21	÷	32	4	43	GTO 22
11	g x=0	22	g π	33	0	44	GTO 00

There are two types of settings

I^n : GTO 00, x, enter, n, R/S .

II : GTO 28, x, R/S .

For $0 < x \leq 0.75$ use II

For $x \geq 0.75$ use I^{30}

That gives "almost" 4 correct decimals.(6 correct decimals for $x \leq 0.25$ or ≥ 1.25 . If 5 correct decimals are wanted for all x then proceed as follows;

$0 < x \leq 0.5$ use II
$0.5 \leq x \leq 0.75$ use I^{200}
$0.75 \leq x \leq 1.00$ use I^{100}
$x \geq 1.00$ use I^{30} .

Any degree of accuracy can be obtained for $x \geq 0.25$ by using I^n with n sufficiently large ; e.g. n=800 for x close to 0.25 gives 6 correct decimals

Running time = n/100 minutes.)

The register has been used as follows,

Program lines 1-27:

STO 0 Q_j
STO 1 n-j
STO 2 x
STO 3-7 not used

Program lines 28-43:

STO 0 x
STO 1-7 not used

VII THE CHI-SQUARE DISTRIBUTION (eccentric).

If $X_1, X_2, \ldots, X_\nu$ are independent normal $(\delta_i, 1)$ then $Z = \sum_1^\nu X_i^2$ is distributed as the eccentric chi-square distribution with ν degrees of freedom and eccentricity $\lambda = \Sigma\,\delta_i^2$.

The program below is based upon

$$\Pr(Z > z) = 1 - \Gamma_\nu(z,\lambda) = \sum_{i=0}^{\infty} b(i,\tfrac{\lambda}{2})(1-\Gamma_{\nu+2i}(z))$$

where b is the Poisson density,

$$b(i,\varkappa) = \frac{\varkappa^i}{i!} e^{-\varkappa} \quad , \quad \Gamma_\nu(z) = \Gamma_\nu(z,0)$$

and where ν is presumed even.

Since

$$1 - \Gamma_\nu(z) = B(\tfrac{\nu}{2} - 1, \tfrac{z}{2})$$

where B is the Poisson cumulative distribution function, we have

$$1 - \Gamma_\nu(z,\lambda) = \sum_{i=0}^{\infty} b(i,\tfrac{\lambda}{2}) \sum_{j=0}^{\frac{\nu}{2}-1+i} (\tfrac{z}{2})^j\, e^{-\frac{z}{2}}/j\,!$$

which is the basis for the program below.

For ν odd, interpolation is proposed.

(We have also for ν odd

$$1-\Gamma_\nu(z,\lambda) = \sum_{i=0}^{\infty} b(i,\tfrac{\lambda}{2}) \sum_{j=0}^{\beta-1+i} (\tfrac{z}{2})^{\beta+i-j-\frac{1}{2}} \frac{e^{-\frac{z}{2}}}{(\beta+i-j-\frac{1}{2})!} + 2(1-G(\sqrt{z}))$$

where $\beta = \frac{\nu-1}{2}$. A program could be worked out for the sum, relying on tables (or manual procedure) for the last term. This has not been done below).

A. Program chi-square distribution (eccentric) $\Gamma_\nu(z;\lambda)$.

(= $Pr(Z_\nu(\lambda) \leq z)$, $EZ_\nu(\lambda) = \nu+\lambda$). Con Amore stopping.

1	STO 1	12	STO +2	23	STO × 0	34	STO -7
2	CHS	13	RCL 3	24	RCL 0	35	1
3	g e^x	14	RCL 7	25	GTO 12	36	STO +3
4	STO 5	15	f x≥y	26	RCL 2	37	STO +6
5	R ↓	16	GTO 26	27	STO -2	38	RCL 6
6	STO 3	17	1	28	RCL 5	39	STO ÷5
7	R ↓	18	STO +7	29	×	40	RCL 1
8	STO 4	19	RCL 7	30	STO +7	41	STO ×5
9	CHS	20	f int	31	f pause	42	RCL 4
10	g e^x	21	STO ÷0	32	f pause	43	GTO 09
11	STO 0	22	RCL 4	33	RCL 3		

If ν is an even number, set

$\frac{z}{2}$, enter, $\frac{\nu}{2} - 1$, enter, $\frac{\lambda}{2}$, R/S .

Observe the window display during pause until the figure begins decreasing (it may first increase) and becomes insignificantly small. Set R/S during pause,

RCL 7, fractional part = $1 - \Gamma_\nu(z;\lambda)$
RCL 6 = m = number of terms in series for Γ
RCL 5 = b_m = Poisson factor in last term .

Inaccuracy $< b_m \frac{\lambda/2}{m-\lambda/2}$.

(For derivation see page 41).

If ν is an odd number then use either linear interpolation between $\nu-1$ and $\nu+1$ or more accurate interpolation by means of

$$f(\nu) = 1.125\ \tfrac{1}{2}\,(f(\nu-1) + f(\nu+1)) - 0.125\ \tfrac{1}{2}(f(\nu-2)+f(\nu+2)) .$$

In the central case (λ=0), it may be covenient to replace 31 f pause by 31 GTO 00. Setting as above with $\lambda = 0$. The result is displayed in the window.

B. Program for the Chi-square distribution (eccentric).

Automatic stopping after m terms.

The program under A is used with the following changes and additions (for m=30 terms),

31	GTO 44
32	g NOP
44	3
45	0
46	RCL 6
47	f x≥y
48	GTO 00
49	GTO 33

Setting and RCL as under A .

Example. (Program under A). $z = 18.307$, $\nu = 10$, $\lambda = 3.71$.
The window displays in turn

0.00782 , 0.03096 , 0.05199 , 0.05097 , 0.03362 , 0.01624 , 0.00609 , 0.00185 , 0.00047 , 0.00010 , 0.000021 , $3.4 \circ 10^{-6}$, $5.4 \circ 10^{-7}$. R/S was pressed when last figure was in display. RCL 7 gives 16.20015 , hence

$$1 - \Gamma_{10}(18.307 \; ; 3.71) = 0.20015 .$$

RCL 6 gives m=12 terms and RCL 5 = $5.4 \circ 10^{-7}$.

Hence

$$\text{Inaccuracy} < 5.4 \circ 10^{-7} \frac{3.7}{12-3.7} < 3 \circ 10^{-7} .$$

The register is used as follows,

STO	0	$b(j,z/2)$	4	$3/2$
	1	$\lambda/2$	5	$b(j,\lambda/2)$
	2	$B(j,z/2)$	6	i
	3	$\nu/2-1+i$	7	j, $\Sigma\, b(1-\Gamma)$

Note the double use of STO 7 , which is possible since j is an integer whereas $0 \leq \Sigma < 1$. (Hence RCL 7 has to be followed by either f int or g frac in the program.)

VIII. THE STUDENT DISTRIBUTION (eccentric)

If X, Z are independent, X normal $(\delta,1)$ and Z chi-square distributed with ν degrees of freedom. Then $T = X\sqrt{\nu}/\sqrt{Z}$ is Student distributed with ν degrees of freedom and eccentricity δ.

We have for $t > 0$

$$Pr(T>t) = 1 - G_\nu(t,\delta) = \int_0^\infty \Gamma_\nu\left(\left(\frac{x}{t}\right)^2\nu\right)g(x-\delta)dx ,$$

which may also be written

$$1 - G_\nu(t,\delta) = G(\delta) - \int_0^\infty \Gamma_\nu^*\left(\left(\frac{x}{t}\right)^2\nu\right)g(x-\delta)dx = G(\delta) - I$$

where $\Gamma_\nu^* = 1 - \Gamma_\nu$. For the integral we have by Simpson's formula,

$$I = K\left[\sum_{i=0}^{\infty} A_i\Gamma_\nu^*(2ci^2)g\left(\frac{i}{m}-\delta\right) + g(\delta)\right]$$

where m is an integer, $c = \nu/2(mt)^2$, $K = 1/3m$ and $A_1 = 4$, $A_2 = 2$, $A_3 = 4$ etc.

The program below is based on this formula.

Program for Student (eccentric) distribution $1 - G_\nu(t,\delta)$.

1	1	13	STO +1	25	RCL 5	37	4
2	STO +0	14	RCL 3	26	GTO 13	38	RCL 0
3	RCL 0	15	RCL 1	27	g frac	39	2
4	g x^2	16	f $x \geq y$	28	RCL 0	40	÷
5	RCL 2	17	GTO 27	29*	(m=) 8	41	g frac
6	×	18	1	30	÷	42	f y^x
7	STO 6	19	STO +1	31	RCL 4	43	×
8	CHS	20	RCL 1	32	-	44	STO +7
9	g e^x	21	f int	33	g x^2	45	f pause | g x=0
10	2	22	STO ÷5	34	g e^x	46	f pause | GTO 00
11	÷	23	RCL 6	35	f $\sqrt{}$	47	0
12	STO 5	24	STO ×5	36	÷	48	STO 1
						49	GTO 01

(*) Line 29 m=8 means that equidistance in Simpson's formula is $\frac{1}{m} = 0.125$.)

If ν even, set

$c = \frac{\nu}{2(mt)^2}$(e.g. m=8), STO 2, $\frac{\nu}{2} - 1$, STO 3, δ, STO 4, R/S .

