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CENCORED EXPONENTIAL MODELS

by

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ABSTRACT

Let \( X_1, X_2, \ldots, X_n \) be independent observations from a censored exponential model. Using the principle of analytical continuation we establish conditions ensuring the existence of a complete and sufficient statistic. The remainder of the paper is devoted to the particular case of censored gamma distributions. It is shown that the answer to the completeness problem as well as the structure of the space of UMVU estimators depends heavily on whether the shape parameter is rational or irrational. Thus there is not a complete and sufficient statistic if and only if \( p \) is rational \( \frac{r}{s} \) where the positive integers \( r \) and \( s \) are relatively prime and \( n > s \). In this case the set of integers \( n > s \) so that a given non-constant \( g \) has a UMVU estimator based on \( n \) observations is a finite set with gaps of length \( s \).

I. Introduction

Suppose the probability of death within the infinitesimal interval \( (x, x+dx) \) is
\[
\frac{\lambda}{P} x^{p-1} e^{-\lambda x} dx
\]
where \( \lambda > 0 \) is unknown while \( p > 0 \) is a known constant. Inference on \( \lambda \) based on the observed lifespans of \( n \) randomly chosen individuals may be based on the sum of observations which is a complete and sufficient statistic. If, however, our experiment is obtained by only observing the times of death before a fixed time \( t \), then the total number of deaths recorded together with the sum of lifelengths of individuals dying before time \( t \) constitutes a minimal sufficient statistic. It was shown by the author [3] that this statistic is not complete when \( p = 1 \) and \( n \geq 2 \). Generalizing this result K. Unni [7] proved that this holds for any integer \( p \geq 1 \). Unni
proved furthermore that the statistic is complete when \( p \) is irrational. The remaining case where \( p \) is rational, say \( p = \frac{r}{s} \) where the positive integers \( r \) and \( s \) are relatively prime, was resolved in [6] where it was shown that this statistic is complete when \( n \leq s \) and not complete when \( n > s \).

The proof of completeness in [6] was a simple application of the principle of analytical continuation. It will be shown in section 2 how this principle may be used to establish completeness of the minimal sufficient statistic in censored exponential models.

The remainder of this paper, i.e. section 3, treats the particular case of censored gamma distributions.

We begin by giving various characterizations of unbiased estimators of zero. These characterizations establish again the completeness results in [3], [6] and [7]. As the space of unbiased estimators of zero is formidable it is not surprising that the set of UMVU estimators is very restricted. It is interesting that the algebra of events with UMVU estimable probabilities does not depend on \( p \), for fixed \( n \), provided we restrict attention to the non complete case. This indicates possibilities of generalizations as well as of simplifications.

The basic tool in establishing properties of UMVU estimators is the following widely known idea. Suppose \( L_1, \ldots, L_m \) and \( L \) are linear functionals on a linear space. Then \( L \) is a linear combination of \( L_1, \ldots, L_m \) if and only if \( L \) vanishes whenever \( L_1, \ldots, L_m \) all vanish.

Call a function \( g \) UMVU estimable if there is an integer \( n \geq 0 \) such that \( g \) has a UMVU estimator based on \( n \) observations. It is shown, in the non complete cases, that the set of integers \( n \) such that a UMVU estimable function \( g \) has a UMVU estimator based on \( n \) observations is finite with gaps of length at least \( s \).
It is a consequence of the Rao-Blackwell theorem that in situations where a complete and sufficient statistic based on \( n \) observations is available for all \( n \) (or, [5], equivalently for arbitrarily large \( n \) ), the set of integers \( n \) such that a given function \( g \) is UMVU estimable on the basis of \( n \) observations is, and may be, any tail \( \{N,N+1,\ldots\} \) of the non-negative integers. What other sets, besides these and those found in this paper, are possible?

The results obtained here tell a story, and we believe it is an interesting one, about UMVU estimation in "non regular" models. They also tell something about the fragility of the property of completeness and of the structure of the space of UMVU estimators. Thus, although the distributions are strongly continuous in \( p \), the situation when \( p \) is rational is entirely different from the situation where \( p \) is irrational.

It is known, see [5], that any separable model is a mixture of models admitting boundedly complete and sufficient statistics. We don't know any manageable expression for a mixing distribution in this case. If we knew, then we might be able to see how the structure of the space of UMVU estimators is related to the mixing distribution.

II Censored exponential models

The models described in the introduction are particular cases of the following type of models.

Let \( Y \) be a random variable whose distribution \( Q_\lambda \) depends on an unknown parameter \( \lambda \). Here \( \lambda \) is a , and may be any, point in a known set \( \Lambda \).
Suppose \( Y \) is observable (or is observed) if and only if \( Y \) belongs to some set \( S \). If \( Y \) should happen to fall outside \( S \) then it is only recorded that \( Y \) did not occur in \( S \). Let us here assume that this is done by noting down the symbol \( § \) which does not belong to the set \( S \). The actual observation is therefore not \( Y \) but \( X \) where

\[
(2.1) \quad X = \begin{cases} 
Y & \text{if } Y \in S \\
§ & \text{if } Y \notin S 
\end{cases}
\]

Our actual observations \( X_1, \ldots, X_n \) are then independent observations of a variable \( X \) whose distribution \( P_\lambda \) is given by:

\[
(2.2) \quad P_\lambda(B) = Q_\lambda(B); \ B \in 2^k \cap S, \ P_\lambda(§) = Q_\lambda(S^c).
\]

Consider now \( n \) independent observations \( X_1, \ldots, X_n \) of \( X \). Let \( Y_i, \ i = 1, \ldots, n \) be the corresponding non observable variables. It is then easily seen that, under general conditions, if a symmetric statistic \( s_j(Y_1, \ldots, Y_j) \) is sufficient for \( Y_1, \ldots, Y_j \) when \( j \leq n \) then the statistic \( (D, T) \) where \( D = \#\{i: 1 \leq i \leq n, X_i \in S\} \) and \( T = s_D(\{X_j: X_j \in S\}) \) is sufficient for \( (X_1, \ldots, X_n) \). We are, however, not permitted (as we shall see) to conclude that \( (D, T) \) is complete when \( n \geq 2 \), even if \( s_j(Y_1, \ldots, Y_j) \) is complete for all \( j \). If \( n = 1 \) then there is no problem since a sub model of a complete model is automatically complete.

We shall here be interested in the situation where both \( \lambda \) and \( Y \) are in \( R^k \) and where the distribution \( Q_\lambda \), for each \( \lambda \in \Lambda \), has density

\[
(2.3) \quad c(\lambda)e^{-\langle \lambda, Y \rangle}; \ y \in R^k
\]

w.r.t. some measure \( \mu \) on \( R^k \).

Let us also, in order to avoid trivialities, assume that
\[ \mu(S) > 0 \text{ and that } \mu(S^c) > 0. \] Choose a \( \lambda_0 \in \Lambda \) and put
\begin{equation}
H = Q_{\lambda_0}
\end{equation}

Then:
\begin{equation}
dQ_{\lambda}/dH = \frac{c(\lambda)}{c(\lambda_0)} e^{-\langle \lambda - \lambda_0, y \rangle}; \quad y \in \mathbb{R}^k
\end{equation}

We may here put \( s_j(Y_1, \ldots, Y_j) = Y_1 + \ldots + Y_j \) so that the statistic \((D, T)\) defined by:
\begin{equation}
(D, T) = \sum (1, X_j) : X_j \in S
\end{equation}
is sufficient.

This may also be seen from the factorization criterion and the fact that the distribution of \((X_1, \ldots, X_n)\) satisfies:
\begin{equation}
dP^n_{\lambda}/dP^n_{\lambda_0} = [c(\lambda)/c(\lambda_0)]^{-D(x)} e^{-\langle \lambda - \lambda_0, T(x) \rangle} \frac{Q_{\lambda}(S^c)^{-n-D(x)}}{H(S^c)}; \quad x \in [SU[S]^n.
\end{equation}

It follows that the distribution of \((X_1, \ldots, X_n)\) as well as the distribution of \((D, T)\) is exponential. Furthermore \((D, T)\) is minimal sufficient provided the map \( \lambda \mapsto \log \int_{S^c} e^{\langle \lambda, x \rangle} \mu(dx) \) does not coincide with an affine function on \( \Lambda \). (This holds in particular if \( \Lambda \) has interior points and \( \mu \) has at least two points of support in \( S^c \).)

Clearly \( D \) is binomially distributed according to \( n \) trials and success parameter \( Q_{\lambda}(S) \) i.e.:
\begin{equation}
P_{\lambda}(D = d) = \binom{n}{d} Q_{\lambda}(S)^d Q_{\lambda}(S^c)^{n-d}; \quad d = 0, 1, \ldots, n
\end{equation}

The conditional distribution of \( T \) given \( D \) is \( V_{\lambda}^{D^*} \) where, for each \( j = 0, 1, \ldots \), \( V_{\lambda} \) is the conditional distribution of \( X \) given that \( X \in S \). Putting \( K = V_{\lambda_0} \) we see that this distribution has density:
Together (2.8) and (2.9) determine the distribution of \((D,T)\).

In order to describe the expectation operator in a convenient way we shall need a few more notations. These are:

\[(2.10) \quad \tau = \text{the conditional distribution of } Y \text{ given that } Y \notin S\]

\[(2.11) \quad A(\lambda) = \int_{S^C} e^{-\langle \lambda, x \rangle} \mu(dx); \lambda \in \Lambda\]

and

\[(2.12) \quad \hat{\nu}(\lambda) = \int_{S^C} e^{-\langle \lambda - \lambda_0, x \rangle} \nu(dx) \quad \text{where } \nu \text{ is a finite (possibly signed) measure on } \mathbb{R}^k \text{ and } \lambda \in \mathbb{R}^k\]

is such that the integral exists and is finite.

It follows that:

\[(2.13) \quad \tau = \alpha H - \beta K \text{ where } \alpha = H(S^C)^{-1} \text{ and } \beta = H(S)/H(S^C)\]

\[(2.14) \quad \hat{A}(\lambda) = c(\lambda_0)/c(\lambda); \lambda \in \Lambda\]

and

\[(2.15) \quad \hat{\tau}(\lambda) = A(\lambda)/A(\lambda_0); \lambda \in \Lambda\]

Furthermore, by the convolution rule for Laplace transforms:

\[(2.16) \quad \hat{\nu}(\lambda) = \hat{\nu}_1(\lambda) \ast \hat{\nu}_2(\lambda) \text{ when } \nu = \nu_1 \nu_2\]

and \(\hat{\nu}_1(\lambda)\) and \(\hat{\nu}_2(\lambda)\) are defined according to (2.12).

Note also that, by for example the completeness theorem for exponential families, that a finite measure \(\nu\) is determined by
the restriction of \( \hat{\eta} \) to any sphere contained in the domain of \( \hat{\rho} \).

Finally a random variable \( \delta(D,T) \) is integrable if and only if \( \delta(d,\cdot) \) is \( V^*_{\lambda} \) integrable when \( d = 0,1,\ldots,n \) and \( \lambda \in \Lambda \).

If \( S \) is bounded then this is equivalent to:

\[
(2.17) \quad \int |\delta(d,t)|K^*(dt) < \infty; \quad d = 0,1,\ldots,n.
\]

An estimator \( \delta \) with finite expectation may be represented by the vector \( \delta = (\delta_0, \delta_1, \ldots, \delta_n) \) where, for each \( d = 0,1,\ldots,n \), \( \delta_d = \delta(d,\cdot) \) and \( \int \delta_d(t)V^*_{\lambda}(dt) < \infty; \quad \lambda \in \Lambda \). Denote this vector space by \( F_n \).

