Abstract

Let $x_i$ be the total claim amount of an insurance policy in calendar year $i$. We assume that the $x_i$'s are conditionally independent given an unknown random parameter $\theta$, and that

$$E(x_i|\theta) = \alpha_i m(\theta) + \beta_i \quad EV(x_i|\theta) = \phi_i$$

$$E(m(\theta)) = 0 \quad V(m(\theta)) = 1$$

for all $i$. In the present paper it is under these assumptions shown how to calculate the credibility estimator of $m(\theta)$ by recursive updating. We also give estimators for the unknown parameters $\alpha_i$, $\beta_i$, and $\phi_i$ based on portfolio data. Some generalizations of the model will be described. Finally we mention some related models.
1. Introduction

In credibility models of insurance experience rating it is usually assumed that for a given policy the risk characteristics are generated by an unknown random parameter $\theta$ describing how this policy may differ from other similar policies in the portfolio. In the simplest case we assume that the total claim amounts from different years are conditionally independent and identically distributed with mean $m(\theta)$ and variance $s^2(\theta)$, given $\theta$. The assumption of identical distribution is in many cases rather unrealistic. One very important reason is inflation. And factors influencing the risk may change; for instance, motor insurance claim amounts may be influenced by improved roads and increased traffic.

In the present paper we shall modify the model by assuming that the total claim amount from calendar year $i$ has mean $a_i m(\theta) + \beta_i$ and variance $s^2_i(\theta)$ given $\theta$, and we shall propose estimators for $a_i$, $\beta_i$, and $\varphi_i = E(s^2_i(\theta))$ based on portfolio data.

The model will be generalized into two directions: 1) to the case of estimating loss ratios and 2) to the case when $m(\theta)$ develops randomly over time.

2. Preliminaries

2 A. With a few minor exceptions we use the notation of Sundt (1979a). All displayed moments are assumed to exist.

2 B. Let $m$ be an unknown random variable. We shall say that an estimator $m^{(1)}$ is a better estimator of $m$ than another estimator $m^{(2)}$ if

$$E(m^{(1)}-m)^2 < E(m^{(2)}-m)^2,$$

that is, we use quadratic loss.
Let \(x_1, \ldots, x_n\) be observable random variables. We shall call an estimator \(\hat{m}\) of \(m\) a linear estimator of \(m\) (based on \(x_1, \ldots, x_n\)) if \(\hat{m}\) may be written \(\hat{m} = g_0 + \sum_{i=1}^{n} g_i x_i\), where \(g_0, g_1, \ldots, g_n\) are non-random numbers. The best linear estimator of \(m\) will be called the credibility estimator of \(m\) (based on \(x_1, \ldots, x_n\)).

3. A model with identical policies

3 A. We consider an insurance policy that has been in force since calendar year \(c\) inclusive. To get convenient notation we shall assume that one insurance year covers one calendar year. Let \(x_i\) be the total claim amount of the policy in year \(i\). We shall assume that \(x_c, x_{c+1}, \ldots\) are conditionally independent given an unknown random parameter \(\theta\), and that for all \(i\)

\[
E(x_i | \theta) = \alpha_i m(\theta) + \beta_i.
\]  

(1)

For the present we shall assume that the \(\alpha_i\)'s and \(\beta_i\)'s are non-random numbers. Without loss of generality we let

\[
E(m(\theta)) = 0 \quad \text{and} \quad V(m(\theta)) = 1.
\]  

(2)

We introduce \(\phi_i = E[x_i | \theta]\). It is assumed that different \(x_i\)'s are positively correlated, that is, that \(\alpha_i\) is positive for all \(i\).

Formula (1) says that the policy has a risk element \(m(\theta)\) that remains unchanged as time passes, and that the conditional means of the claim amounts are linear transformations of \(m(\theta)\). The coefficients and constant terms are the same for all policies in the portfolio and can be estimated from portfolio data. An assumption like \(E(x_i | \theta) = \alpha_i m(\theta)\) would be natural to take care of inflation, and in (1) we have
added a constant term $\beta_i$ that gives further flexibility.

