

A UNIFIED THEORY OF DOMINATION AND  
SIGNED DOMINATION WITH APPLICATION  
TO EXACT RELIABILITY COMPUTATIONS.

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A UNIFIED THEORY OF DOMINATION AND SIGNED DOMINATION WITH APPLICATION TO EXACT RELIABILITY COMPUTATIONS.

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Abstract.

Domination theory plays an important part in the study of network reliability. In the present report we review the most important previous results in this field. However, the theory is presented in a more general setting in order to arrive at extensions. Several new , simplified proofs of old results as well as new, general theory are given. Especially, a generalized domination theorem is presented.

COHERENT STRUCTURES; CLUTTERS; NETWORKS; DOMINATION; MATROIDS;  
INCLUSION-EXCLUSION THEOREM; PIVOTAL DECOMPOSITIONS.

## 1. Introduction.

The concept of domination has played an important part in the study of network reliability. Several different methods and algorithms have been developed using this as a basis. There are mainly two directions in these works. One is based on the Inclusion-Exclusion theorem, and is mainly related to directed networks. Key references here are: Satyanarayana and Prabhakar (1978), and Satyanarayana (1982). The other direction is based on the Factoring theorem, and is restricted to undirected networks. The main results on this field are given in Satyanarayana and Chang (1983).

The efficiency of these methods makes it tempting to apply the same techniques to more general systems. However, the network based algorithms require some sort of graph representation of the given system. Hence, in order to use these methods, one has to find such a representation.

In Huseby (1983) it is shown how to express a given system as a 2-terminal undirected network problem. However, it is also shown that only a small subclass of systems are representable in such a way. Hence, in order to extend the network methods to more general classes of systems, one should try to modify the algorithms to be less dependent on the specific network representation.

In order to achieve this goal, it is necessary to establish a more general theoretical basis for the methods. In a sense the work of Barlow (1982) represents a first step in this direction.

In the present paper we shall also focus on general domination theory. This will be done by reviewing the most important earlier

results. However, these will be introduced in a more general setting in order to obtain new results as well. We also provide new simplified proofs for the old results.

The paper starts out by introducing the basic definitions and results needed throughout the text. This is carried out in Section 2. Instead of focusing on structure functions, as in standard reliability textbooks, (such as Barlow and Proschan (1981)) we have chosen to work on the family of minimal path sets. This is done in order to obtain a more set oriented approach which has certain theoretical advantages. We have also used a slightly different way of introducing the concept of domination and signed domination compared with earlier work.

In Section 3 we study coherency properties of minors. The results obtained here are of great importance in many later proofs. The most interesting result in this section is that every complex coherent structure (Definition 3.3.) can be decomposed into two coherent structures by using pivotal decomposition with respect to a suitable component. This result is a complete generalization of a result on K-terminal undirected network systems given in Satyanarayana and Chang (1983), and plays an important part in the study of the factoring algorithm.

Domination and signed domination were originally defined by introducing so-called formations of minimal path sets, and all earlier results have been developed with this as a basis. However, in Section 4 we provide an entirely new way of expressing these concepts. This new expression has great theoretical advantages in order to simplify the proofs of old results and obtain new useful results. Section 4 includes a new proof of the signed domination

theorem (first proved in Barlow (1982)) and two "multiplication" rules for signed domination.

The first results concerning the Inclusion-Exclusion method and directed graphs was developed without having the signed domination theorem. By application of this result it turns out that the proofs can be made considerably simpler. This is carried out in Section 5. In this section we also provide a slight generalization of a formula given in Satyanarayana (1982). This is proved by applying one of the multiplication rules developed in Section 4. In Section 6 we turn to another important theorem of network reliability known as the domination theorem. This result is mainly used in the study of the factoring algorithm. In Satyanarayana and Chang (1983) this theorem is proved to be true for the case of  $K$ -terminal undirected network systems. It is, however, known to be true for the case of  $k$ -out-of- $n$  systems as well. (See Barlow (1982)). In this section we provide a generalized version of the domination theorem covering all previously known cases. This theorem is based on some basic results of the theory of matroids. Especially, the work of Lehman (1964) is of great importance. Some of these results as well as some more motivation are given in

Huseby (1983). Section 6 also provides new classes of structures where the domination theorem is valid.

In Section 7 the well-known factoring algorithm is presented and its properties are investigated in the light of the new general results obtained in Section 3 and Section 6.

## 2. Basic definitions and results.

In this section we list the basic concepts needed in the theory.

Definition 2.1. A clutter is an ordered pair  $(E, P)$  where  $E$  is a non-empty finite set and  $P$  is a family of incomparable subsets of  $E$ ; i.e. no set in  $P$  is a subset of another set in  $P$ .

If  $P = \{\emptyset\}$ , i.e. the only set in  $P$  is the empty set,  $(E, P)$  is called a 1-clutter, while if  $P = \emptyset$ , i.e.  $P$  contains no sets,  $(E, P)$  is called a 0-clutter. We say that  $(E, P)$  is trivial if it is a 1-clutter or a 0-clutter.

Defintion 2.2. Let  $(E, P)$  be a clutter, and let  $e \in E$ . We say that  $e$  is relevant if  $e \in P$  for at least one  $P \in P$ . If  $e$  is not relevant, it is said to be irrelevant.

A coherent structure is a clutter  $(E, P)$  where all the elements of  $E$  are relevant, i.e.  $\bigcup_{P \in P} P = E$ .

In reliability theory the elements of a clutter  $(E, P)$  are interpreted as edges or components being either functioning or failed. To indicate the state of a component  $e \in E$ , we assign a binary variable,  $x_e$ , defined by:

$$x_e = \begin{cases} 1 & \text{if component } e \text{ is functioning} \\ 0 & \text{if component } e \text{ is failed.} \end{cases}$$

Similarly, we assign a binary variable  $\phi$  indicating the state of a system given by:

$$\phi = \begin{cases} 1 & \text{if the system is functioning.} \\ 0 & \text{if the system is failed.} \end{cases}$$

$\phi$  is assumed to be a function of the component states, and is

called the structure function of the system.

Now, let  $A \subseteq E$  be the set of functioning components in the system and interpret  $P$  as the family of minimal path sets. We then get:

$$(2.1) \quad \phi = \phi(A) = \begin{cases} 1 & \text{if } P \subseteq A \text{ for some } P \in \mathcal{P} \\ 0 & \text{otherwise} \end{cases}$$

We observe that if  $(E, P)$  is a 1-clutter, we get that  $\phi(A) = 1$  for all  $A \subseteq E$ , while if  $(E, P)$  is a 0-clutter,  $\phi(A) = 0$  for all  $A \subseteq E$ .

It is a well-known fact of reliability theory that  $\phi$  can be expressed as a multilinear function of the component states. (See Barlow and Proschan (1981).) That is, let  $E = \{1, \dots, n\}$  and  $\underline{x} = (x_1, \dots, x_n)$ , then  $\phi$  can be written on the form:

$$(2.2) \quad \phi = \phi(\underline{x}) = \sum_{B \subseteq E} \delta(B) \prod_{i \in B} x_i$$

where  $\delta$  is a suitably chosen function denoting the coefficients of each term. ( $\delta(B)=0$  if the term corresponding to the set  $B$  does not occur in the expression.) Thus, if  $A \subseteq E$  is the set of functioning components, then  $\prod_{i \in B} x_i = 1$  if  $B \subseteq A$  and zero otherwise.

Hence,  $\phi$  can be expressed as:

$$(2.3) \quad \phi = \phi(A) = \sum_{B \subseteq A} \delta(B).$$

In this text we shall apply both (2.2) and (2.3) as expressions for the structure function.

The function  $\delta$  is called the signed domination function of the clutter and will be of great importance in the theory we are about



to develop. Especially we define:

Definition 2.3. Let  $(E, P)$  be a clutter, and let  $\delta$  be the signed domination function of the clutter. The signed domination of the clutter,  $d(P)$ , is defined by:

$$d(P) = \delta(E).$$

The domination of the clutter,  $D(P)$ , is defined by:

$$D(P) = |\delta(E)|. \text{ (=The absolute value of } \delta(E)\text{.)}$$

Another perhaps more commonly used way of defining domination is by introducing the concept of formations. To get a better understanding of domination we include this approach as well.

Definition 2.4. Let  $(E, P)$  be a clutter

A formation is a family of minimal path sets  $F \subseteq P$  such that:

$$\bigcup_{P \in F} P = E$$

We say that a formation is odd if  $|F|$  is odd and even if  $|F|$  is even. ( $|F|$  denotes the cardinality of  $F$ .)

The following proposition provides an alternative expression for the signed domination (and the domination):

Proposition 2.5. Let  $(E, P)$  be a clutter. Then we have:

$$d(P) = \text{The number of odd formations minus the number of even formations.}$$

Proof: Apply the principle of Inclusion-Exclusion.  $\square$

Corollary 2.6. Let  $(E, P)$  be a clutter. If  $E$  contains irrelevant components, then  $d(P)$  is zero.

Proof: If  $E$  contains irrelevant components, then the clutter has no formations. Hence, by Proposition 2.5.  $d(P) = 0 - 0 = 0$ .  $\square$

We shall later see that the converse statement of Corollary 2.6 is false. In fact, there exists an important class of coherent structures, related to cyclic directed networks, having zero domination.

In the next definition we introduce the concept of restriction and contraction:

Definition 2.7. Let  $(E, P)$  be a clutter and let  $e \in E$ . We then define

$P_{+e}$  = The family of minimal sets of the form  $P - e$  where  $P \in P$ .

$P_{-e}$  = The family of all sets in  $P$  which do not contain  $e$ .

It is easy to see that if  $P$  is interpreted as the minimal path sets of a system with component set  $E$ , then we have:

$P_{+e}$  = The family of minimal path sets of the system given that  $e$  is functioning.

$P_{-e}$  = The family of minimal path sets of the system given that  $e$  is failed.

(Hence, the notations ' $P_{+e}$ ' and ' $P_{-e}$ ' are motivated.)

The clutter  $(E-e, P_{+e})$  is called the contracted clutter of  $(E, P)$  with respect to the component  $e$ , while the clutter  $(E-e, P_{-e})$  is called the restricted clutter of  $(E, P)$  with respect to the component  $e$ .

