STATISTICAL RESEARCH REPORT No 5
Institute of Mathematics Mai 1976
University of Oslo

DEFICIENCIES IN LINEAR NORMAL EXPERIMENTS.

by

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CONTENT

Summary

Let $X_1,\ldots,X_n$ be independent and normally distributed variables, such that $0 < \text{var } X_i = \sigma^2$, $i=1,\ldots,n$ and $E(X_1,\ldots,X_n)' = A'\beta$ where A is an $k \times n$ matrix with known coefficients and $\beta = (\beta_1,\ldots,\beta_n)'$ is an unknown column matrix. σ^2 may be known or unknown. Denote the experiment obtained by observing $X_1,\ldots,X_n$ by $\mathcal{E}_A$. Let A and B be matrices of dimension $n_A \times k$ and $n_B \times k$.

The deficiency $\delta(\mathcal{E}_A, \mathcal{E}_B)$ is computed when σ^2 is known, and for some cases, including the case $BB' - AA'$ positive semidefinit and AA' nonsingular, also when σ^2 is unknown.

2. Introduction and basic facts.

Definition. An experiment is a pair $\mathcal{E} = ((\mathcal{X},\mathcal{A}); (P_\theta, \theta\in\Theta))$ where $(\mathcal{X},\mathcal{A})$ is a measurable space and $(P_\theta,\theta\in\Theta)$ is a family of probability measures over $(\mathcal{X},\mathcal{A})$.

For two experiments $\mathcal{E}$ and $\mathcal{F}$ indexed by the same parameter set Θ Le Cam defined in [2] the deficiency $\delta(\mathcal{E},\mathcal{F})$ of $\mathcal{E}$ relative to $\mathcal{F}$. The Δ- distance between $\mathcal{E}$ and $\mathcal{F}$ is the number $\Delta(\mathcal{E},\mathcal{F}) = \max(\delta(\mathcal{E},\mathcal{F}), \delta(\mathcal{F},\mathcal{E}))$.

If $\delta(\mathcal{E},\mathcal{F}) = 0$, we say that $\mathcal{E}$ is more informative than $\mathcal{F}$ and write this $\mathcal{E} \geq \mathcal{F}$. If also $\delta(\mathcal{F},\mathcal{E}) = 0$, we say that $\mathcal{E}$ and $\mathcal{F}$ are equivalent and write this $\mathcal{E} \sim \mathcal{F}$.

For $\mathcal{E}, \mathcal{F}, \mathcal{G}$ experiments with the same parameter set Θ the following relations hold

$$0 \leq \delta(\mathcal{E},\mathcal{F}) \leq 2 \qquad \delta(\mathcal{E},\mathcal{E}) = 0$$
$$\delta(\mathcal{E},\mathcal{F}) \leq \delta(\mathcal{E},\mathcal{G}) + \delta(\mathcal{G},\mathcal{F})$$

In particular Δ is a pseudometric.

Let $\mathcal{E} = ((\mathcal{X},\mathcal{A}), (P_\theta\ \theta\in\Theta))$ and $\mathcal{F} = ((\mathcal{Y},\mathcal{B}), Q_\theta\in\Theta)$ be two experiments such that $(P_\theta,\theta\in\Theta)$ is deminated, $\mathcal{Y}$ is a Borel subset of a complete separable metric space and $\mathcal{B}$ is the class of Borel subsets of $\mathcal{Y}$. Then Le Cam [2] has shown that $\delta(\mathcal{E},\mathcal{F}) = \inf_{M\in\mathcal{M}} \sup \|P_\theta M - Q_\theta\|$ where $\mathcal{M}$ is the set of all Markov kernels from $(\mathcal{X},\mathcal{A})$ to $(\mathcal{Y},\mathcal{B})$.

In this paper we will exploit certain symmetric properties of the experiments $\mathcal{E}$ and $\mathcal{F}$ to be able to substitute the class $\mathcal{M}$ in the above expression with a smaller class consisting of "invariant" Markov kernels.

Let G be a group of transformations acting on Θ, $\mathcal{X}$, $\mathcal{Y}$ such that $x \to g(x)$, $y \to g(y)$ are measurable $g\in G$ and

$P_\theta g^{-1} = Pg_\theta$, $Q_\theta g^{-1} = Qg_\theta$, $g \in G$, $\theta \in \Theta$. A Markov kernel is called invariant if $M(g(B)|g(x)) = M(B|x)$ $g \in G$, $B \in \mathcal{B}, x \in \chi - N$ where $P_\theta(N) = 0$, $\theta \in \Theta$. Let $\mathcal{M}_G$ be the set of invariant Markov kernels from $(\chi, \mathcal{A})$ to $(\mathcal{Y}, \mathcal{B})$. It then follows from [5] that the following conditions are sufficient for $\delta(\mathcal{E}, \mathcal{F}) = \inf_{M \in \mathcal{M}_G} \sup_\theta \| P_\theta M - Q_\theta \|$

(i) $(P_\theta, \theta \in \Theta)$ is dominated, $\mathcal{Y}$ is a Borel subset of a complete separable metric space and $\mathcal{B}$ is the class of Borel subsets of $\mathcal{Y}$.

(ii) The families $(P_\theta, \theta \in \Theta)$ and $(Q_\theta, \theta \in \Theta)$ are invariant

(iii) There exists a σ-algebra $\mathcal{G}$ in G such that the maps $(x,g) \to g(x)$, $(y,g) \to g(y)$ are respectively $\mathcal{A} \times \mathcal{G}$ and $\mathcal{B} \times \mathcal{G}$ measurable.

(iv) There exists a σ-finite measure τ on $(G, \mathcal{G})$ such that $\tau(B) = 0$ implies $\tau(Bg) = 0$, $B \in \mathcal{B}$, $g \in G$.

(v) The group G has an invariant mean. **If in addition:**

(vi) There exists one $M \in \mathcal{M}_G$ so that $M(g(B)|g(x)) = M(B|x)$ $B \in \mathcal{B}$, $g \in G$, $x \in \chi$, $\mathcal{M}_G$ may be substituted with $\mathcal{M}_{Go} = \{M \in \mathcal{M} | M(g(B)|g(x)) = M(B|x)\ B \in \mathcal{B}, g \in \mathcal{G}, x \in \chi\}$ i.e. we can restrict attention to invariant Markov kernels with $\emptyset$ as exeptional set.

A sufficient condition for (v) to hold is that $\mathcal{G}$ is solvable.

Suppose $\mathcal{E}_P = ((\chi, \mathcal{A}), (P_\theta, \theta \in \Theta))$ where $\Theta = \chi$ is a second countable locally compact topological group which is Hausdorf, $\mathcal{A}$ is the Borel subsets of χ, and the P_θ's are given by $P_\theta(A) = P(A\theta^{-1})$ $A \in \mathcal{A}$ $\theta \in \chi$ where P is a probability measure. Then $\mathcal{E}_P$ is called a translation experiment. If $\mathcal{E}_Q = ((\mathcal{X},\mathcal{A}), (Q_\theta, \theta\in\Theta))$ is another translation experiment, let $g\in G$ be of the form $(x,g) \to x\theta^{-1}$ where $\theta\in\Theta$. Then the conditions (ii), (iii)and (vi) are satisfied, and $\mathcal{X}$ is a complete separable metric space. If we let τ be the Haar measure on $(\mathcal{X},\mathcal{A})$, also (v) is seen to be satisfied. Hence $\delta(\mathcal{E}_P, \mathcal{E}_Q) = \inf_{M\in\mathcal{M}_{Go}} \sup_\theta \|P_\theta M - Q_\theta\|$ provided $(P_\theta, \theta\in\Theta)$ is dominated and $\mathcal{X}$ is solvable. Torgersen [5] has shown that in this case every invariant Markov kernel with $\emptyset$ as the exeptional set may be written $M(B|x) = N(Bx^{-1})$ where N is a probability measure over $(\mathcal{X},\mathcal{A})$ and that $\delta(\mathcal{E}_P,\mathcal{E}_Q) = \inf \|N * P - Q\|$ where $N * P\ (A) = N\times P\ (\{(x_1,x_2)|x_1x_2 \in A\})$.

The following result, also from [5]and valid under the same conditions, gives a direct method to determine δ for translation experiments. If N_o is a least favourable distribution for all level $\alpha \in [0,1]$ for testing $H: P_\theta"\theta\in\Theta$ against Q where $P_\theta"(A) = P(\theta^{-1}A)\ \theta\in\Theta,\ A\in\mathcal{A}$, then $\delta(\mathcal{E}_P,\mathcal{E}_Q) = \|N_o * P - Q\|$.