3 B. In this subsection we are going to describe how to calculate $\tilde{m}_t$ and $\tilde{x}_t$, the credibility estimators of $m(\theta)$ and $x_t$ based on $x_c, x_{c+1}, \ldots, x_{t-1}$. We are going to give the formulae on recursive form as described in Sundt (1980a).

Let $y_i = (x_i - \beta_i)/a_i$. Then $y_c, y_{c+1}, \ldots$ are independent given $\theta$, and

\[ E(y_i|\theta) = m(\theta) \quad \text{and} \quad \text{EV}(y_i|\theta) = \frac{\psi_i}{\alpha_i^2}. \]

Let $\psi_t = \nu(\tilde{m}_t - m(\theta))$.

As the $y_i$'s are linear transformations of the $x_i$'s, $\tilde{m}_t$ must be the credibility estimator of $m(\theta)$ based on $y_c, \ldots, y_{t-1}$, and formula (11) in Sundt (1980a) gives

\[
\psi_t = \frac{\psi_{t-1}}{\alpha_{t-1}^2} \left( \frac{\varphi_{t-1}}{\psi_{t-1}^2 + \alpha_{t-1}^2} \right) \]

\[
\tilde{m}_t = \frac{\psi_{t-1}}{\alpha_{t-1}^2} y_{t-1} + \frac{\varphi_{t-1}}{\psi_{t-1}^2 + \alpha_{t-1}^2} \tilde{m}_{t-1} \]

\[
\tilde{m}_c = 0 \quad \text{and} \quad \psi_c = 1. \]

We rewrite (3) and (4) as

\[
\psi_t = \frac{\psi_{t-1} \varphi_{t-1}}{\alpha_{t-1}^2 \psi_{t-1} + \varphi_{t-1}} \]

\[
\tilde{m}_t = \frac{\alpha_{t-1}^2 \psi_{t-1} \psi_{t-1} + \varphi_{t-1}}{\alpha_{t-1}^2 \psi_{t-1} + \varphi_{t-1}} \tilde{m}_{t-1} + \frac{\varphi_{t-1}}{\alpha_{t-1}^2 \psi_{t-1} + \varphi_{t-1}} x_{t-1} - \beta_{t-1} \]

\[
\tilde{m}_t = \frac{\alpha_{t-1}^2 \psi_{t-1} \psi_{t-1} + \varphi_{t-1}}{\alpha_{t-1}^2 \psi_{t-1} + \varphi_{t-1}} \tilde{m}_{t-1} + \frac{\varphi_{t-1}}{\alpha_{t-1}^2 \psi_{t-1} + \varphi_{t-1}} x_{t-1} - \beta_{t-1} \]
When we have \( \tilde{m}_t \), we can easily find \( \tilde{x}_t \) by
\[
\tilde{x}_t = \alpha_t \tilde{m}_t + \beta_t. 
\] (7)

3 C. The \( \alpha_i \)'s, \( \beta_i \)'s, and \( \varphi_i \)'s are supposed to be unknown and therefore have to be estimated from portfolio data. We assume that we have a portfolio of independent policies that satisfy the conditions of subsection 3 A and have the same \( (\alpha_i, \beta_i, \varphi_i) \)'s. Suppose that \( k_1N \) policies have been in force in both years \( k \) and \( l \), and let \( k_1x_{ij} \) denote the total claim amount of policy \( i \) in year \( j \) (\( i=1, \ldots, k_1N; j=k, l \)).

We introduce \( k_1\bar{x}_j = k_1N \sum_{i=1}^{k_1} k_1x_{ij} \).

