

STATISTICAL RESEARCH REPORT  
Institute of Mathematics  
University of Oslo

December  
No 4 1979

AN UPPER BOUND FOR DEFICIENCIES BETWEEN  
EXPERIMENTS IN TERMS OF THEIR CONVEX EXTENSIONS  
APPLICATION TO DEFICIENCIES W.R.T.  
STRONGLY PRECOMPACT EXPERIMENTS

by

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## 1. INTRODUCTION AND STATEMENTS OF RESULTS

In his paper [5], D.W. Müller presents an interesting criterion for asymptotic comparison of experiments. The criterion may be viewed as a generalization of a result of Le Cam, see [3], which (roughly) states that information is never gained by passing to weak\* limits of the individual distributions. It is then assumed that the parameter set is finite (or that only restrictions to finite parameter sets are considered). Müller shows that the conclusion still holds if we assume that the weak\* limits constitutes a strongly precompact experiment and that weak\* convergence is uniform in the unknown parameter.

These results may be expressed in terms of the deficiencies introduced by Le Cam in [2] - see also his book [3] and Torgersen [8]. Related to the deficiency is the  $\Delta$ -distance, see [3], for experiments. The  $\Delta$ -distance between experiments  $E$  and  $F$ , having the same parameter set is the largest of the deficiency of  $E$  w.r.t.  $F$  and the deficiency of  $F$  w.r.t.  $E$ .

Criteria for  $\Delta$ -convergence of experiments towards strongly precompact experiments were obtained by D. Lindae in his thesis [4]. He showed that  $\Delta$ -convergence is then implied by  $\Delta$ -convergence for restrictions to finite sub parameter sets and a condition of uniform shape convergence. The latter condition is implied by  $\Delta$ -convergence for pairs (i.e. restrictions to two point parameter sets).

Although the condition of strong precompactness is quite restrictive it is, nevertheless, satisfied in many situations. The condition is, see Le Cam [3] or Siebert [6], equivalent to the condition that  $E$  may be  $\Delta$  approximated as closely as we wish by experiments having finite sample space (and involving

only a finite number of distributions). It is then, by Torgersen [7] to be expected that deficiencies w.r.t. strongly precompact experiments may, in a uniform way, be reduced to deficiencies for  $k$ -decision problems for sufficiently large  $k$ .

The purpose of this note is to describe some general properties of deficiencies, which imply Müller's result, and which are of interest in themselves.

In order to explain the content, let us introduce some notations. An experiment  $E = (P_\theta : \theta \in \Theta)$  with parameter set  $\Theta$  is a map  $\theta \rightsquigarrow P_\theta$  from  $\Theta$  to an (abstract)  $L$ -space such that  $P_\theta \geq 0$ ;  $\theta \in \Theta$  and  $\|P_\theta\| = 1$ ;  $\theta \in \Theta$ . Let  $\Lambda$  be the set of prior distributions on  $\Theta$  with finite support. The convex extension of  $E$  is the experiment  $\hat{E} = (P_\lambda : \lambda \in \Lambda)$  where  $P_\lambda = \sum_{\theta} P_\theta \lambda(\theta)$ ;  $\lambda \in \Lambda$ . We refer to Le Cam [2] and [3], Heyer [1] and Torgersen [7] and [8] for some of the basic facts on experiments. Any experiment is equivalent to an experiment  $E = (P_\theta : \theta \in \Theta)$  where the  $P_\theta$ 's are probability measures on some measurable space. If  $E$  and  $F$  are experiments with the same parameter set, then the deficiency of  $E$  w.r.t.  $F$  (for  $k$ -decision problems) will be denoted by  $\delta(E, F)$  ( $\delta_k(E, F)$ ).

If  $S$  is any set then  $\#S =$  the cardinality of  $S$  if  $S$  is finite and  $= \infty$  if  $S$  is infinite.

We begin by establishing a simple inequality for deficiencies w.r.t.  $k$ -decision problems.

Theorem 1. If  $E$  and  $F$  are experiments with the same parameter set  $\Theta$  then:

$$\delta_k(\hat{E}, \hat{F}) = \delta_k(E, F) \leq (k-1) \sup\{\delta_k(\hat{E}|_{\Lambda'}, \hat{F}|_{\Lambda'}) : \Lambda' \subseteq \Lambda; \#\Lambda' \leq k\}$$

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Remark: If  $k \leq 2$  then the inequality is, see [7], in fact an equality.