The purpose of this paper is to use the above results to compute the deficiencies between linear normal experiments. These experiments may be described as follows: Let A be a known $k\mathrm{x}n_A$ matrix and $\mathcal{E}_A$ the experiment given by the independent normally distributed variables $X_1,\dots,X_{n_A}$ with $\mathrm{var}\ X_i = \sigma^2\ i = 1,\dots,n_A$ and $E(X_1,\dots,X_{n_A})' = A'\beta$ where $\beta = (\beta_1,\dots,\beta_k)' \in \mathbb{R}^k$. To avoid trivialities we shall assume $n_A \geq k \geq 1$.

The parameter set is $]-\infty,\infty[^k$ if σ^2 is known, and

$]-\infty, \infty[^k \times \;]0, \infty[$ if σ^2 is unknown.

From theorem 3.1 in [1] it follows that if A and B are matnces of dimension $k \times n_A$ and $k \times n_B$, then $\mathcal{E}_A \geq \mathcal{E}_B$ if and only if $AA' - BB'$ is positve semidefinite when σ^2 is known, and $\mathcal{E}_A \geq \mathcal{E}_B$ if and only if $AA' - BB'$ is positive semidefinite and $n_A \geq n_B + \text{rank}\,(AA' - BB')$ when σ^2 is unknown. Then $\mathcal{E}_A \sim \mathcal{E}_B$ if and only if $AA' = BB'$ if σ^2 is known, and $\mathcal{E}_A \sim \mathcal{E}_B$ if and only if $AA' = BB'$ and $n_A = n_B$ if σ^2 is unknown.

In the computation of $\delta(\mathcal{E}_A, \mathcal{E}_B)$ we may therefor choose the experiments within the equivalence classes determined by AA', BB' when σ^2 is known, and (AA', n_A), (BB', n_B) when σ^2 is unknown.

3. The case of known variance G^2

Proposition 3.1 Suppose $AA' = I$, $BB' = \Delta$ where Δ is a $k \times k$ diagonal matrix with diagonal elements $\Delta_1, \dots, \Delta_k \geq 0$. Then $\delta(\mathcal{G}_A, \mathcal{G}_B) = E|1 - \prod_{\Delta_i > 1} \sqrt{\Delta_i} \exp(\frac{1}{2}(\Delta_i - 1) Y_i^2|$ where $Y_1, \dots, Y_k$ are independent and identically $N(0,1)$ distributed.

Proof

We may choose $A = \begin{pmatrix} 1 \dots 0 & 0 \dots 0 \\ \vdots & \vdots \\ 0 \dots 1 & 0 \dots 0 \end{pmatrix}$ where the last $n_A - k$ columns consist of only zeros, and

$$B = \begin{pmatrix} \sqrt{\Delta_1} \dots 0 & 0 \dots 0 \\ \vdots \quad \ddots & \vdots \\ 0 \quad \sqrt{\Delta_k} & 0 \dots 0 \end{pmatrix}$$ where the last $n_B - k$ columns consist of only zeros. Without loss of generality we may assume that $\Delta_1, \dots, \Delta_l > 0$ and $\Delta_{l+1} = \dots = \Delta_k = 0$ $0 \leq l \leq k$. This means that $\mathcal{G}_A$ is given by the independent, normally distributet variables $X_1, \dots, X_{n_A}$ where

$$E X_i = \begin{cases} \beta_i & i = 1, \dots, k \\ 0 & i = k+1, \dots, n_A \end{cases} \qquad \text{var } X_i = \sigma^2 \quad i = 1, \dots, n_A$$

Similarly $\mathcal{G}_B$ is given by the independent, normally distributet variables $Y_1, \dots, Y_{n_B}$ where

$$EY_i = \begin{cases} \sqrt{\Delta_i}\, \beta_i & i = 1, \dots, l \\ 0 & i = l+1, \dots, n_B \end{cases} \qquad \text{var } Y_i = \sigma^2 \quad i = 1, \dots, n_B$$

By sufficiency $X_{k+1}, \dots, X_{n_A}$ may be deleted from $\mathcal{G}_A$ and

$Y_{l+1}, \ldots, Y_{n_B}$ may be deleted from $\mathcal{E}_B$. Furthermore, in the same way as in the proof of proposition 2.1 in [1], it may be shown that $X_{l+1}, \ldots, X_{n_A}$ may be deleted in $\mathcal{E}_A$. Finally we may replace $Y_1, \ldots, Y_l$ with $Z_1, \ldots, Z_l$ where $Z_i = \frac{Y_i}{\sqrt{\Delta_i}}$ $i = 1, \ldots, l$.

Now $\mathcal{E}_A$ and $\mathcal{E}_B$ are translation experiments for addition in $\mathbb{R}^l$. Since addition in $\mathbb{R}^l$ is commutative and $\mathcal{E}_A$ and $\mathcal{E}_B$ are both dominated, we may use the method indicated in section 2 to find $\delta(\mathcal{E}_A, \mathcal{E}_B)$. Let $P_\beta, \beta \in \mathbb{R}^l$ be the measure defined by $X_1, \ldots X_l$ independent and normally distributed with $EX_i = \beta_i$, $\operatorname{var} X_i = \sigma^2, i = 1, \ldots l$ and Q be the measure defined by $Y_1, \ldots, Y_l$ independent and normally distributed with $EY_i = 0$, $\operatorname{var} Y_i = \frac{\sigma^2}{\Delta_i}, i = 1, \ldots, l$. Then the least favourably distribution N_o for testing $H: P_\beta, \beta \in \mathbb{R}^l$ against the alternative $K: Q$ is given by the independent variables $U_1, \ldots, U_l$ where $U_i = 0$ with probability 1 if $\Delta_i \geq 1$ and U_i is $N(0, \sigma\sqrt{\frac{1}{\Delta_i} - 1})$ distributed if $\Delta_i < 1$. Hence $\delta(\mathcal{E}_A, \mathcal{E}_B) = \| N_o * P_o - Q \|$. But $N_o * P_o$ has density $\prod_{\Delta_i < 1} \frac{\sqrt{\Delta_i}}{\sigma} \varphi\left(\frac{\sqrt{\Delta_i} x_i}{\sigma}\right) \prod_{\Delta_i \geq 1} \frac{1}{\sigma} \varphi\left(\frac{x_i}{\sigma}\right)$ with respect to the Lebesgues measure in $\mathbb{R}^l$. □

<u>Proposition 3.2</u> If rank $A' = k$, then $\delta(\mathcal{E}_A, \mathcal{E}_B) = E\left|1 - \prod_{\Delta_i > 1} \exp\left(-\frac{1}{2}(\Delta_i - 1) Y_i^2\right)\right|$ where $\Delta_1, \ldots, \Delta_k$ are the solution of $\det[BB' - \lambda AA'] = 0$, and $Y_1, \ldots, Y_k$ are independent $N(0,1)$ distributed.

<u>Proof</u> Since BB' is positive semidefinit, there exists a $k \times k$ nonsingular matrix F such that $F'AA'F = I$ and

$F'BB'F = \Delta$ where $\Delta_1, \dots, \Delta_k \geq 0$ and $\Delta_1, \dots, \Delta_k$ are the solutions of $\det(BB' - \lambda AA') = 0$.

Let $\tilde{A} = F'A$, $\tilde{B} = F'B$. If P_β and Q_β are, respectively, the probability measures inn $\mathcal{G}_A$ and $\mathcal{G}_B$ corresponding to the parameter value β, then since $A'F\beta = \tilde{A}\beta$, $\delta(\mathcal{G}_A, \mathcal{G}_B) = \inf_M \sup_\beta \| P_\beta M - Q_\beta \| - \inf_M \sup_{F\beta} \| P_{F\beta} M - Q_{F\beta} \| = \delta(\mathcal{G}_{\tilde{A}}, \mathcal{G}_{\tilde{B}}) =$

$$E \left| 1 - \prod_{\Delta_i > 1} \sqrt{\Delta_i} \exp\left(-\tfrac{1}{2}(\Delta_i - 1) Y_i^2\right) \right|$$

<u>Proposition 3.3</u> If row $[B'] \not\subset$ row $[A']$, then $\delta(\mathcal{G}_A, \mathcal{G}_B) = 2$.