The obvious estimator of \( \beta_k \) is
\[
\hat{\beta}_k = k_1\bar{x}_k. 
\]

Let
\[
\hat{\alpha}_{kl} = \frac{1}{k_1N-1} \sum_{i=1}^{k_1N} (k_1x_{ik} - k_1\bar{x}_k)(k_1x_{il} - k_1\bar{x}_l). 
\]

We easily see that
\[
\alpha_{kl} = E(\hat{\alpha}_{kl}) = \begin{cases} \varphi_k + \alpha_k^2 & k=l \\ \alpha_k \alpha_l & k \neq l \end{cases}. 
\]

As
\[
\alpha_k^2 = \frac{\alpha_r \alpha_s}{\alpha_{rs}} 
\]
for \( r < s < k \), we estimate \( \alpha_k \) by
\[
\hat{\alpha}_k = \sqrt{\sum_{r<s<k} k_{rs} \alpha_r \alpha_s} \left/ \sqrt{\sum_{r<s<k} k_{rs} \alpha_{rs}} \right. , 
\]

where the \( k_{rs} \)'s are non-random weights. One could for instance choose \( k_{rs} \propto rsN \).
\( \psi_k \) can now be estimated by
\[
\hat{\psi}_k = \hat{\alpha}_{kk}^2 - \hat{\alpha}_k^2.
\]

3 D. Let us now return to the situation of subsections 3 A-B. We see that at the end of year \( t-1 \) we can get estimates of \((\alpha_i, \beta_i, \phi_i)\) for \( i = c, \ldots, t-1 \) by the method described in the previous subsection. Hence we may estimate \( \tilde{m}_t \) by putting these estimates into (5) and (6). However, by (7) we see that to estimate \( \tilde{x}_t \) we also need estimates of \( \alpha_t \) and \( \beta_t \). Unfortunately, these quantities cannot be estimated from the available data unless we introduce some more structure. The author believes that because of the uncertainty by the choice of such structure it should be used to construct estimators \( \alpha_t^* \) and \( \beta_t^* \) to be used only in formula (7), but that in recursion (5)-(6) we should use estimators \( \hat{\alpha}_i, \hat{\beta}_i, \) and \( \hat{\phi}_i \) as developed in subsection 3 C.

The choice of additional structure seems to depend very much on the actual situation, and we shall therefore restrict ourselves to some vague general suggestions.

We shall for the rest of subsection 3 D assume that the \( \alpha_i \)'s and \( \beta_i \)'s are random variables independent of the \( \theta \)'s of the portfolio. Then all expectations and covariances introduced in subsections 3 A-C becomes the analogous conditional quantities given the \( \alpha_i \)'s and \( \beta_i \)'s.

One possibility is to assume that \( E(\alpha_i) \) and \( E(\beta_i) \) have known parametric forms \( a(i; \gamma) \) and \( b(i; \gamma) \). Then we may find an estimator \( \hat{\gamma} \) of the unknown parameter vector \( \gamma \) based on the available \( \hat{\alpha}_i \)'s and \( \hat{\beta}_i \)'s and estimate \( \alpha_t \) and \( \beta_t \) by \( \alpha_t^* = a(t; \hat{\gamma}) \) and \( \beta_t^* = b(t; \hat{\gamma}) \). Because of the approximate nature of the assumption of parametric forms
a(i; γ) and b(i; γ) we ought to give more weight to the most recent \( \hat{\alpha}_i \)'s and \( \hat{\beta}_i \)'s than to the older ones when constructing the estimator \( \hat{\gamma} \).

The special case where the \( (\alpha_i, \beta_i) \)'s are independent and identically distributed, is closely related to the model described in Sundt (1979b). In this case we may estimate \( \alpha_t \) and \( \beta_t \) by

\[
\alpha^*_t = \sum_{i \leq t} t v_i \hat{\alpha}_i \quad \beta^*_t = \sum_{i \leq t} t w_i \hat{\beta}_i ,
\]

where the \( t v_i \)'s and \( t w_i \)'s are non-random weights.

Another approach is to make some martingale assumption.

We shall give a few cases.