Observe the figure displayed in window until insignificant, then press R/S .

$$\text{RCL } 7 = \Sigma\, A\Gamma^* \quad , \quad \text{RCL } 0 = \text{number of terms.}$$

We can now compute

$$1 - G_\nu(t,\delta) = G(\delta) - \frac{4(\text{RCL } 7) + e^{-\frac{\delta^2}{2}}}{3m\sqrt{2\pi}}$$

By changing program line 45-46 above to 45 g x=0 , 46 GTO 00 , we obtain automatic stopping, sometimes, however, with long running time.

If ν odd, then use either linear interpolation between $\nu-1$ and $\nu+1$ or more accurate interpolation by means of

$$f(\nu) = 1.125\tfrac{1}{2}(f(\nu-1)+f(\nu+1)) - 0.125\tfrac{1}{2}(f(\nu-2)+f(\nu+2)) .$$

If high degree of accuracy is wanted, then we may use m=20 instead of m=8. Use program lines 29 2, 30 0, followed by the program lines 30-44 above shifted to 31-45 , then 46-49 as before.

Example 1. (m=8), ν=6, t = 0.7176 (level = 0.25), δ = 0.25, $G(\delta)$ = 0.59871 has been taken from table.

With running time 1 min. 10 sec. the last term = $5.16 \cdot 10^{-7}$, RCL 7 = 3.74151, $1 - G_\nu(t,\delta)$ = 0.33382.

With eccentricity = - 0.25 we get $1 - G_\nu(t,\delta)$ = 0.17886. Hence $1 - G_\nu(t,\delta) + 1 - G_\nu(t,-\delta)$ = 0.51268, agreeing precisely with what can be found from program for the Fisher distribution.

$\delta = 0$ we get $1 - G_\nu(t,0) = 0.249988$ agreeing well with table value = 0.25 .

Example 2. (m=20). ν=100, t = 1.9840 (level = 0.025), δ=4. From tables $G(\delta)$ = 0.9999683. With 45 minutes running time the last term is 0 and RCL 7 = 0.851464576 , $1 - G_\nu(t,\delta)$ = 0.977320388. We also find $1 - G_\nu(t,-\delta)$=0.000000030. Adding the two results gives 0.977320418, whereas the program for the Fisher distribution gives 0.977320430.

The register in the program above has been used as follows:

STO 0	i	STO 4	δ
1	B, j	5	b
2	c	6	ci^2
3	$\frac{\nu}{2} - 1$	7	$\Sigma A \Gamma_\nu^*$

b and B is the density and cumulative Poisson function, respectively.

Some comments on the program.

For the double use of one STO see comment on the last lines of p.27.

By means of "14 RCL 3, 15 RCL 1, 16 f $x \geq y$, 17 GTO 27, 27 g frac" ; $B(\frac{\nu}{2} - 1, ci^2)$ is determined by $j + B(j, ci^2)$ being smallest possible, but $\geq \frac{\nu}{2} - 1$. This gives $j = \frac{\nu}{2} - 1$ if $0 \leq B < 1$. However, due to inaccuracy in the 9-th decimal B may be = 1 instead of < 1 . Then we would get $j = \frac{\nu}{2} - 2$ and B = frac (RCL 1) = 0 , which is seriously wrong and may lead to gross errors.

To avoid this calamity B has been divided by 2, see program lines 10, 11, 13.

The Simpson coefficients $A_1 = 4$, $A_2 = 2$ etc. had to be reproduced with as few program lines as possible. This is done by program lines 37-42 based on

$$\frac{1}{2}A_i = 4^{\text{fraction of } \frac{i}{2}} = \begin{cases} \sqrt{4} = 2 & \text{if } i \text{ odd} \\ 4^0 = 1 & \text{if } i \text{ even} \end{cases}$$

which requires the 6 program lines we can afford to use.

$$\frac{1}{4}A_i = \frac{1}{2} + \text{fraction of } \frac{i}{2}$$

would require 7 lines. A third possibility RCL 0, 2, ÷ , g frac, g x=0,2,4,× requires 8 lines and wipes out $\Gamma^* g$ stored in the stock.

IX. THE CENTRAL FISHER DISTRIBUTION

A special program for the central Fisher distribution is strictly not needed, since in the next chapter we shall give a program for the eccentric Fisher distribution which can be used also when the eccentricity is zero. However, the present program has a very convenient input and may be useful in case one has to work extensively with the central Fisher distribution.

If F is Fisher distributed with μ and ν degrees of freedom, then if μ is even,

$$1 - G_{\mu,\nu}(f) = \sum_{j=o}^{x} \binom{n}{j} p^{x}(1-p)^{n-j} = B_n(x;\ p)$$

where

$$n = \frac{\mu+\nu}{2} - 1 \quad , \quad x = \frac{\mu}{2} - 1 \quad , \quad p = f\,\frac{\mu}{\nu} \,/\, (1+f\,\frac{\mu}{\nu})$$

ν need not be even. Thus this relation is also true when n is not an integer and B_n not a proper cumulative distribution function.

The program below is based on the above equation and has the same structure as that for the binomial distribution in chapter II.

Program for the central Fisher distribution $G_{\mu,\nu}(f)$.

1	STO 2	12	RCL 2	23	RCL 3	34	RCL 3
2	R ↓	13	RCL 3	24	g x=0	35	RCL 1
3	STO 3	14	+	25	GTO 42	36	f x≥ y
4	×	15	1	26	RCL 4	37	GTO 42
5	RCL 2	16	-	27	RCL 2	38	1
6	÷	17	STO 2	28	×	39	STO +1
7	STO 4	18	f y^x	29	RCL 1	40	STO -2
8	1	19	STO 0	30	÷	41	GTO 26
9	STO 1	20	STO 7	31	STO ×0	42	RCL 7
10	+	21	1	32	RCL 0	43	GTO 00
11	g 1/x	22	STO -3	33	STO +7		

μ must be even. Set

f, enter, $\frac{\mu}{2}$, enter, $\frac{\nu}{2}$, R/S .

$1 - G_{\mu,\nu}(f)$ is displayed in the window.

Since $\frac{1}{F}$ is Fisher distributed with ν and μ degrees of freedom, the above program can be used directly if one of the degrees of freedom is even.

If both degrees of freedom are odd, then one can interpolate linearly between $\mu-1$ and $\mu+1$ or more accurately by means of

$$f(\mu) = 1.125\tfrac{1}{2}(f(\mu-1)+f(\mu+1)) - 0.125\tfrac{1}{2}(f(\mu-2)+f(\mu+2)) .$$

The central Student distribution can also be treated by means of the above program. If T_μ is Student distributed with μ degrees of freedom the $1/T_\mu^2$ is Fisher distributed with μ and 1 degrees of freedom, hence

$$G_\mu(t) = Pr(T_\mu \leq t) = \tfrac{1}{2} + \tfrac{1}{2}(1-G_{\mu,1}(\tfrac{1}{t^2})) .$$

Hence the input is

t, g x^2, g 1/x , μ , enter 0.5 , R/S .

(We can not utilize that T_μ^2 is Fisher distributed with 1 and μ degrees of freedom, since this would require interpolation between 0 and 2 .)

If the above program is in the machine anyhow, we may also use it on the central chi-square distribution of a variable Z . Let Z have μ degrees of freedom. Then Z/μ is Fisher distributed with μ and ∞ degrees of freedom or, approximately with μ and $\nu = 20 \cdot 10^5$ degrees of freedom (a higher value of ν would lead to excessive accumulation of errors). Hence the setting should be

Z , enter, μ, $\div$, $\mu/2$, enter, EEX, 5, R/S .

Example 1. μ=24, ν=25, f=1.96433 .
Set 1.96433, enter, 12, enter, 12.5, R/S .
1 - G = 0.049997 .

Example 2. Student distribution with μ=20, t=2.0860 . Set 2.0860, g x^2 , g 1/x, 10, enter 0.5, R/S .
We get $1 - G_{\mu,1} = 0.950004$ and

$G_\mu = 0.975002$.

Example 3. Student distribution with μ=19, t=2.0930.
Using the program on μ=16, 18, 20, 22 we get

μ	$1 - G_\mu$
16	0.9736804
18	46092
20	53487
22	59512

Linear interpolation gives 0.974979 , four point interpolation gives 0.974993. (Linear interpolation on the inverse of the degrees of freedom 1/18 and 1/20 gives 0.974998.)

Example 4. The chi-square distribution with 10 degrees of freedom, $z = 18.307$. Set

1.8307, enter, 5, enter, EEX, 5, R/S .

We get $1 - \Gamma = 0.050008447$.