The operations of contraction and restriction are called minor operations, and we say that a clutter is a minor of  $(E, P)$  if it can be obtained from  $(E, P)$  by performing a (finite) sequence of minor operations. Especially, we say that a clutter is a subclutter of  $(E, P)$  if it is a minor obtained from  $(E, P)$  by performing restrictions only. A subclutter obtained from  $(E, P)$  by performing restrictions with respect to a set  $C$  of components will be denoted by  $(E-C, P_{-C})$ .

The following example provides a geometric interpretation of the minor operations:

Example 2.8. Let  $(E, P)$  be the 2-terminal undirected network system  $G$ , shown in Figure 2.1, which is functioning if the nodes  $S$  and  $T$  can communicate through the network.

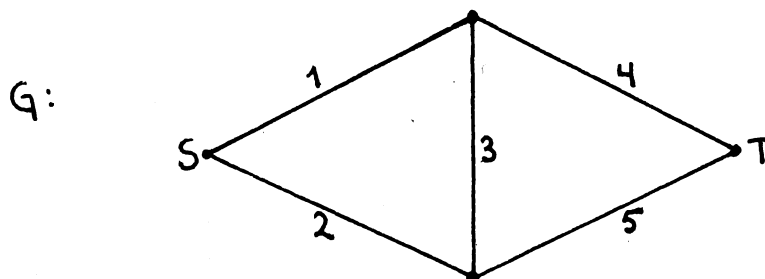


Figure 2.1.

Here we have:  $E = \{1, 2, 3, 4, 5\}$  and the family of minimal path sets is  $P = \{\{1, 4\}, \{1, 3, 5\}, \{2, 3, 4\}, \{2, 5\}\}$ .

Now, we select  $e = 4$ , and get:

$$P_{+4} = \{\{1\}, \{2,3\}, \{2,5\}\} \text{ and } P_{-4} = \{\{1,3,5\}, \{2,5\}\}.$$

We observe that  $(E-4, P_{+4})$  and  $(E-4, P_{-4})$  corresponds to the network systems  $G_{+4}$  and  $G_{-4}$  shown in Figure 2.2.

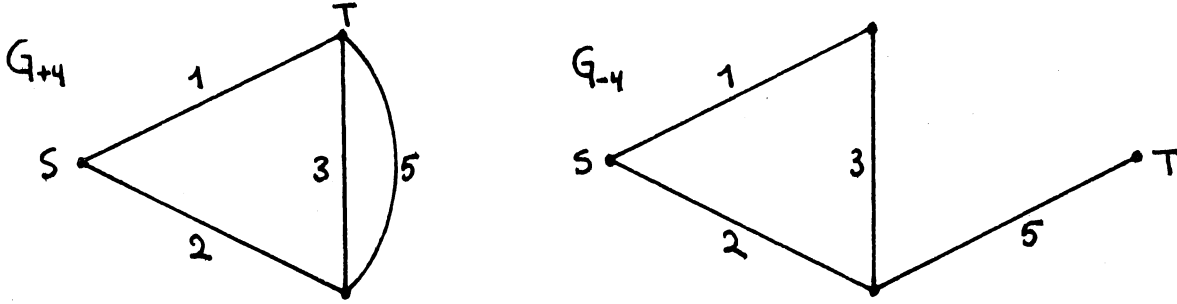


Figure 2.2.

As illustrated in Example 2.8 we have the following geometrical interpretation of the minor operations, assuming that the clutter can be represented as an undirected network system:

Proposition 2.9. Let  $(E, P)$  be a clutter which can be represented as an undirected network system  $G$ , and let  $e \in E$ .

Then  $(E-e, P_{+e})$  can be represented as an undirected network system  $G_{+e}$  obtained from  $G$  by deleting  $e$  and identifying the endpoints of  $e$ .

Similarly  $(E-e, P_{-e})$  can be represented as an undirected network system  $G_{-e}$  obtained from  $G$  by deleting  $e$ .  $\square$

The next proposition expresses the minor operations in terms of the structure function:

Proposition 2.10. Let  $(E, P)$  be a clutter with structure function  $\phi$ , and let  $e \in E$  and  $C \subseteq E$ .

Denote the structure function of  $(E-e, P_{+e})$  by  $\phi_{+e}$ , the structure function of  $(E-e, P_{-e})$  by  $\phi_{-e}$  and the structure function of  $(E-B, P_{-C})$  by  $\phi_{-C}$ . Then we have

- (i)  $\phi_{+e}(A) = \phi(A \cup e)$  for all  $A \subseteq E-e$ .
- (ii)  $\phi_{-e}(A) = \phi(A)$  for all  $A \subseteq E-e$

As a generalization of (ii) we have:

- (iii)  $\phi_{-C}(A) = \phi(A)$  for all  $A \subseteq E-C$ .  $\square$

Corollary 2.11. Let  $(E, P)$  be a clutter with structure function  $\phi$  and signed domination function  $\delta$ , Then we have:

$$d(P_{-C}) = \delta(E-C) \quad \text{for all } C \subseteq E.$$

Proof: Let  $C \subseteq E$ , and let  $\phi_{-C}$  and  $\delta_{-C}$  be the structure function and the signed domination function of  $(E-C, P_{-C})$  respectively.

Since by Proposition 2.10.  $\phi_{-C}(A) = \phi(A)$  for all  $A \subseteq E-C$ , by (2.3) we get:

$$\sum_{B \subseteq A} \delta_{-C}(B) = \sum_{B \subseteq A} \delta(B) \quad \text{for all } A \subseteq E-C$$

Hence, it is easy to obtain that:  $\delta_{-C}(B) = \delta(B)$  for all  $B \subseteq E-C$ . Especially,  $d(P_{-C}) = \delta_{-C}(E-C) = \delta(E-C)$  as stated.  $\square$

The last concept we shall introduce in this section is the concept of minimal cut sets. A cut set of a clutter  $(E, P)$  is a set  $C$  such that  $C \cap P \neq \emptyset$  for all  $P \in P$ . A cut set is minimal if it cannot be reduced and still be a cut set. If the elements of  $E$  are interpreted as components being either functioning or failed and  $P$  is interpreted as the family of minimal path sets, then obviously a cut set is a set of components whose failure is sufficient to cause system failure.

It is a well-known fact of reliability theory that path sets and cut sets are in a sense "dual" to each other. (See Barlow and Proschan (1981)). Especially, we have the following proposition:

Proposition 2.12. Let  $(E, \mathcal{P})$  be a clutter and let  $\mathcal{C}$  be the family of minimal cut sets. Then  $(E, \mathcal{P})$  is coherent if and only if we have:

$$\bigcup_{C \in \mathcal{C}} C = E. \quad \square$$

We close this section by including a result concerning minor operations and minimal cut sets. (Observe the duality between this result and Definition 2.7.)

Proposition 2.13. Let  $(E, \mathcal{P})$  be a clutter and let  $\mathcal{C}$  be the family of minimal cut sets. For an arbitrary component  $e \in E$ , denote by  $\mathcal{C}_{+e}$  and  $\mathcal{C}_{-e}$  the families of minimal cut sets of  $(E-e, \mathcal{P}_{+e})$  and  $(E-e, \mathcal{P}_{-e})$  respectively.

Then we have:

$\mathcal{C}_{+e}$  = The family of all sets in  $\mathcal{C}$  which do not contain  $e$ .

$\mathcal{C}_{-e}$  = The family of minimal sets of the form  $C-e$  where  $C \in \mathcal{C}$ .  $\square$

### 3. Coherency properties of minors.

In the study of signed domination functions, many results can be deduced by investigating the minors of a given structure. Especially, in light of Corollary 2.6., we shall study the coherency properties of minors. In order to do so we shall introduce some useful concepts and definitions.

Definition 3.1. Let  $(E, P)$  be a clutter and let  $C$  be the family of minimal cut sets. Assume that we have:  $E = \{1, \dots, n\}$ ,

$$P = \{P_1, \dots, P_p\} \text{ and } C = \{C_1, \dots, C_c\}.$$

Now define:

$$S_i = \{j: i \in P_j\} \text{ and } K_i = \{j: i \in C_j\}, i=1, \dots, n.$$

$S = \{S_1, \dots, S_n\}$  is called the family of transposed minimal path sets

$K = \{K_1, \dots, K_n\}$  is called the family of transposed minimal cut sets.

Example 3.2. Let  $(E, P)$  be the 2-terminal network system  $G$ , shown in Figure 3.1., which is functioning

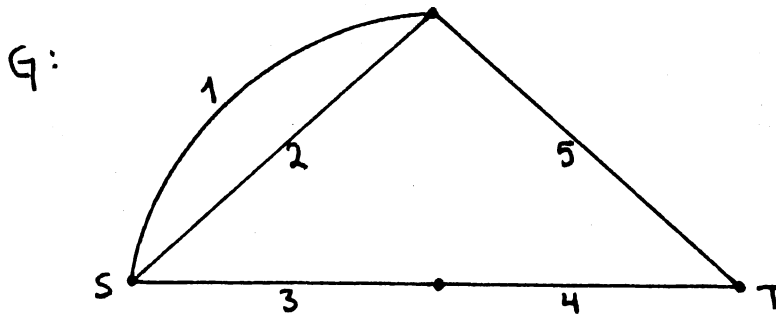


Figure 3.1.

if the nodes  $S$  and  $T$  can communicate through the network. Thus in this case we have:

$$P = \{P_1, P_2, P_3\} \text{ where } P_1 = \{1, 5\}, P_2 = \{2, 5\} \text{ and } P_3 = \{3, 4\}.$$

$$C = \{C_1, C_2, C_3, C_4\} \text{ where } C_1 = \{1, 2, 3\}, C_2 = \{1, 2, 4\},$$

$$C_3 = \{3, 5\} \text{ and } C_4 = \{4, 5\}.$$

Hence we get that  $S = \{S_1, \dots, S_5\}$  and  $K = \{K_1, \dots, K_5\}$  is given by:

$$S_1 = \{1\}, S_2 = \{2\}, S_3 = \{3\}, S_4 = \{3\} \text{ and } S_5 = \{1, 2\}$$

$$K_1 = \{1, 2\}, K_2 = \{1, 2\}, K_3 = \{1, 3\}, K_4 = \{2, 4\} \text{ and } K_5 = \{3, 4\}.$$

We observe that we have  $S_3 = S_4$  and  $K_1 = K_2$ .