Suppose that \( \{\alpha_i\} \) is a martingale. Then

\[
E(\alpha_t | \alpha_{t-1}, \alpha_{t-2}, \ldots) = \alpha_{t-1} ,
\]

and \( \alpha^*_t = \hat{\alpha}_{t-1} \) would be a natural estimator of \( \alpha_t \). This solution seems intuitively very sound; as we have no data for the next year, the best we can to is to use what we have found for the present year. The same approach could be used to estimate \( \beta_t \).

Now let \( \eta_i = \beta_i - \beta_{i-1} \) and assume that \( \{\eta_i\} \) is a martingale. Then

\[
E(\eta_t | \eta_{t-1}, \eta_{t-2}, \ldots) = \eta_{t-1} = \beta_{t-1} - \beta_{t-2} , \tag{8}
\]

and \( \eta^*_t = \hat{\beta}_{t-1} - \hat{\beta}_{t-2} \) would be a reasonable estimator of \( \eta_t \).

As \( \beta_t = \eta_t + \beta_{t-1} \), we estimate \( \beta_t \) by \( \beta^*_t = \eta_t + \hat{\beta}_{t-1} = 2\hat{\beta}_{t-1} - \hat{\beta}_{t-2} \). The present martingale assumption can be interpreted as a very weak assumption of linear trend in the \( \beta_i \)'s. An analogous approach could of course be used in the estimation of \( \alpha_t \).

Now suppose that \( \{\delta_i\} \) given by \( \delta_i = \alpha_i / \alpha_{i-1} \) is a martingale. Then
\( E(\delta_t | \delta_{t-1}, \delta_{t-2}, \ldots) = \delta_{t-1} = \alpha_{t-1}/\alpha_{t-2}, \)

and \( \delta_t^* = \hat{\alpha}_{t-1}/\hat{\alpha}_{t-2} \) would be a reasonable estimator of \( \delta_t. \)

As \( \alpha_t = \delta_t \alpha_{t-1} \), we estimate \( \alpha_t \) by \( \alpha_t^* = \delta_t^* \alpha_{t-1} = \hat{\alpha}_{t-1}/\hat{\alpha}_{t-2}. \)

The quantity \( \delta_i \) could be thought of as a rate of inflation, and the martingale assumption would then say that the expected inflation of next year is equal to the inflation of the present year.

If \( \delta_i \) is interpreted as rate of inflation, it would be natural to assume that it is also related to the \( \beta_i \)'s. Let \( \varepsilon_t = \beta_t - \delta_t \beta_{t-1}. \) We assume that

\( E(\varepsilon_t | \beta_{t-1}, \delta_{t-1}, \delta_{t-2}, \delta_{t-3}, \ldots) = 0. \) \( (9) \)

Then \( \beta_t \) could be estimated by

\[ \beta_t^* = \delta_t^* \beta_{t-1} = \frac{\hat{\alpha}_{t-1} \hat{\beta}_{t-1}}{\alpha_{t-2}}. \]

The assumption (9) says, roughly speaking, that the expected claim amounts of next year are equal to the expected claim amounts of the present year increased by a multiplicative inflation, and in addition we get an additive element, which according to our present knowledge has expectation zero.

As an intermediate case between (8) and (9) we could assume that

\[ E(\varepsilon_t | \beta_{t-1}, \delta_{t-1}, \beta_{t-2}, \delta_{t-2}, \ldots) = \varepsilon_{t-1}. \]

Then \( \beta_t \) could be estimated by

\[ \beta_t^* = \frac{\hat{\alpha}_{t-1} \hat{\beta}_{t-1}}{\alpha_{t-2}} \left( 2 \beta_{t-1} - \frac{\hat{\alpha}_{t-1} \hat{\beta}_{t-2}}{\alpha_{t-2}} \right). \]
3 E. We have now proposed several estimators of $\alpha_t$ and $\beta_t$ based on claim data from before year $t$. However, the insurance company may also possess additional information that ought to be incorporated into the estimators $\alpha_t^*$ and $\beta_t^*$. For instance, in motor insurance one ought to use greater estimated values of $\alpha_t$ and $\beta_t$ than indicated by the available data if it is known that the speed limits are to be increased in year $t$. And the company ought to incorporate available prognoses about inflation.

