ABSTRACT

The present report represents an entry MULTISTATE
COHERENT SYSTEMS which is to appear in the Encyclopedia
of Statistical Sciences, Vol. 5 published by Wiley in
1984. It gives a summary of the present state of the art
of multistate theory also aiming to standardize the ter-
minology.

MULTISTATE COHERENT SYSTEMS

One inherent weakness of traditional reliability theory, see COHERENT STRUCTURE THEORY, is that the system and the components are always described just as functioning or failed. Fortunately, by now this theory is being replaced by a theory for multistate systems of multistate components. This enables one for instance in a power generation system to let the system state be the amount of power generated, or in a pipeline system the amount of oil running through a crucial point. In both cases the system state is possibly measured on a discrete scale. The papers [1], [4], [8] initiating the research in this area came in the late seventies. Here we summarize the theory starting out from two recent papers [2], [7].

Let the set of states of the system be $S = \{0,1,\ldots,M\}$. The $M+1$ states represent successive levels of performance ranging from the perfect functioning level M down to the complete failure level 0. Furthermore, let the set of components be $C = \{1,2,\ldots,n\}$ and the set of states of the i th component $S_i (i=1,\ldots,n)$ where $\{0,M\} \subseteq S_i \subseteq S$. Hence the states 0 and M are chosen to represent the endpoints of a performance scale which might be used for both the system and its components.

If x_i (i=1,....,n) denotes the state or perfor-
mance level of the i th component and $\underline{x}=(x_1,...,x_n)$, it
is assumed, see COHERENT STRUCTURE THEORY, that the
state ϕ of the system is given by the structure
function $\phi=\phi(\underline{x})$. A series of results in multistate
reliability theory can be derived for the following
systems:

Definition 1   A system is a <u>multistate monotone system</u>
(MMS) iff its structure function ϕ satisfies:

(i) $\phi(\underline{x})$ is non-decreasing in each argument

(ii) $\phi(\underline{O})=0$ and $\phi(\underline{M})=M$ $(\underline{O}=(0,...,0),\underline{M}=(M,...,M))$.
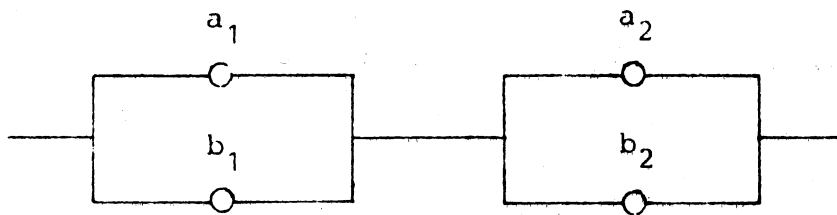

Figure 1. Example of an MMS

As a simple example of an MMS consider the network
of Figure 1. Here component 1 (2) is the parallel module
of the branches a_1 and b_1 (a_2 and b_2). Let (i=1,2)
$x_i=3$ if two branches work and 1 (0) if one (no) branch
works. The state of the system is given in Table 1.

		3	0	2	3
Component 2		1	0	1	2
		0	0	0	0
			0	1	3

Component 1

Table 1. State of system in Figure 1.

Note for instance that the state 1 is a critical one both for each component and the system as a whole in the sense that the failing of a branch leads to the 0 state. In binary theory the functioning state comprises the states $\{1,2,3\}$ and hence just a rough description of the system's performance is possible.

DETERMINISTIC PROPERTIES OF MULTISTATE SYSTEMS

We start by generalizing each of the concepts "minimal path set" and "minimal cut set" in coherent structure theory.[*] In the following $\underline{y} < \underline{x}$ means $y_i \leq x_i$ for $i=1,\dots,n$ and $y_i < x_i$ for some i.