It is easy to see that this is due to the fact that the components 3 and 4 are "series" components and that the components 1 and 2 are "parallel" components, where the words "series" and "parallel" are interpreted in the usual geometric way. Extending this to general clutters we may define as follows:

Definition 3.3. Let  $(E, P)$  be a clutter and let  $e, f \in E$ , and let  $S_e, S_f$  and  $K_e, K_f$  be the corresponding transposed minimal path and cut sets, respectively.

We say that  $e$  and  $f$  are in series if  $S_e = S_f$ .

Similarly, we say that  $e$  and  $f$  are in parallel if  $K_e = K_f$ .

If  $(E, P)$  contains no series or parallel components, and  $(E, P)$  contains more than one relevant component, we say that  $(E, P)$  is complex.

The main result of this section is that if  $(E, P)$  is a complex coherent structure then there exists a component  $e \in E$  such that both  $(E - e, P_{+e})$  and  $(E - e, P_{-e})$  are coherent structures.

Before we can prove this, we need some preliminary results. We start by proving a proposition concerning coherency of minors and transposed minimal path and cut sets.

Proposition 3.4. Let  $(E, P)$  be a clutter, and let  $S = \{S_1, \dots, S_n\}$  and  $K = \{K_1, \dots, K_n\}$  be the families of transposed minimal path and cut sets respectively. Finally, let  $i, j \in E$ .

Then we have:



- (i)  $i$  is irrelevant with respect to  $(E-j, P_{-j})$   
 if and only if  $S_i \subseteq S_j$ .
- (ii)  $i$  is irrelevant with respect to  $(E-j, P_{+j})$   
 if and only if  $K_i \subseteq K_j$ .

Proof: (i) Assume that  $S_i \subseteq S_j$ . Then by Definition 2.7., we have

$$\begin{aligned} P_{-j} &= \{P_k \in P : j \notin P_k\} = \{P_k \in P : k \notin S_j\} \\ &\subseteq \{P_k \in P : k \notin S_i\} \quad (\text{since } S_i \subseteq S_j) \\ &= \{P_k \in P : i \notin P_k\} \end{aligned}$$

Hence,  $i \notin P_k$  for all  $P_k \in P_{-j}$ , and thus by Definition 2.2  $i$  is irrelevant with respect to  $(E-j, P_{-j})$  as stated.

Assume conversely that  $S_i \not\subseteq S_j$ . That is, there exists  $k \in S_i$  (i.e.  $i \in P_k$ ) such that  $k \notin S_j$ .

Hence,  $j \notin P_k$  and thus by Definition 2.7,  $P_k \in P_{-j}$ . So, since  $i \in P_k$ , by Definition 2.2,  $i$  is relevant with respect to  $(E-j, P_{-j})$ , as stated.

(ii) is proved similarly by applying Proposition 2.12 and Proposition 2.13. instead of Definition 2.2. and Definition 2.7.  $\square$

As a corollary we get the following:

Corollary 3.5. Let  $(E, P)$  be a coherent structure, and let  $S = \{S_1, \dots, S_n\}$  and  $K = \{K_1, \dots, K_n\}$  be the families of transposed minimal path and cut sets respectively. Finally, let  $j \in E$ .

Then  $(E-j, P_{+j})$  and  $(E-j, P_{-j})$  are both coherent structures if and only if  $S_i \not\subseteq S_j$  and  $K_i \not\subseteq K_j$  for all  $i \neq j$ .  $\square$

The next proposition provides another useful result on coherency of minors.

Proposition 3.6. let  $(E, P)$  be a coherent structure where  $P$  is the family of minimal cut sets. Let  $j \in E$  and let  $A$  be a fixed non-empty subset of  $E-j$ .

- (i) Assume that  $j \in P$  if and only if  $P \cap A \neq \emptyset$  for all  $P \in P$ . Then  $(E-j, P_{+j})$  is coherent while  $(E-j, P_{-j})$  is non-coherent.
- (ii) Assume that  $j \in C$  if and only if  $C \cap A \neq \emptyset$  for all  $C \in \mathcal{C}$ . Then  $(E-j, P_{-j})$  is coherent while  $(E-j, P_{+j})$  is non-coherent.

Proof: (i) The situation is illustrated in Figure 3.2.

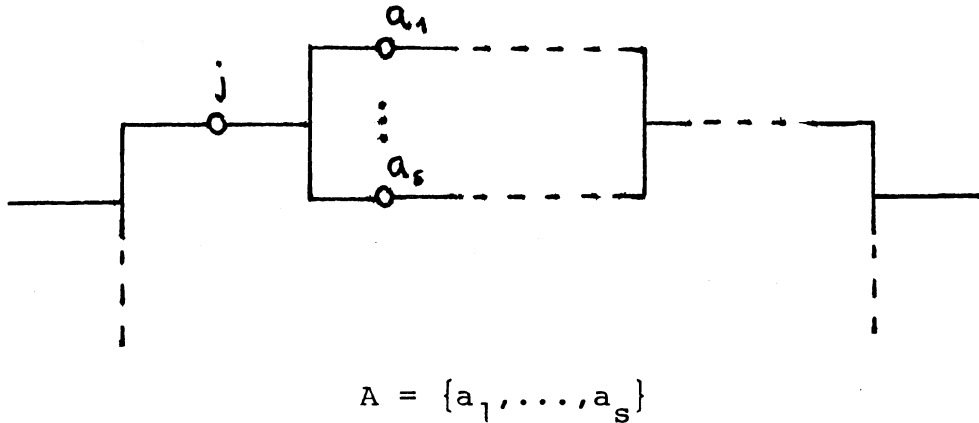


Figure 3.2.

By considering Figure 3.2. we see that if  $j$  is failed, then the components in  $A$  are irrelevant. Hence, we get that  $(E-j, P_{-j})$  is noncoherent.

Moreover, by considering all possible cases it is easy to see that all sets of the form  $P-j$ , where  $P \in P$ , are incomparable. Hence,

$$P_{+j} = \{P-j: P \in P\}.$$

Thus, since  $(E, P)$  is coherent we get that:

$$\bigcup_{P \in P_{+e}} P = \bigcup_{P \in P} (P-e) = (\bigcup_{P \in P} P) - e = E - e$$

So, we conclude that  $(E-e, P_{+e})$  is coherent. (ii) is proved similarly.  $\square$

An important special case of Proposition 3.6. arises when  $A$  contains a single component  $i$ .

If  $i$  and  $j$  are in series, we observe that  $(E-j, P_{+j})$  is obtained from  $(E, P)$  by replacing  $i$  and  $j$  by the single component  $i$ . Similarly, if  $i$  and  $j$  are in parallel we obtain  $(E-j, P_{-j})$  from  $(E, P)$  by replacing  $i$  and  $j$  by the single component  $i$ . The process of replacing series and parallel components by single components is called s-p-reduction. If a coherent structure can be reduced to a single component by performing s-p-reductions, we say that the structure is an s-p-structure.

The following easy corollary provides all necessary results on coherency and s-p-reduction.

Corollary 3.7. Let  $(E, P)$  be a clutter, and let  $i, j \in E$ . Then we have:

(i) If  $i$  and  $j$  are in series, then  $i$  is irrelevant with respect to  $(E-j, P_{-j})$ , i.e.  $(E-j, P_{-j})$  is noncoherent. Moreover, if  $i$  and  $j$  are relevant with respect to  $(E, P)$ , then  $i$  is relevant with respect to  $(E-j, P_{+j})$ . Especially, if  $(E, P)$  is coherent, then  $(E-j, P_{+j})$  is coherent as well.

(ii) If  $i$  and  $j$  are in parallel, then  $i$  is irrelevant with respect to  $(E-j, P_{+j})$ , i.e.  $(E-j, P_{+j})$  is noncoherent. Moreover, if  $i$  and  $j$  are relevant with respect to  $(E, P)$ , then  $i$  is relevant with respect to  $(E-j, P_{-j})$ . Especially, if  $(E, P)$  is coherent, then  $(E-j, P_{-j})$  is coherent as well.

(iii) If  $(E, P)$  is a nontrivial clutter, i.e.  $(E, P)$  contains at least one relevant component, then every s-p-reduction of  $(E, P)$  is nontrivial.

Proof: (i) and (ii) follow by application of Proposition 3.4 and Proposition 3.6, while (iii) is a consequence of (i) and (ii).  $\square$

We now turn to the proof of the main result of this section.

Throughout this section, let  $(E, P)$  be a coherent structure, and let  $S = \{S_1, \dots, S_n\}$  and  $K = \{K_1, \dots, K_n\}$  be the families of transposed minimal path and cut sets respectively.

Lemma 3.8. Let  $i, j \in E$ . Then we have:

- (i) If  $S_i \subseteq S_j$ , then  $K_i$  and  $K_j$  are incomparable.
- (ii) If  $K_i \subseteq K_j$ , then  $S_i$  and  $S_j$  are incomparable.

Proof: (i) Assume that  $S_i \subseteq S_j$ , i.e.  $i \in P \Rightarrow j \in P$  for all  $P \in \mathcal{P}$  (See Figure 3.3.)

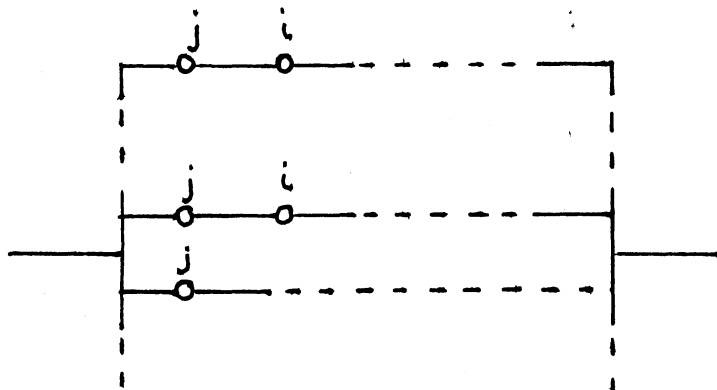


Figure 3.3.

Now, let  $C$  be a minimal cut set, and assume that  $j \in C$ . Then by the minimality of  $C$ ,  $i \notin C$ . (See Figure 3.3.) Hence  $K_i \cap K_j = \emptyset$ .

Furthermore, since  $(E, P)$  is coherent, obviously both  $K_i$  and  $K_j$  are nonempty. (See Proposition 2.12.) Hence, since they are disjoint, we conclude that  $K_i$  and  $K_j$  are incomparable.