<u>Proof</u> row $[B'] \not\subset$ row $[A']$ implies $(\text{row } [B'])^\perp \not\supset (\text{row } A')^\perp$. Let $\beta_o \in (\text{row } A')^\perp$. $\beta_o \notin (\text{row } B')^\perp$. Then $A'\beta_o = 0, B'\beta_o \neq 0$ and $\delta(\mathcal{G}_A, \mathcal{G}_B) = \inf_M \sup_\beta \| P_\beta M - Q_\beta \| \geq \inf_M \sup_{t \in \mathbb{R}} \| P_{t\beta_o} M - Q_{t\beta_o} \| = \inf_M \sup_t \| P_o M - Q_{t\beta_o} \|$. But $\| P_o M - Q_{t\beta_o} \| \underset{t \to \infty}{\to} 2$ for all Markov kernels M, so that $\delta(\mathcal{G}_A, \mathcal{G}_B) = 2$. □

Suppose $BB' - AA'$ is positive semidefinit and rank $A' = k$. If F has the same meaning as in the foregoing proof, then $Y'(\Delta - I) = Y'F'F'^{-1}(\Delta - I)F^{-1}FY = Z'(BB' - AA')Z$ where $Z = FY$. Furthermore $EZZ' = EFYY'F' = FF' = (AA')^{-1}$ and

$\frac{\det(BB')}{\det(AA')} = \frac{\det(F'BB'F)}{\det(F'AA'F)} = \Delta_1 \dots \Delta_k$ so that we may write

$\delta(\mathcal{G}_A, \mathcal{G}_B) = E \left| \frac{\det(BB')}{\det(AA')} \exp\left[-\tfrac{1}{2} Z'(BB' - AA')Z\right] - 1 \right|$ where Z is multivariate normal with mean zero and covariance matrix $(AA')^{-1}$. This is the result given by Le Cam in [3]

Suppose next that row $[B'] \subset$ row $[A']$ and let $V_1', \dots, V_r'$ be a basis for row $[A']$, $0 \leq r \leq k$. Then as in the proof of theorem 3.1 in [1], we may write $A = VS$ where $V = (V_1', \dots, V_r')$ is a $k \times r$ matrix and S is a $r \times n_A$ matrix of rank r.

Similarly $B = VT$ with T a $r \times n_B$ matrix. By writing $\alpha = V'\beta$ so that $A'\beta = S'V\beta = S'\alpha$ and $B'\beta = T'\alpha$, it follows that $\delta(\mathcal{G}_A, \mathcal{G}_B) = \inf_M \sup_\beta \| P_\beta M - Q_\beta \| = \inf_M \sup_\alpha \| P'_\alpha M - Q'_\alpha \| = \delta(\mathcal{G}_S, \mathcal{G}_T)$ where P_α' and Q_α' are, respectively, the measures in $\mathcal{G}_S$ and $\mathcal{G}_T$, corresponding to the parameter value α. The following result is then an immediate consequence of proposition 3.2

<u>Proposition 3.4</u> If row $[B'] \subset$ row $[A']$,

$$\delta(\mathcal{G}_A, \mathcal{G}_B) = E\left| 1 - \prod_{\Delta_i > 1} \sqrt{\Delta_i} \exp\left(-\tfrac{1}{2}(\Delta_i - 1) Y_i^2\right) \right|$$

where $\Delta_1, \ldots, \Delta_r$ are the solution of $\det(TT' - \lambda SS') = 0$ and $A = VS$, $B = VT$, rank $S = r$, $V = (V_1', \ldots, V_r')$ with $V_1', \ldots, V_r'$ a basis for row $[A']$.

If row $[A'] \neq$ row $[B']$ then either row $[A'] \not\subset$ row $[B']$ or row $[B'] \not\subset$ row $[A']$ so that $\delta(\mathcal{G}_B, \mathcal{G}_A) = 2$ or $\delta(\mathcal{G}_A, \mathcal{G}_B) = 2$. Consequently $\Delta(\mathcal{G}_A, \mathcal{G}_B) = 2$.

Suppose next that row $[A'] =$ row $[B']$, and let V, S, T have the same meaning as in proposition 3.4 If then λ is a solution of $\det(TT' - \lambda SS') = 0$, λ^{-1} is a solution of $\det(SS' - \lambda TT') = 0$. Nothing that $E\left| 1 - \prod_{\Delta_i < 1} \sqrt{\Delta_i} \exp\left(-\tfrac{1}{2}(\Delta_i^{-1} - 1) Y_i^2\right) \right| = E\left| 1 - \prod_{\Delta_i < 1} \sqrt{\Delta_i} \exp\left(-\tfrac{1}{2}(\Delta_i - 1) Y_i^2\right) \right|$, this gives together with proposition 3.4:

<u>Theorem 3.1</u> If row $[A'] \neq$ row $[B']$, then $\Delta(\mathcal{G}_A, \mathcal{G}_B) = 2$

If row A' = row B' and $A = VS$, $B = VT$ where $V = (V_1', .. V_r')$ and $V_1', \ldots, V_r'$ is a basis for row $[A']$, then $\Delta(\mathcal{G}_A, \mathcal{G}_B) = \max\left(E\left| 1 - \prod_{\Delta_i > 1} \exp\left(-\tfrac{1}{2}(\Delta_i - 1) Y_i^2\right)\right|, E\left| 1 - \prod_{\Delta_i < 1} \exp\left(-\tfrac{1}{2}(\Delta_i - 1) Y_i^2\right)\right|\right)$

where $\Delta_1, \ldots, \Delta_r$ are the solutions of $\det(TT' - \lambda SS') = 0$ and $Y_1, \ldots, Y_r$ are independent and identically $N(0,1)$ distributed.

Consider now linear normal e periments where σ^2 is unknown. By fixing the parameter σ^2 we obtain experiments for which δ can be found by the methods of this section. This means that a δ computed for known σ^2 always gives a lower bound for the corresponding δ with σ^2 unknown.

From theorem 2.1 it then follows that the Δ-distance is 2 between the experiments given by $X_1,..X_n$ independent and normally distributed with var $X_i = \sigma^2$, $E X_i = \alpha + \beta ti$ $i = 1,..,n$, and $Y_1,...,Y_n$ independent and normally distributed with var $Y_i = \sigma^2$, $EY_i = \alpha + \beta t_i + \gamma t_i^2$ $i = 1,..,n$ whether σ^2 is known or not. The Δ-distance is thus of no help if we want to determine the amount of information obtained by observing $Y_1,.., Y_n$ instead of $X_1,..,X_n$.

4. The case of unknown variance σ^2

Some of the notations which will be used in this section are:

If (X,τ) is a topological space, let $\mathcal{B}(X) = \sigma(\{B \mid B \in \tau\})$ be the Borel sets in X.

$P_{l,n_1,\beta_1,\dots,\beta_l,\sigma^2}$, $Q_{l,n_2,\beta_1,\dots,\beta_l,\sigma^2}$ are the probability measures over $(\mathbb{R}^l \times \mathbb{R}^+, \mathcal{B}(\mathbb{R}^l \times \mathbb{R}^+))$ given by $X_1,\dots,X_l$, S independent $X_i \sim N(\beta_i, \sigma)$ $i=1,\dots,l$, $S/\sigma^2 \sim \chi^2_{n_1}$ and by $Y_1,\dots,Y_i,T$ independent $Y_i \sim N(\beta_i \frac{\sigma}{\sqrt{\Delta_i}})$, $i=1,\dots,l$, $T/\sigma^2 \sim \chi^2_{n_2}$ where $\Delta_1,\dots,\Delta_l > 0$ are known.

$P'_{l,\beta_1,\dots,\beta_l,\sigma^2}$, $Q'_{l,\beta_1,\dots,\beta_l,\sigma^2}$ are the probability measures over $(\mathbb{R}^l, \mathcal{B}(\mathbb{R}^l))$ given by $X_1,\dots,X_l$ independent $X_i \sim N(\beta_i,\sigma)$ $i = 1,\dots l$ and by $Y_1,\dots,Y_l$ independent $Y_i \sim N(\beta_i, \frac{\sigma}{\sqrt{\Delta_i}})$ $i = 1,\dots,l$

$$\varphi(x) = (2\pi)^{-\frac{1}{2}} \exp(-\frac{x^2}{2}) \ , \ \Phi(x) = \int_{-\infty}^{x} \varphi(u)\, du$$

$$\gamma_{n;t}(x) = (\Gamma(\tfrac{n}{2})\, 2^{n/2})^{-1} x^{n/2-1} \exp(-\tfrac{x}{2t})\, t^{-n/2}$$

$$x > 0 \quad t > 0 \quad \Gamma_{n,t}(x) = \int_0^x \gamma_{n,t}(u)\, du$$

$\#(S)$ is the number of elements in S if S is finite.