Consider now the expectation of an integrable random variable \( \delta(D,T) \). By the results mentioned above:

\[
E_\lambda \delta(D,T) = \sum_d \left( \frac{n}{d} \right) Q_\lambda(S)^d Q_\lambda(S^c)^{n-d} \int \delta_d(t)V^*_{\lambda}(dt) =
\]

\[
= \left[ \frac{c(\lambda)}{c_0(\lambda)} \right]^n \sum_d \left( \frac{n}{d} \right) \left( \frac{A(\lambda)}{A_0(\lambda)} \right)^{n-d} H(S)^d H(S^c)^{n-d} \int \delta_d(t) e^{-(\lambda-A_0(t))K^*(dt)}; \lambda \in \Lambda.
\]

Hence, by (2.15):

\[
(2.19) \quad E_\lambda \delta(D,T) = c(\lambda)^n \sum_{d=0}^{n} \hat{\tau}(\lambda)^{n-d} \delta_d(\lambda); \quad \lambda \in \Lambda
\]

where

\[
(2.20) \quad d\sigma_j/dK^{j*}_t = \delta_j(t)(\frac{n}{j}) H(S)^j H(S^c)^{n-j}/c(\lambda)^n; \quad j = 0,1,\ldots,n.
\]

Here \( \sigma = (\sigma_0, \sigma_1, \ldots, \sigma_n) \) is, and may be, any sequence of finite measures such that \( \hat{\delta}_j(\lambda) \) is defined for any \( \lambda \in \Lambda \) and such that \( \sigma_j \ll K^{j*} \) when \( j \in \{0,1,\ldots,n\} \).

Denote the vector space of such sequences by \( M_n \).

(2.19) tells us that \( E_\lambda \delta(D,T)/c(\lambda)^n \) is the transform of the measure

\[
\sum_{j=0}^{n} \tau^{(n-j)} \ast \sigma_j
\]
Substituting $\tau = \alpha H - \beta K$ where $\alpha$ and $\beta$ are defined in (2.13) this measure may be expressed as:

\( \sum_{j=0}^{n} \tau(n-j)* \sigma_j = \sum_{j=0}^{n} H(n-j)* R_j \) 

where

\( R_j = \sum_{d=0}^{j} \alpha^{n-d}/(n-j)^{(d)} \) \( H(n-d)* \sigma_d; \quad j = 0, 1, \ldots, n \).

Here \( R = (R_0, \ldots, R_n) \) is, and may be, any sequence in \( M_n \). \( \sigma \) may be recovered from \( R \) by:

\( \sigma_d = \sum_{j=0}^{d} \frac{1}{\alpha^{n-j}} \left( \sum_{d-j}^{n-j} H(d-j)* R_j \right); \quad d = 0, 1, \ldots, n. \)

An estimator \( \delta \) with finite expectation may, as we have seen, be represented as a vector \( \delta = (\delta_0, \delta_1, \ldots, \delta_n) \) (allowing ourselves a slight abuse of notations) or as a sequence \( \sigma \) in \( M_n \) or as a sequence \( R \) in \( M_n \). It is easily seen that the maps: \( \delta \rightarrow \sigma \) and \( \delta \rightarrow R \) from \( F_n \) to \( M_n \) and the map \( \sigma \rightarrow R \) from \( M_n \) to \( M_n \) are all isomorphisms.

We shall need:

**Proposition 2.1** Consider the representations \( \sigma \) and \( R \) of an integrable estimator \( \delta \). Then the following conditions are equivalent:

(i) \( \delta \) is bounded

(ii) There is a constant \( M < \infty \) so that:

\[ |\sigma_d| \leq MK^d; \quad d = 0, 1, \ldots, n \]

(iii) There is a constant \( M < \infty \) so that:

\[ |R_d| \leq MK^d; \quad d = 0, 1, \ldots, n \]

**Proof:** This is an immediate consequence of (2.22) and (2.23).
The unbiased estimators of zero are characterized by

Theorem 2.2 Let \( \delta(D,T) \) be an estimator with finite expectations and representations \( \sigma = (\sigma_0, \ldots, \sigma_n) \in M_n \) and \( R = (R_0, \ldots, R_n) \in M_n \). Then the following conditions are equivalent:

(i) \( \delta \) is an unbiased estimator of zero

\[
\sum_{j=0}^{n} \delta_j(\lambda) \frac{A(\lambda)^{-j}}{A(\lambda)} = 0 ; \quad \lambda \in \Lambda
\]

(ii) \( \sum_{j=0}^{n} R_j(\lambda) \frac{\sigma(\lambda)^{-j}}{\sigma(\lambda)} = 0 ; \quad \lambda \in \Lambda
\]

If \( \Lambda \) has interior points, then (ii) and (iii) may be replaced by, respectively

(ii') \( \sum_{j=0}^{n} \gamma^{(n-j)} \sigma_j = 0 \)

and

(iii') \( \sum_{j=0}^{n} H^{(n-j)} R_j = 0 \)

Proof: This follows from (2.14), (2.15), (2.19) and (2.21).

Corollary 2.3 Suppose \( S \) is bounded, \( \mu(S^c) = \infty \) and that \( 0 \in \Lambda \). Then \( \delta(0,0) = 0 \) for any unbiased estimator \( \delta(D,T) \) of zero.

Proof: By assumption \( A(\lambda) = \int_{S^c} e^{-\langle \lambda, x \rangle} \mu(dx) \to \infty \) as \( \lambda \to 0 \).

Letting \( \lambda \to 0 \) in (ii) we find \( \sigma_0([0]) = \sum_{j=0}^{n} \delta_j(0)0^j = 0 \), i.e. \( \delta(0,0) = 0 \).

The theorem yields the following criterions for completeness.

Theorem 2.4 The following conditions are equivalent:

(i) \( (D,T) \) is complete
(ii) \( \sigma \in M_n \) & \( \sum_{j=0}^{n} \hat{\sigma}_j(\lambda)/A(\lambda)^j = 0 ; \lambda \in \Lambda \implies \sigma = 0 

(iii) \( R \in M_n \) & \( \sum_{j=0}^{n} \hat{R}_j(\lambda)c(\lambda)^j = 0 ; \lambda \in \Lambda \implies R = 0 

If \( \Lambda \) has interior points then (ii) and (iii) may be replaced by, respectively,

(\textit{ii}') \( \sigma \in M_n \) & \( \sum_{j=0}^{n} (n-j)^* \tau_j = 0 \implies \sigma = 0 

and

(\textit{iii}') \( R \in M_n \) & \( \sum_{j=0}^{n} (n-j)^* P_j = 0 \implies R = 0 

\hfill 

\textbf{Corollary 2.5} Suppose \( S \) is bounded. Then each of the following conditions implies that \( (D,T) \) is complete:

(i) If \( f_0 = \text{constant}, f_1, \ldots, f_n \) are entire functions on \( k \) such that \( \sum_{j=0}^{n} j f_j(\lambda)/A(\lambda)^j = 0 ; \lambda \in \Lambda \) then \( f_0 = f_1 = \ldots = f_n = 0 \).

(ii) If \( f_0 = \text{constant}, f_1, \ldots, f_n \) are entire functions on \( k \) such that \( \sum_{j=0}^{n} j f_j(\lambda)c(\lambda)^j = 0 ; \lambda \in \Lambda \) then \( f_0 = f_1 = \ldots = f_n = 0 \).

\hfill 

\textbf{Proof:} In this case the functions \( \hat{R}_j(\lambda) \) and \( \hat{\sigma}_j(\lambda) \) are all entire.

It might at a first glance appear that conditions like (i) and (ii) are too much to hope for. Consider, however, the case where \( c \) is analytic on \( \Lambda \subseteq \mathbb{Z}^k \), where \( G \) is any region in \( \mathbb{Z}^k \) and where \( c_1, \ldots, c_s \) are different analytical functions on \( G \).
which all are obtained from $c$ on $\Lambda$ by finite chains of direct analytical continuations. Then an identity:

\[(2.24) \sum_{j=0}^{\infty} f_j(\lambda)c(\lambda)^j = 0 ; \lambda \in \Lambda\]

where $f_0, f_1, \ldots, f_n$ are entire implies the $s$ identities:

\[(2.25) \sum_{j=0}^{\infty} f_j(\lambda)c_k(\lambda)^j = 0 ; \lambda \in \Gamma; k = 1, \ldots, s.\]

If $n \leq s$ then (2.25) implies that $f_0 = f_1 = \ldots = f_n = 0$ and by the corollary, $(D,T)$ is complete provided $S$ is bounded.

This follows from the algebraic fact that:

\[(2.26) \begin{vmatrix} z_1, \ldots, z_n \\ z_1^2, \ldots, z_n^2 \\ \vdots \\ z_n^1, \ldots, z_n^2 \end{vmatrix} = \Pi(z_j - z_i) \text{ where } z_0 = 0\]

and the product is taken over all pairs $(i,j)$ such that $0 \leq i < j \leq n$. Thus this determinant differs from zero if and only if the numbers $z_1, \ldots, z_n$ are all distinct and different from zero.

As an example consider the case where $k = 1$, where $\Lambda = [0, \infty]$, where $\lambda_c = 1$ and where $\hat{H}$ is of the form:

\[(2.27) \hat{H}(\lambda) = \Pi U(\lambda) ; \lambda > 0\]

where $U$ is an entire function.

This is the case if, for example, $H$ is the convolution of a gamma distribution with shape parameter $p > 0$ and scale parameter $1$ and a distribution whose Laplace transform is entire.

By the above considerations and corollary 2.5 $(D,T)$ is complete provided $S$ is bounded and provided $p$ is either irrational or rational $= \frac{r}{s}$ where $r$ and $s$ are relatively prime and $n \leq s$. 
If $H$ actually is the gamma distribution with shape parameter $p > 0$ and scale parameter $1$, then, as we shall see in the next section, these conditions are necessary as well.

### III Fixed point censoring of gamma-distributions

We shall in this section assume that $Y$ is distributed according to a gamma distribution with known shape parameter $p > 0$ and unknown scale parameter $\lambda > 0$. Then $Q_\lambda$ has density:

$$
(3.1) \quad \frac{\lambda^p}{\Gamma(p)} y^{p-1} e^{-\lambda y}, \; y > 0 \; \text{w.r.t. Lebesgue measure.}
$$

The observations from $Y$ are assumed to be censored at a fixed time point. We may, without loss of generality, assume that the observations are censored at time $t = 1$. (If they are censored at $t = a$ replace $\lambda$ by $a \lambda$ and apply the results proved for $a = 1$.) This is clearly a particular case of the exponential models described in the previous section and we shall use the notations introduced there, with $\lambda_0 = 1$.

$S = [0,1]$ is now the censoring set and we may and shall, since $1 \notin S$, put $S = 1$.