(ii) is proved similarly.  $\square$

Lemma 3.9. Let  $i, j, k \in E$ . Then we have:

(i) If  $K_i \subseteq K_j$  and  $S_j \subseteq S_k$ , then  $S_i \subseteq S_k$ .

(ii) If  $S_i \subseteq S_j$  and  $K_j \subseteq K_k$ , then  $K_i \subseteq K_k$ .

Proof: (i) Assume that  $K_i \subseteq K_j$  and  $S_j \subseteq S_k$ , and choose some index  $s \in S_i$ , i.e.  $i \in P_s$  where  $P_s$  is a minimal path set.

We shall prove that  $s \in S_k$ , i.e.  $k \in P_s$  as well.

It is easy to see that it is possible to find a minimal cut set  $C_t$  such that  $P_s \cap C_t = i$ . (See Huseby, (1983)).

Since  $K_i \subseteq K_j$  (by the assumption), this implies that  $j \in C_t$  as well.

We have assumed  $S_j \subseteq S_k$ , so by the proof of Lemma 3.8, we know that  $K_j \cap K_k = \emptyset$ . Hence,  $j \in C_t$  implies  $k \notin C_t$ . Moreover,  $j \in P \Rightarrow k \in P$  for all  $P \in P$ . Hence,  $C' = (C_t - j) \cup k$  = the set obtained from  $C_t$  by replacing  $j$  by  $k$ , is also a cut set, and so there exists a minimal cut set  $C \subseteq C'$ .

Since  $j \notin C'$ , we know that  $j \notin C$ . Hence, since  $K_i \subseteq K_j$  (by the assumption) this implies that  $i \notin C$ .

Furthermore, obviously  $k \in C$  since we have:  $C-k \subseteq C'-k = C_t-j \subseteq C_t$ , i.e.  $C-k$  cannot be a cut set by the minimality of  $C_t$ .

Thus  $C \subseteq (C_t-i) \cup k$  and since  $C_t \cap P_s = i$  this implies that  $C \cap P_s = k \cap P_s \subseteq k$ .

However, since  $C$  is a cut set,  $C \cap P_s$  must be nonempty implying that  $C \cap P_s = k$ , i.e.  $k \in P_s$ . Hence  $s \in S_k$  implying that  $S_i \subseteq S_k$  as stated.

(ii) is proved similarly.  $\square$

We now introduce an ordering on the set  $E$  as follows.

Definition 3.10. Let  $i, j \in E$ .

If  $S_i \subseteq S_j$  or  $K_i \subseteq K_j$ , then we say that  $i < j$ . (This ordering is well-defined since by Lemma 3.8., we cannot have  $S_i \subseteq S_j$  and  $K_j \subseteq K_i$ , or  $S_j \subseteq S_i$  and  $K_i \subseteq K_j$ .)

Lemma 3.11. The ordering defined in Definition 3.10 is:

- (i) Antisymmetric. ( $i < j$  implies  $j > i$ )
- (ii) Transitive ( $i < j$  and  $j < k$  implies  $i < k$ )

Moreover, if  $(E, P)$  is complex, then the ordering is

- (iii) Reflexive ( $i < j$  and  $j < i$  implies  $i=j$ , i.e.  $i$  and  $j$  are the same component.)

Proof:

- (i) This follows directly from Definition 3.10.
- (ii) Assume that  $i < j$  and  $j < k$ .

If  $S_i \subseteq S_j$  and  $S_j \subseteq S_k$  or  $K_i \subseteq K_j$  and  $K_j \subseteq K_k$ , then obviously  $i < k$  by Definition 3.10.

If  $K_i \subseteq K_j$  and  $S_j \subseteq S_k$  or  $S_i \subseteq S_j$  and  $K_j \subseteq K_k$ , then  $i < k$  by Lemma 3.9 and Definition 3.10.

(iii) Assume that  $(E, P)$  is complex.

That is,  $(E, P)$  contains no series or parallel components.

(Definition 3.3.)

Hence, by Definition 3.3,  $S_i \not\subseteq S_j$  and  $K_i \not\subseteq K_j$  for all  $i \neq j$ .

Thus we cannot have  $i < j$  and  $j < i$  unless  $i=j$  as stated.  $\square$

An ordering which is antisymmetric, transitive and reflexive, is called a partial ordering. The following proposition states a well-known property of partial orderings. (See Graver and Watkins (1977))

Proposition 3.12. Let  $E$  be a finite partially ordered set. Then  $E$  contains at least one minimal element. That is, there exists  $j \in E$  such that no other element  $i \in E$  satisfies  $i < j$ .  $\square$

Now, we can prove the main result of this section:

Theorem 3.13. Let  $(E, P)$  be a complex coherent structure. Then there exists a component  $j \in E$  such that both  $(E-j, P_{+j})$  and  $(E-j, P_{-j})$  are coherent.

Proof: Since  $(E, P)$  is complex, by Lemma 3.11, the ordering defined in Definition 3.10 is a partial ordering. Hence, by Proposition 3.12 there exists at least one minimal component  $j \in E$  with respect to this ordering.

Since  $j$  is minimal,  $S_i \not\subseteq S_j$  and  $K_i \not\subseteq K_j$  for all components  $i \neq j$ .

Hence, by Corollary 3.5, both  $(E-j, P_{+j})$  and  $(E-j, P_{-j})$  are coherent.  $\square$

#### 4. Basic properties of signed domination

The main result of this section is the so-called "signed domination theorem" which was proved for all coherent structures in Barlow (1982). In this paper we provide a considerably simplified proof of this result. In order to do so we need an alternative expression for the signed domination function. We start by proving the following simple lemma.

Lemma 4.1. Let  $C$  and  $A$  be two (fixed) subsets of a given set  $E$ . Then we have:

$$\sum_{C \subseteq B \subseteq A} (-1)^{|A|-|B|} = \begin{cases} 1 & \text{if } C = A \\ 0 & \text{otherwise.} \end{cases}$$

Proof: We consider three possible cases.

Case 1.  $C \not\subseteq A$ .

In this case there exists no  $B$  satisfying  $C \subseteq B \subseteq A$ . Hence the sum consists of zero terms, so obviously the value of the sum must be zero.

Case 2.  $C = A$ .

In this case  $\sum_{C \subseteq B \subseteq A} (-1)^{|A|-|B|} = (-1)^{|A|-|A|} = 1$

Case 3.  $C \subset A$ .

Since  $C$  is a proper subset of  $A$ , there exists an element  $e \in A-C$ .



We now split the sum in two parts as follows:

$$\begin{aligned}
 \sum_{\substack{C \subseteq B \subseteq A \\ e \notin B}} (-1)^{|A|-|B|} &= \sum_{\substack{C \subseteq B \subseteq A \\ e \notin B}} (-1)^{|A|-|B|} + \sum_{\substack{C \subseteq B \subseteq A \\ e \in B}} (-1)^{|A|-|B|} \\
 &= \sum_{\substack{C \subseteq B \subseteq A \\ e \notin B}} (-1)^{|A|-|B|} + \sum_{\substack{C \subseteq B \subseteq A \\ e \notin B}} (-1)^{|A|-|B \cup e|} \\
 &= \sum_{\substack{C \subseteq B \subseteq A \\ e \notin B}} (-1)^{|A|-|B|} - \sum_{\substack{C \subseteq B \subseteq A \\ e \notin B}} (-1)^{|A|-|B|} = 0.
 \end{aligned}$$

Hence, we conclude that the lemma is true in all three cases.  $\square$

We can now develop the alternative expression for signed domination function. The proof is based on a method called Möbius inversion.

**Theorem 4.2.** Let  $(E, P)$  be a clutter with structure function  $\phi$  and signed domination function  $\delta$ . Then we have:

$$\delta(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} \phi(B) \quad \text{for all } A \subseteq E.$$

**Proof:** By (2.3) we know that  $\phi$  can be expressed as  $\phi(B) = \sum_{C \subseteq B} \delta(C)$

Hence we get:

$$\begin{aligned}
 \sum_{B \subseteq A} (-1)^{|A|-|B|} \phi(B) &= \sum_{B \subseteq A} (-1)^{|A|-|B|} \sum_{C \subseteq B} \delta(C) \\
 &= \sum_{C \subseteq E} \delta(C) \sum_{C \subseteq B \subseteq A} (-1)^{|A|-|B|} = \delta(A) \quad \text{as stated.} \quad \square
 \end{aligned}$$

(The last equality follows by Lemma 4.1.)

From a numerical point of view the formula given in Theorem 4.2. is quite useless since the number of terms in the sum grows exponentially with the cardinality of the set  $A$ . However, the formula appears to be useful in order to develop some general

results on signed domination. Especially, the following theorem is easy to prove by applying Theorem, 4.2.

Theorem 4.3. (The signed domination theorem).

Let  $(E, \rho)$  be a clutter with  $|E| > 2$  and let  $e \in E$ . Then we have:

$$d(\rho) = d(\rho_{+e}) - d(\rho_{-e})$$

Proof: Applying Theorem 4.2. we can write:

$$\begin{aligned} d(\rho) &= \delta(E) = \sum_{B \subseteq E} (-1)^{|E|-|B|} \phi(B) \\ &= \sum_{\substack{B \subseteq E \\ e \in B}} (-1)^{|E|-|B|} \phi(B) + \sum_{\substack{B \subseteq E \\ e \notin B}} (-1)^{|E|-|B|} \phi(B) \\ &= \sum_{B \subseteq E-e} (-1)^{|E|-|B \cup e|} \phi(B \cup e) + \sum_{B \subseteq E-e} (-1)^{|E|-|B|} \phi(B) \end{aligned}$$

We now apply Proposition 2.10. and get:

$$\begin{aligned} &= \sum_{B \subseteq E-e} (-1)^{|E-e|-|B|} \phi_{+e}(B) + \sum_{B \subseteq E-e} (-1)^{|E-e|+1-|B|} \phi_{-e}(B) \\ &= \sum_{B \subseteq E-e} (-1)^{|E-e|-|B|} \phi_{+e}(B) - \sum_{B \subseteq E-e} (-1)^{|E-e|-|B|} \phi_{-e}(B) \\ &= d(\rho_{+e}) - d(\rho_{-e}), \text{ as stated. } \square \end{aligned}$$

(The last equality follows by Theorem 4.2.)

We observe that Theorem 4.3. is true for all clutters, not only coherent structures. Our result is thus a slight generalization of the signed domination theorem given in Barlow (1982).