The register in the above program has been used as follows,

STO	0	b_n	4	$f\ \mu/\nu$
	1	j	5	not used
	2	$\mu/2$ and $\frac{\mu+\nu}{2}-j$	6	not used
	3	$\nu/2$ and $\nu/2-1$	7	$B_n = 1-\Gamma$

X THE FISHER DISTRIBUTION (eccentric)

Program A and program B below, which are identical except for one program line, are based respectively on the formula

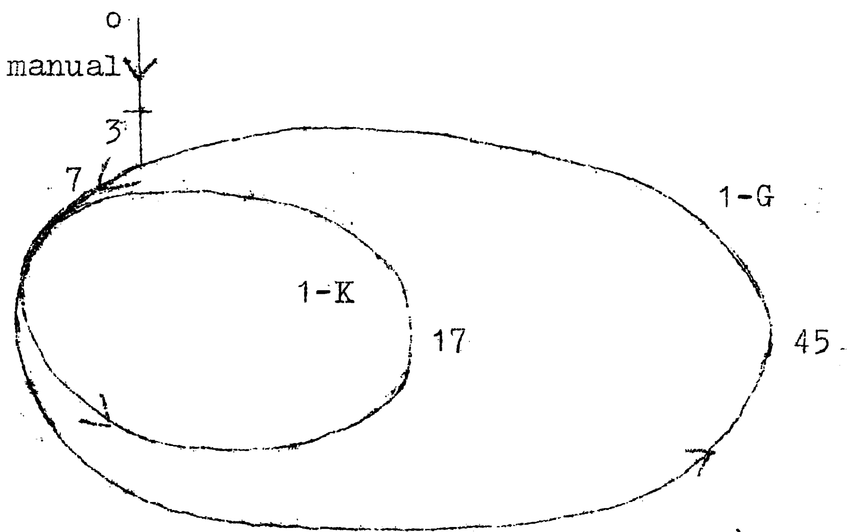
$$1 - G_{\mu,\nu}(f;\lambda) = \sum_{i=o}^{\infty} b(i;\lambda/2)(1-K_{\mu+2i,\nu}(f\ \mu/\nu)) \qquad \text{(A)}$$

$$G_{\mu,\nu}(f;\lambda) = \sum_{i=o}^{\infty} b(i;\lambda/2)(1-K_{\nu,\mu+2i}(\nu/f\ \mu)) \qquad \text{(B)}$$

where

$$1 - K_{\mu,\nu}(g) = \sum_{j=o}^{\mu/2-1} b_n(j;g/(1+g)) \ ; \ n = \frac{\mu+\nu}{2} - 1 \ .$$

Both programs contain two loops, like this,



(the figures are the number of program lines)

In program A $\mu+2i$ rounds are run in the small loop between one round at the time in the large loop. In program B ν rounds in the small loop are run.

Thus the running time for each term in 1 - G is increasing in case of program A , but kept constant in case of program B .

Hence if μ is much less than ν , the running time for program A is shorter than for program B. If μ is not much smaller than ν and in particular if λ is large, the opposite is the case. If high accuracy is wanted B has a relative advantage.

Program A assumes that μ is an even number whereas program B assumes that ν is even. Hence only program B can be used on the equal tailed Student test.

Program (eccentric) Fisher distribution $G_{\mu,\nu}(f;\lambda)$.

Program A.

1	CHS	13	f int	25	÷	37	RCL 2
2	g e^x	14	RCL 7	26	STO ×0	38	STO - 7
3	STO 5	15	f x≥y	27	RCL 0	39	STO + 1
4	RCL 4	16	GTO 29	28	GTO 11	40	1
5	g frac	17	1	29	RCL 2	41*)	STO + 2
6	RCL 1	18	STO +7	30	g frac	42	STO + 1
7	1	19	STO -1	31	STO -2	43	STO + 4
8	-	20	RCL 3	32	RCL 5	44	RCL 4
9	f y^x	21	RCL 1	33	×	45	f int
10	STO 0	22	×	34	STO +7	46	STO ÷ 5
11	STO +2	23	RCL 7	35	f pause	47	RCL 6
12	RCL 2	24	f int	36	f pause	48	STO × 5
						49	GTO 04

Program A. μ must be an even number. Set

1. $\frac{\mu+\nu}{2}$, STO 1, $\frac{\mu}{2}-1$, STO 2 ,
2. f $\frac{\mu}{\nu}$, STO 3, 1, +, g 1/x, STO 4,
3. $\frac{\lambda}{2}$, STO 6, R/S .

The window display in the pause is observed until it starts decreasing (it may first increase) and became insignificantly small. Then press R/S and $1 - G_{\mu,\nu}(f,\lambda)$ = fractional part of RCL 7 .

m=number of terms = integer part of RCL 4 . Last b = RCL 5 .

$$\text{Inaccuracy} < \text{last b} \frac{\lambda/2}{m-\lambda/2} .$$

Program B. As program A but 41 STO + 3 is replaced by 41 g NOP . ν must be an even number. Set

1. $\frac{\mu+\nu}{2}$, STO 1, $\frac{\nu}{2}-1$, STO 2 ,
2. $\frac{1}{f}\frac{\nu}{\mu}$, STO 3, 1, +, g 1/x, STO 4 ,
3. $\lambda/2$, STO 6, R/S .

Observe window display as by program A, but be sure to press R/S during display. Write down the window display = last term in 1-G . $G_{\mu,\nu}(f;\lambda)$ = fractional part of RCL 7 , m=no. of terms = integer part of RCL 4 .

$$\text{Inaccuracy} < \text{last term} \frac{\lambda/2}{m-\lambda/2} .$$

If T is Student distributed with ν degrees of freedom and eccentricity δ , then T^2 is Fisher distributed with $(1,\nu)$ degrees of freedom and eccentricity δ^2 . Thus program B can be used to find $Pr(|T| > t)$.

Set

1. $\frac{\nu+1}{2}$, STO 1, $\frac{\nu}{2}$ -1, STO 2 ,

2. ν/t^2 , STO 3, 1, +, g 1/x, STO 4 ,

3. $\delta^2/2$, STO 6, R/S .

RCL 7 gives $Pr(|T| < t)$.

Example 1. (Program A), $\mu = 4$, $\nu = 6$, $f = 4.5337$, $\lambda = 4$.
The window displays are as follows (f fix 5) ,
0.00677 , 0.02762 , 0.04526 , 0.04343 , 0.02869 , 0.01427 , 0.00566 , 0.00186 , 0.00052 , 0.00013 , 0.00003 , 0.00001 , $9.25 \circ 10^{-7}$ = last term.

RCL 7 = 13,17424 , hence 1 - G = 0.17424 .

RCL 4 = 12.24850 , hence m = 12 , RCL 5 = $1.16 \circ 10^{-6}$ = last b .
Hence inaccuracy $< 1.16 \circ 10^{-6} \circ 2/(12-2) = 1.2 \circ 10^{-7}$.

Example 2. (Program A). $\mu = 2$, $\nu = 6$, $f = 5.1433$, $\lambda = 4$.
The window displays are as follows,
0.00677 , 0.03918 , 0.07157 , 0.07044 , 0.04599 , 0.02220 , 0.00847 , 0.00267 , 0.00072 , 0.00017 , 0.00003 , 0.00001 , $1.10 \circ 10^{-6}$ = last term. We get 1 - G = 0.26821 .

$$\text{Inaccuracy} < 1.16 \circ 10^{-6} \frac{2}{12-2} = 2.3 \circ 10^{-7} .$$

Note that the program works differently when $\mu/2 - 1 = 0$ and $\mu/2 - 1 > 0$. Hence both examples 1 and 2 should be used to test correct setting of the program.

Example 3. Student's distribution with 6 degrees of freedom, $t = 2.4469$, eccentricity $\delta = 2$. Last term = $5.58 \circ 10^{-7}$.
$Pr(|T| < t) = 0.60894$, m = 10, inaccuracy $< 1.4 \circ 10^{-7}$.

If t is the $1-\epsilon$ fractile for $\delta = 0$,
then

$$\bar{\beta}_{2\epsilon}(\delta) = 1 - G_\nu(t,\delta)$$

is the power of the Student equal tailed test with level 2ϵ. Let $\beta_\epsilon(\delta)$ be the power of upper tailed test with level ϵ. Then we have

$$\bar{\beta}_{2\epsilon}(\delta) \cdot \epsilon \leq \beta_\epsilon(\delta) \leq \bar{\beta}_{2\epsilon}(\delta)$$

$$\bar{\beta}_{2\epsilon}(\delta) - \Delta \leq \beta_\epsilon(\delta) \leq \bar{\beta}_{2\epsilon}(\delta)$$

where

$$\Delta = \Gamma_\nu(z)\ G(-\delta) + G(-\delta - t\sqrt{\tfrac{z}{\nu}})(1-\Gamma_\nu(z))$$

for any z. Hence we choose z to make Δ small.
Sometimes $z = \infty$, $\Delta = G(-\delta)$ is good enough.
(Derivation : We have $\beta_{2\epsilon}(\delta) = Pr(T < -t) + Pr(T > t)$
$= \int_0^\infty G(-\delta - t\sqrt{\tfrac{y}{\nu}})\ d\Gamma_\nu(y) + \beta_\epsilon(\delta)$. Write $\int_0^\infty = \int_0^z + \int_z^\infty$).

Thus in the above example $t = 2.4469$ corresponds to $2\epsilon = 0.05$.
We get

$$0.3886 < \beta_\epsilon(\delta) < 0.3911 .$$

Hence program B is useful also for the one tailed Student test.

Since μF is approximately chi-square distributed when ν is large, program A can also be used on the chi-square distribution. Choose $\nu = 10^5$. Then $z/10^5$ is set instead of $F\mu/\nu$.

Example. $z = 18.307$, $\mu = 10$, $\lambda = 3.71$.
Set $\frac{10+10^5}{2}$, STO 1 , 4 , STO 2 , $18.307 \cdot 10^{-5}$, STO 3 , 1 , + , g 1/x , STO 4 , 3.71/2 , STO 6 .
$1 - \Gamma_6(18.307 ; 3.71) = 0.20017$ (correct value = 0.20015).

Example. $\mu = \nu = 24$, $\lambda = 16$, $f = 1.9838$ (level = 0.05). $1 - G = 0.3351019$ after 6 minutes on program B. On program A the running time is 13 minutes. (Since $\mu=\nu$, the running time for the first term would be the same with both programs. Since the running time for the terms is constant on program B, the first term would take $6/24 = \frac{1}{4}$ min. on both programs, whereas the i-th term would have running time of the term $\frac{1}{4} + a\,i$ on program A.