Thus our observations $X_1, X_2, \ldots, X_n$ are independent and identically distributed such that:

$$
(3.2) \quad P_\lambda(X_i < x) = \int_0^x \frac{\lambda^p}{\Gamma(p)} t^{p-1} e^{-\lambda t} dt \quad \text{if } x < 1
$$

while

$$
(3.3) \quad P_\lambda(X_i = 1) = \int_1^\infty \frac{\lambda^p}{\Gamma(p)} t^{p-1} e^{-\lambda t} dt
$$

$$
= \frac{\lambda^p}{\Gamma(p)} A(\lambda)
$$
where

\[ A(\lambda) = \int_1^\infty t^{p-1} e^{-\lambda t} \, dt. \]

The gamma distribution with density:

\[ \Gamma(q)^{-1} x^{q-1} e^{-x}; \quad x > 0 \text{ w.r.t. Lebesgue measure} \]

will be denoted by \( H_q \) so that, with the notations of section 2:

\[ H = H_p \]

Furthermore:

\[ H_q(\lambda) = \lambda^{-q}; \quad q > 0 \]

so that

\[ H_q^n = H_{nq}; \quad q > 0, \quad n = 0, 1, \ldots \]

The distributions \( K \) and \( T \) are obtained from \( H \) by conditioning on, respectively, \([0,1]\) and \([1,\infty[.\]

The density of \( K_j^* \), when \( j \geq 1 \), may be specified as proportional to \( x^{j-1} e^{-x} \) on \([0,1]\). It is continuous and positive on \([0,j[ \) and it is bounded from above and from below by functions of the type constant \( \cdot (j-x)^{j-1} \) on \([j-\frac{1}{2}, j]\). It follows that an estimator \( \delta(D,T) \) has finite expectation if and only if:

\[ \int_0^j \delta(j,t)t^{j-1} e^{-j} \, dt < \infty; \quad j = 1, \ldots, n. \]

If \( p \) is irrational or if \( p \) is rational \( = \frac{r}{s} \) where \((r,s) = 1 \) and \( n < s \) then we saw in the last section that \((D,T)\) is complete. Hence any estimator \( \delta(D,T) \) with everywhere finite variance is UMVU in this case.

It remains to consider the case of a rational \( p = \frac{r}{s} \) where the positive integers \( r \) and \( s \) are relatively prime and where \( s < n \).
We will from here on, unless otherwise stated, assume that \( p \) and \( n \) satisfy these conditions.

If \( a \) and \( b \) are integers such that \( a-b \) is divisible by \( s \) then we shall write this

\[
a = b
\]

The largest integer which is not greater than the number \( x \) will be denoted by \( [x] \).

Let, for each \( j=1,2,\ldots,s \), \( I_j \) be the set of integers \( k \) in \([1,n]\) such that \( k = j \). The largest number in \( I_j \) will be denoted by \( n_j \) so that:

\[
(3.9.2) \quad n_j = j + \left[ \frac{n-1}{s} \right]s \quad ; \quad j=1,\ldots,s
\]

If \( \rho \) is any finite measure on \([0,\infty]\) and \( t \geq 0 \) is an integer then we put:

\[
(3.10) \quad L_t(\rho) = \int \frac{(-x)^t}{t!} e^x \rho(dx)
\]

provided this integral exists and is finite.

Let \( \delta(D,T) \) be any unbiased estimator of zero. By corollary 2.3 \( R_0 = \sigma_0 = 0 \), i.e. \( \delta_0 = 0 \), for any unbiased estimator of zero. Condition (ii') in theorem 2.2 implies that:

\[
(3.11) \quad \sigma_n = - \sum_{j=0}^{n-1} \tau(n-j) \delta_j
\]

Hence \( \sigma_n \) is concentrated on \([1,n]\) so that \( \delta(n,t) = 0 \) for (Lebesgue) almost all \( t \) on \([0,1]\).

It follows that there are no restrictions, except for integrability conditions on the values of a UMVU estimator on the set \( B_0 \) defined by:
\[ B_0 = \{(0,0)\} \cup \{(n,t) : 0 \leq t < 1\} \]

Note that "\((D,T) = (0,0)\)" is the condition that all observations are censored while "\((D,T) \in \{(n,t) : 0 \leq t < 1\}\)" is the condition that the sum \(\sum_{i=1}^{n} Y_i\) (which is sufficient for \((Y_1, \ldots, Y_n)\)) would not have been censored if it had been considered as a single observation.

Thus any square integrable variable \(\varphi(D,T)\) which is constant when \((D,T) \in B_0\) is UMVU, and it will be shown later that these are the only UMVU estimators. The class of functions on \(\Lambda\) which possesses UMVU estimators is therefore a very restricted class. Furthermore, as we shall see, a non constant function of \(\lambda\) is UMVU estimable on the basis of \(n\) observations only for \(n\) belonging to a finite set of integers.

Before establishing these results and results on completeness we shall have to labour a bit with the unbiased estimators of zero.

We begin by studying the sequences \(R\) in \(M_n\) which represent unbiased estimators of zero.

**Proposition 3.1** A sequence \(R = (0,R_1,\ldots,R_n)\) in \(M_n\) represents an unbiased estimator of zero if and only if:

\[ (3.12) \sum H_{n-j-k}^*(R_k : k \in I_j) = 0 \quad ; \quad j=1, \ldots, s \]

**Proof:** By theorem 2.2, \(R = (0,R_1,\ldots,R_n)\) is an unbiased estimator of zero if and only if

\[ (3.13) \sum_{j=1}^{n} R_j(\lambda) ^{j+\lambda P} = 0, \ \lambda > 0 \]

Putting \(\lambda = e^z\) and applying the principle of analytical continuation we see that (3.13) is equivalent to:
Replacing $z$ with $z + 2a\pi i$ where $a$ is a given integer, writing $\zeta = e^{2\pi iP}$, and then replacing $e^z$ with $\lambda$ we find that (3.14) is equivalent to:

$$\sum_{j=1}^{n} \hat{R}_j(\lambda)e^{jPz} = 0 ; \ z \in \mathbb{Z}$$

(3.15)

$$\sum_{j=1}^{n} R_j(\lambda)\lambda^j\zeta^{aj} = 0 ; \ \lambda > 0$$

If $j_1, j_2 \in I_j$ then $\zeta^{aj_1} = \zeta^{aj_2}$. Hence (3.15) may be written:

(3.16)

$$\sum_{j=1}^{s} \zeta^{aj} \sum_{k \in I_j} \hat{R}_k(\lambda)\lambda^pk = 0 ; \ \lambda > 0$$

Hence, by inserting $a=1,2,\ldots,s$ and applying (2.26) we find that $(0, R_1, \ldots, R_n)$ is an unbiased estimator of zero if and only if

(3.17)

$$\sum_{k \in I_j} \hat{R}_k(\lambda)\lambda^pk = 0 ; \ \lambda > 0 ; \ j=1,2,\ldots,s$$

and this is just the transform version of (3.12).

**Corollary 3.2** Let $R_i$, for each $i=1,\ldots,n-s$, be an absolutely continuous and finite measure on $[0,i]$. Then $(0, R_1, \ldots, R_{n-s})$ may be extended to an unbiased estimator $(0, R_1, \ldots, R_n)$ of zero if and only if

(3.18)

$$\sum_{k \in I_j} H^{(n_j-k)*}\hat{R}_k ; k \in I_j ; k < n_j = 0 ; \ j=1,\ldots,s$$

If $R_1, \ldots, R_{n-s}$ satisfies these conditions then the extension is unique and it is given by:

(3.19)

$$R_{nj} = -\sum_{k \in I_j} H^{(n_j-k)*}\hat{R}_k ; k < n_j ; \ j=1,\ldots,s$$

**Remark:** (3.19) implies that $R_{nj} = 0$ when $n_j \leq s$. 

(3.18) may be expressed in terms of the moments of $R_1, \ldots, R_{n-s}$ as follows:

**Proposition 3.3** Let $R_i$, for each $i=1, \ldots, n-s$, be an absolutely continuous and finite measure on $[0,i]$. Then $(0,R_1, \ldots, R_{n-s})$ may be extended to an unbiased estimator $(0,R_1, \ldots, R_n)$ of zero if and only if

$$
\Sigma \{I_{m-hr}(R_j+hs) : h=0,1, \ldots, [m/r] \} = 0 ; \quad j=0, \ldots, s, \quad m=0, \ldots, \lfloor \frac{n-j}{s} \rfloor - 1
$$

If these conditions are satisfied then $R_{nj}$ is supported by $[0, nj-s]$ when $nj > s$.

**Remark 1** Note that (3.20) is no requirement at all when $nj \leq s$.

**Remark 2** Suppose $S$ is an unbiased estimator of zero such that $\delta_d ; d \leq n-s$ are all bounded. Then $\delta$ is bounded. This may be seen as follows: By (2.20) $|\sigma_d| \leq \text{constant} \cdot k^{d^*} ; d \leq n-s$. Hence, by (2.22), $|R_d| \leq \text{constant} \cdot k^{d^*} , d \leq n-s$. It remains then, by proposition 2.1 and corollary 2.2, to show that $|R_{nj}| \leq \text{constant} \cdot k^{nj*}$ when $nj > s$. But then $H_{nj-k}* \leq \text{constant} \cdot k^{nj-k*}$ on $[0, nj-s]$. (Actually $\sigma_d = \delta_d$ holds on $[0,1]$.) Hence the conclusion follows from (3.19).

**Proof of the proposition:** We may, by remark 1, restrict attention to pairs $(j,m)$ such that $nj > s$ and $m < [(n-j)/s]r$. The density of $\Sigma \{H_{nj-k}*R_k : k \in I_j ; k < nj \}$ on $[nj, \infty]$ may be written:
(3.21) \[ \sum_{k=j, \ldots, n_j} \int \Gamma((n_j-k)p)^{-1} (y-x)^{(n_j-k)p-1} e^{-(y-x)R_k} (dx) \]

It follows that this measure is supported by \([0, n_j]\) if and only if:

(3.22) \[ \sum_{k=j, \ldots, n_j} \int \Gamma((n_j-k)p)^{-1} (y-x)^{(n_j-k)p-1} e^{xR_k} (dx) = 0; y > 0 \]

or equivalently

(3.23) \[ \sum_{k=j, \ldots, n_j} \int \frac{1}{\Gamma((n_j-k)p-1-t)} (n_j-k)p-1-t \]

The proof is now completed by using corollary 3.2 and by checking that (3.20) merely states that the coefficients of the polynomial in (3.23) all vanish.

Proposition 3.3 may be phrased directly in terms of the estimator as follows:

Theorem 3.4 Let \( \delta_i \), for each \( i=1, \ldots, n-s \), be a \( K^iK^* \) integrable function. Then \((0, \delta_1, \ldots, \delta_{n-s})\) may be extended to an unbiased estimator \((0, \delta_1, \ldots, \delta_n)\) \( \in F_n \) of zero if and only if

(3.24) \[ \sum_{a=1}^{j+[m/r]} \int \Pi_{j,m,a}(x) e^{x\delta_a(x)K^*(dx)} = 0; j=1, \ldots, s, \]

where \( \Pi_{j,m,a} \) is the polynomial defined by:

(3.25) \[ \Pi_{j,m,a}(x) = (-1)^a \frac{m+[(j-a)/s]x}{x} \sum_{x=0}^{m} (-1)^x \frac{x^x}{x!} \]

\[ [m-x] \]

\[ (-1)^{hs} \Gamma((p-A)^{hs}((-1)^{hs}(n-a)_{j+hs-a}) L_{m-hs-k} \]

\[ (-1)^{hs} \Gamma((p-A)^{hs}((-1)^{hs}(n-a)_{j+hs-a}) L_{m-hs-k} \]

\[ h = -[(j-a)/s] \]
Remark: If \( n_j \leq s \) then (3.24) is no requirement at all.