In the following sections we shall see how this result can be used in order to obtain more efficient algorithms for reliability computations. Another useful theorem obtained by applying Theorem 4.2, is the following "multiplication rule" for signed domination:

Theorem 4.4. Let  $(E_1, P_1)$  and  $(E_2, P_2)$  be two disjoint clutters, i.e.  $E_1 \cap E_2 = \emptyset$ .

(i) Let  $(E, P)$  be the series connection of  $(E_1, P_1)$  and  $(E_2, P_2)$ , i.e.  $E = E_1 \cup E_2$  and  $P \in \mathcal{P}$  if and only if  $P = P_1 \cup P_2$  where  $P_1 \in P_1$  and  $P_2 \in P_2$ .

Then,  $d(P) = d(P_1) \cdot d(P_2)$

(ii) Let  $(E, P)$  be the parallel connection of  $(E_1, P_1)$  and  $(E_2, P_2)$ , i.e.  $E = E_1 \cup E_2$  and  $P \in \mathcal{P}$  if and only if  $P \in P_1$  or  $P \in P_2$ . Then,  $d(P) = -d(P_1) \cdot d(P_2)$ .

Proof: Let  $\phi_1, \phi_2$  and  $\phi$  be the structure functions of  $(E_1, P_1)$ ,  $(E_2, P_2)$  and  $(E, P)$  respectively.

(i) Since  $(E, P)$  is the series connection of  $(E_1, P_1)$  and  $(E_2, P_2)$ , for all  $A_1 \subseteq E_1$  and  $A_2 \subseteq E_2$  we have:

$$\phi(A_1 \cup A_2) = \phi_1(A_1) \cdot \phi_2(A_2).$$

Hence, by Theorem 4.2. we get:

$$\begin{aligned} d(P) &= \sum_{A \subseteq E} (-1)^{|E| - |A|} \phi(A) \\ &= \sum_{A_1 \subseteq E_1} \sum_{A_2 \subseteq E_2} (-1)^{|E_1 \cup E_2| - |A_1 \cup A_2|} \phi(A_1 \cup A_2) \\ &= \sum_{A_1 \subseteq E_1} \sum_{A_2 \subseteq E_2} (-1)^{|E_1| - |A_1|} \phi_1(A_1) \cdot (-1)^{|E_2| - |A_2|} \phi_2(A_2) \\ &= \sum_{A_1 \subseteq E_1} (-1)^{|E_1| - |A_1|} \phi_1(A_1) \cdot \sum_{A_2 \subseteq E_2} (-1)^{|E_2| - |A_2|} \phi_2(A_2) \\ &= d(P_1) \cdot d(P_2), \quad \text{as stated.} \end{aligned}$$

(ii) Since  $(E, P)$  is the parallel connection of  $(E_1, P_1)$  and  $(E_2, P_2)$ , for all  $A_1 \subseteq E_1$  and  $A_2 \subseteq E_2$  we have:

$$\phi(A_1 \cup A_2) = \phi_1(A_1) + \phi_2(A_2) - \phi_1(A_1) \cdot \phi_2(A_2).$$

Hence, by Theorem 4.2. we get:

$$\begin{aligned} d(P) &= \sum_{A \subseteq E} (-1)^{|E|-|A|} \phi(A) \\ &= \sum_{A_1 \subseteq E_1} (-1)^{|E_1|-|A_1|} \phi_1(A_1) \cdot \sum_{A_2 \subseteq E_2} (-1)^{|E_2|-|A_2|} \\ &\quad + \sum_{A_2 \subseteq E_2} (-1)^{|E_2|-|A_2|} \phi_2(A_2) \cdot \sum_{A_1 \subseteq E_1} (-1)^{|E_1|-|A_1|} \\ &\quad - \sum_{A_1 \subseteq E_1} (-1)^{|E_1|-|A_1|} \phi_1(A_1) \cdot \sum_{A_2 \subseteq E_2} (-1)^{|E_2|-|A_2|} \phi_2(A_2) \end{aligned}$$

However, by Lemma 4.1.  $\sum_{\emptyset \subseteq A_1 \subseteq E_1} (-1)^{|E_1|-|A_1|} = \sum_{\emptyset \subseteq A_2 \subseteq E_2} (-1)^{|E_2|-|A_2|} = 0$ .

Thus, the two first terms vanish and by applying Theorem 4.2. to the last terms we get:

$$d(P) = -d(P_1) \cdot d(P_2) \quad \text{as stated. } \square$$

The next theorem provides a formula for computing the signed domination of dual clutter.

Theorem 4.5. Let  $(E, P)$  be a clutter, and let  $(E, C)$  be the corresponding dual clutter.

Then we have:  $d(C) = (-1)^{|E|+1} d(P)$ .

Proof: Let  $\phi$  and  $\phi^D$  be the structure functions of  $(E, P)$  and  $(E, C)$  respectively.

By standard results on dual structures we have:

(See Barlow and Proschan (1981).)

$$\phi^D(A) = 1 - \phi(E-A) \quad \text{for all } A \subseteq E.$$

Hence, by Theorem 4.2. and Lemma 4.1., we get:

$$\begin{aligned} d(C) &= \sum_{A \subseteq E} (-1)^{|E|-|A|} \phi^D(A) = \sum_{A \subseteq E} (-1)^{|E|-|A|} (1 - \phi(E-A)) \\ &= \sum_{A \subseteq E} (-1)^{|A|-|A|} - \sum_{A \subseteq E} (-1)^{|E|-|A|} \phi(E-A) \\ &= 0 - \sum_{B \subseteq E} (-1)^{|E|-|B|} \phi(B) \\ &= -(-1)^{|E|} \sum_{B \subseteq E} (-1)^{|E|-|B|} \phi(B) \\ &= (-1)^{|E|+1} d(P) \quad \text{as stated. } \square \end{aligned}$$

We close this section by giving some simple applications of these results.

Proposition 4.6. Let  $(E, \mathcal{P})$  be a coherent structure where  $\mathcal{C}$  is the family of minimal cut sets. Let  $j \in E$  and let  $A$  be a fixed non-empty subset of  $E - j$ .

(i) Assume that  $j \in P$  if and only if  $P \cap A \neq \emptyset$  for all  $P \in \mathcal{P}$ .

Then we have:  $d(P) = d(P_{+j})$ , and thus  $D(P) = D(P_{+j})$ .

(ii) Assume that  $j \in C$  if and only if  $C \cap A \neq \emptyset$  for all  $C \in \mathcal{C}$ .

Then we have:  $d(P) = -d(P_{+j})$ , and thus  $D(P) = D(P_{+j})$ .

Proof: (i) Assume that  $e \in P$  if and only if  $P \cap A \neq \emptyset$  for all  $P \in \mathcal{P}$ . Then, by Proposition 3.6,  $(E-j, \mathcal{P}_{+j})$  is coherent while  $(E-j, \mathcal{P}_{-j})$  is noncoherent.

Hence, by Theorem 4.3. and Corollary 2.6, we get:

$$d(P) = d(P_{+j}) - d(P_{-j}) = d(P_{+j}) - 0 = d(P_{+j}).$$

$$\text{Moreover: } D(P) = |d(P)| = |d(P_{+j})| = D(P_{+j}).$$

(ii) is proved similarly.  $\square$

Corollary 4.7. Let  $(E, P)$  be a coherent structure and let  $i, j \in E$ .

(i) If  $i$  and  $j$  are in series, then we have:

$$d(P) = d(P_{+j}), \text{ and thus } D(P) = D(P_{+j}).$$

(ii) If  $i$  and  $j$  are in parallel, then we have:

$$d(P) = -d(P_{-j}), \text{ and thus } D(P) = D(P_{-j}).$$

Proof: Apply Corollary 3.7., Theorem 4.3. and Corollary 2.6.  $\square$

Corollary 4.8. The domination of a coherent structure remains invariant to s-p-reductions.  $\square$

Corollary 4.9. The domination of an s-p-structure is one.

Proof: Let  $(E, P)$  be a coherent structure, and let  $(E^r, P^r)$  be the clutter obtained from  $(E, P)$  by performing all possible s-p-reductions. Then, by Corollary 3.7.  $(E^r, P^r)$  is a coherent structure as well.

Thus, if  $(E, P)$  is an s-p-structure, we have:

$$(E^r, P^r) = (\{e\}, \{P\}), \text{ where } P = \{e\}, \text{ and } e \in E.$$

Hence, obviously  $D(P^r) = 1$  by Proposition 2.5. However, by Corollary 4.8.,  $D(P) = D(P^r)$ . So we conclude that  $D(P) = 1$ , as stated.  $\square$

### 5. Signed domination and directed graphs.

The results given in this section are mainly taken from Satyanarayana and Prabhakar (1978), Willie (1980) and Satyanarayana (1982).

However, we shall obtain these results by applying Theorem 4.3. This leads to a new, simple and unified proof, handling the most general case directly.

The class of system we shall consider are so-called source-to-K-terminal (SKT) systems. The following example describes such a system.

Example 5.1. Consider the directed network illustrated in Figure 5.1.

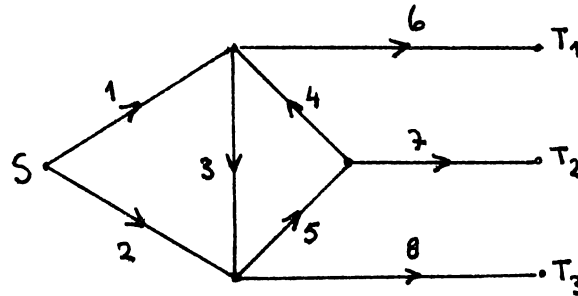


Figure 5.1.

The system is functioning if the source node  $S$  can send communication to the terminals  $T_1$ ,  $T_2$  and  $T_3$  through the network.

Hence, the minimal path sets are :  $P_1 = \{1, 2, 5, 6, 7, 8\}$ ,  $P_2 = \{1, 3, 5, 6, 7, 8\}$  and  $P_3 = \{2, 4, 5, 6, 7, 8\}$ . This is a source-to-K-terminal (SKT) system, where  $K = \{T_1, T_2, T_3\}$  is the set of terminals. We observe that the network contains a directed cycle, consisting of the components  $\{3, 4, 5\}$ . In general a directed graph containing directed cycles will be called a cyclic graph while

directed graphs not containing such cycles will be called acyclic graphs. The main results in this section is strongly related to these two concepts.