Hence $13 = \sum_{0}^{24} (\frac{1}{4}+ai) = 6 + 300\,a$, $a = 0.0233$. Hence the running time per term on program A increases from 15 seconds to $\frac{1}{4} + a \circ 24 = 49$ seconds).

If λ is large the first terms in $1 - G$ would be very small because $b(i;\lambda/2)$ is small. To shorten running time one could then choose an i_o such that b is negligable for $i < i_o$ and then set

1. $\frac{\mu+\nu}{2} + i_o - 1$, STO 1 , $\frac{\mu}{2} + i_o - 1$ (program A) or $\frac{\nu}{2} - 1$ (program B) , STO 2.
2. As above,
3. $b_o = (\frac{\lambda}{2})^{i_o} e^{-\frac{\lambda}{2}} / i_o!$, STO 5 , $\frac{\lambda}{2}$, STO 6 , i_o , STO +4 , GTO 04 , R/S .

Example. $\mu = \nu = 8$, $\lambda = 80$, $f = 3.43813$. Without the above method of shortening, the running time will be 10 minutes and gives $1 - G = 0.99397$ on program B. To shorten running time choose $i_o = 15$. Then $b_o = 3.4883575 \circ 10^{-6}$ and after 5 minutes running time we get 0.99396 .

The register in the above program A has been used as follows,

STO	0	$b_n(j,p)$	4	$1-p, i$
	1	$\frac{\mu+\nu}{2} + j$	5	b
	2	$\frac{\mu}{2} - 1 + i$	6	$\lambda/2$
	3	$f\ \mu/\nu$	7	j and $\Sigma(1-K)b$

For comment on the double use of STO 4 and STO 7 see the last lines on page 27. We shall evaluate the accuracy by programs A and B. The remainder term after m terms in the series used, is

$$R_m = \sum_{i=m-1}^{\infty} b_i(1-K_i)$$

where the meaning of b_i and K_i is clear from the equations on page 35. . By program A R_m is added to , by program B subtracted from the final result in STO 7 . By program A, $1 - K_i$ is an increasing function of i and goes to 1 . By program B, $1 - K_i$ is a decreasing function which goes to 0 .

We have also

$b_m(1-K_m)$ = last term = window display when stopping.

b_m = RCL 5

m = integer part of RCL 4.

We obviously have

$$R_m < 1 - B(m;\tfrac{\lambda}{2}) = b(m;\tfrac{\lambda}{2}) \sum_{m+1}^{\infty} \frac{(\frac{\lambda}{2})^{i-m}}{(m+1)\dots(m+i)} .$$

Since $m + j > m$ we have if $m > \lambda/2$,

$$R_m < b(m;\lambda/2)\,\frac{\lambda/2}{m-\lambda/2} = \text{RCL } 5\ \frac{\lambda/2}{m-\lambda/2}$$

which is true both for program A and program B. (We could also have made use of $m + j > j$ and obtain

$$R_m < \text{RCL } 5 \circ e^{\lambda/2}$$

which is true also if $m \leq \lambda/2$.)

In the case of program B we also have

$$R_m < (1-K_m)b_m\,\frac{\lambda/2}{m-\lambda/2} = \text{window}\ \frac{\lambda/2}{m-\lambda/2} .$$

November 1976

Program for bivariate normal distribution on HP-25 calculator.

by Erling Sverdrup

1. The standardized binormal distribution.

Let (X, Y) be binormally distributed with

$EX = EY = 0$ (1) , $\mathrm{var}\, X = \mathrm{var}\, Y = 1$ (2)

and correlation coefficient ρ. Then

$$\Pr((X \le x) \cap (Y \le y)) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{x}\int_{-\infty}^{y} e^{-\frac{v^2 - 2\rho v w + w^2}{2(1-\rho^2)}}\, dv\, dw = G_\rho(x, y) \quad (3)$$

Since $(-X, -Y)$ has the same distribution as (X, Y) and $(-X, Y)$ has correlation coefficient $-\rho$, we have

$$\Pr((X > x) \cap (Y > y)) = G_\rho(-x, -y) \quad (4)$$

$$\Pr((X > x) \cap (Y < y)) = G_{-\rho}(-x, y) \quad (5)$$

Furthermore,

$$\Pr((X > x) \cap (Y < y)) = G(y) - G_\rho(x, y) \quad (6)$$

where $G(y) = \Pr(Y \le y)$ is the gaussian integral. Hence

$$G_{-\rho}(x, y) = G(y) - G_\rho(-x, y) \quad (7)$$

and also

$$G_\rho(x, y) = G_\rho(y, x) \quad (8)$$

2. Application of the binormal distribution.

Nylon ... breaking strength Z ... normal ... $\bar{Z} = 9 = 8.320$ kg and standard deviation $\sigma = 0.130$. When measuring Z there is a measurement error V which is normal and independent of Z, $EV = 0$, standard deviation of V is $\sigma_0 = 0.043$. Thus the measured strength is $Z' = Z + V$.

The smallest breaking strength that is tolerated is $a = 8.10$. Taking into account measurement error it is decided to accept nylon lines if and only if measured breaking strength is $\geq b = 8.15$. Then the buyers risk is

$$P = Pr\,(\text{accepting unacceptable line}) =$$

$$= Pr\left((Z < a) \cap (Z' \geq b)\right) \qquad (9)$$

However (Z, Z') is binormal with

$$EZ = EZ' = \xi, \quad \operatorname{var} Z = \sigma^2, \quad \operatorname{var} Z' = \sigma^2 + \sigma_0^2 = \tau^2$$

$$\operatorname{cov}(Z, Z') = \sigma^2$$

Thus the standardized variables

$$X = \frac{Z - \xi}{\sigma}, \quad Y = \frac{Z' - \xi}{\tau} \qquad (10)$$

are jointly normal with correlation coefficient

$$\rho = \sigma / \sqrt{\sigma^2 + \sigma_0^2} \qquad (11)$$

Hence

$$P = \Phi_{-\rho}(A, -B) \qquad (12)$$

where

$$A = \frac{a - \xi}{\sigma}, \quad B = \frac{b - \eta}{\tau} \qquad (13)$$

Introducing the values given above, we get

$$P = G_{-0.4494}(-1.6923, 1.2418) \qquad (14)$$

3. An Auxiliary Function

We shall make use of

$$T(x, a) = \frac{1}{2\pi} \int_0^a \frac{e^{-\frac{x^2}{2}(1+v^2)}}{1+v^2} \, dv \qquad (15)$$

We see that

$$T(x, a) = T(-x, a), \quad T(x, a) = -T(x, -a), \quad T(x, 0) = 0 \qquad (16)$$

$$T(0, a) = \frac{1}{2\pi} \operatorname{arctg} a, \quad T(x, \infty) = \frac{1}{2}(1 - G(x)) \qquad (17)$$

The program below will give high accuracy if $|a| \leq 2$. If $|a| > 2$ we compute $T(xa, 1/|a|)$ and make use of the relation

$$T(x, a) = \frac{1}{2} G(x) + \frac{1}{2} G(xa) - G(x) G(xa) - T(xa, \frac{1}{|a|}), \qquad (18)$$

valid for $a > 0$.

There is the following relation between the binormal distribution and the auxiliary function

$$G_\rho(x, y) = \frac{1}{2} G(x) + \frac{1}{2} G(y) - T(x, a_\rho(\frac{y}{x})) - T(y, a_\rho(\frac{x}{y})) - \frac{0}{\frac{1}{2}} \qquad (19)$$

where

$$a_\rho(t) = (t-\rho)/\sqrt{1-\rho^2} \qquad (20)$$

and the upper choice in the last term in (19) is made if $xy > 0$ or $xy = 0$ and $x+y > 0$, otherwise the lower choice is made.

4. Program and manual procedure

The program and manual procedure is based upon

(i), a program for $T(x, a)$ given by (15) using Simpsons formula with 2nc+1 = 11 terms and assuming $a \leq 2$,

(ii), using manually equation (18) if either $a_\rho(\frac{y}{x})$ or $a_\rho(\frac{x}{y})$ is > 2

(iii), using (19) and (20) to obtain $L_\rho(x, y)$.

The binormal distribution. Program for the auxiliary function:

1	$g\ x^2$	13	RCL 2	25	RCL 3	37	−
2	2	14	RCL 0	26	×	38	STO 3
3	÷	15	×	27	STO + 7	39	GTO 13
4	CHS	16	$g\ x^2$	28	1	40	RCL 7
5	STO 1	17	1	29	STO − 2	41	RCL 0
6	$g\ e^x$	18	+	30	3	42	×
7	STO 7	19	STO 5	31	g π	43	g π
8	1	20	RCL 1	32	RCL 2	44	6
9	0	21	×	33	g x = 0	45	×
10	STO 2	22	$g\ e^x$	34	GTO 40	46	÷
11	1	23	RCL 5	35	×		
12	STO 3	24	÷	36	f cos		

Set once for all (after shifting to "run"): g rad.

For each (x, a) set: $|a|/_{10}$, STO 0, $|x|$, R/S.