Proof: We may, by the above remark, restrict attention to pairs \((j, m)\) such that \( n_j > s \). By (2.22):

\[
L_t(R_K) = \sum_{a=1}^{k} \alpha^{n-k} \binom{n-a}{n-k} L_t((-\beta K)(k-a)^*\sigma_a)
\]

Furthermore:

\[
L_t(K^{a^*}\sigma_b) = \int \frac{(-1)^t}{t!} (x+y)^t e^{x+y} K^{a^*}(dx)\sigma_b(dy)
\]

so that:

\[
(3.27) \quad L_t(K^{a^*}\sigma_b) = \sum_{\kappa=0}^{t} L_t(K^{a^*}) L_{t-\kappa}(\sigma_b)
\]

Inserting this in (3.26) we get:

\[
(3.28) \quad L_t(R_K) = \sum_{a=1}^{k} \alpha^{n-k} \binom{n-a}{n-k} (-\beta)^{k-a} \sum_{\kappa=0}^{t} L_t(K^{a^*}) L_{t-\kappa}(\sigma_a)
\]

It follows that (3.20) may be written:

\[
(3.29) \quad \sum_{h=0}^{[m/r]} \sum_{a=1}^{j+[m/r]s} \alpha^{n-j-hs} \binom{n-a}{n-j-hs} (-\beta)^{j+hs-a} \sum_{\kappa=0}^{m-hr} L_t(K^{j+hs-a^*}) L_{m-hr-\kappa}(\sigma_a) = 0
\]

Interchanging the order of summations on \( h \) and on \( a \) we find that the two first summation signs may be replaced by

\[
(3.30) \quad \sum_{a=1}^{[m/r]} \sum_{h=0}^{[m/r]s} \binom{n-a}{n-j-hs} (-\beta)^{j+hs-a} \sum_{\kappa=0}^{m-hr} L_t(K^{j+hs-a^*}) L_{m-hr-\kappa}(\sigma_a) = 0
\]

By (2.20):

\[
(3.31) \quad L_{m-hr-k}(\sigma_a) = \binom{r}{a} A(1)^{n-a} (\Gamma(p)-A(1))^{a} \int \frac{(-x)^{m-hr-k}}{(m-hr-k)!} e^{-x} \delta_{a}(x) K^{a^*}(dx)
\]

Inserting (3.31) in (3.29) and interchanging summation as indicated in (3.30) we see that (3.29) may be written:
where

\[
\Pi_{j,m,a}(x) = \sum_{h=-\frac{(j-a)}{s}}^{\frac{m}{s}} \frac{m-hr}{m-hr-x} \sum_{\kappa=0}^{\infty} I_{\kappa}(K(j+hs-a^*)) \frac{(-x)^{m-hr-\kappa}}{(m-hr-\kappa)!}
\]

Interchanging the order of the summation on \( h \) and on \( \kappa \) in (3.32) we get:

\[
\tilde{\Pi}_{j,m,a}(x) = \frac{m+[(j-a)/s]r}{(m-hr-\kappa)!} (-x)^{\kappa}
\]

\[
\sum_{h=\frac{m-hr}{(j-a)/s}}^{(m-hr-x) (\Gamma(p)-A(1)) hs}^{\infty} I_{\kappa}(K(j+hs-a^*)) \frac{(-x)^{m-hr-\kappa}}{(m-hr-\kappa)!}
\]

\[
\sum_{\kappa=0}^{\infty} I_{m-hr-\kappa}(K(j+hs-a^*)) \frac{(-1)^{m-hr-\kappa} \kappa}{\kappa!}
\]

Interchanging the order of the summation on \( h \) and on \( \kappa \) in (3.32) we get:

\[
\tilde{\Pi}_{j,m,a}(x) = \frac{m+[(j-a)/s]r}{(m-hr-\kappa)!} (-x)^{\kappa}
\]

\[
\sum_{h=\frac{m-hr}{(j-a)/s}}^{(m-hr-x) (\Gamma(p)-A(1)) hs}^{\infty} I_{\kappa}(K(j+hs-a^*)) \frac{(-x)^{m-hr-\kappa}}{(m-hr-\kappa)!}
\]

\[
\sum_{\kappa=0}^{\infty} I_{m-hr-\kappa}(K(j+hs-a^*)) \frac{(-1)^{m-hr-\kappa} \kappa}{\kappa!}
\]
Cancelling factors which does not depend on \( a \), using (3.32) and (3.33) we find that \( \Pi_{j,m,a} \) in (3.32) may be replaced by \( \Pi_{j,m,a} \) as defined in (3.25).

The only property of the polynomials \( \Pi_{j,m,a} \) which we shall utilize is:

**Proposition 3.5**

\[
\text{degree}(\Pi_{j,m,a}) = m + \left[ \frac{j-a}{s} \right]r ; \quad j=1, \ldots, s, \quad m=0, \ldots, \lfloor (n-j)/s \rfloor r-1, \quad a=1, \ldots, j+ \lfloor m/r \rfloor s
\]

**Remark:** Denote, for each integer \( b \) in the range of \( a \) i.e. in \([1,n-s]\), the set of pairs \((j,m)\) such that

\[
1 \leq j \leq s, \quad 0 \leq m < \lfloor n-j/s \rfloor r \quad \text{and} \quad j + \lfloor m/r \rfloor = b
\]

by \( C_b \). Let also for each integer \( b \), \( \bar{b} \) be the unique integer in \([1,s]\) such that \( b = \bar{b} \).

Then \( C_b \) consists of the \( r \) pairs \((\bar{b}, r \frac{b-\bar{b}}{s} + h) ; h=0, 1, \ldots, r-1 \). If \( a \) is a fixed integer in \([1,b]\) then \( \text{degree}(\Pi_{j,m,a}) = \lfloor (b-a)/s \rfloor r + h \) when \((j,m) \in C_b\) and \( m = r(b-\bar{b})/s + h \). It follows that \( \text{degree}(\Pi_{j,m,a}) \) for a fixed \( a \leq b \) runs through the set \([\lfloor (b-a)/s \rfloor r + h ; h=0, \ldots, r-1]\) when \((j,m) \) runs through \( C_b \). It follows in particular that these polynomials \( \Pi_{j,m,a} ; (j,m) \in C_b \) are linearly independent.
Proof of the proposition: The absolute value of the leading term \( x^{m+[\frac{(j-a)}{s}]r} \) in (3.25) is 
\[
\frac{n}{a} \Gamma (\frac{1}{r}) (\Gamma (p)-A(1))^{h} (\frac{n-a}{j+h-a}).
\]
where \( k = m+[\frac{(j-a)}{s}]r \) and \( h = -[\frac{(j-a)}{s}] \).
This number is, since \( j+h \geq a \), not zero.

A simple consequence of theorem 3.4 is:

Proposition 3.6  Let \( a_0 \in \{1,2,\ldots,n-s\} \) and let \( \delta_{a_0}^{(0)} \in L_{1}(K^{a_0}) \)
be such that:
\[
(3.35) \quad \int x^{t} e^{x} d_{a_0}^{(0)}(x) K^{a_0} (dx) = 0; \quad t=0,1,\ldots,\left[\frac{n-a_0}{s}\right] r-1.
\]

Then there exist an unbiased estimator \( \delta \) of zero such that:
\[
(3.36) \quad \delta_{a_0} = \delta_{a_0}^{(0)} \quad \text{and} \quad \delta = 0 \quad \text{when} \quad 1 \leq a_0 \leq n-s \quad \text{and} \quad a \neq a_0.
\]

Proof: (3.24) may in this case be written:
\[
(3.37) \quad \int \Pi_{j,m,a_0}(x) e^{x} d_{a_0}^{(0)}(x) K^{a_0} (dx) = 0 \quad \text{when} \quad j+\left[\frac{m}{r}\right]s \geq a_0.
\]

If \( j+\left[\frac{m}{r}\right]s \geq a_0 \) then, by the remark after proposition 3.5, degree \( (\Pi_{j,m,a_0}) \) is at most \( \left[\frac{(n-a_0)}{s}\right] r-1 \).

We are now in a position to resolve the problem of completeness.

Theorem 3.7  Let \( X_1, X_2, \ldots, X_n \) be independent and identically distributed such that:
\[
(3.38) \quad P(X_1 \leq x) = \int_{0}^{x} \lambda^{p} e^{-\lambda x} dx ; \quad 0 \leq x < 1
\]
and
\[
(3.39) \quad P(X_1 = 1) = \int_{1}^{\infty} \lambda^{p} e^{-\lambda x} dx
\]

Suppose \( \lambda > 0 \) is unknown and that \( p > 0 \) is known. Then the experiment \( \mathcal{E}_{n}^{(p)} \) defined by \( X_1, \ldots, X_n \) admits a complete and sufficient statistic if and only if it admits a boundedly complete
and sufficient statistic and this is the case if and only if either

i) \( p \) is irrational
or

ii) \( p \) is rational \( = \frac{r}{s} \) where the integers \( r \) and \( s \) are relatively prime and \( n \leq s \).

Remark: Only a fraction of the results established so far in this section is actually needed, [6], to prove this. On the other hand these results are needed in order to obtain the promised characterization of UMVU estimators.

Proof of the theorem: The fact that i) and ii) implies completeness of \((D,T)\) was established at the end of section 2. Suppose next that \( p = \frac{r}{s} \) where \((r,s) = 1\) and \( n \geq s+1 \). Let \( \delta_{n-s}^{(o)} \) be a non vanishing polynomial of degree \( r \) such that

\[
\int x^t e^{n-s}(x)(x)^{n-s}(dx) = 0 ; \quad t=0,1,\ldots,r-1.
\]

By proposition 3.6 there is an unbiased estimator \( \delta \) of zero such that \( \delta_{n-s} = \delta_{n-s}^{(o)} \) and \( \delta_a = 0 ; a < n-s \). By remark 2 after proposition 3.3, \( \delta \) is a non vanishing and essentially bounded estimator of zero. It is a consequence of the theory of sufficiency that the model admits a (boundedly) complete and sufficient statistic if and only if \((D,T)\) is (boundedly) complete.

Another simple consequence of proposition 3.6 is:

**Proposition 3.8** If \( \varphi = (\varphi_0, \ldots, \varphi_n) \in F_n \) is a UMVU estimator then \( \varphi_a \) is constant when \( 1 \leq a \leq n-s \).
Proof: Let $1 \leq a_0 \leq n-s$. Let $\delta^{(o)}_{a_0} \in L_\infty(K^{a_0^*})$ satisfy \eqref{3.35}. Then, as argued in the proof of theorem 3.7, $(0, \ldots, 0, \delta^{(o)}_{a_0})$ may be extended to a bounded estimator $(0, \ldots, 0, \delta^{(o)}_{a_0}, \delta^{(o)}_{a_0+1}, \ldots, \delta^{(o)}_{n})$ of zero. Then $\varphi\delta$ is also an unbiased estimator of zero and $(\varphi\delta)_a = 0$ when $a \leq n-s$ and $a \neq a_0$. By theorem 3.4
\[ \int \varphi^{(o)}_{a_0}(x)\delta^{(o)}_{a_0}(x)e^x\Pi_{j,m,a_0}(x)K^{a_0^*}(dx) = 0 \] when $j + [m/r]s \geq a_0$. In particular this holds when $j = a_0$ and $m = r(a_0 - j)/s$, and then, by proposition 3.5, $\Pi_{j,m,a_0}$ is a constant $\neq 0$.

Hence $\int \varphi^{(o)}_{a_0}(x)\delta^{(o)}_{a_0}(x)e^xK^{a_0^*}(dx) = 0$ when $\delta^{(o)}_{a_0} \in L_\infty(K^{a_0^*})$ satisfies \eqref{3.35}. It follows that $\varphi^{(o)}_{a_0}$ is a polynomial of degree $<(n-a_0)/s]r$. The same reasoning with $j = j_1 = n$ and $m = m_1 = (n-s-j_1)p+r-1$ shows that $\varphi^{(o)}_{a_0}\Pi_{j_1,m_1,a_0}$ is also a polynomial of degree $<(n-a_0)/s]r$ although degree $\Pi_{j_1,m_1,a_0} = [n-a_0]/s]r-1$.