In order to apply Theorem 4.3. to SKT systems, we shall first study the effects of minor operations on such systems.

Example 5.2. Consider the directed network illustrated in Figure 5.2.

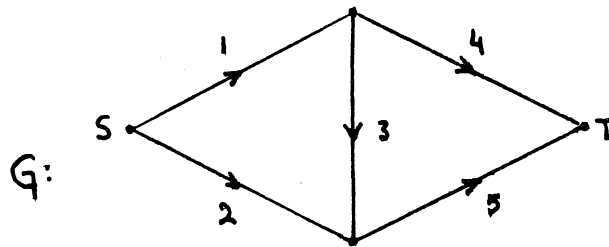


Figure 5.2.

The system is said to be functioning if  $S$  can send communication to  $T$ . So, this is an SKT system with  $K = \{T\}$ .

The clutter describing this system is given by  $(E, P)$  where  $E = \{1, 2, \dots, 5\}$  and  $P = \{\{1, 4\}, \{1, 3, 5\}, \{2, 5\}\}$ .

Let us first perform contraction and restriction with respect to the edge 1. We get:

$$P_{+1} = \{\{4\}, \{3, 5\}, \{2, 5\}\} \text{ and } P_{-1} = \{\{2, 5\}\}.$$

It is easy to see that the minor clutters,  $(E-1, P_{+1})$  and  $(E-1, P_{-1})$  can be represented by the networks  $G_{+1}$  and  $G_{-1}$  (See Figure 5.3.) respectively.



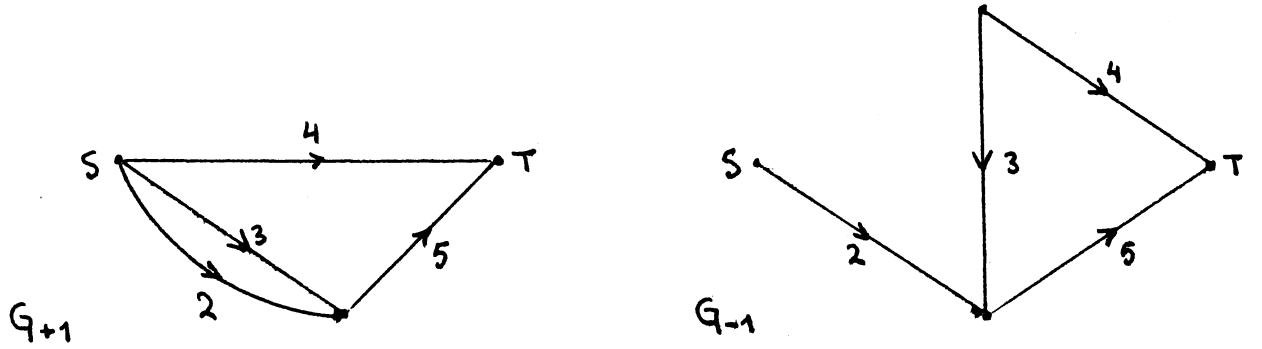


Figure 5.3.

Thus, we have geometric interpretation of the minor operations which is similar to the one for undirected networks.

However, if we perform contraction and restriction with respect to the edge 3, this will not be true.

In this case we get:

$$P_{+3} = \{\{1,4\}, \{1,5\}, \{2,5\}\} \text{ and } P_{-3} = \{\{1,4\}, \{2,5\}\}.$$

We observe that  $(E-3, P_{-3})$  can be represented by a network obtained from  $G$  by deleting the edge 3, while  $(E-3, P_{+3})$  is nonrepresentable.

The following obvious proposition, which we state without proof, characterize this problem.

Proposition 5.3. Let  $(E, P)$  be an SKT system represented by a directed network  $G$ , and let  $e \in E$ . Furthermore, let  $G_{+e}$  be the directed network obtained from  $G$  by deleting  $e$  and identifying the endpoints of  $e$ , and let  $G_{-e}$  be the directed network obtained from  $G$  by deleting  $e$ .

Then  $(E-e, P_{-e})$  can be represented by  $G_{-e}$ .

Moreover,  $(E-e, P_{+e})$  can be represented by  $G_{+e}$  if and only if

$e$  can be replaced in  $G$  by an undirected component without altering  $(E, P)$ .

In this case we say that  $e$  is one-way relevant. Components which are not one-way relevant are said to be two-way relevant.  $\square$

The results we are about to develop in this section will all rely on the following very simple observation:

Lemma 5.4. Let  $(E, P)$  be an SKT system, and let  $S$  be the source node. Then any component directed into  $S$  is irrelevant.  $\square$

Corollary 5.5. Let  $(E, P)$  be an SKT system, and let  $E_s \subseteq E$  be the set of components coming out from the source node. Then each component in  $E_s$  is one-way relevant.  $\square$

Corollary 5.6. Let  $(E, P)$  be an SKT system where all the components are relevant, (i.e.  $(E, P)$  is a coherent structure.), and let  $E_s \subseteq E$  be the set of components coming out from the source node. Then none of the components in  $E_s$  is included in any cycle in the network.  $\square$

Having established these basic results, we now turn to the two main theorems of this section.

Theorem 5.7. Let  $(E, P)$  be a cyclic SKT system, (i.e. the corresponding directed network is cyclic.) Then  $d(P) = 0$ .

Proof: If the system contains irrelevant components, then by Corollary 2.6,  $d(P) = 0$ . Hence, we may assume that  $(E, P)$  is a coherent structure.

Let  $n = |E|$ . The proof is by induction on  $n$ .

Obviously, there exists no cyclic, coherent SKT system having only one component. Thus, in order to initiate the induction process, we start out by simply assuming that  $n$  is as small as possible without specifying this number. That is, any SKT system having less than  $n$  components is either acyclic or noncoherent.

Let  $e$  be a component coming out from the source node. By Corollary 5.5  $e$  is one-way relevant.

Thus, by Proposition 5.4. both  $(E-e, P_{+e})$  and  $(E-e, P_{-e})$  are SKT systems (possibly degenerated) having  $(n-1)$  components.

Furthermore, by Corollary 5.6  $e$  is not included in any cycle in the network, implying that  $(E-e, P_{+e})$  and  $(E-e, P_{-e})$  are cyclic. Thus, by the assumption that  $n$  is as small as possible, we conclude that  $(E-e, P_{+e})$  and  $(E-e, P_{-e})$  are noncoherent.

However, by Corollary 2.6, this implies that:

$$d(P_{+e}) = d(P_{-e}) = 0$$

Hence, by Theorem 4.3. we get:

$$d(P) = d(P_{+e}) - d(P_{-e}) = 0 - 0 = 0.$$

Assume now that the theorem is true for every  $m < n_0$ , and let  $n = n_0 + 1$ .

We then choose  $e$ , a component coming out from the source node.

By the same arguments as above we get that both  $(E-e, P_{+e})$  and  $(E-e, P_{-e})$  are cyclic SKT systems having  $n_0$  components.

Thus, by the induction hypothesis, we must have:

$$d(P_{+e}) = d(P_{-e}) = 0$$

Hence, applying Theorem 4.3. once again, we get:

$$d(P) = d(P_{+e}) - d(P_{-e}) = 0 - 0 = 0, \text{ as stated. } \square$$

The next result concerns the case of acyclic SKT systems, (i.e. SKT systems where the corresponding directed network is acyclic.)

Before we can prove this result we need the following concepts:

The indegree of a node is the number of components directed into the node. A node with indegree zero is called a root in the network.

The following lemma is a well-known property of acyclic graphs, so we state this result without proof. (It can be proved by introducing a partial ordering on the nodes and using the existence of minimal elements of finite partial ordered sets. For more details, see Graver & Watkins (1977) Ch. IIB.)

Lemma 5.8. Any acyclic directed network has at least one root.  $\square$

We can now prove the theorem.

Theorem 5.9. Let  $(E, p)$  be an acyclic SKT system with only relevant components, and no isolated nodes. Let  $n_0$  be the number of components and  $v$  be the number of nodes in the corresponding directed network. Then we have:

$$d(P) = (-1)^{n-v+1}$$

Proof: The proof is by induction on  $n$ . It is very easy to see that the theorem is true when  $n = 1$ .

Assume now that the theorem is true for every  $m < n_0$  and let  $n = n_0 + 1$ . So, especially  $n > 2$ .

Let  $G$  be the network of the system, and denote by  $G-S$  the network obtained from  $G$  by deleting the source node,  $S$ , and all the components directed out from  $S$ .

Since  $G$  is assumed to be acyclic, obviously  $G-S$  must be acyclic as well.

Hence, by Lemma 5.8.  $G-S$  has at least one root, say node  $N$ .

Since the system has only relevant components and no isolated nodes,  $G$  must contain at least one component directed from  $S$  to  $N$ , say component  $e$ .

By Corollary 5.5.  $e$  is one-way relevant, so by Proposition 5.3., both  $(E-e, P_{+e})$  and  $(E-e, P_{-e})$  are SKT systems having  $n-1$  components. We also know that  $(E-e, P_{+e})$  contains  $v-1$  nodes while  $(E-e, P_{-e})$  contains  $v$  nodes. Finally, since  $(E, P)$  is acyclic, obviously  $(E-e, P_{+e})$  and  $(E-e, P_{-e})$  must be acyclic as well.

Let  $E_S$  be the set of components directed from  $S$  to  $N$ , (so  $e \in E_S$ ), let  $E_N$  be the set of components coming out from  $N$ , and let  $K$  be the set of terminal nodes.

If  $|E_S| > 2$ , then  $e$  is in parallel with the rest of the components in  $E_S$ . Thus,  $(E-e, P_{-e})$  cannot contain any isolated nodes. Moreover, by Corollary 3.7.,  $(E-e, P_{+e})$  is noncoherent while  $(E-e, P_{-e})$  is coherent.

Hence, by Theorem 4.3., Corollary 2.6. and the induction hypothesis, we get:

$$d(P) = 0 - d(P_{-e}) = -(-1)^{(n-1)-v+1} = (-1)^{n-v+1} \text{ as stated.}$$

If  $|E_S| = 1$ , i.e.  $E_S = \{e\}$ , we consider two possible cases:

Case 1.  $N \in K$

Since  $E_s = \{e\}$  and  $N$  is a root in  $G-S$ ,  $\{e\}$  is the only path from  $S$  to  $N$  in  $G$ . Thus, since  $N \in K$ , we must have  $e \in P$  for all  $P \in \mathcal{P}$ . Since we have assumed that  $|E| = n > 2$ , this implies that for all  $P \in \mathcal{P}$ , we have that  $e \in P$  if and only if  $P \cap (E-e) \neq \emptyset$ .