Then $T(x, |a|)$ is shown in the window and

$$T(x, a) = T(x, |a|)\cdot \mathrm{sgn}\, a$$

The complete procedure to find $G_\rho(x, y)$ goes as follows

A. Find $a = a_\rho\left(\frac{y}{x}\right) = \left(\frac{y}{x} - \rho\right)/\sqrt{1-\rho^2}$ ($\sqrt{1-\rho^2}$ may be set in STO 4 for later use). If $|a| \le 2$ run program for $T(x, |a|)$. If $|a| > 2$ run program for $T(ax, \frac{1}{|a|})$ and compute

$$T(x, |a|) = \tfrac{1}{2} G(|x|) + \tfrac{1}{2} G(|xa|) - G(|x|)G(|xa|) - T\left(ax, \tfrac{1}{|a|}\right)$$

In either case set $T(x, a) = T(x, |a|)\, \mathrm{sgn}\, a$ in STO 6.

B. Repeat the procedure under A with (x, y) replaced by (y, x). Set the result in STO + 6.

C. Find $\frac{1}{2} G(x)$ and STO 4

D. Find $\frac{1}{2} G(y)$, RCL 4, +, RCL 6, −.

E. If $xy < 0$ or $xy = 0$ and $x + y > 0$, the $G_\rho(x, y)$ is shown in the window. Otherwise subtract 0.5 and $G_\rho(x, y)$ is shown in the window.

(F. The G-s in A, B, C, D can be found from tables or [illegible] — from [illegible]

[illegible]

[illegible])

Returning to the example in section 2 above, we want to find

$$P = G_{-0.4494}(-1.6923,\ 1.2418)$$

We find

$$T(x, a_{-f}(\tfrac{y}{x})) = T(1.6923,\ 0.68648) = 0.018019 \quad (\text{STO } 6)$$

$$T(y, a_{-f}(\tfrac{x}{y})) = T(1.2418,\ -1.316205) = -0.051464 \quad (\text{STO}+6)$$

$$G(-1.6923) = 1 - 0.954704 = 0.045296$$

divide by 2 and STO 4

$$G(1.2418) = 0.892843$$

2, ÷, RCL4, +, RCL6, −. Then 0.5015142, hence

$$P = 0.001514$$

Program on HP 25 desk calculator for the distribution of the multiple correlation coefficient.

by Erling Sverdrup

1. The distribution of the multiple correlation coefficient

Given a random vector $X = (X_1, \cdots X_p)'$, the multiple correlation coefficient between X_1 and $(X_2, \ldots, X_p)$ is defined as

$$\rho_{1\cdot 2\cdots p} = \max_{\alpha} \rho(X_1, \sum_2^p \alpha_i X_i) \qquad (1)$$

where $\rho(\cdot,\cdot)$ denote the ordinary correlation coefficient between two variables.

Suppose that X is normally distributed with $EX = \xi$ and covariance matrix $= \sigma = (\sigma_{ij})$, where $\sigma_{ij} = \text{cov}(X_i, X_j)$. Define Σ_{12}, Σ_{22} by means of

$$\sigma = \begin{pmatrix} \sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}, \quad \text{then } \rho_{1\cdot 2\cdots p} = \sqrt{\frac{\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}}{\sigma_{11}}} \qquad (2)$$

Note that $1 \geq \rho_{1\cdot 2\cdots p} \geq 0$, and for $p=2$ is $\rho_{1\cdot 2} = |\rho|$, where ρ is the ordinary correlation coefficient between X_1 and X_2.

Let $X_\alpha = (X_{1\alpha}, \cdots, X_{p\alpha})'$, $\alpha = 1, 2, \cdots, N$ independent observations of X and let

$$\bar{X} = \frac{1}{N}\sum_\alpha X_\alpha, \qquad a = \sum_\alpha (X_\alpha - \bar{X})(X_\alpha - \bar{X})' \qquad (3)$$

Write

$$a = \begin{pmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad (4)$$

Then

$$R_{1\cdot 2\cdots p} = \sqrt{\frac{A_{12}\, A_{22}^{-1}\, A_{21}}{a_{11}}} \tag{5}$$

is a maximum likelihood estimate of $\rho_{1\cdot 2\cdots p}$.
Below we write $R_{1\cdot 2\cdots p} = R$, $\rho_{1\cdot 2\cdots p} = \rho$.

It is known (see T. W. Anderson (1958)) that in the conditional distribution of all $X_{1\alpha}$ given all $(X_{2\alpha}, \ldots, X_{p\alpha})$ is $R^2/(1-R^2)$ eccentric chi-square-ratio distributed with $p-1$ and $N-p$ degrees of freedom and eccentricity a certain L. On the other hand $L/\frac{\rho^2}{1-\rho^2}$ is central chi-square distributed with $N-1$ degrees of freedom.

From this we find that $H(r^2\,|\,\rho^2, N, p) =$

$$= P_r(R^2 \le r^2) = \sum_{i=0}^{\infty} K_{p-1+2i,\,N-p}\left(\frac{r^2}{1-r^2}\right)\cdot\binom{\frac{N-1}{2}+i}{i}\frac{\rho^{2i}}{i!}(1-\rho^2)^{\frac{N-1}{2}} \tag{6}$$

where $K_{\mu,\nu}(g)$ is the cumulative distribution function of the central chi-square ratio with μ and ν degrees of freedom. Note that if $\rho = 0$, then $(N-p)R^2/(1-R^2)(p-1)$ is Fisher distributed with $p-1$ and $N-p$ degrees of freedom.

In the program we make use of the fact that

$$K_{\mu,\nu}(g) = 1 - K_{\nu,\mu}\left(\frac{1}{g}\right) \tag{7}$$

and that if $\frac{\nu}{2} = \frac{N-p}{2}$ is an integer then

$$1 - K_{\nu,\mu}\left(\frac{1}{g}\right) = \sum_{j=0}^{\frac{\nu}{2}-1}\binom{\frac{\mu+\nu}{2}-1}{j}\left(\frac{1}{1+g}\right)^{j}\left(\frac{g}{1+g}\right)^{\frac{\nu}{2}-1-j} \tag{8}$$

($\frac{\mu+\nu}{2} = \frac{N-1}{2}$ need not be an integer).

We also make use of

$$q_i = \binom{\frac{N-1}{2}+i-1}{i} \rho^{2i} (1-\rho^2)^{\frac{N-1}{2}} = \frac{\frac{N-1}{2}+i-1}{i} \rho^2 q_{i-1}, \quad q_0 = (1-\rho^2)^{\frac{N-1}{2}} \tag{9}$$

We shall evaluate the inaccuracy when using $t+1$ terms in (6). We write (6)

$$H(r^2 \mid \rho^2, N, p) = \sum_0^\infty L_i = \sum_0^t L_i + \sum_{t+1}^\infty L_i \tag{10}$$

Since $K_{\mu,\nu}$ is a decreasing function of μ we have for $i > t$,

$$K_{p-1+2i,\,N-p}\left(\frac{r^2}{1-r^2}\right) \leq K_{p-1+2t,\,N-p}\left(\frac{r^2}{1-r^2}\right) \tag{11}$$

We also have

$$q_{i+1} \leq \frac{\frac{N-1}{2}-1}{t+1} \rho^2 q_i$$

hence for $j \geq 0$,

$$q_{t+j} \leq \left[\left(\frac{N-3}{2(t+1)}+1\right)\rho^2\right]^j q_t \tag{12}$$

Combining with (11) we get

$$L_{t+j} \leq L_t \left[\left(\frac{N-3}{2(t+1)}+1\right)\rho^2\right]^j \tag{13}$$

Hence

$$\sum_{t+1}^\infty L_i \leq \frac{L_t}{1 - \frac{N-1+2t}{2t+2}\rho^2} \tag{14}$$

provided the last term in the denominator is < 1.

2. The application of the multiple correlation coefficient.

As an example of the usefulness of the program consider the regression of $X_{1\alpha}$ with respect to $X_\alpha^{(2)} = (X_{2\alpha}, \cdots, X_{p\alpha})'$, i.e.

$$E(X_{1\alpha} \mid X_\alpha^{(2)}) = \beta X_\alpha^{(2)} \qquad (15)$$

where $\beta = (\beta_2, \cdots, \beta_p) = \Sigma_{12} \Sigma_{22}^{-1}$ (see (2)).
The estimate of β is $\hat{\beta} = A_{12} A_{22}^{-1}$.

We are interested in β and use the multiple comparison approach. We then know that we can only discover significant and interesting features of the components of β if $(N-p)R^2/(1-R^2)(p-1) =$

$$= \frac{N-p}{p-1} \hat{\beta} A_{22} \hat{\beta}' / (a_{11} - \hat{\beta} A_{22} \hat{\beta}') > f \qquad (16)$$

where f is a high fractile of the Fisher distribution with $p-1$ and $N-p$ degrees of freedom.

The performance, i.e. the probability of discovering a significant feature if $\beta \neq 0$ is then given by

$$1 - G_{p-1, N-p}(f, L) \qquad (17)$$

where $G_{\mu,\nu}(f, L)$ is the cumulative Fisher distribution with eccentricity L and

$$L = \beta A_{22} \beta' / \sigma_{11\cdot 2}, \quad \sigma_{11\cdot 2} = \sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} \qquad (18)$$

Thus we must specify the factor matrix A_{22} given from $X_\alpha^{(2)} = (X_{2\alpha}, \cdots, X_{p\alpha})'$; $\alpha = 1, 2, \cdots, N$; in order to study the performance as a function of β.