This is only possible if $\varphi^{(o)}_{a_0}$ is constant.

Our next result on unbiased estimators of zero is:

**Proposition 3.9** Let $1 \leq a_0 < a_0 + s \leq n-s$ and let $\delta^{(o)}_{a_0} \in L_\infty(K^{a_0^*})$ satisfy
\[ (3.40) \int e^x x^t\delta^{(o)}_{a_0}(x)K^{a_0^*}(dx) = 0 \quad t=0,1,\ldots,r-1. \]

Then there exists a bounded unbiased estimator $\delta$ of zero such that:
\[ (3.41) \delta^{(o)}_{a_0} = \delta^{(o)}_{a_0} \quad \text{and} \quad \delta_a = 0 \text{ if } a \neq a_0 \text{ and } a < a_0 + s. \]

**Note on notation:** In order to simplify the writing we shall in the following permit ourselves to use inconsistent notations as $\int f(x)e^v(dx)$ instead of the more proper: $\int f(x)e^xv(dx)$. 

Proof of the proposition: The requirements given in theorem (3.4) may now be written:

\[ \int \Pi_{j,m,a} e^{x \delta_{a_0}} dK_0 = 0 \; ; \; j + [m/r]s \in [a_0, a_0 + s[ \]

\[ \int \Pi_{j,m,a} e^{x \delta_{a_0}} dK_0 + \int \Pi_{j,m,a+se^{x \delta_{a_0} + s} dK_0 = 0 ; \; j + [m/r]s = a_0 + s \]

(3.42)

\[ \int \Pi_{j,m,a} e^{x \delta_{a_0}} dK_0 + \int \Pi_{j,m,a+se^{x \delta_{a_0} + s} dK_0 + \]

\[ + \int \Pi_{j,m,a+se^{x \delta_{a_0} + s} dK_0 = 0 ; \; j + [m/r]s = a_0 + s + 1 \]

If \( j + [m/r]s < a_0 + s \) then degree \( \Pi_{j,m,a} \leq r-1 \) so that the first set of requirements are automatically satisfied. The next set of requirements may be satisfied for some polynomial \( \delta_{a_0 + s} \) of degree \( < r \) and then the third set (if it is there) is satisfied by some polynomial \( \Pi_{a_0 + s + 1} \) of degree \( < r \) and so on. The reason that we can proceed like this is that the polynomials \( \Pi_{j,m,a} \) with \( j + [m/r]s \) fixed are linearly independent and of degree \( < r \) when \( j + [m/r]s = a \). The proof is now completed by applying remark 2 after proposition 3.3.

Corollary 3.10 If \( \phi \) is a UMVU estimator and \( a_1, a_2 \in \{1, 2, \ldots, n-s\} \) is such that \( a_1 = a_2 \) then \( \phi_{a_1} = \phi_{a_2} \).

Proof: It suffices to show that \( \phi_{a_0} = \phi_{a_0 + s} \) when \( 1 \leq a_0 < a_0 + s \leq n-s \). Let \( \delta_{a_0} \) satisfy the requirements of the proposition and let \( \delta \) be an extension as described there. Let \( j + [m/r]s = a_0 + s \). Then \( \phi \delta \) is also an unbiased estimator of zero.
so that, by (3.24) and proposition 3.8:

\[
(3.43) \quad \varphi_{a_0} \int \Pi_{j,m,a_0} \delta_0 \delta(a_0) e^x dK_{a_0}^* + \varphi_{a_0+s} \int \Pi_{j,m,a_0+s} \delta_0 \delta(a_0+s) e^x dK_{a_0+s}^* = 0
\]

and

\[
(3.44) \quad \int \Pi_{j,m,a_0} \delta_0 \delta(a_0) e^x dK_{a_0}^* + \int \Pi_{j,m,a_0+s} \delta_0 \delta(a_0+s) e^x dK_{a_0+s}^* = 0
\]

Hence \((\varphi_{a_0} - \varphi_{a_0+s}) \int \Pi_{j,m,a_0} \delta_0 \delta(a_0) e^x dK_{a_0}^* = 0\) when \(j + [m/r]s = a_0 + s\) and \(\delta_0\) is a polynomial such that \(\int x \delta_0 e^x dK_{a_0}^* = 0\); \(t = 0, 1, \ldots, r-1\). It follows that \((\varphi_{a_0} - \varphi_{a_0+s}) \Pi_{j,m,a_0} \) is a polynomial of degree \(< r\). This, since degree \((\Pi_{j,m,a_0}) \geq r\) imply that \(\varphi_{a_0} = \varphi_{a_0+s}\). 

Let us so consider unbiased estimators \(\delta = \delta(D,T)\) of zero such that \(\delta_a = 0\) when \(a < n-s\). The last condition is by (2.20), (2.22) and (2.23) equivalent to:

\[
(3.45) \quad \sigma_a = R_a = 0 \quad ; \quad a = 0, 1, \ldots, n-s-1.
\]

Hence, by (2.22)

\[
(3.46) \quad R_{n-s+h} = a^{s-h} s^h (\frac{s}{h}) (-\beta)^h \sigma_{n-s} + \sum_{d=n-s+1}^{n-s+h} \frac{a^{s-h} (s-h) (n-d) (-\beta)^h \sigma_{d}^*}{d-n-s+1} ;
\]

\(h = 0, \ldots, s\)

In particular

\[
(3.47) \quad R_{n-s} = a^s \sigma_{n-s}
\]

and by (3.19) and (3.8):

\[
(3.48) \quad R_n = R_{s} * R_{n-s} = R \sigma_{n-s}
\]

By (2.23) and (3.19)

\[
(3.49) \quad \sigma_d = \frac{s}{n-d} (\beta)^{d-n+s} \sigma_{n-s} \quad ; \quad n-s+1 \leq d < n
\]
By (3.47), (3.48) and (3.49):

$$(3.50) \quad \sigma_n = (\beta K)^{s*} \sigma_{n-s} + R_n = [(\beta K)^{s*} - (\alpha H)^{s*}] \sigma_{n-s}$$

Assume also that $\delta_{n-s}$ is a polynomial. By remark 2 after proposition 3.3 this implies that $\delta$ is essentially bounded. The only other requirements we need to impose are, by (3.24) that:

$$(3.51) \quad \int x^t \delta_{n-s}(x)e^{xK(n-s)^*}(dx) = 0 \quad ; \quad t=0, \ldots, r-1$$

Let $\varphi$ be any UMVU estimator. Then, since $\varphi_{n-s}$ is constant by proposition 3.8, $\varphi \delta$ has the same properties. By (3.49) and (3.50) $\sigma_d(\varphi \delta) = \varphi_{n-s} \sigma_d(\delta)$ for all $d$. Hence $(\varphi - \varphi_{n-s}) \delta = 0$ so that

$$(3.52) \quad (\varphi_d - \varphi_{n-s}) \delta_d = 0 \quad ; \quad d=n-s+1, \ldots, n-s$$

Let $d \in \{n-s+1, \ldots, n-1\}$ and let $z$ be a number in $[0, d]$ which is not an integer. Then, as we shall see, $\delta_{n-s}$ may be chosen so that $\delta_d \neq 0$ in a neighbourhood of $z$. Then by (3.52):

**Proposition 3.11** If $\varphi$ is a UMVU estimator then $\varphi_d = \varphi_{n-s}$ when $n-s \leq d < n$.

**Completion of the proof of proposition 3.11:** Let $z$ and $d$ be as above. It follows from (3.50) that $\sigma_d$ has a continuous density (if not otherwise specified a density is always w.r.t. Lebesgue measure) $\sigma_d^1$ which in $z$ takes the value:

$$(n-s)^{Az} \int_{n-d}^{s} \beta^{d-n+s} \int_{z-d+n-s}^{d} [K(d-n+s)]' (z-x) \sigma_{n-s}(dx)$$

Suppose (3.53) vanishes whenever $\delta_{n-s}$ satisfies (3.51).

Then by (2.20):
\[ (3.54) \quad I \left( x \right) = \frac{r-1}{t=0} \sum c_t x^t \]  
\[ (z-(d-n+s))^+(n-s)\Lambda z] \]

for some constants \( c_0, \ldots, c_{r-1} \).

We shall now need the fact that the density \( (K^m)^* \) can not be specified as a function which is analytic in any point in \( \{1, \ldots, m\} \). This may be seen as follows:

The relation \( \delta K = \alpha H - \tau \) (i.e. 2.13) implies that:

\[ (3.55) \quad \delta^m K^m = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \alpha^{m-k} \tau^k H(m-k)^* \]

Now \( \tau^k H(m-k)^* \) is, for each \( k \), supported by \( \lceil k, \infty \rceil \) and its density \( \tau^k H(n-k)^* \) may be specified as an analytic function on \( \lceil k, \infty \rceil \). If \( i \in \{1, \ldots, m\} \) and \( 0 < \varepsilon < 1 \) then the density of \( \delta^m K^m \) is \( \sum_{k=0}^{i-1} (-1)^k \binom{m}{k} \alpha^{m-k} \tau^k H(m-k)^* \) on \( \lceil i-\varepsilon, i \rceil \) while it is \( \sum_{k=0}^{i} \) on \( \lceil i, \varepsilon \rceil \). If the density \( (K^m)^* \) could be specified as a function which is analytic in \( \lceil i-\varepsilon, i+\varepsilon \rceil \) then it would have to coincide on \( \lceil i-\varepsilon, i \rceil \) and \( \lceil i, i+\varepsilon \rceil \) with respectively \( \sum_{k=0}^{i-1} \) and \( \sum_{k=0}^{i} \). Now \( \sum_{k=0}^{i-1} \) is analytic on \( \lceil i-1, \infty \rceil \) so that it would have to coincide with \( (K^m)^* \) on \( \lceil i-\varepsilon, i+\varepsilon \rceil \). It follows that \( \sum_{k=0}^{i-1} = \sum_{k=0}^{i} \) on \( \lceil i, i+\varepsilon \rceil \) so that \( (\tau^i H(m-i)^*) = 0 \) on \( \lceil i, i+\varepsilon \rceil \) contradicting the fact that \( (\tau^i H(m-i)^*)(\lceil i, i+\varepsilon \rceil) > 0 \). This proves our assertion concerning the behaviour of \( (K^m)^* \) on \( \{1, 2, \ldots, m\} \).