Case 2.  $N \notin K$ .

In this case obviously  $E_N$  must be nonempty since otherwise  $e$  would be irrelevant, contradicting the assumption that  $(E, \mathcal{P})$  is coherent. Applying this, it is easy to see that for all  $P \in \mathcal{P}$ , we have that  $e \in P$  if and only if  $P \cap E_N \neq \emptyset$ .

We see that in both cases we may apply Proposition 3.6. (i) and get that  $(E-e, \mathcal{P}_{+e})$  is coherent while  $(E-e, \mathcal{P}_{-e})$  is noncoherent. Moreover, since we have assumed that  $(E, \mathcal{P})$  contains no isolated nodes, obviously  $(E-e, \mathcal{P}_{+e})$  cannot contain any isolated nodes either.

Hence, we may apply Theorem 4.3., Corollary 2.6. and the induction hypothesis and get:

$$d(P) = d(P_{+e}) - 0 = (-1)^{(n-1)-(v-1)+1} = (-1)^{n-v+1} \quad \text{as stated.} \quad \square$$

As an illustration we shall now briefly outline how Theorem 5.7. and Theorem 5.9. can be used in order to calculate the reliability of SKT systems.

Let  $(E, \mathcal{P})$  be an SKT system with structure function  $\phi = \phi(\underline{x})$  where  $\underline{x} = (x_1, \dots, x_n)$  is the component state vector. Furthermore, let  $\delta$  be the signed domination function of  $(E, \mathcal{P})$ , and let  $\underline{X}$  be the random vector corresponding to  $\underline{x}$ .

Since, by (2.2)  $\phi$  is given by:

$$\phi = \phi(\underline{x}) = \sum_{B \subseteq E} \delta(B) \prod_{i \in B} x_i$$

we get that the reliability of the system can be written as:

$$\begin{aligned} h_\phi &= E\phi(\underline{X}) = \sum_{B \subseteq E} \delta(B) P\left(\prod_{i \in B} X_i = 1\right) \\ &= \sum_{B \subseteq E} d(P_{-(E-B)}) P\left(\prod_{i \in B} X_i = 1\right), \text{ applying Corollary 2.11.} \end{aligned}$$

Now, in general we could proceed by applying Proposition 2.5. in order to determine  $h_\phi$ . This method is equivalent to the well-known Inclusion-Exclusion method, (See Barlow and Proschan (1981)) and we observe that in order to proceed like this, we need to know the family  $P$ . Determining  $P$  is in general an exponential time problem. Moreover, when  $P$  is found, there are still great computational tasks left before arriving at  $h_\phi$ . Hence, this procedure is of limited value when the system under consideration is large. However, since  $(E, P)$  is an SKT system, obviously  $(B, P_{-(E-B)})$  is an SKT system as well, for all  $B \subseteq E$ . Hence,  $d(P_{-(E-B)})$  may be computed by applying Theorem 5.7. and Theorem 5.9. Thus  $h_\phi$  may be rewritten as:

$$h_\phi = \sum_{B \in \mathcal{B}} (-1)^{|B| - v(B) + 1} P\left(\prod_{i \in B} X_i = 1\right),$$

where we define:

$$\mathcal{B} = \{B \subseteq E : (B, P_{-(E-B)}) \text{ is acyclic}\}$$

$v(B)$  = The number of nodes in  $(B, P_{-(E-B)})$

Satyanarayana and Prabhakar (1978) and Satyanarayana (1982) provides an algorithm which efficiently generates the family  $\mathcal{B}$  and hence  $h_\phi$  may be computed.

We close this section by providing an easy generalization of

Theorem 5.7. and Theorem 5.9. to network communication systems with more than one source node. Such systems will be called multi-source-to-K-terminal systems, (MSKT systems). We still assume that the underlying network is directed and we say that an MSKT system is functioning if every terminal can get communication from at least one source node.

The following theorem generalizes Theorem 5.7. and Theorem 5.9. to MSKT systems.

Theorem 5.10. Let  $(E, P)$  be a coherent MSKT system with  $n$  components and  $v$  nodes. Moreover assume that  $s$  of the nodes are source nodes.

Then we have:

$$d(P) = \begin{cases} (-1)^{n-v+s} & \text{if the network is acyclic} \\ 0 & \text{if the network is cyclic.} \end{cases}$$

Proof: Let  $G$  be the underlying directed network, let  $S_1, \dots, S_s$  be the source nodes, and  $T_1, \dots, T_k$  the terminal nodes.

We then add a new artificial source node  $S$ , and artificial components  $a_1, \dots, a_s$  from  $S$  to  $S_1, \dots, S_s$  respectively. (See Figure 5.4.).

Let  $\tilde{G}$  denote the extended network, and  $A = \{a_1, \dots, a_s\}$

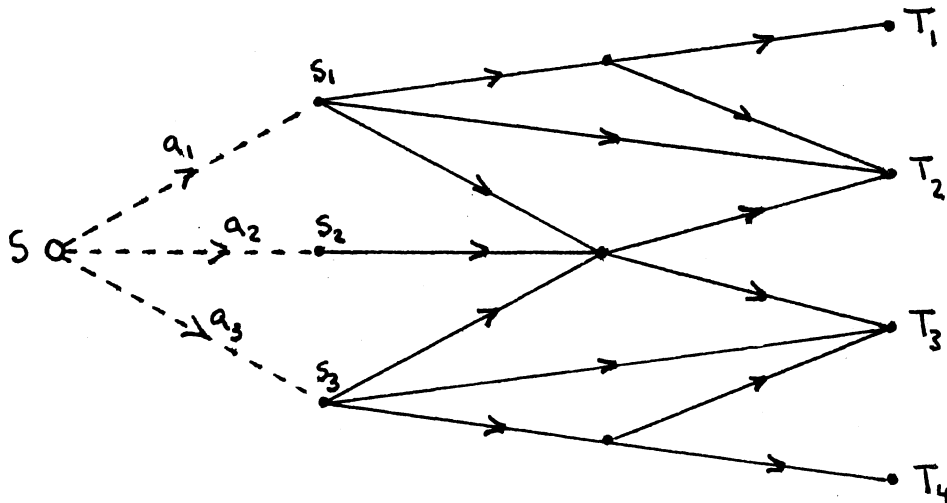


Figure 5.4.



We observe that  $\tilde{G}$  is acyclic if and only if  $G$  is acyclic.

Now, let  $(A, p^*)$  be the SKT system which is functioning if  $S$  can send communication to  $S_1, \dots, S_s$ , and let  $(AUE, \tilde{p})$  denote the series connection of  $(A, p^*)$  and  $(E, p)$ .

Thus,  $(AUE, \tilde{p})$  is functioning if and only if  $S$  can send communication to  $S_1, \dots, S_s$  and  $T_1, \dots, T_k$ .

Hence,  $(AUE, \tilde{p})$  is an SKT system with  $\tilde{G}$  as the underlying network.

So, since  $\tilde{G}$  has  $n + s$  components and  $v+1$  nodes, by Theorem 5.7 and Theorem 5.9 we get:

$$d(\tilde{p}) = \begin{cases} (-1)^{(n+1)-(v+1)+1} & \text{if } G \text{ is acyclic} \\ 0 & \text{if } G \text{ is cyclic.} \end{cases} = (-1)^{n-v+s}$$

Moreover, by Theorem 5.9.  $d(p^*) = (-1)^{s-(s+1)+1} = 1$ . Hence, by Theorem 4.4. (i) we get:

$$d(p) = \frac{d(\tilde{p})}{d(p^*)} = \begin{cases} (-1)^{n-v+s} & \text{if } G \text{ acyclic} \\ 0 & \text{if } G \text{ is cyclic. } \square \end{cases}$$

For the time being the algorithm of Satyanarayana, and Prabhakar (1978) has not been generalized to MSKT systems. However, by using methods similar to those in Satyanarayana (1982), it is probably easy to carry out this extension.

## 6. The domination theorem.

The domination theorem which will be proved in this section, plays an important part in the study of the so-called "factoring algorithm" for reliability computations.

However, it turns out that this theorem is not true for all coherent structures, Until now it has only been proved to be true for K-terminal undirected network systems and k-out-of-n systems.

(See Satyanarayana and Chang, (1983) and Barlow (1982).)

We shall present a generalized version of this result which is valid under a certain regularity condition. This is done by establishing a relation between coherent structures and matroids.

Especially, we shall prove that the regularity condition is satisfied for both K-terminal undirected network systems and k-out-of-n systems. Hence, our result is in fact a generalization of the previously known results.

Before we can prove the domination theorem, we shall review some basic results from matroid theory. We start by the definition of a matroid.

Definition 6.1. A matroid is a clutter  $(F, M)$  satisfying:

(6.2) if  $A, B \in M$ ,  $A \neq B$ , and  $j \in A \cap B$ , then there exist a set  $C \in M$  such that  $C \subseteq (A \cup B) - j$ .

Example 6.3. Let  $G$  be an undirected graph with edge set  $F$ , and let  $M$  be the collection of minimal circuits in the graph. Then it is easy to see that  $(F, M)$  is a matroid, (called the circuit matroid of the graph  $G$ .)

Especially, let  $G$  be the graph shown in Figure 6.1. with edge set  $F = \{1, 2, 3, 4, 5\}$ .

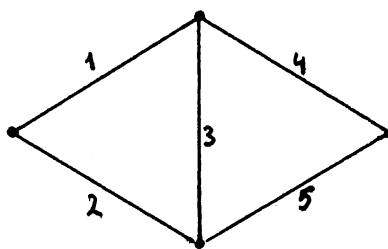


Figure 6.1.

In this case we get:  $M = \{A, B, C\}$  where  $A = \{1, 2, 3\}$ ,  $B = \{3, 4, 5\}$  and  $C = \{1, 2, 4, 5\}$ .

We observe that  $C = \{1, 2, 4, 5\} \subseteq (A \cup B) - \{3\}$ , and so on.

Example 6.4. Let  $H$  be a matrix (of numbers from a given field). Furthermore, let  $F$  be the set of columns, and let  $M$  be the family of minimal linearly dependent subsets of  $F$ . Then  $(F, M)$  can be shown to be a matroid.