This would be natural to do if the factors could be designed. However, in the non-designable situation it is more natural to take into account that by chance the pattern of $X^{(2)}_\alpha$; $\alpha = 1, 2, \cdots N$ may be good or poor, i.e. to study the average performance which is just $H(r^2 \mid \rho^2, N, p)$ where

$$r^2 = \frac{g}{1+g} \quad , \quad g = f\,\frac{p-1}{N-p} \quad , \quad \rho^2 = \beta \Sigma_{22} \beta' / \sigma_{11} \qquad (19)$$

3. The multiple correlation coefficient. Program for values of the cumulative distribution function.

1	RCL 4	13	GTO 26	25	GTO 08	38	STO+4
2	g frac	14	1	26	RCL 6	39	RCL 1
3	RCL 1	15	STO+7	27	STO−6	40	RCL 4
4	1	16	STO−1	28	RCL 5	41	f int
5	−	17	RCL 3	29	×	42	÷
6	f y^x	18	RCL 1	30	STO+7	43	RCL 2
7	STO 0	19	×	31	f pause	44	g frac
8	STO+6	20	RCL 7	32	f pause	45	×
9	RCL 2	21	f int	33	RCL 2	46	STO×5
10	f int	22	÷	34	f int	47	1
11	RCL 7	23	STO×0	35	STO−7	48	STO+1
12	f $x \geq y$	24	RCL 0	36	STO+1	49	GTO 01
				37	1		

If $N-p$ even, set

ρ^2, STO 2, $1-\rho^2$, (enter), $\frac{N-1}{2}$, STO 1, f y^x, STO 5,

$\frac{N-p}{2}-1$, STO+2, R^2, STO 4, $\frac{1}{R^2}-1$, STO 3. R/S

The window display is observed until it starts decreasing (it may first increase) and become insignificantly small. Then press R/S during pause, L_t = last term is shown; RCL 7, fractional part is $H(r^2|\rho^2, N, p)$; RCL 4, integer part is number of terms t. Then

$$\text{inaccuracy} < \frac{L_t}{1-\frac{N-1+2t}{2t+2}\rho^2}$$

(provided fraction in denominator is less than 1).

If $N-p$ odd, replace N by $N+1$ and by $N-1$ and interpolate between $H(r^2|\rho^2, N+1, p)$ and $H(r^2|\rho^2, N-1, .)$

Example: $N=20$, $p=8$, $\rho=0.4$, $R=0.8$

After less than 3 minutes running time we press R/S when $3.3\cdots 10^{-9}=L_t$ is shown in the window, RCL 7 gives $H=0.8[illegible]44944C1$, RCL 4 gives $t=1[illegible]$. Pressing [illegible] shows L_t again and we compute inaccuracy $< 4.2 \cdot 10^{-9}$

Example $N = 63$, $p = 13$, $\rho = 0.5$. 0.95 and 0.05 fractiles, according to Biometrika tables II (1972), are 0.737 and 0.489. Running the program with $R = 0.737$ we get when stopping with window $= 5.398\cdots 10^{-7}$, RCL 4 = 32 = t, RCL 7 = = 0.9507 = H, inaccuracy $< 1.0 \cdot 10^{-6}$.

With $R = 0.489$ we get window $= 3.58\cdots 10^{-6}$, RCL 4 = 17, RCL 7 = 0.0499, inaccuracy $< 1.0 \cdot 10^{-6}$.

—

If $p = 2$ then, since R is the absolute value of the ordinary correlation coefficient, we have

$$H(r^2 \mid \rho^2, N, 2) = F(r \mid \rho, N) - F(-r \mid \rho, N)$$

where F denotes the cumulative distribution function of the ordinary correlation coefficient, $F(r \mid \cdots) = \Pr(R \le r)$. F has been tabulated by F.N. David (1954). The table values below are taken from her tables.

For $p = 2$ it is known that for N large

$$W = \sqrt{N-3}\left(Z - \zeta - \frac{\rho}{2(N-1)}\right),$$

where $Z = \frac{1}{2}\log_e \frac{1+R}{1-R}$, $\zeta = \frac{1}{2}\log_e \frac{1+\rho}{1-\rho}$, is approximately normal (0,1).

We have

r	ρ	N	p	H: program	H: table	H: norm. appr.
[illegible]	0.6	10	2	0.376454853	0.3746	0.38744
0.7	0.6	[illegible]	2	0.95416832	0.95417	0.95404

4 References

T. W. Anderson : Introduction to multivariate statistical Analysis (1958). New York. John Wiley & sons

F. N. David : Tables of the correlation coefficient (1954) Cambridge at The University Press

E. S. Pearson and H. O. Hartley : Biometrika tables for Statisticians (1972) volume II. Cambridge at The University Press.

1976

Tolerance limit on HP-25 desk calculator.

by Erling Sverdrup

Let $P_r(X \leq x) = F_\theta(x)$ and $X_1, \dots, X_n$ independent observations of X. We want to find an upper tolerance limit for F_θ; we want to find an $L(X_1, \dots, X_n)$ such that with probability γ at least a proportion P of the probability mass is less than $L(X_1, \dots, X_n)$, i.e.

$$P_r\left(F_\theta(L(X_1, \dots, X_n)) > P\right) = \gamma \qquad (1)$$

for all θ. Hence

$$P_r\left(L(X_1, \dots, X_n) > F_\theta^{-1}(P)\right) = \gamma \qquad (2)$$

Hence a tolerance band for F_θ is really the same confidence interval for a fractile $F_\theta^{-1}(P)$.

Let in particular X be normal (ξ, σ). Then $F_\theta(x) = F_{\xi,\sigma}(x) = G\left(\frac{x-\xi}{\sigma}\right)$, where G is the Gaussian integral. Thus we want an $L(X_1, \dots, X_n)$ such that

$$L(X_1, \dots, X_n) > F_{\xi,\sigma}^{-1}(P) = \xi + \sigma G^{-1}(P)$$

It is natural to choose $L = \bar{X} + kS$ where $\bar{X}$ and S^2 are the usual unbiased estimates of ξ and σ^2. Hence

$$\frac{\bar{X} - \xi + kS}{\sigma} > G^{-1}(P) \qquad (3)$$

Introducing $Y = -(\bar{x} - \xi)\sqrt{m}/\sigma$ and $Z = S^2(m-1)/\sigma^2$

we get

$$-(Y/\sqrt{m}) + k\sqrt{Z/(m-1)} > G^{-1}(P)$$

or

$$T = \frac{Y + \sqrt{m}\, G^{-1}(P)}{\sqrt{Z}} \sqrt{m-1} < k\sqrt{m} \qquad (4)$$

Here T is eccentric Student distributed with with $m-1$ degrees of freedom and eccentricity $\delta =$ $= \sqrt{m}\, G^{-1}(P)$. Denote this distribution by $G_{m-1}(t, \delta)$ and we get

$$G_{m-1}(k\sqrt{m}, \sqrt{m}\, G^{-1}(P)) = \gamma \qquad (5)$$

from which k can be determined by means of a program of $G_\nu(t, \delta)$ (see below)

We try out different values of k, and start by roughly assuming $Z \approx m-1$ in (4), we then get a first trial value

$$k_0 = \frac{G^{-1}(\gamma)}{\sqrt{m}} + G^{-1}(P) \qquad (6)$$

Example $m = 9$, $P = 0.90$, $\gamma = 0.95$. Hence

$$G_8(3k, 3.84465) = 0.95$$

We get from (6), $k_0 = 1.8$

We then compute as follows,

k_2	1.8	2.2	2.4	2.5	2.45
G_1	0.48	0.91	0.94389	0.9547	0.94960

$k_2 = 2.45$ may be good enough for practical performances. However, to get a more accurate value

k_2	2.452	2.454	2.453	2.4537	2.4538
G_1	0.949816	0.950030	0.949993	0.949998	0.950009

Hence $\underline{k_2 = 2.4537}$.

Re Stat. Res. Rep. No 7, 1976: Programs on HP-25 etc by E. Sverdrup. The following is an improved program for the eccentric Student, see page 29.

1	RCL 0	13	RCL 1	25	STO − 1	37	STO + 0
2	g x^2	14	f x≥y	26	g frac	38	RCL 0
3	RCL 2	15	GTO 25	27	RCL 0	39	2
4	×	16	1	28	(m=)8	40	÷
5	STO 6	17	STO + 1	29	÷	41	g frac
6	CHS	18	RCL 1	30	RCL 4	42	−
7	g e^x	19	f int	31	−	43	×
8	2	20	STO ÷ 5	32	g x^2	44	STO + 7
9	÷	21	RCL 6	33	g e^x	45	f pause
10	STO 5	22	STO × 5	34	f √	46	f pause
11	STO + 1	23	RCL 5	35	÷	47	GTO 01
12	RCL 3	24	GTO 11	36	1	48	

If ν is even set

1, STO 0, $\frac{\nu}{2(m+1)^2}$ (e.g. m=8), STO 2, $\frac{\nu}{2}-1$, STO 3, δ; STO 4, R/S

When figure displayed during pause insignificant, then press R/S

RCL 7 = $\Sigma A\Gamma^*$, RCL 0 = number of terms

We can now compute

$$G_\nu(t,\delta) = 1 - G(\delta) + \frac{8(\text{RCL }7) + e^{-\frac{\delta^2}{2}}}{3m\sqrt{2\pi}}$$

Automatic stopping after $A\Gamma^*$ = last term $< 10^{-m}$ is obtained by making the following changes:

45 EEX   Many results ..
46 m(=5)
47 CHS   ... stop almost
48 f x<y   at once (small i) when
49 GTO 01   g($\frac{1}{m}$ − d) Will occur if δ large > 0

RCL 7 when the calculator stops. Then proceede as above.