Suppose first that $AA' = I$, $BB' = \Delta$ where Δ is a diagonal matrix with diagonal elements $\Delta_1,\dots,\Delta_k \geq 0$. Without loss of generality we may assume that $\Delta_1,\dots,\Delta_{l-m} \geq 1$, $0 < \Delta_{l-m+1},\dots,\Delta_l < 1$ and $\Delta_{l+1} = \dots = \Delta_k = 0$ where $k \geq l \geq 0$ In the same manner as in section 3 we may consider a situation where

$$\overset{\varphi}{\mathcal{G}}_A = ((\mathbb{R}^l \times \mathbb{R}^+, \ (\mathbb{R}^l \times \mathbb{R}^+), (P_{l,n_A-k,\beta_1,\ldots,\beta_l,\sigma^2}, (\beta_1,\ldots,\beta_l,\sigma^2) \in \mathbb{R}^l \times \mathbb{R}^+)) \quad \text{if} \quad n_A > k$$

$$\overset{\varphi}{\mathcal{G}}_A = ((\mathbb{R}^l, \ (\mathbb{R}^l)), (P'_{l,\beta_1,\ldots,\beta_l,\sigma^2}, (\beta_1,\ldots,\beta_l,\sigma^2) \in \mathbb{R}^l \times \mathbb{R}^+)) \quad \text{if} \quad n_A = k$$

$$\overset{\varphi}{\mathcal{G}}_B = ((\mathbb{R}^l \times \mathbb{R}^+, \ (\mathbb{R}^l \times \mathbb{R}^+), (Q_{l,n_B-l,\beta_1,\ldots,\beta_l,\sigma^2}, (\beta_1,\ldots,\beta_l,\sigma^2) \in \mathbb{R}^l \times \mathbb{R}^+)) \quad \text{if} \quad n_B > l$$

$$\overset{\varphi}{\mathcal{G}}_B = ((\mathbb{R}^l, \ (\mathbb{R}^l)), Q'_{l,\beta_1,\ldots,\beta_l,\sigma^2}, (\beta_1,\ldots,\beta_l,\sigma^2) \in (\mathbb{R}^l \times \mathbb{R}^+)) \quad \text{if} \quad n_B = l$$

The reduction is quite analogous with what was done in section 3 except that sufficiency now gives that $X_{k+1},\ldots,X_{n_A}$ must be replaced with $S = \sum_{i=k+1}^{n_A} X_i^2$ when $n_A > k$ and that $Y_{l+1},\ldots,Y_{n_B}$ must be replaced with $T = \sum_{i=l+1}^{n_B} Y_i^2$ when $n_B > l$.

Consider now the group $\mathbb{R}^l \times \mathbb{R}^+$ with group operation $xy = (y_1 + \sqrt{y^1}x_1,\ldots,y_l + \sqrt{y^1}x_l, x^1y^1)$ if $x = (x_1,\ldots,x_l,x^1)$, $y = (y_1,\ldots,y_l,y^1) \in \mathbb{R} \times \mathbb{R}^+$ IT may be shown that this group is solvable and consequently has an invariant mean. With the standard topology for $\mathbb{R}^l \times \mathbb{R}^+$ the group operation is continuous. Hence $\mathbb{R}^l \times \mathbb{R}^+$ is a topological group.

<u>Proposition 4.1</u> If $n_A = k$ and $n_B = l$, $\delta(\overset{\varphi}{\mathcal{G}}_A, \overset{\varphi}{\mathcal{G}}_B) = 2$

<u>Proof</u> Let the group G be given by

$$g(x_1,\ldots,x_l) = (\sqrt{g^1}\, x_1 + g_1,\ldots,\sqrt{g^1}\, x_l + g_l)$$

$$g(y_1,\ldots,y_l,t) = (\sqrt{g^1}y_1 + g_1,\ldots, \sqrt{g^1}y_l + g_l, g^1 t)$$

$$g(\beta_1,\ldots,\beta_l,\sigma^2) = (\sqrt{g^1}\beta_1 + g_1,\ldots, \sqrt{g^1}\beta_l + g_l, g^1\sigma^2)$$

where $(g_1,\ldots,g_l, g^1) \in \mathbb{R}^l + \mathbb{R}^+$. It may be verified that the assumptions (i) - (v) given in section 2 are satisfied so that we may restrict attention to the set of invariant Markov kernels $\mathcal{M}_G$. It is furthermore not difficult to show that every $M \in \mathcal{M}_G$ must have ϕ as exeptional set i.e. $\mathcal{M}_G = \mathcal{M}_{Go}$

Assume $\delta(\mathcal{E}_A, \mathcal{E}_B) = \delta < 2$ and let $\varepsilon > 0$ so that $\delta + \varepsilon < 2$. Then there exists $M \in \mathcal{M}_{G_0}$ so that

$$\| P'_{1,\beta_1,\ldots,\beta_l,\sigma^2} M - Q_{1,n_B-1,\beta_1,\ldots,\beta_l,\sigma^2} \| < \varepsilon + \delta$$

$$(\beta_1,\ldots,\beta_l,\sigma^2) \in \mathbb{R}^l \times \mathbb{R}^+$$

Suppose $B_1 x \ldots x B_l \times B \subset K$ where K is compact and $B_i \in \mathcal{B}(\mathbb{R})$ $i=1,\ldots,l$, $B \in \mathcal{B}(\mathbb{R}^+)$. Then

$$M(B_1 x \ldots B_l \times B | x_1,\ldots,x_l) = M(\sqrt{g^1}B_1 + g_1 x \ldots x \sqrt{g^1}B_l + g_l x g^1 B |$$
$$\sqrt{g^1}x_1 + g_1,\ldots,\sqrt{g^1}x_l + g_l)$$

Now let $g^1 \to 0$. Then $\sqrt{g^1}B_1 + g_1 x \ldots x \sqrt{g^1}B_l + g_l \times g^1 B \to \emptyset$ so that $M(B_1 x \ldots x B_l x B \mid x,,\ldots,x_l) = 0$ which is a contradiction since $\mathbb{R}^l \times \mathbb{R}^+$ is σ-compact and probability measures on metric spaces are regular. □

<u>Proposition 4.2</u> If $n_A = k$, $n_B = 1$,

$$\delta(\mathcal{E}_A, \mathcal{E}_B) = \| P'_{1,0,\ldots,0,1} - Q'_{1,0,\ldots,0,1} \|$$

<u>Proof</u> The proof is analogous to a part of the proof of proposition 2.1 i [1].

Let G be the group given by

$$g(x_1,\ldots,x_l) = (\sqrt{g^1}x_1 + g_1,\ldots, \sqrt{g^1}x_l + g_l)$$

$$g(y_1,\dots,y_l) = (\sqrt{g^1}y_1 + g_1,\dots,\sqrt{g^1}y_l + g_l)$$
$$g(\beta_1,\dots,\beta_l,\epsilon^2) = (\sqrt{g^1}\beta_1 + g_1,\dots,\sqrt{g^1}\beta_l + g_l, g^1\sigma^2)$$
$$(g_1,\dots,g_l,g^1) \in \mathbb{R}^l \times \mathbb{R}^+$$

It is easily verified that

$$\delta(\mathcal{E}_A, \mathcal{E}_B) = \inf_{M\in\mathcal{M}_G} \sup_{(\beta_1,\dots,\beta_l,\sigma^2)} \|P'_{1,\beta_1,\dots,\beta_l,\sigma^2} M - Q'_{1,\beta_1,\dots,\beta_l,\sigma^2}\| = \inf_{M\in\mathcal{M}_G} \|P'_{1,0,\dots,0,1}M - Q'_{1,0,\dots,0,1}\|.$$

Suppose $M\in\mathcal{M}_G$. Since $M(\cdot|x_1,\dots,x_l)$ is a probability measure over a complete separable metric space, $M(\cdot|x_1,\dots,x_l)$ is regular. Thus, for $\epsilon > 0$ there exists K compact so that $M(K|x_1,\dots,x_l) > 1-\epsilon$. Let $\{x_1,\dots,x_l\}\cup K \subset \prod_{i=1}^{l} [a_i,b_i]$.
Then

$$M(\prod_{i=1}^{l}[a_i,b_i]|x_1,\dots,x_l) = M(\prod_{i=1}^{l}\sqrt{g^1}[a_i,b_i]+g_i|\sqrt{g^1}x_1+g_1,\dots,\sqrt{g^1}x_l+g_l)$$
$$= M(\prod_{i=1}^{l}\sqrt{g^1}([a_i,b_i]+x_i|x_1,\dots,x_l) > 1-\epsilon$$ by inserting $g_i=x_i-\sqrt{g^1}x_i$

$i = 1,\dots,l$. Now let $g^1 \to 0$. Then $M(\{x_1,\dots,x_l\}| x_1,\dots,x_l) > 1-\epsilon$, so that $M(B|x_1,\dots,x_l) = I_B(x_1,\dots,x_l)$ $B\in\mathcal{B}(\mathbb{R}^l)$. □

Let us now consider the case where $n_A > k$. Fist we need a lemma.