3 F. As we have seen in the two previous subsections, there are several approaches that can be used to find estimators $\alpha_t^*$ and $\beta_t^*$. Experience and knowledge would probably give the actuary some idea that some of the approaches are better than others in his actual situation. However, when the claim data from year $t$ are available, one ought to examine different choices of $\alpha_t^*$ and $\beta_t^*$ and see if some approaches seem to be better than others. It seems that the function

$$Q_t(\alpha_t^*, \beta_t^*) = \sum (x_t - \alpha_t^* m_t - \beta_t^*)^2,$$

where the sum is taken over all policies that have been in force in year $t$, is useful in this connection; the estimators $\alpha_t^*$ and $\beta_t^*$ that minimize $Q_t$ would be preferable. If this analysis indicates that one approach of finding estimators $\alpha_t^*$ and $\beta_t^*$ is better than the others, it would be natural to use this approach for the estimators $\alpha_{t+1}$ and $\beta_{t+1}$.

Instead of $Q_t$, one could of course minimize the functions $(\hat{\alpha}_t - \alpha_t^*)^2$ and $(\hat{\beta}_t - \beta_t^*)^2$, but as we essentially want to fit the estimated $\tilde{x}_t$'s to the $x_t$'s, it seems more natural to minimize $Q_t$. 
4. Credibility for loss ratios

4 A. We shall now modify the model of Section 3 to credibility estimation of loss ratios. Our approach is a generalization of a model by Buhlmann and Straub (Bühlmann & Straub (1970), Bühlmann (1971)).

We consider an insurance portfolio that has been ceded since calendar year \( c \) inclusive. It is assumed that one reinsurance year covers one calendar year. Let \( p_i \) be the direct insurance risk premium of year \( i \) and \( s_i \) the total reinsurance claims of the same year. Then the observed loss ratio of year \( i \) is \( x_i = s_i / p_i \). It is assumed that the \( x_i \)'s are conditionally independent given an unknown random parameter \( \theta \), and that assumptions (1) and (2) of subsection 3 A are satisfied with positive \( \alpha_i \)'s. We further assume that

\[
EV(x_i | \theta) = \frac{\varphi_i}{p_i}.
\]

Let \( \tilde{m}_t \) and \( \tilde{x}_t \) be the credibility estimators of \( m(\theta) \) and \( x_t \) based on \( x_c, \ldots, x_{t-1} \), and let \( \psi_t = \nabla(\tilde{m}_t - m(\theta)) \). The situation is obviously the same as in subsection 3 B, and we get

\[
\psi_t = \frac{\psi_{t-1} \varphi_{t-1}}{p_t^{-1} \alpha_{t-1}^2 \psi_{t-1}^{-1} \varphi_{t-1}}
\]

\[
\tilde{m}_t = \frac{p^{-1} \alpha_{t-1}^2 \psi_{t-1}^{-1} \varphi_{t-1}}{p_t^{-1} \alpha_{t-1}^2 \psi_{t-1}^{-1} \varphi_{t-1}} + \frac{x_{t-1} \beta_{t-1}}{\alpha_{t-1}} + \frac{\varphi_{t-1}}{p_t^{-1} \alpha_{t-1}^2 \psi_{t-1}^{-1} \varphi_{t-1}} \tilde{m}_{t-1}
\]

\[
\tilde{m}_c = 0 \quad \psi_c = 1
\]

\[
\tilde{x}_t = \alpha_t \tilde{m}_t + \beta_t.
\]
4B. The difference from the model of Section 3 appears when we are going to estimate the \((\alpha_i, \beta_i, \omega_i)\)'s by data from a portfolio of ceded portfolios as the different ceded portfolios have different amounts of direct insured premiums.