Suppose now that \( z \) is such that \( \lceil 0, n-s \rceil \notin \lceil z-(d-n+s) \rceil +, (n-s)\Lambda z \). Then the left hand side of (3.54) vanishes when \( x \in \lceil 0, n-s \rceil \) and \( x \notin \lceil (z-(d-n+s))^+, (n-s)\Lambda z \rceil \). Hence \( c_0 = \ldots = c_{r-1} = 0 \) so that \( (K(d-n+s)^*)'(z-x) \) when \( x \in \lceil 0, n-s \rceil \cap \lceil z-(d-n+s) \rceil +, (n-s)\Lambda z \rceil = \lceil z-d+n-s \rceil +, (n-s)\Lambda z \rceil \).
As $x$ runs through this set, $z-x$ runs through a non degenerate sub interval of $]0,d-n+s[$ and this contradicts the fact that $(K(d-n+s)^* )$ is positive on $]0,d-n+s[$. Hence

$$(3.56) \quad ]0,n-s[ \subseteq [z-(d-n+s), (n-s)Az]$$

so that $n-s < z < d-n+s$. Then (3.54) states that 

$$(K(d-n+s)^*)^t(z-x) = \sum c_t x^t e^x \text{ on } ]0,n-s[. \quad \text{If } x \in ]0,n-s[ \text{ then } (z-x) \in ]z-(n-s), z[ \subset ]0,d-n+s[. \quad \text{It follows, since } z > n-s \geq 1 \quad \text{and the length of the interval } ]z-(n-s), z[ \text{ is } n-s \geq 1, \text{ that } (K(d-n+s)^* )^t \text{ is analytic at some positive integer } i < d-n+s \text{ and this contradicts the fact on convolutions } K^m* \text{ established above.}$$

The remaining step for establishing the promised characterization of the UMVU estimators is:

**Proposition 3.12** If $\varphi$ is a UMVU estimator and $n_j > s$ then

$$\varphi_{n_j} = \varphi_{n_j-s} \text{ when } n_j < n \text{ while } \varphi_{n} = \varphi_{n-s} \text{ on } ]1,n[.]$$

**Proof:** Fix a $j$ such that $n_j > s$ and put $\delta_a = 0$ if $0 \leq a \leq n-s$ and $a \neq n_j-s$. In order that $\delta$ should satisfy (3.24) we must ensure that

$$(3.57) \quad \int \Pi_{j^t,m,n_j-s} \delta_{n_j-s} e^{x} dK^{(n_j-s)*} = 0 ; \quad j^t + [\frac{m}{r}] \leq n_j-s$$

Then degree $(\Pi_{j^t,m,n_j-s}) \leq r-1$. It follows that (3.57) is equivalent to:

$$(3.58) \quad \int x^t \delta_{n_j-s} e^{x} dK^{(n_j-s)*} = 0 ; \quad t=0, \ldots, r-1$$

Suppose (3.57) is satisfied for a polynomial $\delta_{n_j-s}$ and that $\delta_a = 0$ if $a \leq n-s$ and $a \neq n_j-s$. Then, as before, $\delta_a; a \leq n-s, \delta_{n_j-s}$ are components of a bounded unbiased estimator $\delta$ of zero. Clearly:
(3.59) \[ R_1 = R_2 = \ldots = R_{n_j} = 0 = \sigma_1 = \sigma_2 = \ldots = \sigma_{n_j} = 1 \]

By corollary 3.2:

(3.60) \[ R_{n-s+d} = 0 \text{ when } n-s+d \neq n_j \text{ and } 1 \leq d \leq s \]

By (2.23):

(3.61) \[ \sigma_{n_j-s} = \alpha^{n_j-n-s} R_{n_j-s} \text{ and } \sigma_{n_j} = \sum_{k=n_j-s}^{n_j} \frac{1}{n-k} (n-k) \beta^k (n_j-k)^* R_k \]

Hence, by (2.22):

\[
\sigma_{n_j} = \sum_{k=n_j-s}^{n_j} \alpha^{n-k} R_{n_j-s} \]

so that:

(3.62) \[ \sigma_{n_j} = \nu_j \sigma_{n_j-s} \text{ where} \]

(3.63) \[ \nu_j = c_j (\beta^k)^* - (\alpha H)^* \text{ and } c_j = (-1)^{n-n_j} \frac{1}{n-n_j+s} (n-n_j+s-1) \]

Note that the coefficient \( c_j \) of \((\beta^k)^*\) equals 1 if and only if \( n_j = n \). Hence, since \((\beta^k)^* = (\alpha H)^*\) on \([0,1]\), \( \sigma_n \) is supported by \([1,n]\).

Suppose now that \( \varphi \) is UMVU. Then \( \varphi \delta \) has the same properties as \( \delta \) has, and by (3.62) and proposition 3.8:

\[ \sigma_{n_j}(\varphi \delta) = \varphi_{n_j-s} \sigma_{n_j}(\delta) \text{ so that} \]

\[ \sigma_{n_j}(\varphi_{n_j-s} \cdot \delta_{n_j}) = 0 \text{ i.e.} \]

(3.64) \[ (\varphi_{n_j} - \varphi_{n_j-s}) \delta_{n_j} = 0 \]
Thus $\varphi_{n_j} = \varphi_{n_j-s}$ (a.e.) on any set where $\delta_{n_j} \to 0$. The proof will now be completed by showing that there to any non integer $z$ in $]0,n_j[\] if $n_j < n$ and in $]1,n[\] if $n_j = n$, corresponds a $\delta_{n_j}$ which differs from zero in a neighbourhood of $z$. This requirement is obviously satisfied whenever $\sigma_{n_j}$ has a continuous density which does not vanish in $z$.

We may at first note that $\nu_j$ has a continuous density $\nu_j'$ on $]0,\infty[\] and that a continuous density $\sigma_{n_j}'$ of $\sigma_{n_j}$ may be specified by

$$\sigma_{n_j}'(z) = \text{constant} \int_0^z \nu_j'(z-x)\delta_{n_j-s}dK \quad (n_j-s)^*$$

(3.65)

Suppose now that (3.58) for a polynomial $\delta_{n_j-s}$ implied $\sigma_{n_j}'(z) = 0$. Then:

$$\nu_j'(z-x) = \sum_{t=0}^{r-1} c_t x^t e^{x} ; \ x \in ]0,n_j-s[ \]$$

for some constants $c_0, c_1, \ldots, c_{r-1}$.

Suppose first that $z < n_j-s$. Then choices of $x$ in $]z,n_j-s[\]$ show that $c_0 = \cdots = c_{r-1} = 0$. Hence $\nu_j'(z-x) = 0$ when $x \in ]0,z[\] i.e. $\nu_j'(z) = 0$ when $z \in ]0,\zeta[\]$. If $z < 1$ then this would, by (3.62), imply that $(\alpha H)^{s*} = 0_j (\eta K)^{s*} = c_j (\alpha H)^{s*}$ on $[0,z]$ so that $c_j = 1$. Then, as we have seen, $n_j = n$ and our assumptions implied that $z > 1$ in this case. If $z > 1$ then we might conclude that $(K^{s*})'$ is analytic in $1$, and this is, as we saw in the proof of proposition 3.11, impossible.

Consider next the case $z > n_j-s$. Then $z > 1$ and $\nu_j'(z-x) = \sum_{t=0}^{r-1} c_t x^t e^{x}$ when $0 < x < n_j-s$, or equivalently $\nu_j'(z) = \sum_{t=0}^{r-1} c_t x^t e^{-x} ; 0 < x < z$. This would again imply the impossibility that $(K^{s*})'$ is analytic in $1$. 
Altogether we have shown that the polynomial \( \delta_{n_j} \) may be chosen so that \( \delta_{n_j} \) differs from zero in a neighbourhood of \( z \).

Consider now an arbitrary UMVU estimator \( \varphi \). By proposition 3.8, \( \varphi_{n-s} \) is a constant, say = b. By propositions 3.8 and 3.11 \( \varphi_d \) is constant when \( 1 \leq d < n \). By proposition 3.12 \( \varphi_n \) is a constant on \([1,n]\). Propositions 3.11 and 3.12 and corollary 3.10 imply that these constants all equal b. This together with the remarks preceding proposition 3.1, and the Rao-Blackwell theorem, provide the following characterization of the UMVU estimators.

**Theorem 3.13** Suppose \( p = \frac{r}{s} \) where the positive integers r and s are relatively prime and that \( s < n \). Then an estimator \( \varphi \) with everywhere finite variance is a UMVU estimator of its expectation if and only if \( \varphi \) is a function of \((D,T)\) which is constant on \([1 \leq D < n] \cup [D=n, 0 \leq T < 1]\).

**Remark:** It follows in this, as in the "complete case" that a quadratically integrable function of a UMVU estimator is itself a UMVU estimator. That this is not always the case is shown by Bahadur in [1].

We shall now turn to the set of functions of \( \lambda \) possessing UMVU estimators. The basic assumption in this section that \( p \) is rational shall still be in force. We will, however, permit ourselves to consider both the "incomplete situation" \( s < n \) as well as the "complete situation" \( s \geq n \). As \( n \) is allowed to vary we will indicate dependence on \( n \) by writing \((D_n,T_n)\) instead of \((D,T)\).

Denote by \( \mathcal{G}_n \) the smallest \( \sigma \)-algebra in \((D_n,T_n)\) space containing \( \{(0,0)\}, \{(d,t): 1 \leq d < n, 1 \leq t \leq d\} \cup \{(n,t): 1 < t \leq n\} \) and all measurable sub sets of \( \{(n,t): 0 \leq t < 1\} \). Then our last theorem
simply states that a quadratically integrable variable based on 
\((X_1, \ldots, X_n)\) where \(n \geq s+1\) is UMVU if and only if it is of the form \(\phi(D_n, P_n)\) where \(\phi\) is \(\mathcal{G}_n\) measurable.

Let us begin by describing the functions of \(\lambda\) which possesses \(\mathcal{G}_n\) measurable unbiased estimators.

**Proposition 3.14** \(\mathcal{G}_n\) is complete and a function \(g(\lambda)\) has a \(\mathcal{G}_n\) measurable and unbiased estimator if and only if there are constants \(u\) and \(v\) and a function \(w\) on \([0,1]\) so that

\[(3.67)\]
\[g(\lambda) = u + v(t) + \lambda w(t) \tag{3.67}\]

or, equivalently:

\[(3.68)\]
\[g(\lambda)/\lambda^p = uH(\lambda) + vA(1) + \lambda w(t) \tag{3.68}\]

where \(\lambda\) is the measure on \([0,1]\) with density \(w(t)\).

If these conditions are satisfied then the, necessarily unique, \(\mathcal{G}_n\) measurable and unbiased estimator is given by:

\[\delta(0,0) = u + v \Gamma(p)\]

\[(3.69)\]
\[\delta(d,t) = u \quad \text{when} \quad 1 \leq d < n \quad \text{or,} \quad d = n \quad \text{and} \quad t \geq 1\]
\[\delta(n,t) = u + \Gamma(n)p \cdot w(t) \quad \text{when} \quad 0 \leq t < 1 \tag{3.69}\]

**Remark 1:** The assumption that \(p\) is rational does not play any role in this proposition. The proposition is valid for any \(p > 0\) and any \(n \geq 1\). If \(n \geq 2\) then \(u, v\) and \(w\) are unique.

**Remark 2:** It was shown by Bahadur [1] that the \(\sigma\)-algebra induced by the bounded UMVU estimators is, provided the model is dominated, complete and essentially contained in any sufficient \(\sigma\)-algebra.
(i.e. necessary in Bahadur's terminology). Thus the completeness of \( \mathcal{C}_n \), established in proposition 3.14, is, when \( n > s \), a consequence of this result of Bahadur.

**Proof of the proposition:** Suppose \( g(\lambda) = E_\lambda \delta \) where \( \delta \) is \( \mathcal{C}_n \) measurable. Then there is a constant \( u \) so that \( \delta(D_n,T_n) = u \) when \( 1 \leq D_n < n \) or \( D_n = n \) and \( T_n > 1 \). Hence, by 2.18

\[
E_\lambda \delta = 1 + \frac{\delta(0,0) - u}{\Gamma(p)^n} \left( \int_0^\infty x^{p-1}e^{-x}dx \right)^n + \lambda^{np} \int_0^1 e^{-\lambda t}(\delta(n,t) - u) t^{np-1}dt
\]

and this is the desired form with

\[
v = (\delta(0,0) - u)/\Gamma(p)^n \quad \text{and} \quad w(t) = (\delta(n,t) - u)/\Gamma(np)
\]

This shows also that a \( g \) of the form (3.67) has \( \delta(D,T) \) as an unbiased estimator of zero where \( \delta \) is given by (3.69).