Comments. Compared to program in Stat. Res. Rep. No 7 − 1976 the following is changed.

(i). 47 0, 48 STO 1 has be replaced by 25 STO − 1

(ii). The Simpson coeff A_i is found from $\frac{1}{4}A_i = 1 - \text{frac}\,\frac{i+1}{2}$ which gives 6 lines. Combinding with 1 STO + 0 makes for saving 1 line, see line 36 − 42.

Program for the inverse Gaussian integral $x = G^{-1}(P)$ on HP25 (Newton iteration).

1	STO 2	13	g x^2	25	RCL 1	37	STO×7
2	RCL 7	14	f y^x	26	g π	38	RCL 5
3	STO 1	15	STO 7	27	×	39	STO−7
4	÷	16	1	28	f cos	40	RCL 2
5	STO 3	17	STO−1	29	−	41	Enter
6	g x^2	18	RCL 0	30	×	42	g x^2
7	2	19	RCL 1	31	STO+7	43	g e^x
8	÷	20	g x^2	32	GTO 16	44	f √
9	CHS	21	f y^x	33	STO+7	45	RCL 7
10	g e^x	22	f x=y	34	RCL 3	46	×
11	STO 0	23	GTO 33	35	3	47	f pause
12	RCL 1	24	3	36	÷	48	−
						49	GTO 01

After switch to "run", set once for all g rad.

(Note that the program is obtained by changes and additions (line 1-2, 37-49) of the program I for the normal distribution of november 1976)

If the P-fractile is wanted, then set

Im (= e.g 8), STO 4, $(P - \frac{1}{2})\sqrt{2\pi}$, STO 5, x_0, R/S

where x_0 is a trial value.

Observe the number in the window during pause. When insignificant then

RCL 2 = x.

(We could also set 47 −, 48 f pause, then the iterated values of x will be shown in the window)

Example: P = 0.95. Set 8, STO 4, $0.45\sqrt{2\pi}$, STO 5, 2, R/S. In the window we get

0.505, −0.134, −0.016, −0.00021, $4.3 \cdot 10^{-8}$

x = RCL 2 = 1.64485

Udregninger [?]

Januar 1977

Generation of random normal variables on HP 2?

Set program:

1	g π	13	RCL 3	25	STO 6	37	RCL 2
2	+	14	f ln	26	f cos	38	X
3	5	15	2	27	X	39	RCL 1
4	f y^x	16	X	28	RCL 2	40	+
5	g frac	17	CHS	29	X	41	R/S
6	STO 3	18	f √	30	RCL 1	42	RCL 4
7	g π	19	STO 5	31	+	43	GTO 1
8	+	20	RCL 4	32	R/S		
9	5	21	g π	33	RCL 6		
10	f y^x	22	X	34	f sin		
11	g frac	23	2	35	RCL 5		
12	STO 4	24	X	36	X		

Set once for all

g rad

If normally distributed variables with expectation ξ and variance σ are wanted, then set

$$\xi,\ STO\ 1,\ \sigma,\ STO\ 2,\ U_0,\ R/S,$$

where U_0 is a random number $0 \le U_0 < 1$. Pressing R/S repeatedly gives $X_1, X_2, X_3, \dots$ which are independent normal (ξ, σ).

Example, $\xi = 1$, $\sigma = 2$. Choose $U_0 = 0.192743568$ we get, 1.36, −2.94, −1.21, 4.23, −0.91, 0.56, 2.62, −0.09, 1.17, 3.40, ...

The program is based upon the following: Let $U_0, U_1, \dots$ be independent and uniformly distributed over the interval (0,1). Then $X_0, X_1, X_2, X_3, \dots$ where

$$X_{2n} = \sqrt{-2\ln U_{2n}}\cos 2\pi U_{2n+1}, \quad X_{2n+1} = \sqrt{-2\ln U_{2n}}\sin 2\pi U_{2n+1}$$

$n = 0, 1, 2, \dots$ are independent normal (0,1). U_m is generated from

$$U_m = \text{fractional part of } (\pi + U_{m-1})^5;\ m = 1, 2, \dots$$

The program has not been extensively tested

November 1976

PROGRAM FOR THE NORMAL DISTRIBUTION

by

Erling Sverdrup

Preface

It appears that the program and procedure for the Normal Distribution in my paper "Programs on HP-25 Desk Calculator of Mathematical Functions for Use in Statistics", (Statistical Research Report No. 7, 1976), although useful and adequate, could be replaced by considerably better programs and procedures. Thus the manual application of

$$G(x) = (y^{16}+4y^9+2y^4+4y+1)\ \frac{x}{12\sqrt{2\pi}} + 0.5\ ,$$

where $y = \exp(-x^2/32)$, is almost universaly useful and very convenient. It is easy to remember, or could be mentally reconstructed from the Simpson's formula, and one almost never slips up during the operation. Paper and pencil are not needed.

The program based on Simpson's formula with an arbitrary choice of the number $2m+1$ of terms is also superior to the program given before. It gives almost 8 decimals accuracy up to 5.5-6 times the standard deviation from the mean and the running time is short.

Thus the text presented below could replace Chapter VI in the above cited publication.

NORMAL DISTRIBUTION

Introduction

Manual routines or programs are not so much in need for the normal distribution since convenient tables are easily available.

However, if you have the small calculator on your desk anyhow, the manual routine with a 5 term Simpson's formula in A below is easy to perform and gives almost 4 correct decimals, for arguments, up to 3 times the standard deviation from the mean.

Two complete programs are also given below. The first one is based on Simpson's formula and allows an arbitrary number 2m+1 of terms to be set. It is recommended for all practical purposes with 2m+1 = 5 or 9 , or occasionally 2m+1 = 21 and gives high accuracy even in extreme cases.

The explicit formula is

$$G(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{v^2}{2}} dv = \frac{1}{2} + \sum_{i=o}^{2m} A_i \; y^{(2m-i)^2} \frac{x}{6m\sqrt{2\pi}}$$

where $y = e^{-x^2/8m^2}$ and

$$A_1 = A_{2m} = 1 \quad , \quad A_1 = A_3 = \ldots A_{2m-1} = 4 \quad , \quad A_2 = A_4 = \ldots = A_{2m-2} = 2 \; .$$

A_i is reproduced in the program by means of

$$A_i = 3 - \cos \pi i = 3 - \cos(2m-i)\pi \quad ; \quad i=1,2,\ldots,2m-1 \; .$$

A program based on the continuous fraction formula

$$1 - G(x) = G'(x) \cdot \cfrac{1}{x + \cfrac{1}{x + \cfrac{2}{x + \cfrac{3}{x + \cdots + \cfrac{n-1}{x + \frac{n}{x}}}}}}$$

is also given. It has the advantage of giving an absolute control of the accuracy.

Since $G(-x) = 1 - G(x)$ it is assumed below that $x > 0$.

A. Manual routines for $G(x) = \int_{-\infty}^{x} g(v)dv$,

where, $g(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.

This procedure is strongly recommended for all practical purposes because it is easy to remember and easy to use. Paper and pencil are not needed during the operation.

Assume $x > 0$ and use

$$G(x) = (y^{16} + 4y^9 + 2y^4 + 4y + 1) \frac{x}{12\sqrt{2\pi}} + 0.5$$

where

$$y = e^{-\frac{x^2}{32}} .$$

Store x , then compute y and store it. Then compute the polynomial in y and finally $G(x)$. Thus for $x = 1.96$ we get $G(x) = 0.974961$. It gives almost 4 correct decimals for x up to 3. For $x \geq 3$ it will usually be enough to know that $1 - G(x) \leq 1 - G(3) = 1.3 \cdot 10^{-3}$. However,

$$1 - G(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{x}{x^2+1}$$

may also be used for $x > 3$.

B. Program I (Simpson) for $G(x) = \int_{-\infty}^{x} g(v)dv$, where $g(v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}$.

1	STO 2	13	g x^2	25	RCL 1	37	g π
2	≷	14	f y^x	26	g π	38	2
3	STO 1	15	STO 7	27	×	39	×
4	÷	16	1	28	f cos	40	f √
5	STO 3	17	STO -1	29	-	41	÷
6	g x^2	18	RCL 0	30	×	42	RCL 7
7	2	19	RCL 1	31	STO +7	43	×
8	÷	20	g x^2	32	GTO 16	44	2
9	CHS	21	f y^x	33	STO +7	45	g 1/x
10	g e^x	22	f x=y	34	RCL 3	46	+
11	STO 0	23	GTO 33	35	3		
12	RCL 1	24	3	36	÷		

Set once for all (after shifting from PRGM to RUN)

g rad.

For each $x > 0$ set

2m (e.g.=4 or 8) , enter, x , R/S.

G(x) is shown in the window.

Running time is 3+2(2m+1) seconds.

Examples:

x	Table values	2m+1 = 3	2m+1 = 5	2m+1 = 9	2m+1 = 21
0.25	0.5987063	0.5987067			
1.5	0.9331928	0.9332524	0.9332077	0.9331938	
1.96	0.9750021		0.9749609	0.9750000	
2.5	0.9937903		0.9936579	0.9937816	0.9937901
3.0	0.9986501		0.9984794	0.9986415	0.9986499
3.5	0.9997674		0.9991835	0.9997623	0.9997672
4.0	0.9999683			0.9999665	0.9999683
4.5	0.9999966			0.9999961	0.9999966
5.0	0.9999997133				0.999999710
5.5	0.9999999810				0.999999981
5.75	0.9999999955				0.999999996
6.0	0.9999999990				0.999999999

Thus 2m = 4 or 8 are good enough for all practical purposes.