We assume that we have a portfolio of independent ceded portfolios that satisfy the conditions given in subsection 4A and have the same \((\alpha_i, \beta_i, \omega_i)\)'s. Suppose that \(k_1N\) portfolios have been ceded both calendar years \(k\) and \(l\), and let \(k_1x_{ij}\) denote the observed loss ratio of portfolio \(i\) in year \(j\) (\(i = 1, \ldots, k_1N; j = k, l\)).

Let

\[
k_1x_j = \sum_{i=1}^{k_1N} k_1a_{ij} k_1x_{ij},
\]

where the \(k_1a_{ij}\)'s are non-random weights. Then

\[
E((k_1x_{ik} - \bar{x}_k)(k_1x_{il} - \bar{x}_l)) = \left\{ \begin{array}{ll}
    k_1c_i \phi_k + kk_1a_i^2 & \text{k = 1} \\
    k_1c_i a_i & \text{k \neq 1}
\end{array} \right.
\]

with

\[
k_1c_i = \frac{1 - 2k_1a_{ik}}{kk_1P_i} + \frac{k_1a_{ik}}{kk_1P_i},
\]

\[
k_1a_{ik} = 1 - k_1a_{ik} - k_1a_{il} + \sum_{r=1}^{k_1N} k_1a_{rk} k_1a_{rl}.
\]

Let

\[
\hat{a}_{k1} = \sum_{i=1}^{k_1N} k_1b_i (k_1x_{ik} - \bar{x}_k)(k_1x_{il} - \bar{x}_l),
\]

where the constants \(k_1b_i\) are chosen so as to satisfy

\[
\sum_{i=1}^{k_1N} k_1b_i k_1c_i = 1.
\]

Then we have

\[
a_{k1} = E(\hat{a}_{k1}) = \left\{ \begin{array}{ll}
    k_1c_i \phi_k + k_1a_i^2 & \text{k = 1} \\
    k_1a_i & \text{k \neq 1}
\end{array} \right.
\]
with
\[ k^C = \sum_{i=1}^{N} k^{bi} k^{ci}, \]

and \( \alpha_k \) may be estimated by
\[ \hat{\alpha}_k = \sqrt{\frac{1}{r<s<k} k^{wr} \hat{\alpha}_{rk} \hat{\alpha}_{sk}}, \]

where the \( k^{wr} \)'s are non-random weights, e.g. proportional to \( \sqrt{rsP_r rsP_s} \), where
\[ rsP_j = \sum_{i=1}^{N} rsP_{ij}. \]

\( \varphi_k \) can now be estimated by
\[ \hat{\varphi}_k = \frac{\hat{\alpha}_{kk} - \hat{\alpha}_{k}^2}{k^C}. \]

As choice of \( k^{a_{ij}} \) and \( k^{b_i} \) we propose
\[ k^{a_{ij}} = \frac{k^{P_{ij}}}{k^{P_j}}, \quad k^{b_i} = \frac{1}{g_{kl}}k^{P_{ik}}k^{P_{il}} \]

with
\[ g_{kl} = \sum_{i=1}^{N} k^{P_{ik}} k^{P_{il}} k^{C_i} \]

(cf. Sundt (1980b), subsection 3B).

As
\[ v = \sum_{i=1}^{N} k^{P_{ik}} a_k + \varphi_k k^{X_{ik}} \]
\[ g_{kl} = \sum_{i=1}^{N} k^{P_{ik}} a_k^2 + \varphi_k \]

is the best linear unbiased estimator of \( \beta_k \) based on the available claim amounts (see Sundt (1978)). We propose to
estimate $\beta_k$ by

$$\hat{\beta}_k = \frac{k^N \sum_{i=1}^{N} kkP_{ik} \alpha_k + \phi}{k^N \sum_{i=1}^{N} kkP_{ik} \alpha_k + \phi_k}$$

For estimation of $\alpha_t$ and $\beta_t$ by claim data from before year $t$, we refer to subsections 3D-F.