<u>Lemma 4.1</u> Let $\mathcal{E}_i=(\chi_i,\mathcal{A}_i),(P_{\theta_i,\theta_3}(\theta_i,\theta_3)\in\Theta_i\times\Theta_3))$ $i=1,2$ $\mathcal{F}_j=((\mathcal{Y}_j,\mathcal{B}_j),(Q_{\theta_j,\theta_3},(\theta_j,\theta_3)\in\Theta_j\times\Theta_3))$ $j=1.2$ be four experiments such that $(Q_{\theta_j,\theta_3}(\theta_j,\theta_3)\in\Theta_j\times\Theta_3)$ $j=1,2$ are diminated and $\mathcal{Y}_j$ $j=1,2$ are Borel subsets of complete separable metric spaces and $\mathcal{B}_j$ $j=1,2$ are the classes of Borel subsets of $\mathcal{Y}_j$ $j=1,2$.

Let $\tilde{\mathcal{G}}=((\mathcal{X}_1 \times \mathcal{X}_2, \mathcal{A}_1 \times \mathcal{A}_2), (P_{\theta_1,\theta_3} \times P_{\theta_1,\theta_3}$

$(\theta_1,\theta_2,\theta_3) \in \Theta_1 \times \Theta_2 \times \Theta_3))$

$\tilde{\mathcal{F}} =((\mathcal{Y}_1 \times \mathcal{Y}_2, \mathcal{B}_1 \times \mathcal{B}_2), Q_{\theta_1,\theta_3} \times Q_{\theta_2,\theta_3}$

$(\theta_1,\theta_2,\theta_3) \in \Theta_1 \times \Theta_2 \times \Theta_3))$

Then if $\mathcal{G}_2 \geq \mathcal{F}_2$, $\delta(\tilde{\mathcal{G}},\tilde{\mathcal{F}}) \leq \delta(\mathcal{G},\mathcal{F})$.

Proof From the assumptions it follows that there exists a Markov kernel M_2 from $(\mathcal{X}_2,\mathcal{B}_2)$ to $(\mathcal{Y}_2,\mathcal{A}_2)$ such that $P_{\theta_2,\theta_3} M_2 = Q_{\theta_2,\theta_3}$ $(\theta_2,\theta_3)\in\Theta_2 \times \Theta_3$. If M_1 is a Markov kernel from $(\mathcal{X}_1,\mathcal{A}_1)$ to $(\mathcal{Y}_1,\mathcal{B}_1)$, Then $M_1 \times M_2$ is a Markov kernel from $(\mathcal{X}_1 \times \mathcal{X}_2, \mathcal{A}_1 \times \mathcal{A}_2)$ to $(\mathcal{Y}_1 \times \mathcal{Y}_2, \mathcal{B}_1 \times \mathcal{B}_2)$ and $\| P_{\theta_1,\theta_2} \times P_{\theta_2,\theta_3} M_1 \times M_2 -$

$$Q_{\theta_1,\theta_2} \times P_{\theta_2,\theta_3} \| = \|P_{\theta_1,\theta_3} M_1 - Q_{\theta_1,\theta_3} \|$$

Proposition 4.3 If $n_A - k \geq n_B - 1 + m \geq 0$, then

$\delta(\mathcal{G}_A, \mathcal{G}_B) = \|P'_{1-m,0,\ldots,0,1} - Q'_{1-m,0,\ldots,0,1}\|$ □

Remark If $n_A - k = n_B - 1 = m = 0$ proposition 4.2 and 4.3 give the same result.

Proof Let $n_1=n_A-k$, $n_B-1=n_2$. The proof will be carried out only for $n_1,n_1 > 0$, the proofs of the cases $n_1=0$, $n_2 > 0$ and $n_1 > 0$, $n_2=0$ are quite analoguous.

$\delta(\mathcal{E}_A, \mathcal{E}_B) = \inf_M \sup_{(\beta_1,\ldots,\beta_l,\sigma^2)} \|P_{l,n_1,\beta_1,\ldots,\beta_l,\sigma^2}M - Q_{l,n_1,\beta_1,\ldots,\beta_l,\sigma^2}\| \geq$

$\inf_M \sup_{(\beta_1,\ldots,\beta_l,\sigma^2)} \|P_{l,n_1,\beta_1,\ldots,\beta_{l-m},0,\ldots,0,1}M - Q_{l,n_2,\beta_1,\ldots,\beta_{l-m},0,\ldots,0,1}\|$

$= \delta(\mathcal{E}'_A, \mathcal{E}'_B)$ where

$\mathcal{E}_A'$ is given by $X_1,\ldots,X_l,S$ independent, $X_i \sim N(\beta_i,1)$ $i=1,\ldots,l$ $S \sim \chi^2_{n_1}$ and $\mathcal{E}_B'$ is given by $Y_1,\ldots,Y_l,T$ independent $Y_i \sim N(\beta_i,\frac{1}{\sqrt{\Delta_i}})$ $i=1,\ldots,l$ $T \sim \chi^2_{n_2}$. By sufficiency S may be deleted in $\mathcal{E}_A'$ and T in $\mathcal{E}_B'$. Then proposition 3.1 gives $\delta(\mathcal{E}_A, \mathcal{E}_B) \geq \delta(\mathcal{E}_A', \mathcal{E}_B') = \|P'_{l-m,0,\ldots,0,1} - Q'_{l-m,0,\ldots,0,1}\|$.

But by lemma 4.1 the other equality also must hold since we may write $\mathcal{E}_A = \tilde{\mathcal{E}}$, $\mathcal{E}_B = \tilde{\mathcal{F}}$ with $\mathcal{E}_1$ given by $X_1,\ldots,X_{l-m}$, $\mathcal{E}_2$ given by $X_{l-m+1},\ldots,X_l,S$, $\mathcal{F}_1$ given by $Y_1,\ldots,Y_{l-m}$ and $\mathcal{F}_2$ given by $Y_{l-m+1},\ldots,Y_l$, T. Then the assumptions of the lemma are satisfied. In particular $\mathcal{E}_2 \geq \mathcal{E}_2$ follows from proposition 2.1 in [1]. □

Suppose now that $n_A > k$ and $n_B > l$. With the group operation defined in the beginning of this section and with the standard topology $\mathbb{R}^l \times \mathbb{R}^+$ becomes a locally topological group which is Hausdorf and statisfies the second axiom of countability.

Let $X_1,\ldots,X_l,S$ be independent $X_i \sim N(0,1)$ $i=1,\ldots,l, S \sim \chi^2_{n_{A-k}}$ Then $(X_1,\ldots,X_l,S)(\beta_1,\ldots,\beta_l,\sigma^2) = (\sigma X_1+\beta_1,\ldots,\sigma X_l+\beta_l,\sigma^2 S)$ and $P_{l,n_A-k,\beta_1,\ldots,\beta_l,\sigma^2}(B) = P_{l,n_A-k,0,\ldots,0,1}((X_1,\ldots,X_l,S)(\beta_1,\ldots,\beta_l,\sigma^2) \in B) = P_{l,n_A-k,0,\ldots,0,1}(B(\beta_1,\ldots,\beta_l,\sigma^2)^{-1})$ $B \in (\mathbb{R}^l \times \mathbb{R}^+)$. Similarly $Q_{l,n_B-l,\beta_1,\ldots,\beta_l,\sigma^2}(B) = Q_{l,n_B-l,0,\ldots,0,1}(B(\beta_1,\ldots,\beta_l)^{-1})$ $B \in (\mathbb{R}^l \times \mathbb{R}^+)$ so that $\mathcal{E}_A$ and $\mathcal{E}_B$ are transtation experiments.