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The approximation result for strongly precompact experiments which we shall need is well known - see e.g. LeCam[3] or Siebert [6].

Proposition 2. Let  $E = (P_\theta : \theta \in \Theta)$  be strongly precompact. Then there is to each  $\epsilon > 0$  an experiment  $\tilde{E} = (\tilde{P}_\theta : \theta \in \Theta)$  on the same sample space such that:

- i)  $\|P_\theta - \tilde{P}_\theta\| < \epsilon; \theta \in \Theta$
  - ii)  $\tilde{E}$  is equivalent to an experiment with finite sample space
  - iii) The set  $\{\tilde{P}_\theta : \theta \in \Theta\}$  is finite
- 

Remark. Two experiments  $E$  and  $F$  are called equivalent if  $\delta(E, F) = \delta(F, E) = 0$ .

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Proposition 2 implies the following condition for uniform approximation of deficiencies.

Corollary 3. If  $E$  is strongly precompact and  $\epsilon > 0$  then there is a positive integer  $k$  so that

$$\delta(F, E) \leq \delta_k(F, E) + \epsilon$$

for all experiments  $F$ . It follows in particular that if  $E_n; n \in \mathbb{N}$  is a net of experiments such that  $\delta_k(E_n, E) \rightarrow 0$ ;  $k = 1, 2, \dots$  where  $E$  is strongly precompact then  $\delta(E_n, E) \rightarrow 0$ .

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Approaching the set up considered by Müller let us assume that we are given

A (abstract) L-space  $L$  with dual  $M = L^*$

An experiment  $E = (P_\theta : \theta \in \Theta)$  in  $L$

A net  $E_n = (P_{\theta,n} : \theta \in \Theta)$ ;  $n \in \mathbb{N}$  of experiments in  $L$ .

Thus the  $P_\theta$ 's and the  $P_{\theta,n}$ 's are all probabilities (i.e. non-negative normalized elements) in  $L$ .

Finally we are given a  $w(M,L)$  dense subvector lattice of  $M$  containing the unit of  $M$ . Here  $w(M,L)$  is the smallest topology on  $M$  making each  $S$  in  $L$  a continuous linear functional on  $M$ .

If  $\Theta$  is finite then the following proposition is contained in LeCam [3].

Proposition 4. Suppose  $\sup_\theta |P_{\theta,n}(v) - P_\theta(v)| \rightarrow 0$  when  $v \in V$  and that  $E$  is strongly precompact. Then:

$$\sup\{\delta(E_n | \theta', E | \theta') : \#\theta' \leq r\} \rightarrow 0$$

when  $r < \infty$ .

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Remark. In order to apply theorem 1 and then corollary 3 let us note that  $E_n; n \in \mathbb{N}$  and  $E$  satisfy the conditions of proposition 4 if and only if  $\hat{E}_n; n \in \mathbb{N}$  and  $\hat{E}$  do.

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Here is "almost" Müller's criterion.

Theorem 5. Suppose  $\sup_\theta |P_{\theta,n}(v) - P_\theta(v)| \rightarrow 0$  when  $v \in V$  and that  $E$  is strongly precompact. Then

$$\delta(E_n, E) \rightarrow 0$$

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It follows from Lelam's randomization criterion [2], that, under the hypothesis of the theorem, there is a net  $T_n; n \in N$  of transitions from  $L$  to  $L$  such that  $\sup_{\theta} \|P_{\theta, n} T_n - P_{\theta}\| \rightarrow 0$ .

Assume for the final result of this note that  $L$  is the  $L$ -space of finite measures (with total variation norm) on the Borel class of a complete separable metric space  $\chi$ . Then, [3], the transitions  $T_n$  above may, provided the experiments  $E_n$  are dominated be replaced by Markov kernels. If, however  $V$  is the class of bounded continuous functions, then, by Müller [5], this holds for any sequence  $E_n; n = 1, 2, \dots$  of experiments satisfying the assumptions of the theorem. This is, with a slight modification, the content of:

Corollary 6. Let  $E_n = (P_{\theta, n}; \theta \in \Theta); n = 1, 2, \dots$  and  $E = (P_{\theta}; \theta \in \Theta)$  be experiments such that the  $P_{\theta}$ 's and the  $P_{\theta, n}$ 's are all probability measures on the Borel class,  $\mathcal{B}$ , of a complete reparable metric space  $\chi$ . Let  $V$  be a vector lattice of bounded uniformly continuous functions on  $\chi$  which contains the constants and which separates the probability distributions on  $\mathcal{B}$ .