5. Estimation when $\theta$ varies with time

5A. In subsection 3A we assumed that the claim amounts $x_c, x_{c+1}, \ldots$ of an insurance policy depended on an unknown random parameter $\theta$. Now we are going to assume that this $\theta$ is a sequence $(\theta_c, \theta_{c+1}, \ldots)$ of unknown random parameters and that $x_i$ depends on $\theta$ only through $\theta_i$, that is, we allow the individual risk characteristics of the policy to change as time passes. This is a very natural assumption; e.g., in motor insurance a car owner's driving abilities are not constant. We shall assume that $x_c, x_{c+1}, \ldots$ are independent given $\theta$, and replace assumptions (1) and (2) by

$$E(x_i | \theta) = \alpha_i m(\theta_i) + \beta_i$$

$$E(m(\theta_i)) = 0$$

$$C(m(\theta_i), m(\theta_j)) = \rho |i-j|.$$  \hspace{1cm} (10)

Assumptions similar to (10) have been studied by Sundt (1980a). It is assumed that the $\alpha_i$'s are positive, and that $\rho \in <0,1]$. 

In the same way as in subsection 3B we find

\[
\psi_t = \rho^2 \frac{\psi_{t-1} \varphi_{t-1}}{\alpha_{t-1}^2 \psi_{t-1} + \varphi_{t-1}} + 1 - \rho^2
\]

\[
\tilde{m}_t = \rho \left( \frac{\alpha_{t-1}^2 \psi_{t-1}}{\alpha_{t-1}^2 \psi_{t-1} + \varphi_{t-1}} \frac{x_{t-1} - \beta_{t-1}}{\alpha_{t-1}} + \frac{\varphi_{t-1}}{\alpha_{t-1}^2 \psi_{t-1} + \varphi_{t-1}} \tilde{m}_{t-1} \right)
\]

\[
\tilde{m}_c = 0 \quad \psi_c = 1
\]

\[
\tilde{x}_t = \alpha_t \tilde{m}_t + \beta_t
\]

5B. We are now going to develop estimators of the \( \alpha_i \)'s, \( \beta_i \)'s, \( \varphi_i \)'s, and \( \rho \). Assume that we have a portfolio of independent policies that satisfy the conditions given in the previous subsection and have the same \( (\alpha_i, \beta_i, \varphi_i) \)'s and \( \rho \). Let \( k \in \mathbb{N}, k l x_j, \hat{\beta}_k \), and \( \hat{\alpha}_{kl} \) be defined as in subsection 3C, and let

\[
a_{kl} = E(\hat{\alpha}_{kl}) = \begin{cases} \varphi_k + \alpha_k^2 & k = 1 \\ \rho \mid \{k-1\} \alpha_k \alpha_1 & k \neq 1 \end{cases}
\]

We estimate \( \beta_k \) by \( \hat{\beta}_k \).

As for all \( r \)

\[
\frac{\alpha_{r-3,r-1} \alpha_{r-2,r}}{\alpha_{r-3,r-2} \alpha_{r-1,r}} = \frac{\alpha_{r-3,r} \alpha_{r-2,r-1}}{\alpha_{r-3,r-2} \alpha_{r-1,r}} = \rho^2,
\]

we suggest to estimate \( \rho \) by

\[
\hat{\rho} = \sqrt{\frac{\sum_r w_r (\tilde{\alpha}_{r-3,r-1}^2 \tilde{\alpha}_{r-2,r} + \tilde{\alpha}_{r-3,r}^2 \tilde{\alpha}_{r-2,r-1})}{2 \sum_r w_r \tilde{\alpha}_{r-3,r-2} \tilde{\alpha}_{r-2,r-1}}},
\]

where the non-random weights \( w_r \) could e.g. be chosen proportional to \( r^{-3/2} N \).
We also have