Since $\{P_{l,n_A-k,\beta_1,\ldots,\beta_l,\sigma^2} | (\beta_1,\ldots,\beta_l,\sigma^2) \in \mathbb{R}^l \times \mathbb{R}^+\}$ and $\{Q_{l,n_B-l,\beta_1,\ldots,\beta_l,\sigma^2} | (\beta_1,\ldots,\beta_l,\sigma^2) \in \mathbb{R}^l \times \mathbb{R}^+\}$ are dominated and $\mathbb{R}^1 \times \mathbb{R}^+$ is solvable, the method described in section 2, may be applied.

If $B \in \mathcal{B}(\mathbb{R}^1 \times \mathbb{R}^+)$, then $P''_{1;n_A-k,\beta_1,..,\beta_1,\sigma^2}(B)=$

$P_{1,n_A-k;0,..,0,1}((\beta_1,..,\beta_1,\sigma^2)^{-1}B) = P_{1n_A-k,0,..,0,1}((X_1,..,X_1,S)$
$\in(\beta_1,..,\beta_1,\sigma^2)^{-1}B) = P_{1,n_A-k,0,..0,1}((\beta_1,..,\beta_1,\sigma^2)(X_1,..,X_1,S)\in B)=$

$P_{1,n_A-k;0,..,0,1}((X_1 + \beta_1\sqrt{S},..,X_1+\beta_1\sqrt{S},S\,\sigma^2) \in B)=$
$\int I_B(x_1 + \beta_1\sqrt{s},..,x_1 + \beta_1\sqrt{s},s\,\sigma^2) \prod_{i=1}^{1} \varphi(x_i)\,\gamma\, n_A-k,1(s)dx_1..dx_1,ds =$
$\int_B \prod_{i=1}^{1} (x_i-\beta_i\sqrt{\frac{s}{\sigma^2}}) \frac{1}{\sigma^2} \gamma_{n_A-k,1}(\frac{s}{\sigma^2})dx_1...dx_1,ds.$ Thus

$P''_{1,n_A-k,\beta_1,..;\beta_1;\sigma^2}$ has density $\prod_{i=1}^{1} \varphi(x_i - \beta_i\sqrt{\frac{s}{\sigma^2}})\frac{1}{\sigma^2}\gamma_{n_A-k,1}(\frac{s}{\sigma^2})$

with respect to the Lebesgues measure.

Similarly $Q_{1,n_A=1,0,..,0,1}$ has density $\prod_{i=1}^{1}\sqrt{\Delta_i}\varphi(x_i\sqrt{\Delta_i})\gamma_{n_B-1}(s)$

with respect to the Lebesgues measure.

<u>Proposition 4.4</u> If m=0 i.e. $\Delta_1,..,\Delta_1 \geq 1$ and

$n_B-1 > n_A-k \geq 1$, then $\delta(\mathcal{E}_A, \mathcal{E}_B) = \|P''_{1,n_A-k,0,..,0,\frac{n_B-1}{n_A-k}} - Q_{1,n_B-1,0,..0,1}\| =$

$\int | \prod_{i=1}^{1} (x_i)\,\gamma_{n_A-k,\frac{n_B-1}{n_A-k}}(s) - \prod_{i=1}^{1}\sqrt{\Delta_i}\,\varphi(x_i\sqrt{\Delta_i})\,\gamma_{n_B-1,1}(s)|dx_1..dx_1,ds$

<u>Proof</u> Let $n_A-k=n_1, n_B-1=n_2$ and let $\delta_x(B)=I_B(x)$ We must show that $N_o=\delta_o x..x\,\delta_o \times \delta_{\frac{n_2}{n_1}}$ is a least favourable distribution for testing.

$H= \{P''_{1;n_1,\beta_1,..,\beta_1,\sigma^2} : (\beta_1,..,\beta_1,\sigma^2) \in \mathbb{R}^1 \times \mathbb{R}^+\}$ against Q at all levels α. Then the proposition will follow from the results given i section 2.

The strongest α-level rest for H_{N_0} against Q is given by:

$$\delta_{N_0}(x_1,\ldots,x_1,s)=1 \Leftrightarrow \prod_{i=1}^{1}\sqrt{\Delta_i}\ \varphi(x_i\sqrt{\Delta_i})\gamma\ n_2,1\ (s)>C\prod_{i=1}^{1}\varphi(x_i)$$

$$\gamma_{n_1,1}\left(s\frac{n_1}{n_2}\right)\frac{n_1}{n_2} \Leftrightarrow \exp\left(-\tfrac{1}{2}\sum_{i=1}^{1}(\Delta_i-1)x_i^2\right)\ s^{\frac{1}{2}(n_2-n_1)}\ \exp$$

$(-s/2(1-{}^{n_1}/n_2)) > C' \Leftrightarrow (x_1,\ldots,x_1,s)\in K$ where

(i) $\alpha = P_{1,n_1,0,\ldots,0,\ {}^{n_2}/n_1}(K)$

(ii) $K_s = \{(x_1,\ldots,x_1) \mid (x_1,\ldots,x_1,s)\in K\} =$

$\{(x_1,\ldots,x_1) \mid \frac{1}{2}\sum_{i=1}^{1}(\Delta_i-1)x_i^2 < -\log C' + \log[S^{\frac{n_2-n_1}{2}}$

$\exp(-\frac{s}{2}(1-{}^{n_1}/n_2))]\}$ is an ellipse which may be degenerate since $\Delta_i = 1$ is possible.

Let $k_3 = \max_s \log s^{\frac{n_2-n_1}{2}} \exp(-s/2(1-{}^{n_1}/n_2))$

(iii) $K_{x_1,\ldots,x_1} = \{s \mid (x_1,\ldots,x_1,s)\in K\} = \langle k_1(x_1,\ldots,x_1),$

$k_2(X_1,\ldots,X_1)\rangle$ where

$$k_1(x_1,\ldots,x_1)^{\frac{n_2-n_1}{2}}\ \exp\left(-\tfrac{1}{2}k_1(x_1,\ldots,x_1)\left(1-\frac{n_1}{n_2}\right)\right) =$$

$$k_2(x_1,\ldots,x_1)^{\frac{n_2-n_1}{2}}\ \exp\left(-\tfrac{1}{2}k_2(x_1,\ldots,x_1)\left(1-\frac{n_1}{n_2}\right)\right)$$

Then $\quad P_{1,n_1,0,\ldots,0,\frac{n_2}{n_1}}(K) =$

$$\int_{-\frac{1}{2}\Sigma(\Delta_i-1)x_i^2 < \log C'+k_3}\ \int_{k_1(x_1,\ldots,x_2)}^{k_2(x_1,\ldots,x_2)} \prod^{1}\varphi(x_i)\ \frac{n_1}{n_2}\ \gamma_{n_1,1}\left(\frac{sn_1}{n_2}\right)\ ds\,dx_1,\ldots,dx_1$$

Let $E_{\beta_1,\ldots,\beta_1,\sigma^2}$ be the expectation taken relative to $P''_{1,n_1,\beta_1,\ldots,\beta_1,\sigma^2}$. Then $P''_{1,n_1,\beta_1,\ldots,\beta_1,\sigma^2}(K) =$

$$E_{\beta_1,\ldots,\beta_1,\sigma^2}\,[I_K(X_1,\ldots,X_1,S)] =$$

$E_{\beta_1,..,\beta_l,\sigma^2}\, E_{\beta_1,..,\beta_l,\sigma^2}\, [I_K(X_1,..,X_l,S)|S]$. But $E_{\beta_1,..,\beta_l,\sigma^2}(I_K(X_1,..,X_l,S)|S)$ is a function of $(X_1,.,,X_l,S)$ only through S. Thus the distribution is independent of $(\beta_1,..,\beta_l)$. Consequently $P''_{l,n_1,\beta_1,..,\beta_l,\sigma^2}(K) =$

$E_{0,..,0,\sigma^2}\, E_{\beta_1,..,\beta_l,\sigma^2}[I_K(X_1,..,X_l,S)|\, S]$. Futhermore $E_{\beta_1,..,\beta_l,\sigma^2}[I_K(X_1,..,X_l,S)|S] \leq E_{0,..,0,\sigma^2}[I_K(X_1,..,X_l,S)|S]$ since K_s is an ellipse with center in $(o,..,o) \in \mathbb{R}^l$, and the probability for $(X_1,..,X_l) \in K_s$ where $(X_1,..,X_l)$ are independent $X_i \sim N(\beta_i \sqrt{\frac{s}{\sigma^2}}\ ,1)$ $i=1,..,l$, is maximized when the center in the ellipse and the distribution coincide.

Thus $P''_{l,n_1,\ \beta_1,..,\beta_l,\sigma^2}(K) \leq$

$P''_{l,n_1,o,..,o,\sigma^2}(K)$, $(\beta_1,..,\beta_l,\sigma^2) \in \mathbb{R}^l \times \mathbb{R}^+$.