Suppose  $\sup_{\theta} |P_{\theta, n}(v) - P_{\theta}(v)| \rightarrow 0$  when  $v \in V$  and that  $E$  is strongly precompact. Then there are Markov kernels  $M_n; n = 1, 2, \dots$  from  $(\chi, \mathcal{B})$  to  $(\chi, \mathcal{B})$  such that:

$$\sup_{\theta} \|P_{\theta, n} M_n - P_{\theta}\| \rightarrow 0.$$

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2. PROOFS

Proof of theorem 1. We may without loss of generality assume that  $E = (P_\theta : \theta \in \Theta)$  and  $F = (Q_\theta : \theta \in \Theta)$  where the  $P_\theta$ 's and the  $Q_\theta$ 's are probability measures on respectively, measurable spaces  $(X, A)$  and  $(Y, B)$ . By LeCam [2]),  $\delta(E, F) = \delta(\hat{E}, \hat{F})$ . By Torgersen [7],  $\delta_k(E, F) = \sup \delta_k(E, \tilde{F})$  where the sup is taken for all restrictions of  $F$  to sub-algebras of  $B$  containing at most  $2^k$  sets. It follows that

$$\delta_k(E, F) = \delta_k(\hat{E}, \hat{F}) ; k = 1, 2, \dots .$$

We may for the remaining part of this proof assume, again without loss of generality - that  $\Theta$  is finite.

Put  $\eta = \sup(\delta_k(\hat{E}|\Lambda', \hat{F}|\Lambda') : \#\Lambda' \leq k)$ . Let  $\Psi_k$  be the class of functions on  $R^\Theta$  which are maximums of  $k$  (or fewer) linear functionals.

It suffices, by Torgersen [7], to show that

$$(\S) \quad \psi(F) - \psi(E) \leq (k-1)\eta \sum_{\theta} (\psi(e^\theta) + \psi(-e^\theta))/2$$

when  $\psi \in \Psi_k$ . Here  $e^\theta$  is the  $\theta$ -th unit vector in  $R^\Theta$  i.e.  $(e^\theta)(\theta') = 1$  or  $= 0$  as  $\theta' = \theta$  or  $\theta' \neq \theta$ .

We may, as both sides of ( $\S$ ) vanishes when  $\psi$  is linear, assume that  $\psi$  is monotonically increasing; i.e. that  $\psi(x) = \bigvee_{i=1}^k \langle a_i, x \rangle$ ;  $x \in R^\Theta$  where  $\bigwedge_{i=1}^k a_i = 0$ .

Put  $A_i = \sum_{\theta} a_i(\theta)$ ;  $i = 1, \dots, k$  and

$$\lambda_i(\theta) = A_i^{-1} a_i(\theta) \text{ or } = [1/\#\Theta] \text{ as } A_i > 0 \text{ or } A_i = 0 .$$

Then  $a_i = A_i \lambda_i$  so that  $\langle a_i, x \rangle = A_i \langle \lambda_i, x \rangle$ .

Put  $\rho(z_1, \dots, z_k) = \bigvee_{i=1}^k A_i z_i$  when  $z_1, \dots, z_k \in R$ .

Let  $S_E$  ( $S_F$ ) be the standard measures of, respectively,  $E$  and  $F$ .

Then

$$\begin{aligned} \psi(F) - \psi(E) &= \int \prod_{i=1}^k \langle a_i, x \rangle S_F(dx) - \int \prod_{i=1}^k \langle a_i, x \rangle S_E(dx) \\ &= \rho(Q_{\lambda_1}, \dots, Q_{\lambda_k}) - \rho(P_{\lambda_1}, \dots, P_{\lambda_k}) \\ &\leq \frac{n}{2} [\rho(-1, 0, \dots, 0) + \rho(1, 0, \dots, 0) + \dots + \rho(0, 0, \dots, -1) + \rho(0, 0, \dots, 1)] \\ &= \frac{n}{2} \sum_i A_i = \frac{n}{2} \sum_{\theta} \sum_i a_{i\theta} \end{aligned}$$

Now  $\psi(e^\theta) = \sum_i a_{i\theta}$  while  $\psi(-e^\theta) = -\sum_i a_{i\theta} = 0$

Hence, utilizing that  $\sum_i a_i = 0$ , we find:

$$\psi(F) - \psi(E) \leq \frac{n}{2}(k-1) \sum_{\theta} \sum_i a_{i\theta} = \frac{n}{2}(k-1) \sum_{\theta} (\psi(e^\theta) + \psi(-e^\theta)).$$