The register in program I above has been used as follows:

STO 0	y
STO 1	2m-i , i=0,1,2,...
STO 2	x
STO 3	x/2m
STO 7	$\Sigma A_i \, y^{(2m-i)^2}$

C. Program B (continuous fraction) for G(x).

1	STO 1	8	1	15	RCL 0	22	g π
2	⪆	9	STO -1	16	g 1/x	23	2
3	STO 2	10	RCL 1	17	RCL 2	24	×
4	÷	11	g x=0	18	g x^2	25	f √
5	STO 0	12	GTO 15	19	g e^x	26	÷
6	RCL 2	13	RCL 0	20	f √		
7	STO +0	14	GTO 04	21	÷		

Assume $x > 0$.

Set x, enter n, R/S.

1 - G(x) is shown in the window.

The program works well when x is large, then only a small n is needed. For $x \geq 1$, set n=30. For $0.5 \leq x \leq 1.0$ set n=200 .

However, any degree of accuracy can be obtained for $x > 0$ by using n sufficiently large; e.g. n=800 for x=0.25 gives 6 correct decimals. 1 - G(x) = 0.401294.

The running time is n/100 minutes.

n odd gives a lower bound , n even an upper bound for 1-G(x).

<u>Examples:</u>

$x = 1.96$ $n = 4$ and $n = 5$ give respectively

$0.02507582 > 1-G(x) > 0.024962441$.

$x = 1.96$, $n = 30$ and 31 give the same

9 decimals $1-G(x) = 0.024997895$.

$x = 1$, $n = 30$ and $n = 31$ give respectively

$0.158668 > 1-G(x) > 0.158664$.

Of course program B should preferably be used for large x .

$x = 3$, $n = 4$ and 5 give respectively

$0.001350205 > 1-G(x) > 0.001349802$.

(The following examples show how the program works when x is small.

$x = 0.25$, $n = 800$ and $n = 801$ give respectively

$0.401294386 > 1-G(x) > 0.401292968$.

$x = 0.25$, $n = 2000$ and 2001 give the same

9 figures , $1-G(x) = 0.401293674$.

$x = 0.10$, $n = 800$ and $n = 801$ give respectively

$0.463659288 > 1-G(x) > 0.456721353$,

whereas $n = 2000$ and $n = 2001$ give respectively

$0.460302225 > 1-G(x) > 0.46004248$.

The true value is 0.4601722.)

MATEMATISK INSTITUTT
Avd. C

March 1977

PROGRAMS ON HP-25 DESK CALCULATOR
FOR THE BINOMIAL AND THE POISSON DISTRIBUTION

by

Erling Sverdrup

1. The Binomial distribution

In Statistical Research Report No. 7, 1976, "Programs on HP-25 desk calculator etc." the present author treated the binomial distribution cursory.

Below program A gives

$$B_n(x;p) = \sum_{j=o}^{x} b_n(j;p) \quad , \text{ where } \quad b_n(x;p) = \binom{n}{x} p^x (1-p)^{n-x}$$

for single values of x in a compact form.

However, this program is only useful for $(1-p)^n > 10^{-99}$, i.e.

$$n < 99/\log_{10} \frac{1}{1-p} = K(p)$$

which means that for $p = \frac{1}{2}$ the program can only be used for $n < 329$. (The accuracy is very good up to this value. For $n = 327$, $p = \frac{1}{2}$, $x = (n-1)/2$ we get $B_n = 0.500000023$, whereas the true value is of course 0.5 .)

Thus a program B is given below which is useful for larger values of n. If $n \geq K(p)$ then in the beginning $(1-p)^n$ is "inflated" by multiplying by 10^{99}. In the end the outcome is deflated by multiplying by 10^{-99}. This goes automatically without the operator having to be concerned with the value of n.

Program B is useful for

$$n < 2\,K(p) .$$

The upper limits to n in the two programs are then as follows:

Upper limits for n

p	Program A	Program B
0.5	328	657
0.25	792	1584
0.10	2163	4327

Thus on program B we got $B_{4327}(432 , 0.1) = 0.498652933$ (and $b_{4327}(432 , 0.1) = 0.020213882$) after seven minutes of running time.

Note that if n is large one should always arrange that $p \leq \frac{1}{2}$, by using $B_n(x;p) = 1 - B_n(n-x-1;1-p)$.

2. The Poisson distribution

The program for the Poisson distribution given in the above cited Research Report is only useful for $e^{-\lambda} > 10^{-99}$, i.e. for $\lambda \leq 227.9$. The program for

$$B(x;\lambda) = \sum_{j=0}^{x} b(j;\lambda) \quad ; \quad b(x;\lambda) = \frac{\lambda^x}{x!}e^{-\lambda}$$

given below, where the same principle for "inflation" and "deflation" as above has been used, is applicable for

$$\lambda \leq 455.9 .$$

The inflation factor is e^{227} .

3. Programs for the Binomial distribution

Program A for the cumulative Binomial distribution.

1	STO 5	9	RCL 3	17	RCL 5	25	RCL 2
2	R ↓	10	g 1/x	18	RCL 1	26	×
3	STO 3	11	1	19	f x≥y	27	RCL 1
4	≶	12	STO +2	20	GTO 33	28	÷
5	STO 2	13	-	21	1	29	STO ×0
6	f y^x	14	STO 4	22	STO +1	30	RCL 0
7	STO 0	15	0	23	STO -2	31	STO +7
8	STO 7	16	STO 1	24	RCL 4	32	GTO 17
						33	RCL 7

Set

n, enter, 1-p, enter, x , R/S .

$B_n(x,p)$ is shown in the window. RCL 0 gives $b_n(x,p)$. The program works if $n < 99/\log \frac{1}{1-p}$.

Example: $B_{100}(20,0.2) = 0.55946$.

Program B for the cumulative Binomial distribution for large n .

1	STO 5	13	$\gtrless$	25	RCL 5	37	STO ×0
2	R ↓	14	g 10^x	26	RCL 1	38	RCL 0
3	STO 3	15	STO 0	27	f $x \geq y$	39	STO +7
4	f log	16	STO 7	28	GTO 41	40	GTO 25
5	$\gtrless$	17	RCL 3	29	1	41	RCL 7
6	STO 2	18	g 1/x	30	STO +1	42	RCL 6
7	×	19	1	31	STO -2	43	STO -6
8	9	20	STO +2	32	RCL 4	44	g 10^x
9	9	21	-	33	RCL 2	45	×
10	CHS	22	STO 4	34	×	46	GTO 00
11	f $x \geq y$	23	0	35	RCL 1	47	STO 6
12	GTO 47	24	STO 1	36	+	48	-
						49	GTO 14

Set

n, enter, 1-p, enter, x, R/S.

$B_n(x;p)$ is shown in the window.

If $n < 99/\log_{10} \frac{1}{1-p}$, then RCL 0 = $b_n(x,p)$

If $n \geq 99/\log_{10} \frac{1}{1-p}$, Then RCL 0 × 10^{-99} = $b_n(x,p)$.

The program works if $n < 198 \log \frac{1}{1-p}$.

Example 1. $B_{101}(50;\frac{1}{2}) = 0.50000$

Example 2. $B_{401}(200;\frac{1}{2}) = 0.50000$.

Both examples should be run to check correct setting. Running time is roughly x seconds. Hence 3 minutes and 20 seconds in example 2.

The register in the above programs has been used as follows:

STO 0	$b_n(j,p)$	STO 4	$p/(1-p)$
STO 1	j	STO 5	x
STO 2	n-j+1	STO 6	99(in program B)
STO 3	1-p	STO 7	$B_n(j,p)$.

4. Program for the cumulative Poisson distribution for large λ.

1	STO 3	11	≷	21	STO +1	31	GTO 37
2	R↓	12	CHS	22	RCL 1	32	RCL 4
3	STO 2	13	g e^x	23	STO ÷0	33	RCL 2
4	2	14	STO 0	24	RCL 2	34	-
5	2	15	STO +7	25	STO ×0	35	g e^x
6	7	16	RCL 3	26	RCL 0	36	STO ×7
7	f x≥ y	17	RCL 1	27	GTO 15	37	0
8	GTO 11	18	f x≥ y	28	RCL 4	38	STO 1
9	STO 4	19	GTO 28	29	g x≠ 0	39	STO 4
10	GTO 12	20	1	30	GTO 32	40	RCL 7
						41	STO -7

Set

λ , enter , x , R/S.

$B(x;\lambda)$ is shown in the window.

If $\lambda > 227$. Set 227 , RCL 2 , - , g e^x , RCL 0 , × gives $b(x;\lambda)$

if $\lambda \leq 227$, RCL 0 gives $b(x,\lambda)$.

Example 1. $B(5;4.68) = 0.67191$.

Example 2. $B(450;450) = 0.51253$.

Both examples should be run to check correct setting of the program.

The register in this program has been used as follows:

STO 0	$b(j,\lambda)$	STO 4	227
STO 1	j	STO 5-6	not used
STO 2	λ	STO 7	$B(j,\lambda)$
STO 3	x .		