__Definition 2__ Let ϕ be the structure function of an MMS and let $j\varepsilon\{1,\dots,M\}$. A vector $\underline{x}$ is said to be a minimal __path (cut) vector to level__ $\underline{j}$ iff $\phi(\underline{x}) \geq j$ and $\phi(\underline{y}) < j$ for all $\underline{y} < \underline{x}$ ($\phi(\underline{x}) < j$ and $\phi(\underline{y}) \geq j$ for all $\underline{y} > \underline{x}$). The corresponding minimal __path (cut) sets to level__ $\underline{j}$ are given by $C^j(\underline{x}) = \{i \mid x_i \geq 1\}$ $(D^j(\underline{x}) = \{i \mid x_i < M\})$.

For the structure function tabulated in Table 1 the minimal path (cut) vectors for instance to level 2 (1) are (3,1) and (1,3) ((3,0) and (0,3)).

We now impose some further restrictions on the structure function ϕ. The following notation is needed: $(\cdot_i, \underline{x}) = (x_1, \dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_n)$,

$$S_{i,j}^0 = S_i \cap \{0, \dots, j-1\} \quad \text{and} \quad S_{i,j}^1 = S_i \cap \{j, \dots, M\}.$$

Definition 3 Consider an MMS with structure function ϕ satisfying

(i) $\min\limits_{1\leqslant i\leqslant n} x_i \leqslant \phi(\underline{x}) \leqslant \max\limits_{1\leqslant i\leqslant n} x_i$.

If in addition $\forall i \in \{1,\ldots,n\}, \forall j \in \{1,\ldots,M\}, \exists (\cdot_i,\underline{x})$ such that

(ii) $\phi(k_i,\underline{x}) \geqslant j$, $\phi(l_i,\underline{x}) < j$ $\forall k \in S^1_{i,j}$, $\forall l \in S^0_{i,j}$, we have a multistate strongly coherent system (MSCS),

(iii) $\phi(k_i,\underline{x}) > \phi(l_i,\underline{x})$ $\forall k \in S^1_{i,j}$, $\forall l \in S^0_{i,j}$, we have a multistate coherent system (MCS),

(iv) $\phi(M_i,\underline{x}) > \phi(0_i,\underline{x})$, we have a multistate weakly coherent system (MWCS).

All these systems are generalizations of a system introduced in [4]. The first one is presented in [7], whereas the two latter for the case $S_i = S(i=1,\ldots,n)$ are presented in [6]. When M=1, all reduce to the established binary coherent system (BCS). The structure function $\min\limits_{1\leqslant i\leqslant n} x_i$ $(\max\limits_{1\leqslant i\leqslant n} x_i)$ is often denoted the multistate series (parallel) structure.

Now choose $j \in \{1,\ldots,M\}$ and let the states $S^0_{i,j}(S^1_{i,j})$ correspond to the failure (functioning) state for the i th component if a binary approach had been applied. Condition (ii) above means that for all components i and any level j, there shall exist a combination of the states of the other components,

$(\cdot_i, \underline{x})$, such that if the i th component is in the binary failure (functioning) state, the system itself is in the corresponding binary failure (functioning) state. Loosely speaking, modifying [2], condition (ii) says that every level of each component is relevant to the same level of the system, condition (iii) says that every level of each component is relevant to the system, whereas condition (iv) simply says that every component is relevant to the system.

For a BCS one can prove the following practically very useful principle: Redundancy at the component level is superior to redundancy at the system level except for a parallel system where it makes no difference. Assuming $S_i = S(i=1,\ldots,n)$ this is also true for an MCS, but not for an MWCS.

We now mention a special type of an MSCS. Introduce the indicators $(j=1,\ldots,M)$

$$I_j(x_i)=1 \ (0) \text{ if } x_i \geq j (x_i < j), \text{ and the indicator}$$
vector $\underline{I}_j(\underline{x})=(I_j(x_1),\ldots,I_j(x_n))$.