□

Proof of proposition 2:

We may, without loss of generality, assume that the  $P_\theta$ 's are probability measures on a common measurable space. By assumption there is a finite subset  $\{\pi_1, \pi_2, \dots, \pi_r\}$  of  $\{P_\theta : \theta \in \Theta\}$  such that to each  $\theta$  in  $\Theta$  corresponds an integer  $v(\theta)$  in  $\{1, 2, \dots, r\}$  such that  $\|P_\theta - \pi_{v(\theta)}\| < \epsilon/2$ . Put  $\mu = \sum_{i=1}^r \pi_i$  and

$f_i = d\pi_i / d\mu$ ;  $i = 1, \dots, r$ . Then there are non-negative measurable functions  $\tilde{f}_1, \dots, \tilde{f}_r$ , all with finite range, such that

$$\int \tilde{f}_i d\mu = 1 \quad \text{and} \quad \int |f_i - \tilde{f}_i| d\mu < \epsilon/2$$

Put  $\tilde{\pi}_i = \tilde{f}_i \mu$  and  $\tilde{P}_\theta = \tilde{\pi}_{v(\theta)}$ . Then  $\tilde{E} = (\tilde{P}_\theta : \theta \in \Theta)$  satisfies conditions i), ii) and iii).

□

Proof of corollary 3:

Let  $\epsilon > 0$  and let  $\tilde{E}$  be as in the proposition. Suppose  $\tilde{E}$  is



equivalent to an experiment with a sample space containing  $k$  points. Then, by Tongersen [7], for any experiment  $F$

$$\begin{aligned} \delta(F, E) &\leq \delta(F, \tilde{E}) + \delta(\tilde{E}, E) \\ &= \delta_k(F, \tilde{E}) + \delta(\tilde{E}, E) \\ &\leq \delta_k(F, \tilde{E}) + \epsilon \\ &\leq \delta_k(F, E) + \delta_k(E, \tilde{E}) + \epsilon \\ &\leq \delta_k(F, E) + 2\epsilon. \end{aligned}$$

Suppose now that  $E_n$ ,  $n \in \mathbb{N}$  is such that  $\delta_i(E_n, E) \rightarrow 0$ ;  $i=1, 2, \dots$

Then  $\limsup_n \delta(E_n, E) \leq \limsup_n \delta_k(E_n, E) + 2\epsilon = 2\epsilon + 0$  as  $\epsilon \rightarrow 0$ .

□

The proof of proposition 4, to be given below, rests on two well known facts on vector lattices. They are:

(i) Let  $M$  be a vector lattice and let  $\mu_1, \mu_2, \dots, \mu_r$  be  $r$  order bounded linear functionals on  $M$  and let  $u \in M_+$ .

Then:

$$\left( \bigvee_{i=1}^r \mu_i \right)(u) = \sup \left\{ \sum_{i=1}^r \mu_i(u_i) : u_1, \dots, u_r \in M_+; \sum_{i=1}^r u_i = u \right\}$$

(ii) Let  $L$  be an (abstract)  $L$ -space with dual  $M = L^*$ . Let  $V$  be a  $w(M, L)$  dense sub vector lattice of  $M$ .

Consider  $r$  elements  $\mu_1, \mu_2, \dots, \mu_r$  of  $L$  and an element  $v$  of  $V_+$ . Then:

$$\left( \bigvee_{i=1}^r \mu_i \right)(v) = \sup \left\{ \sum_{i=1}^r \mu_i(v_i) : v_1, \dots, v_r \in V_+; \sum_{i=1}^r v_i = v \right\}$$

Indication of proof of (i):

If  $r = 2$  then (i) is found in most textbooks containing at least one chapter on vector lattices. The case of a general non-negative integer  $r$  follows by induction. □

Indication of proof of (ii):

$L$  is, by evaluation, isomorphic to a vector lattice of  $M^*$ . Hence, by (i),

$$(\S) \quad \left( \bigvee_{i=1}^r \mu_i \right)(v) = \sup \{ \sum_{i=1}^r \mu_i(u_i) : u_1, \dots, u_r \in M_+; \sum_{i=1}^r u_i = v \}$$

Now the weak closure of any convex subset  $V$  of  $M$  coincides with the closure of  $V$  for the topology  $\beta$  of uniform convergence on intervals of  $L$ . (This may be seen by first showing that the  $\beta$ -continuous linear functionals on  $M$  are in  $L$ .) The advantage of using the latter topology is that it makes the lattice operations on  $M$  continuous.