\[ a_k = \frac{a_{k-1,k}}{a_{k-3,k-2}} \frac{a_{k-3,k-1} a_{k-2,k-1}}{a_{k-3,k-1} a_{k-2,k-1}} \]

and may estimate \( a_k \) by

\[ \hat{a}_k = \frac{\hat{a}_{k-1,k}}{\hat{a}_{k-3,k-2}} \frac{\hat{a}_{k-3,k-1} \hat{a}_{k-2,k-1}}{\hat{a}_{k-3,k-1} \hat{a}_{k-2,k-1}} \]

\( \psi_k \) can now be estimated by

\[ \hat{\psi}_k = \hat{a}_{kk} - \hat{a}_k^2 \]

For the estimation of \( a_t \) and \( \beta_t \) by claim data from before year \( t \), we refer to subsections 3D-F.

5C. As we in Section 4 modified the model of Section 3 to estimation of loss ratios, we can do a similar extension of the present model. In the model assumptions of subsection 5A we then replace \( EV(x_i|\theta) = \omega_i \) by \( EV(x_i|\theta) = \omega_i/p_i \) and get

\[ \psi_t = \rho^2 \frac{\psi_{t-1} \omega_{t-1}}{\hat{a}_{t-1} p_{t-1} \psi_{t-1} + \hat{\phi}_{t-1}} + 1 - \rho^2 \]

\[ \tilde{m}_t = \rho \left( \frac{\alpha_{t-1}^2 p_{t-1} \psi_{t-1}}{\hat{a}_{t-1} p_{t-1} \psi_{t-1} + \hat{\phi}_{t-1}} x_{t-1} - \beta_{t-1} \right) + \frac{\phi_{t-1}}{\hat{a}_{t-1} p_{t-1} \psi_{t-1} + \hat{\phi}_{t-1}} \tilde{m}_{t-1} \]

\[ \tilde{m}_c = 0 \quad \psi_c = 1 \]

\[ \tilde{x}_t = a_t \tilde{m}_t + \beta_t \]

We shall not go any further into this model.

6. Conclusion. Related models

6A. The methods treated in Sections 3 - 5 may seem a bit inconsequent; at the end of year \( t-1 \) we have estimators of \( a_i \) and \( \beta_i \) assuming no connection between \( a_i \)'s and \( \beta_i \)'s from different years, but then for the estimation of \( a_t \) and \( \beta_t \) we suddenly introduce some structure. The
reason for the introduction of this structure is, as argued in subsection 3D, the need of additional assumptions to be able to estimate $\alpha_t$ and $\beta_t$. But as we do not feel too confident about these assumptions, we are willing to use them only when strictly necessary.

6B. Alternatively, we may find it reasonable that for all $i$, $E(x_i | \theta) = \gamma_i' b(\theta)$, where $\gamma_i$ is a known non-random design vector, and $b(\theta)$ is a vector function of $\theta$. Such models were introduced in credibility theory by Taylor (1975) and Hachemeister (1975). Of later contributions to the theory we mention Jewell (1975), Taylor (1977), De Vylder (1977,1978), and Norberg (1980).

6C. These regression models assume that different policies are independent, and that time-heterogeneity occurs in accordance with known design vectors. An opposite approach is to assume that to each calendar year $i$ there is connected an unknown random parameter $\eta_i$ that influences the whole portfolio in that year. The $\eta_i$'s are assumed to be independent and identically distributed, and for a policy with random risk parameter $\theta$ the conditional distribution of $x_t$ given $\theta$ and the $\eta_i$'s is of the form $F(\cdot | \theta, \eta_t)$, where the function $F(\cdot | \theta, \eta_t)$ is independent of the policy and the year. Such models may describe cases where purely random elements influence the whole portfolio; e.g., in motor insurance a winter with extremely icy roads may lead to many accidents. Models of this sort have been treated by Welten (1968) and Sundt (1979b).
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References