Finally if we show that $P''_{l,n_1,o,..,o,\sigma^2}(K) \leq$

$P''_{l,n_1,o,..,o_1\frac{n_2}{n_1}}(K) = \alpha$, $\sigma^2 > o$, theorem 3.7 in [4] will give that N_o is the least favourable distribution.

Let $\alpha(\sigma^2) = P''_{l,n_1,o,..,\sigma^2}(K) =$

$$\int_{\frac{1}{2}\sum_{i=1}^{l}(\Delta_i-1)x_i^2<k_3} \prod_{i=1}^{l} \varphi(x_i)[\Gamma_{n_1,\sigma^2}(k_2(x_1,..,x_l))-\Gamma_{n_1,\sigma^2}(k_1(x_1,..,x_l))]dx,..,dx_l$$

$$= \int_{\frac{1}{2}\sum(\Delta_i-1)x_i^2<k_3} \prod_{i=1}^{l} (x_i)[\Gamma_{n_1,1}(\tfrac{1}{\sigma^2}\, k_2(x_1,..,x_l)) -$$

$$\Gamma_{n_1,1}(\tfrac{1}{\sigma^2}\, k_1(x_1,..,x_l))]dx_1..,dx_l$$

Now $\{P''_{1,n_1,o,..,o,\sigma^2} : \sigma^2 \in \mathbb{R}^+\}$ is an exponentaial family of distributions and $\alpha(\sigma^2) = \int I_K(x_1,..,x_l,s)$

$P_{1,n_1,o,..,o,\sigma^2}(dx_1..ds)$ Hence, by theorem 2.9 in [4] derivation with respect to σ^2 under the intergration sign is permitted.

$$\alpha'(\sigma^2) = \int_{\frac{1}{2}\Sigma(\Delta_i - 1)x_i^2 < k_3} \prod_{i=1}^{l} \varphi(x_i)\ (\tfrac{1}{\sigma}2)^2[k_1(x_1,..,x_l)$$

$$\Gamma_{n_1,1}(\tfrac{1}{\sigma}2\ k_1(x_1,..,x_l))\ k_2(x_1,..,x_l)\ \Gamma_{n_1,1}(\tfrac{1}{\sigma}2\ k_2(x_1,..,x_l))]\ dx_1,..,dx_l\ .$$

By (iii) above $\alpha'(\frac{n_2}{n_1}) = 0$, so that α has an extremal point in $\frac{n_2}{n_1}$.

Consider $f(t) = \Gamma_{n_1,1}(\frac{k_2}{t}) - \Gamma_{n_1,1}(\frac{k_1}{t})$ $k_2 > k_1 > 0$, $t > 0$.

$f(t)$ can have only one extrmal point, t_o. Since $f > 0$ and $\lim_{t\to 0} f(t) = \lim_{t\to\infty} f(t) = 0$, this must be a maximum point and $f'(t) < 0$ $t > t_o$, $f'(t) > 0$ $t < t_o$. These results applied to the intergrand in the expression for $\alpha'(\sigma^2)$, give that $\frac{n_2}{n_1}$ must be a maximum point □

It still remains to condider the case $1 \leq n_A - k < n_B - 1 + m$ and $m > 0$. $\delta(\mathcal{G}_A^{\varphi}, \mathcal{G}_B^{\varphi})$ is not known then.

Suppose now that $0 \leq \text{rank } A = r \leq k$. By the remark at the end of section 3 $\delta(\mathcal{G}_A^{\varphi}, \mathcal{G}_B^{\varphi}) = 2$ if row $[B'] \not\subset$ row $[A']$

If row $[B'] \subset$ row $[A']$ we may write, in the same way as in section 3, $A = VS$, $B = VT$. Then $\delta(\mathcal{G}_A^{\varphi}, \mathcal{G}_B^{\varphi}) = \delta(\mathcal{G}_S^{\varphi}, \mathcal{G}_T^{\varphi})$

If $F'SS'F = I, F'TT'F = \Delta$ with F a nonsingular $r \times r$ matrix, $\text{rank}(B') = \text{rank}(T') = \# \{i \mid \Delta_i > 0\}$. Let $\tilde{S} = F'S$, $\tilde{T} = F'T$. Then $\delta(\mathcal{G}_A, \mathcal{G}_B) = \delta(\mathcal{G}_S, \mathcal{G}_T) = \delta(\mathcal{G}_{\tilde{S}}, \mathcal{G}_{\tilde{T}})$, and the results above may be summarized in the following theorem.

Theorem 4.1

If row $[B'] \not\subset$ row $[A']$, $\delta(\mathcal{G}_A, \mathcal{G}_B) = 2$

If row $[B'] \subset$ row $[A']$, let $A = VS$, $B = VT$ where $V = (V_1', \ldots, V_r')$ and $V_1', \ldots, V_r'$ are a basis for row $[A']$, and let $\Delta_1, \ldots, \Delta_{\text{rank}(A')}$ be the solutions of $\det(TT' - \Delta SS') = 0$. Then

$$\delta(\mathcal{G}_A, \mathcal{G}_B) = \begin{cases} 2 & \text{if } \text{rank}(A') = n_A \quad \text{rank}(B') < n_B \\ E\left| 1 - \prod_{\Delta_i > 0} \exp(-\frac{1}{2}(\Delta_i - 1) Y_i^2) \right| & \text{if } n_A = \text{rank}A', \; n_B = \text{rank } B' \\ E\left| 1 - \prod_{\Delta_i > 1} \sqrt{\Delta_i} \exp(-\frac{1}{2}(\Delta_i - 1) Y_i^2) \right| & \text{if } n_A \geq n_B + \#\{i \mid 0 \leq \Delta_i < 1\} \\ E\left| 1 - \frac{n_B - \text{rank}(B')}{n_A - \text{rank}(A')} \frac{\chi_{n_B - \text{rank}(B'), 1}(S)}{\chi_{n_A - \text{rank}(A'), 1}\left(\frac{n_A - \text{rank}(A')}{n_B - \text{rank}(B')} S\right)} \prod_{\Delta_i > 1} \sqrt{\Delta_i} \exp\left[-\frac{1}{2}(\Delta_i - 1) Y_i 2\right] \right| & \\ \quad \text{if } \#\{i \mid 0 < \Delta_i < 1\} = 0 \text{ and } n_B - \text{rank}(B') \geq n_A - \text{rank}(A') \geq 1 & \end{cases}$$

$Y_1, \ldots, Y_{\text{rank}(A')}$, S are independent $Y_i \sim N(0,1)$, $\frac{n_A - \text{rank}(A')}{n_B - \text{rank}(B')} S \sim \chi^2_{n_A - \text{rank}(A')}$

Proof

$n_A - \text{rank}\ (A') \geq n_B - \text{rank}\ (B') + \#\{i \mid 0 < \Delta_i < 1\}$ is equivalent with $n_A \geq n_B + \text{rank}\ (A') - \text{rank}\ (B') + \{i \mid 0 < \Delta_i < 1\} =$ $n_B + \#\ \{i \mid 0 \leq \Delta_i < 1\}$, so that the third expression for $\delta(\mathcal{G}_A, \mathcal{G}_B)$ in the second half of the theorem follows from proposition 4.3.

Consider now the situation treated by Le Cam for σ^2 known.

Corollary

If AA' is nonsingular and $BB' - AA'$ is positive semidefinit

$$\delta(\mathcal{G}_A, \mathcal{G}_B) = \begin{cases} 2 \quad \text{if } n_A = \text{rank}\ (A') < n_B \\[2ex] E \left| \left(\dfrac{n_B - \text{rank}(A')}{n_A - \text{rank}(A')}\right)^{\frac{n_A - \text{rank}(A')}{2}} \dfrac{\Gamma\left(\frac{n_A - \text{rank}(A')}{2}\right)}{\Gamma\left(\frac{n_B - \text{rank}(A')}{2}\right)} \right. \\[2ex] \dfrac{\det(BB')}{\det(AA')} \left(\dfrac{S}{2}\right)^{\frac{n_B - n_A}{2}} \exp\left(-\tfrac{1}{2} S \dfrac{n_B - n_A}{n_B - \text{rank}(A')} - \tfrac{1}{2} Z'(BB' - \right. \\[2ex] \left. AA')Z\right) - 1 \Big| \qquad \text{if rank}\ (A') < n_A < n_B \\[2ex] E \left| \dfrac{\det\ (BB')}{\det\ (AA')} \exp\ \left(-\tfrac{1}{2} Z'(BB' - AA')Z\right) - 1 \right| \\[2ex] n_A \geq n_B \geq \text{rank}\ (A') \end{cases}$$

where Z is multilinear normal with expectation 0 and covariance matrix $(AA')^{-1}$, and $\dfrac{n_A - \text{rank}(A')}{n_B - \text{rank}(A')} S$ is χ^2-distributed with $n_A - \text{rank}(A')$ degrees of freedom and is indepentdent of Z.