It follows in particular that if  $u_1, \dots, u_r \in M_+$  and  $\sum_{i=1}^r u_i = v$  then there is a net  $(v_{1,n}, v_{2,n}, \dots, v_{r,n})$  in  $V_+^r$  such that  $v_{i,n} \xrightarrow{\beta} u_i; i = 1, \dots, r$ . Put:

$$\begin{aligned} \tilde{v}_{1,n} &= v_{1,n} \wedge v \\ \tilde{v}_{i,n} &= (v_{1,n} + \dots + v_{i,n}) \wedge v - (v_{1,n} + \dots + v_{i-1,n}) \wedge v; \\ & \hspace{20em} 1 \leq i < r \end{aligned}$$

$$\text{and } \tilde{v}_{r,n} = v - \sum_{i=1}^{r-1} \tilde{v}_{i,n}$$

Then  $\tilde{v}_{i,n} \in V_+; i = 1, \dots, r; \sum_{i=1}^r \tilde{v}_{i,n} = v$  and  $\tilde{v}_{i,n} \xrightarrow{\beta} v_i; 1 \leq i \leq r$ .

It follows that (§) still holds if the sup is taken for all

$(u_1, \dots, u_r) \in V_+^r$  such that  $\sum_{i=1}^r u_i = v$ . □

Proof of proposition 4:

Let  $1 < r < \infty$ . It suffices to show that  $T_n \rightarrow 0$  when

$$T_n = \delta((P_{\theta_{1,n},n}, \dots, P_{\theta_{r,n},n}), (P_{\theta_{1,n}}, \dots, P_{\theta_{r,n}})) \rightarrow 0$$

and  $(\theta_{1,n}, \theta_{2,n}, \dots, \theta_{r,n})$ ;  $n \in \mathbb{N}$  is a net in  $\theta^r$ . This, by  $\Delta$ -compactness for finite  $\theta$  and by the sublinear functional criterion (see [7]), will follow if we can show that

$$\liminf_n [\| \sum_{i=1}^k \sum_{j=1}^r a_{ij} P_{\theta_{j,n},n} \| - \| \sum_{i=1}^k \sum_{j=1}^r a_{ij} P_{\theta_{j,n}} \|] \geq 0$$

when all  $a_{ij} \geq 0$ . Suppose, on the contrary, that  $\liminf_n [ ] < 0$ .

Then there is a subnet  $\{n'\}$  and an  $\alpha > 0$  so that

$$\| \sum_{i=1}^k \sum_{j=1}^r a_{ij} P_{\theta_{j,n'},n'} \| - \| \sum_{i=1}^k \sum_{j=1}^r a_{ij} P_{\theta_{j,n'}} \| < -\alpha$$

for all  $n'$ . We may, by strong precompactness, assume without loss of generality that  $\|P_{\theta_{j,n'}} - \pi_j\| \rightarrow 0$ ;  $j = 1, 2, \dots, r$ .

Then

$$\| \sum_{i=1}^k \sum_{j=1}^r a_{ij} P_{\theta_{j,n'}} \| \rightarrow \| \sum_{i=1}^k \sum_{j=1}^r a_{ij} \pi_j \|$$

Hence, for sufficiently large  $n'$ :

$$\| \sum_{i=1}^k \sum_{j=1}^r a_{ij} P_{\theta_{j,n'},n'} \| \leq \| \sum_{i=1}^k \sum_{j=1}^r a_{ij} \pi_j \| - \frac{\alpha}{2}$$

Let  $v \in V$ . Then:

$$\begin{aligned} |P_{\theta_{j,n'},n'}(v) - \pi_j(v)| &\leq |P_{\theta_{j,n'},n'}(v) - P_{\theta_{j,n'}}(v)| + |P_{\theta_{j,n'}}(v) - \pi_j(v)| \\ &\leq \sup_{\theta} |P_{\theta,n'}(v) - P_{\theta}(v)| + \|P_{\theta_{j,n'}} - \pi_j\| \|v\|. \end{aligned}$$

Thus, by uniform convergence:

$$P_{\theta_{j,n'},n'}(v) \rightarrow \pi_j(v).$$

Let finally  $v_1, v_2, \dots, v_k \in V_+$  be such that  $\sum_{i=1}^k v_i = I$ . Then

$$\begin{aligned} & \left\| \sum_{i=1}^k \sum_{j=1}^r a_{ij} \pi_j \right\| - \frac{\alpha}{2} \\ & \geq \liminf_{n'} \sum_{i=1}^k \sum_{j=1}^r a_{ij} P_{\theta_{j,n',n'}}(v_i) \\ & = \sum_{i=1}^k \sum_{j=1}^r a_{ij} \pi_j(v_i) \end{aligned}$$