Definition 4 An MSCS is said to be a <u>binary type</u> <u>multistate strongly coherent system</u> (BTMSCS) iff there exist binary coherent structures ϕ_j, $j=1,\ldots,M$ such that its structure function ϕ satisfies

$$\phi(\underline{x}) \geq j \Leftrightarrow \phi_j(\underline{I}_j(\underline{x}))=1 \text{ for all } j\epsilon\{1,\ldots,M\} \text{ and all } \underline{x}.$$

Choose again $j\epsilon\{1,\ldots,M\}$ and let the states

$s_{i,j}^0 (s_{i,j}^1)$ correspond to the failure (functioning) state for the i th component if a binary approach is applied. By the definition above ϕ_j will from the binary states of the components uniquely determine the corresponding binary state of the system. It is easily checked that the MMS of Figure 1 is an MSCS but not a BTMSCS. In [7] it is shown that if all ϕ_j are identical, the structure function ϕ reduces to the one suggested in [1]. Furthermore, it is indicated that most of the theory for a BCS can be extended to a BTMSCS.

PROBABILISTIC PROPERTIES OF MULTISTATE SYSTEMS

We now concentrate on the relationship between the stochastic performance of the system and the stochastic performance of the components. Let X_i denote the random state of the i th component and let (i=1,...,n; j=0,...,M)

$$\Pr(X_i < j) = P_i(j) \qquad \bar{P}_i(j) = 1 - P_i(j).$$

P_i represents the performance distribution of the i th component. Now if ϕ is a structure function, $\phi(\underline{X})$ is the corresponding random system state. Let (j=0,...,M)

$$\Pr(\phi(\underline{X}) < j) = P(j) \qquad \bar{P}(j) = 1 - P(j).$$

P represents the performance distribution of the system. We also introduce the performance function of the system, h, defined by

$$h = E\phi(\underline{X}).$$

We obviously have the relation

$$h = \sum_{j=1}^{M} \bar{P}(j-1).$$

Hence, for instance bounds on the performance distribution of the system automatically give bounds on h.

We now briefly illustrate how coherent structure theory[*] bounds are generalized to bounds on the performance distribution of an MMS of associated components. First we give the following crude bounds

$$\prod_{i=1}^{n} \bar{P}_i(j-1) \leqslant \bar{P}(j-1) \leqslant 1 - \prod_{i=1}^{n} P_i(j-1)$$

Next we give bounds based on the minimal path and cut vectors. For $j \epsilon \{1,\ldots,M\}$ let $\underline{y}_r^j = (y_{1r}^j,\ldots,y_{nr}^j)$ $r=1,\ldots,n_j$ $(\underline{z}_r^j = (z_{1r}^j,\ldots,z_{nr}^j)$ $r=1,\ldots,m_j)$ be the system's minimal path (cut) vectors to level j and $C^j(\underline{y}_r^j)$ $r=1,\ldots,n_j$ $(D^j(\underline{z}_r^j)$ $r=1,\ldots,m_j)$ the corresponding minimal path (cut) sets to level j. Then

$$\prod_{r=1}^{m_j} [1 - Pr(\bigcap_{i \epsilon D^j(\underline{z}_r^j)} (X_i \leqslant z_{ir}^j))] \leqslant Pr[\phi(\underline{X}) \geqslant j]$$

$$\leqslant 1 - \prod_{r=1}^{n_j} [1 - Pr(\bigcap_{i \epsilon C^j(\underline{y}_r^j)} (X_i \geqslant y_{ir}^j))], \quad j=1,\ldots,M.$$

These bounds are obviously simplified in the case of independent components.

As a simple application of the crude bounds consider the system of Figure 1. Let the probability of a branch working be p, and assume that branches within a component work independently whereas the two components are associated. Then we easily get

$$2p^4 + [1-(1-p)^2]^2 \leqslant h \leqslant 3 - (1-p)^4 -2(1-p^2)^2$$

For p=0 and p=1, we get the obvious results whereas for p=1/2 we have $11/16 \leqslant h \leqslant 29/16$.

As a conclusion it should be admitted that almost all efforts on multistate systems theory have been concentrated on mathematical generalizations of the traditional binary theory. This research has, however, been quite successful. One key area where much research remains is the development of appropriate measures of component importance. Finally, it is a need for several convincing case studies demonstrating the practicability of the generalizations introduced. We know that some are under way.

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(COHERENT STRUCTURE THEORY)

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