It remains to show that \( \mathcal{C}_n \) is complete. Suppose

\[0 = g(\lambda) = E_\lambda \delta(D_n,T_n)\]

where \( g \) is given by (3.67), where \( v \) and \( w \) are given by (3.71) and where \( u \) is the constant value of \( \delta \) on \( \{(d,t) : 1 \leq d < n\} \cup \{(n,t) : 1 < t < n\} \). We know, by (3.11) that \( \delta(n,\cdot) = 0 \) a.e. on \([0,1] \). Hence, by (3.71), \( w \) is a constant so that:

\[
u = v(\int_0^\infty x^{p-1}e^{-x}dx)^n + w\lambda^{np} \int_0^1 t^{np-1}e^{-\lambda t}dt = 0
\]

\[
\lambda \to 0 \quad \text{yield:} \quad u + v \Gamma(p)^n = 0 \]

\[
\lambda \to \infty \quad \text{yield:} \quad u + w \Gamma(np) = 0
\]
Differentiating (3.72) w.r.t. \( \lambda \), multiplying with \( \lambda^{1-p} \) and letting \( \lambda \to 0 \) we find provided \( n \geq 2 \), that \( v = 0 \) so that \( u = v = w = 0 \).

Hence, by (3.71), \( \delta = 0 \). If \( n = 1 \) then, as we argued in section 2, \( (D_n, T_n) \) is complete.

Let \( S_n \) for each \( n \leq s \) be the \( \sigma \)-algebra generated by all measurable subsets of \( (D_n, T_n) \) space i.e. of \( \bigcup_{d=0}^{n} \{(t); 0 \leq t \leq d\} \). Put \( S_n = S_n \) when \( n > s \). By theorems 3.13 and 3.7 and by Rao-Blackwell's theorem a real valued statistic is a UMVU estimator based on \( (X_1, \ldots, X_n) \) if and only if it is square integrable and of the form \( \delta(D_n, T_n) \) where \( \delta \) is \( S_n \) measurable.

Call a function \( g(\lambda) \) UMVU estimable if there is a \( n \geq 0 \) so that \( g \) has a UMVU estimator based on \( n \) observations. Our final results are concerned with the set of integers \( n \) such that a given UMVU estimable \( g \) has a UMVU estimator based on \( n \) observations \( (X_1, \ldots, X_n) \). These are collected in:

**Theorem 3.15** Suppose \( g(\lambda); \lambda > 0 \) is non constant and UMVU estimable. Let \( m \) denote the smallest integer such that \( g \) has a UMVU estimator based on \( m \) observations.

Then \( g \) has UMVU estimators based on \( n \) observations for any \( n \) such that

\[(3.75) \quad m \leq n \leq s\]

\( g \) has a UMVU estimator for some \( n > \max\{m, s\} \) only if \( g \) is of the form

\[(3.76) \quad g(\lambda) = \text{constant} + \lambda^{mp} R_m(\lambda)\]
where \( R_m \) is an absolutely continuous and finite measure on \([0,m]\).

If \( g \) is of this form then the following three conditions are all equivalent:

(i) \( : n > \max\{m,s\} \) and \( g \) has a UMVU estimator based on \( n \) observations

(ii) \( : n > \max\{m,s\} \) and \( g \) has a \( \mathcal{G}_n \) measurable unbiased estimator

(iii) \( : R_m * H^{(n-m)} \) is supported by \([0,1]\).

These conditions imply:

(iv) \( : n \in \{m+s,m+2s,\ldots,m+\lceil k/r \rceil s\} \) where \( k \) is the unique integer such that \( \int x^k e^{x} R_m(dx) \neq 0 \) while \( \int x^j e^{x} R_m(dx) = 0 \) when \( 0 \leq j < k \).

If \( R_m \) is supported by \([0,1]\), and this is always the case when \( m > s \), then all four conditions (i),(ii),(iii) and (iv) are equivalent.

If \( m < s \) and if \( m < s \) are such that \( g \) has a UMVU estimator based on \( n \approx m \) observations but not on \( n \) observations when \( s < n < m \) then \( m = m \) and (i),(ii),(iii) and (iv) remain equivalent when \( m \) is replaced by \( m \).

Remark 1: The measure \( R_m \) in (3.76) is, since \( g \) is not constant, not the zero measure. It follows that the integer \( k \) in (iv) is well defined.

Remark 2: If \( g \) has an UMVU estimator based on \( n \) observations then it may be found by using proposition 3.14 and the representation:
\[ g(\lambda) = u + \lambda R_n H_n^*(\lambda) ; \lambda > 0 \] where \( H_n^* = R_n H^{(n-m)} \).
Proof: The fact that \( g \) has UMVU estimators based on \( n \) observations when \( m \leq n \leq s \) follows from theorem 3.7 and the Rao-Blackwell theorem. Let us from here on distinguish the cases \( m \geq s+1 \) and \( m \leq s \).

1°. The case "\( m \geq s+1 \)":
Suppose that \( g \) has a \( S_n \) measurable unbiased estimator based on \( n \) observations where \( n > m \). By proposition 3.14:

\[
(3.77) \quad g(\lambda) = \lambda^{k_p} \left[ u_k H(\lambda)^k + v A(1)^k + \hat{\kappa}_k(\lambda) \right]
\]

\[
= u_k + b_k \left( \int_{\lambda}^{\infty} x^{k_p-1} e^{-\lambda x} dx \right)^k + \lambda^{k_p} \int_{0}^{1} w_k(x) x^{k_p-1} e^{-\lambda x} dx ;
\]

where \( u_m, v_m, u_n \) and \( v_n \) are constants,

\[
w_k = e^{x} u_k(x)x^{1-k_p} ; \quad x \in [0,1[ \quad \text{and}
\]

\[
(3.78) \quad \int_{0}^{1} u_m(x)^2 x^{1-m_p} dx < \infty
\]

Hence

\[
(3.79) \quad u_k + v_k \Gamma(p)^k = \lim_{\lambda \to \infty} g(\lambda) \quad \text{and} \quad -kv_k \Gamma(p)^{k-1} = \lim_{\lambda \to \infty} \lambda^{1-p} g(\lambda) ;
\]

\[
k = m, n
\]

The equality of the representations (3.77) for \( k=m \) and \( k=n \) imply, and is implied by:

\[
(3.80) \quad \frac{1}{\lambda^{(n-m)p}} \left[ u_m H(\lambda)^m + v_m A(1)^m \hat{\tau}(\lambda)^m + \hat{\kappa}_m(\lambda) \right]
\]

\[
= u_n H(\lambda)^n + v_n A(1)^n \hat{\tau}(\lambda)^n + \hat{\kappa}_n(\lambda) ; \quad \lambda > 0
\]

which is equivalent to:

\[
(3.81) \quad v_n A(1)^m H(n-m)^* \hat{\tau}(n-m)^* + \kappa_n H(n-m)^* = (u_n - u_m) H^* + v_n A(1)^n \hat{\tau}(n)^* + \kappa_n
\]
On \([1, m]\) this may be written

\[(3.82) \quad \kappa_m \ast H{(n-m)}^* = (u_n - u_m) H_n^*\]

Now \(\kappa_m \ast H{(n-m)}^*\) and \(H_n^*\) has both analytic densities on \([1, \infty[.\) It follows that \(\kappa_m \ast H{(n-m)}^* = (u_n - u_m) H_n^*\) on \([1, \infty[\) so that \(v_m A(1)^m (n-m)^* \tau^m* = v_n A(1)^n \tau^m*\) on \([1, \infty[.\) If \(v_m \neq 0\) then this would imply that \(H{(n-m)}^* \tau^m*(\langle m, n \rangle) = 0\) contradicting the fact that any \(x > m\) is a point of support of \(H{(n-m)}^* \tau^m*\).

Thus \(v_m = 0\) so that, by (3.79),

\[(3.83) \quad u_m = u_n, \quad v_m = v_n = 0\]

This proves the claimed representation. We may therefore assume that \(g\) is of the form:

\[(3.84) \quad g(\lambda) = u + \lambda^m \kappa_m^* (\lambda) = \lambda^m [u H^m(\lambda) + \kappa_m(\lambda)] ; \lambda > 0\]

We have also shown that \(g\) has a \(\mathcal{S}_n\) measurable and unbiased estimator if and only if \(\kappa_m \ast H{(n-m)}^*\) is supported by \([0, 1]\). If so, then the \(\mathcal{S}_n\) measurable and unbiased estimator may be deduced from proposition 3.14 and the representation:

\[(3.85) \quad g(\lambda) = \lambda^m [u H^m(\lambda) + \kappa_n(\lambda)]\]

where \(\kappa_n = \kappa_m \ast H{(n-m)}^*\).

We shall now show that a \(\mathcal{S}_n\) measurable and unbiased estimator is quadratically integrable when \((n-m)p \geq 1\). By propositions 3.14 and (3.9) this is equivalent to

\[(3.86) \quad \int \kappa'_n(x)^2 x^{1-np} dx < \infty \quad \text{where} \quad \kappa_n = \kappa_m \ast H{(n-m)}^*\]

or

\[(3.87) \quad \int_0^1 x^{1-np} \left( \int_0^x (x-z)^{n-m) p-1} e^{x \kappa'(m) (z) dz} \right)^2 dx < \infty\]
Utilizing that \((n-m)p > 1\) we see that the integral in (3.87) is
\[
\int_0^1 x^{(n-2m)p+1} \left[ \int_0^1 e^{2z\kappa_m'(z)} dz/x \right]^2 dx \leq \int_0^1 x^{(n-2m)p} e^{2z\kappa_m'(z)} dz dx
\]
\[
\leq e^{2\int_0^1 \kappa_m'(y) (\int_y^1 x^{(n-2m)p} dx) dy} \leq \text{constant} \int_0^1 (\int_0^1 e^{2z\kappa_m'(y)} dy)^2 [1-y(n-2m)p+1] dy.
\]
Using that \(mp-1 \geq (s+1)^2 - 1 > 0\) we find successively:
\[
\int_0^1 \kappa_m'(y)^2 (1-y(n-2m)p+1) dy \leq \int_0^1 \kappa_m'(y)^2 dy = \int_0^1 (\int_0^y (1-y(n-2m)p+1) dy)^2 dy
\]
\[
\leq \int_0^1 y^{mp\kappa_m'(y)^2} < \infty \quad \text{proving (3.86)}.
\]

The condition "\((n-m)p \geq 1\)" is automatically satisfied since, as we now shall show, \(n = m\). This may be deduced from the condition that \(\kappa_m^*H(n-m)^*\) is concentrated on \([0,1]\) as follows:
By the convolution formula for densities this is equivalent to
\[
\int (y-x)^{n-m}p-1 e^{-y-x} \kappa_m(dx) = 0 \quad , \quad y > 1
\]
or equivalently that:
\[
\int (1-zx)^{n-m}p-1 e^{x\kappa_m(dx)} = 0 \quad \text{when} \quad |z| < 1
\]
The last integral is analytical in the open unit circle.
Considering the derivatives in zero we see that (3.89) is equivalent to:
\[
(n-m)p-1 \int x^i e^{x\kappa_m(dx)} = 0 \quad ; \quad i=0,1,\ldots
\]
Defining \(k\) as in (iv) with \(R_m = \kappa_m\) we see that this is equivalent to
\[
n \in \{m+s, m+2s, \ldots, m+[k/r]s\}
\]
Altogether this proves the theorem when \(m \geq s+1\).