Taking supremum of the right hand side for all  $(v_1, \dots, v_k) \in V_+^k$  and that  $v_1 + \dots + v_k = I$  we find

$$\left\| \sum_{i=1}^k \sum_{j=1}^r a_{ij} \pi_j \right\| - \frac{\alpha}{2} \geq \left\| \sum_{i=1}^k \sum_{j=1}^r a_{ij} \pi_j \right\|$$

contradicting the assumption " $\alpha > 0$ ".

□

Proof of the statements in the remark after proposition 4:

The first claim follows from the inequality:

$$|P_{\lambda,n}(v) - P_{\lambda}(v)| \leq \sup_{\theta} |P_{\theta,n}(v) - P_{\theta}(v)|.$$

The second claim follows from the following observation:

Let  $\pi_1, \pi_2, \dots, \pi_r$  be probabilities in  $L$  such that  $\bigwedge_{i=1}^r \|P_{\theta} - \pi_i\| < \frac{\varepsilon}{2}$ ;  $\theta \in \Theta$  and let  $Q_1, Q_2, \dots, Q_s$  be probabilities in  $\langle \pi_1, \pi_2, \dots, \pi_r \rangle$  such that  $\bigwedge_{j=1}^s \|P - Q_j\| < \frac{\varepsilon}{2}$  whenever  $P \in \langle \pi_1, \pi_2, \dots, \pi_r \rangle$ . Then  $\bigwedge_{j=1}^s \|P_{\lambda} - Q_j\| < \varepsilon$ ;  $\lambda \in \Lambda$ .

□

Proof of theorem 5

Suppose  $E_n$ ;  $n \in \mathbb{N}$  and  $E$  satisfies the assumptions of proposition 4. By this proposition

$$\sup\{\delta(\hat{E}_n | \Lambda'; \hat{E} | \Lambda') : \#\Lambda' \leq k\} \rightarrow 0.$$

Hence, by theorem 1;  $\delta_k(E_n, E) \rightarrow 0$ ;  $k = 1, 2, \dots$ . Finally, by corollary 3,  $\delta(E_n, E) \rightarrow 0$ .

□

Proof of corollary 6

Let  $L$  be the  $L$ -space of finite measures on  $\mathcal{B}$ . Then  $V$  is  $w(M,L)$  dense in  $M = L^*$ .

Denote the metric on  $\chi$  by  $d$  and let  $\chi_0 = \{x_1, x_2, \dots\}$  be a countable dense subset of  $\chi$ . Let  $g_n(x) = x_i$  where  $i$  is the smallest integer  $\geq 1$  such that  $d(x, x_i) \leq \frac{1}{n}$ . Thus  $d(g_n(x), x) \leq \frac{1}{n}$ . Then

$$\begin{aligned} |P_{\theta, n} g_n^{-1}(v) - P_{\theta}(v)| &\leq |P_{\theta, n}(v(g_n)) - P_{\theta, n}(v)| + |P_{\theta, n}(v) - P_{\theta}(v)| \\ &\leq \sup\{|v(x) - v(y)| : d(x, y) \leq \frac{1}{n}\} + \sup_{\theta} |P_{\theta, n}(v) - P_{\theta}(v)|. \end{aligned}$$

It follows that  $\tilde{E}_n = (P_{\theta, n} g_n^{-1} : \theta \in \Theta)$ ,  $n = 1, 2, \dots$  and  $E$  satisfy the same conditions as  $E_n$ ;  $n = 1, 2, \dots$  and  $E$ . Hence, by the theorem,  $\delta(\tilde{E}_n, E) \rightarrow 0$ . Now  $\tilde{E}_n$  is dominated and  $\chi$  is complete separable metric. It follows that there are Markov kernels  $U_n$ ;  $n = 1, 2, \dots$  from  $\chi_0$  to  $\chi$  such that  $\sup_{\theta} \|(P_{\theta, n} g_n^{-1})U_n - P_{\theta}\| \rightarrow 0$ .

The corollary follows now by putting  $M_n(B|x) = U_n(B|g_n(x))$ ;  $B \in \mathcal{B}$ ,  $x \in \chi$ ,  $n = 1, 2, \dots$ .

□

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