Proof

Let F be a nonsingular matrix such that $F'AA'F = I$, $F'BB'F = \Delta$. Then, since $BB' - AA'$ is positive semidefinit and since $\Delta_1,..,\Delta_{\text{rank } A}$ are the solutions of $\det(BB' - \Delta AA') = 0$, $\Delta_1,..,\Delta_k \geq 1$. By nothing that AA' nonsingular implies BB' nonsingular, the corollary now follows in the same way as the corresponding result in section 3. $\square$

5. Examples

Example 1

Let $A' = \begin{pmatrix} 1 & t_1 \\ \vdots & \\ 1 & t_{n_A} \end{pmatrix}$ $\quad B' = \begin{pmatrix} 1 & S_1 \\ \vdots & \\ 1 & S_{n_B} \end{pmatrix}$

Then $AA' = \begin{pmatrix} n_A & n_A\bar{t} \\ n_A\bar{t} & \sum_{i=1}^{n_A} ti^2 \end{pmatrix}$

Suppose rank $A' = 2$, i.e. not all $t_1,..,t_{n_A}$ equal. If

$M_A = \sum_{i=1}^{n_A} (t_i - \bar{t})^2$, $M_B = \sum_{i=1}^{n_B} (S_i - \bar{S})$, $\det(BB' - \Delta AA') = 0$

has 2 zeroes Δ_1, Δ_2 given by

$\frac{1}{2n_A MA} [n_B M_A + n_A M_B + n_B n_A(\bar{s}-\bar{t}) \pm (n_B M_A + n_A M_B + n_A n_B(\bar{t}-\bar{s}) - 4n_B n_A M_A M_B)^{\frac{1}{2}}]$

$\delta(\mathcal{E}_A, \mathcal{E}_B)$ may now be computed for σ^2 known, and for σ^2 unknown except when $0 < n_A - 2 < n_B - \text{rank}B + \{i | 0 < \Delta_i < 1\}$ and $\{i | 0 < \Delta_i < 1\} > 0$.

Note that if $\bar{t} = \bar{s}$, $\Delta_1 = \frac{n_B}{n_A}$, $\Delta_2 = \frac{M_B}{M_A}$.

Example 2 If $\mathcal{M}_i$ and $\mathcal{M}_a$ are the minimal informative and the maximal informative experiments respectively, $\delta(\mathcal{E}_A, \mathcal{E}_i)$ and $\delta(\mathcal{E}_a, \mathcal{E}_A)$ give absolute measures of the information in the experiment $\mathcal{E}_A$. Unfortunately for translation experiments on the real line both of these deficiencies are equal to 2 as shown by Torgersen in [5]. Hence $\delta(\mathcal{E}_A, \mathcal{E}_i) =$

$\delta(\mathcal{M}_a, \mathcal{E}_A) = 2$ for the case when σ^2 is known and consequently also for σ^2 unknown.

However, if an experiment is given by the independent, observations $X_1,..,X_n$, deficiencies may be used to compute the information contained in an additional observation.

In the experiments considered in this paper, the observations are not identically distributed, because the distribution of the additional observation is dependent of the choice of the regression coefficients. The question then naturally arises whether deficiencies may be of help to determine the regression coefficients so the additional observation contains as much information as possible.

Let $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ a_{k1} & & a_{kn} \end{pmatrix}$ $B = \begin{pmatrix} a_{11} & \cdots & a_{1n} & t_1 \\ & & & \cdot \\ a_{k1} & \cdots & a_{kn} & t_k \end{pmatrix}$

Suppose rank $A = k$

Then $\det(BB' - \Delta AA') = \det(AA' + tt' - \Delta AA') =$

$\det(tt' - (\Delta-1) AA')$. Since rank $[tt'] = 1$, the solutions of $\det(tt' - \Delta AA') = 0$ are all equal to zero, exept one.

The solution $\Delta = 0$ has multiplicity $k-1$. Hence

$\det(tt' - \Delta AA') = 0 = \det(AA') \det((tt')(AA')^{-1} - \Delta I) =$

$\det(AA')(-\Delta)^{k-1}(\Delta_o - \Delta) = [(-1)^k \Delta^k + (-1)^{k-1} \Delta_o \Delta^{k-1}] \det(AA')$

where Δ_o is the nonzero solution of $\det[(tt')(AA')^{-1} - \Delta I] = 0$

But $\det[(tt')(AA')^{-1} - \Delta I] = (-\Delta)^k + \mathrm{tr}\,(tt')(AA')^{-1}(-\Delta)^{k-1} +$ factors of lower order in Δ. Hence $\Delta_o = \mathrm{tr}\,(tt')(AA')^{-1}$, and $1+ \operatorname{tr}(tt')(AA')^{-1}, 1, \ldots, 1$ are the k solutions of $\det(BB' - \Delta AA') = 0$

Let $X \sim N(0,1)$. Then, by noting that $\operatorname{tr}(tt')(AA')^{-1} = t'(AA')^{-1}t$, $\delta(\mathcal{G}_A, \mathcal{G}_B) = E|1- \sqrt{t'(AA')^{-1}t+1}\exp(-\frac{1}{2}\, t'(AA')^{-1}tX^2)| =$

$$4\left[\Phi\left(\left[(1+t'(AA')^{-1}t)\frac{\log(1+t'(AA')^{-1}t)}{t'(AA')^{-1}t}\right]^{\frac{1}{2}}\right) - \Phi\left(\left(\frac{\log(1+t'(AA')^{-1}t)}{t'(AA')^{-1}t}\right)^{\frac{1}{2}}\right)\right]$$

for σ^2 known. In the above expression we have written the integrand f, and used that $\int|f| = 2\int f^+$.

If Y, S are independent $Y \sim N(0,1)$, $\frac{n-k}{n-k+1}\, S \sim \chi^2{n-k}$,

$$\delta(\mathcal{G}_A, \mathcal{G}_B) = \int|\varphi(x)\gamma_{n-k,\frac{n-k+1}{n-k}}(s) - (1+t'(AA')^{-1}t)^{\frac{1}{2}}\varphi(x(1+t'(AA')^{-1}t)^{\frac{1}{2}})\gamma_{n-k+1}(s)|\,dxds$$

$$= E|1 - \left(\frac{n-k+1}{n-k}\right)^{(n-k)/2}\frac{\Gamma(\frac{n-k}{2})}{\Gamma(\frac{n-k+1}{2})}(1+t'(AA')^{-1}t)\sqrt{\frac{S}{2}}\exp\left(-\frac{1}{2}\frac{S}{n-k+1} - \frac{1}{2}t'(AA')^{-1}t\,Y^2\right)|$$

when σ^2 is unknown.

If σ^2 is known and $s'(AA')^{-1}s > t'(AA')t$ the experiment where the $n+1$ th observation has regression coefficients $(s_1, \ldots, s_k)$ is more informative than the experiment where the $n+1$ th observation has regression coefficients $(t_1, \ldots, t_k)$ Hence $\delta(\mathcal{G}_A, \mathcal{G}_B)$ is increasing in $t'(AA')^{-1}t$ for σ^2 known.

Consider again the situation in example 1. If we weite $t' = (1, t_{n+1})'$, $t'(AA')^{-1}t = \frac{1}{M}(t_{n+1}{}^2 - 2\bar{t}\,t_{n+1} = \frac{1}{n}\sum_{i=1}^{n} t_i{}^2)$

$\bar{t} = \frac{1}{n} \sum_{i=1}^{n} t_i$, so that the minimal increase in $\delta(\mathcal{G}_A, \mathcal{G}_B)$ is obtained by letting $t_{n+1} = \bar{t}$, and in this case

$$\delta(\mathcal{G}_A, \mathcal{G}_B) = 4[\Phi([(n+1) \log (1+\tfrac{1}{n})]^{\frac{1}{2}}) - \Phi((n \log (1 + \tfrac{1}{n})^{\frac{1}{2}})]$$

when σ^2 is known.

Acknowledgment. The problem was suggested by Dosent Erik Torgersen who have also pointed out improvements of several of the proofs.

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