2°. \(m \leq s\). Then \(g\) has, for any \(n=m,m+1,\ldots,s\), a UMVU estimator based on \(n\) observations. Suppose now that \(g\) has a
\[ S_n \] measurable and unbiased estimator where \( n > s \). By (2.19) and (2.21):

\[
(3.92) \quad g(\lambda) = \frac{\lambda^{mp}}{\Gamma(p)^m} \sum_{d=0}^{m} \hat{\tau}(\lambda)^{m-d} \hat{\sigma}_d(\lambda) = \frac{\lambda^{mp}}{\Gamma(p)^m} \sum_{k=0}^{m} \hat{\Sigma}(\lambda)^{m-k} \hat{R}_k(\lambda) ; \lambda > 0
\]

where \((\sigma_0, \ldots, \sigma_m)\) and \((R_0, \ldots, R_m)\) are representations of the UMVU estimator based on \( m \) observations. Square integrability imply:

\[
(3.93) \quad \int_{0}^{j} \left( d\sigma_j / d\kappa_j^* \right)^2 x^j p^{-1} (j-x)^{j-1} dx < \infty ; \quad j=1, \ldots, n
\]

By proposition 3.14 we also have a representation

\[
(3.94) \quad g(\lambda) = \lambda^{np} [u \hat{H}(\lambda)^n + vA(1)^n \tau(\lambda)^n + \hat{\kappa}_n(\lambda)]
\]

for some constants \( u \) and \( v \) and some absolutely continuous and finite measure \( \kappa_n \) on \([0,1] \). The equality of the two representations may be written:

\[
(3.95) \quad \sum_{k=0}^{n} H(n-k)^* \tilde{R}_k = vA(1)^n \tau^n
\]

where \( \tilde{R}_o \) is concentrated in 0 and assigns mass \( \Gamma(p)^{-m} R_o(0) - u \) there while

\[
(3.96) \quad \tilde{R}_k = \Gamma(p)^{-m} R_k \quad k=1, \ldots, m, \quad \tilde{R}_k=0 \quad m<k<n \quad \text{and} \quad \tilde{R}_n = -\kappa_n.
\]

On \( ]m,n[ \) this reduces to \( \sum_{k=0}^{m} H(n-k)^* \tilde{R}_k = 0 \), Hence, since the density of \( \sum_{k=0}^{m} \) may be specified as an analytic function on \( ]m,\infty[ \):

\[
v A(1)^n \tau^n = \sum_{k=0}^{n} H(n-k)^* \tilde{R}_k = \sum_{k=0}^{m} H(n-k)^* \tilde{R}_k = 0
\]

Thus \( v A(1)^n \tau^n \) has no point of support on \( ]n,\infty[ \) so that \( v = 0 \) and (3.95) may be written:
Hence, by theorem 2.2, $\tilde{R}_0, \ldots, \tilde{R}_n$ represents an unbiased estimator of zero based on $(D_n, T_n)$. It follows by corollary 2.3 that $0 = \tilde{R}_o(\{0\})$ so that $R_o(\{0\}) = u\Gamma(p)^m$. Let $j_o$ be the unique number in $\{1, \ldots, s\}$ such that $j_o = n$. Define $n_j; j=1,2,\ldots,s$ by (3.9) so that $n_j = n$. By corollary 3.2:

$$(3.96) \quad \sum_{k=0}^{n} H(n-k)^*\tilde{R}_k = 0$$

so that

$$(3.97) \quad \tilde{R}_{n_j} = -\sum_{k=j}^{n} H(n-j)^*\tilde{R}_n$$

Now $\tilde{R}_{n_j} = 0$ when $n > n_j > s$ so that $\tilde{R}_j = 0$ in this case.

If $n_j < s$ then, by corollary (3.2) again: $\tilde{R}_{n_j} = \tilde{R}_j = 0$. Hence $R_j = 0$ when $j \neq j_o$. If $j_o > m$ then, by (3.92), $g$ is a constant. Thus $j_o \leq m$. Suppose $j_o < m$. Then, by (3.92):

$$g(\lambda) = R_o(\{0\})\Gamma(p)^{-m} + \lambda^{j_o}p^{j_o-m}\Gamma(p)^{-m}R_j^j(\lambda)$$

$$= \frac{\lambda^{j_o}}{\Gamma(p)^{j_o}} \left[ H^{j_o}(\lambda)\Gamma(p)^{j_o-m}R_o(\{0\}) + \Gamma(p)^{j_o-m}R_j^j(\lambda) \right]$$

It follows that $(\Gamma(p)^{j_o-m}R_o, 0, \ldots, 0, \Gamma(p)^{j_o-m}R_j^j)$ represents an unbiased estimator of $g$ which is $\mathbb{S}_{j_o}$ measurable. It suffices, in order to establish square integrability of this estimator, to consider the estimator $(R_o, \ldots, R_j^j)$. The alternative representation, $(\tilde{\sigma}_0, \ldots, \tilde{\sigma}_j)$ described by (2.23) is given by:
\[ (3.99) \quad \tilde{\sigma}_d = a^{-j_0 \alpha_d}(\beta K)_d^* R_0(\{0\}) ; \quad 0 \leq d < j_0 \]

and

\[ (3.100) \quad \sigma_{j_0} = a^{-j_0 \alpha_j}(\beta K)^{j_0^*} R_0(\{0\}) + R_{j_0} \]

By (2.23) again:

\[ (3.101) \quad \sigma_d = a^{-m \alpha_d}(\beta K)^{d^*} R_0(\{0\}) ; \quad 0 \leq d < j_0 \]

and

\[ (3.102) \quad \sigma_{j_0} = a^{-m \alpha_{j_0}}(\beta K)^{j_0^*} R_0(\{0\}) + a^{j_0^*-m} R_{j_0} \]

The square integrability of the \( S_m \) measurable and unbiased estimator \((\sigma_0, \ldots, \sigma_m)\) clearly implies square integrability of \((\tilde{\sigma}_0, \ldots, \tilde{\sigma}_m)\). It follows that \((\Gamma(p)^m_{j_0} R_0, \ldots, \Gamma(p)^m_{j_0} R_{j_0})\) represents a square integrable \( S_{j_0} \) measurable unbiased estimator. Thus \( g \) has a UMVU estimator based on \( j_0 \) observations where \( j_0 < m \), and this contradicts the minimality of \( m \). It follows that \( j_0 = m \) and:

\[ (3.103) \quad g(\lambda) = u + \lambda^{m \alpha} \tilde{R}_m(\lambda) ; \quad \lambda > 0. \]

The condition of square integrability implies that:

\[ (3.104) \quad \int_0^1 (d\tilde{R}_m/dk^*) x^{mp-1} dx < \infty \]

or, since \( k^* = \text{constant} \ H^* \) on \([0,1]\):

\[ (3.105) \quad \int_0^1 x^{1-mp} \tilde{R}_m'(x)^2 dx < \infty \]

Furthermore we see by the above computations that \( g \) admits a \( S_n \) measurable and unbiased estimator if and only if \( H(n-m)^* \tilde{R}_m \) is concentrated on \([0,1]\). If this holds then this
estimator may be deduced from the representation:

\[ g(\lambda) = u + \lambda^{np} \hat{\kappa}_n(\lambda) \]

where \( \hat{\kappa}_n = \hat{H}^{(n-m)\hat{R}_m} \). Proceeding as in the first part of the proof we find that \( \hat{H}^{(n-m)\hat{R}_m} \) is concentrated on \([0,1]\) if and only if:

\[ \int_0^y (y-x)^{(n-m)p-1} e^{-\frac{y-x}{\hat{R}_m}} (dx) = 0 ; \ y > 1 \]

This implies in particular that:

\[ \int_0^m (y-x)^{(n-m)p-1} e^{\frac{y-x}{\hat{R}_m}} (dx) = 0 ; \ y > m \]

or

\[ \int_0^m (1-zx)^{(n-m)p-1} e^{\frac{1-zx}{\hat{R}_m}} (dx) = 0 ; \ |z| < \frac{1}{m} \]

As in the first part of this proof we see that (3.108) is equivalent to:

\[ n \in \{m+s, m+2s, \ldots, m+[k/r)s\} \]

where \( k \) is defined in (iv). If \( R_m \) is concentrated on \([0,1]\) then (3.110) is equivalent to (3.107) and thus it remains only to show that (3.107) implies that the \( S_n \) measurable unbiased estimator is square integrable and, consequently, UMVU.

We must show that

\[ \int_0^1 x^{1-np} \hat{\kappa}_n^2(x) = \infty \]

Now \( g(\lambda) \) may be written:

\[ g(\lambda) = \frac{\lambda^{sp}}{\Gamma(p)^s} [\Gamma(p)^s u \overset{\hat{\kappa}}{\hat{\kappa}}(\lambda)^S + \Gamma(p)^s \hat{R}_m(\lambda) \overset{\hat{H}}{\hat{H}}(\lambda)^{n-m}] \]
Let $\mathcal{R}_0$ be the measure which is concentrated on zero and which assigns mass $\Gamma(p)^s \mu$ to this point. Put $\mathcal{R}_m = \Gamma(p)^s \mathcal{R}_m$ and $\tilde{\mathcal{R}}_j = 0$; $0 < j \leq s$, $j \neq m$. Then, by (2.19) and (2.21), $\mathcal{R}_0, \ldots, \mathcal{R}_s$ represents an unbiased $\mathcal{F}_s$ measurable estimator of $g$. It follows from the Rao-Blackwell theorem that this estimator is UMVU and, consequently, square integrable. If $(\tilde{\sigma}_0, \tilde{\sigma}_1, \ldots, \tilde{\sigma}_s)$ is the alternative representation defined by (2.23) then:

$$\tilde{\sigma}_s = \alpha^{-s}(\beta k)^s \mathcal{R}_0(o) + \alpha^{m-s}(\beta k)^{s-m} * \mathcal{R}_m$$

Square integrability implies:

$$\int_0^1 \left( \frac{d\tilde{\sigma}_s}{dk^s} \right)^2 \alpha^{s-1} dx < \infty$$

i.e.

$$\int_0^1 Q_s'(x) x^{1-s} dx < \infty$$

where

$$Q_s = H(s-m) * \mathcal{R}_m$$

Now:

$$\kappa_n = H(n-m) * \tilde{\mathcal{R}}_m = H(n-s) * H(s-m) * \Gamma(p)^s \mathcal{R}_m$$

$$= \Gamma(p)^s H(n-s) * Q_s$$

(3.111) may therefore be written:

$$\int_0^1 x^{1-np} \left[ \int_0^x (x-z)^{(n-s)p-1} e^{xQ_s(z)} (dz) \right]^2 dx < \infty$$

or equivalently that:

$$\int_0^1 x^{(n-2s)p+1} \left( \int_0^x (1-z)^{(n-s)p-1} e^{2xQ_s'(xz)} (dz) \right)^2 dx < \infty$$

This holds if:
\[
\int_0^1 x^{(n-2s)\rho+1} \left[ \int_0^1 (1-z)^{(n-s)\rho-1} e^{2zx} Q_s'(xz)^2 \, dz \right] \, dx = \\
= \int_0^1 x^{1-s\rho} \left[ \int_0^x (x-z)^{(n-s)\rho-1} e^{2z} Q_s'(z)^2 \, dz \right] \, dx = \\
= \int_0^1 Q_s'(z)^2 \left[ \int_z^1 x^{s\rho+1} (x-z)^{(n-s)\rho-1} \, dx \right] \, dy < \infty
\]

It suffices therefore, since \( sp = r \geq 1 \), to show that

\[
\int_0^1 Q_s'(z)^2 z^{-sp+1} \left[ \int_z^1 (x-z)^{(n-s)\rho-1} \, dx \right] \, dz < \infty
\]

or

(3.117) \[
\int_0^1 Q_s'(z)^2 z^{-sp+1} (1-z)^{(n-s)\rho} \, dz < \infty
\]

This, however, since \( n > s \), follows from (3.114).

The last statement in the theorem follows now by observing that the minimality of \( m \) was not used in the first part of this proof.
References