As illustrated in Example 6.4., the matroid  $(F, M)$  just describes the "dependency structure" of the matrix  $H$ . Thus, the concept of matroids can be understood as an abstraction of the dependency concept described by a set of axioms. It turns out that a lot of the well-known concepts of linear algebra, such as rank, bases, hyperplanes and others, can be generalized to matroids. In this paper, we shall especially need the concept of rank which is defined as follows:

Definition 6.5. Let  $(F, M)$  be a matroid. Then we define:

$$\rho(M) = \text{The rank of } M = \max \{|A| : M \not\subseteq A \text{ for all } M \in M\}$$

Example 6.6. Let  $(F, M)$  be a matroid corresponding to a matrix  $H$ . Then  $\rho(M)$  is just the rank of the matrix  $H$ .

Example 6.7. Let  $(F, M)$  be the circuit matroid of a connected graph with  $v$  nodes.

Then a maximal set of edges not containing a circuit is a spanning tree. (See Graver and Watkins (1977)).

Hence, since all spanning trees of the graph contains  $v-1$  edges, we conclude that  $\rho(M) = v-1$ .

We shall now introduce the matroid concept corresponding to coherency.

Definition 6.8. Let  $(F, M)$  be a matroid. We say that  $(F, M)$  is connected if for every pair of distinct elements  $i$  and  $j$  of  $F$ , there is a set  $M \in M$  containing  $i$  and  $j$ .

Now, just by considering this definition, it is not obvious how connectedness is related to coherency. The next proposition, however, provides an explanation for this.

Let us first introduce some more notation.

Assume that  $M$  is a family of subsets of a given set  $F$ , and let  $e \in F$ .

Then we denote by  $M_e$  the subfamily of  $M$  given by:

$$M_e = \{M \in M : e \in M\}.$$

Proposition 6.9. Let  $(F, M)$  be a matroid, and let  $e$  be a fixed element of  $F$ . Then  $(F, M)$  is connected if and only if:

$$\bigcup_{M \in M_e} M = F$$

Proof: See Welsh (1976).  $\square$

We now easily get the following corollary describing the relation between coherency and connectedness.

Corollary 6.10. Let  $(F, M)$  be a matroid, and let  $e$  be a fixed element of  $F$ . Then the following are equivalent:

- (i)  $(F, M)$  is connected.
- (ii)  $(F, M_e)$  is a coherent structure.
- (iii)  $(F-e, (M_e)_{+e})$  is a coherent structure.

Proof: (i)  $\Leftrightarrow$  (ii) follows directly from Proposition 6.9. and Definition 2.2.

(i)  $\Leftrightarrow$  (iii) follows since  $(M_e)_{+e} = \{M-e: M \in M_e\}$ .

(This is true since  $e \in M$  for all  $M \in M_e$ , and hence all sets of form  $M-e$  ( $M \in M_e$ ) will be incomparable).

Hence, 
$$\bigcup_{M \in (M_e)_{+e}} M = \bigcup_{M \in M_e} (M-e) = \left[ \bigcup_{M \in M_e} M \right] - e, \text{ so } \bigcup_{M \in (M_e)_{+e}} M = F-e \text{ if}$$

and only if 
$$\bigcup_{M \in M_e} M = F. \quad \square$$

The last two result we need from the theory of matroids is the following:

Proposition 6.11. Let  $(F, M)$  be a matroid. Then every minor of  $(F, M)$  is a matroid as well. That is, the class of matroids is closed under minor operations.

Proof: See Welsh (1976).  $\square$

Proposition 6.12. Let  $(F, M)$  be a matroid, and let  $e \in F$ . Then we have:

- (i)  $\rho(M_{+e}) = \rho(M) - 1$
- (ii)  $\rho(M_{-e}) = \rho(M)$  if  $(F, M)$  is connected.

Proof: See Welsh (1976)  $\square$

Now, in order to apply matroid theory to clutters and coherent structures, we must establish some "natural" relationship between these concepts. This problem was studied in Lehman, A. (1964), and in the spirit of Lehman's approach, we now introduce the following definition, which appears to solve our problems.

Definition 6.13. We say that a matroid  $(F, M)$  corresponds to the clutter  $(E, P)$  if:

$$(E, P) = (F-x, (M_x)_{+x})$$

where  $x$  is some fixed artificial component. Furthermore, we say that a clutter is regular if it has a corresponding matroid.

We observe that if  $(F, M)$  corresponds to the clutter  $(E, P)$  where  $E = F-x$ , then  $P$  is obtained from  $M$  by selecting the sets in  $M$  containing  $x$ , (i.e. the subcollection  $M_x$ ), and then deleting  $x$  from each set in  $M_x$ .

Similarly we see that if  $(E, P)$  is a clutter and  $\tilde{P} = \{PUx: P \in P\}$ , then  $(E, P)$  is regular if  $\tilde{P}$  can be extended to a family  $M$  such that  $(EUx, M)$  is a matroid and  $\tilde{P} = M_x$ .

Finally, we get by Corollary 6.10., that a regular clutter is a coherent structure if and only if the corresponding matroid is connected.

As indicated in Definition 6.13. it is not true in general that a clutter has a corresponding matroid. That is, there exists irregular clutters as well as regular ones. Especially, it is possible to show that every MSKT system not being an s-p-structure is irregular (Corollary 6.20).

The following example indicates that at least 2-terminal undirected network systems are regular.

Example 6.14. Consider the 2-terminal undirected network system,  $G$ , shown in Figure 6.2., which functions if node  $S$  and node  $T$  can communicate through the network.

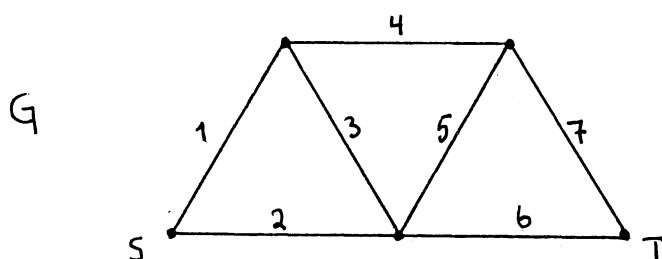


Figure 6.2.

Let  $E = \{1, 2, \dots, 7\}$  be the set of components, and

$\mathcal{P}$  = The family of minimal paths from  $S$  to  $T$ .

In order to find the corresponding matroid of  $(E, \mathcal{P})$ , we introduce  $\tilde{\mathcal{P}} = \{PUx : P \in \mathcal{P}\}$  where  $x \notin E$  is an artificial component. Now, our problem is to extend  $\tilde{\mathcal{P}}$  to a new family  $\mathcal{M}$  such that  $(E \cup x, \mathcal{M})$  is a matroid. This is done as follows:

Assume that the artificial component  $x$  is added to the graph between  $S$  and  $T$ . (See Figure 6.3.)

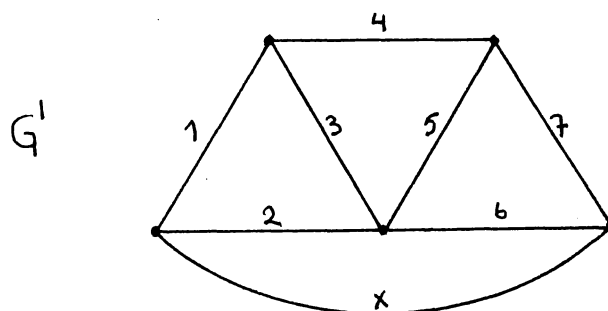


Figure 6.3.

We now define:

$M$  = The family of minimal circuits in the graph  $G'$ .

As in Example 6.3. it is easy to see that  $(E \cup x, M)$  is a matroid. Moreover, since obviously  $\tilde{P}$  is the family of minimal circuits in  $G'$  containing  $x$ , we get that  $\tilde{P} = M_x$ .

Thus, we conclude that  $(E, P)$  is regular.  $\square$

An important question to answer before we can prove the domination theorem is the following: Is a corresponding matroid to a regular clutter unique, or is it possible that a clutter may have several corresponding matroids. The following proposition provides an answer to this question.

Proposition 6.15. Let  $(E, P)$  be a regular clutter, and let  $x$  be an artificial component ( $x \notin E$ ). Then a corresponding matroid is uniquely determined by  $(E, P)$  and  $x$  if and only if  $(E, P)$  is coherent.

Proof: See Lehman, A. (1964).  $\square$

One should notice that the uniqueness of the corresponding matroid depends on the coherency property. That is, a noncoherent structure may in general have more than one corresponding matroid. It is, however, possible to prove the following result:

Proposition 6.16. Let  $(E, P)$  be a regular clutter, and let  $(F, M)$  be a corresponding matroid.

Then for all  $e \in E$  we have

(i)  $(F - e, M_{+e})$  is a corresponding matroid of  $(E - e, P_{+e})$ .



(ii)  $(F-e, M_{-e})$  is a corresponding matroid of  $(E-e, M_{-e})$ .

Proof: (i) Assume that  $F = E \cup x$  where  $x \notin E$ , and let  $e \in E$ . Since  $(F, M)$  corresponds to  $(E, P)$ , by Definition 6.13, we must have:

$$M_x = \{ P \cup x : P \in P \}$$

Furthermore, by Definition 2.7., we have:

$P_{+e}$  = The family of minimal sets of the form  $P - e$ , where  $P \in P$ .

Applying this, we get:

$$\begin{aligned} \{P \cup x : P \in P_{+e}\} &= \text{The family of minimal sets of the form } (P-e) \cup x, \\ &\quad \text{where } P \in P. \\ &= \text{The family of minimal sets of the form } (P \cup x) - e, \\ &\quad \text{where } P \in P. \\ &= \text{The family of minimal sets of the form } M - e \\ &\quad \text{where } M \in M_x \\ &= \text{The family of minimal sets of the form } M - e \\ &\quad \text{where } M \in M \text{ and such that } x \in (M-e). \\ &= (M_{+e})_x \quad (\text{By Definition 2.7.}) \end{aligned}$$

Hence, by Definition 6.13.,  $(F-e, M_{+e})$  corresponds to  $(E-e, P_{+e})$ .

(ii) Is proved similarly.  $\square$

Now, we turn to the domination theorem, and start by proving the following Lemma:

Lemma 6.17. Let  $(E, P)$  be a regular clutter, and let  $(F, M)$  be a corresponding matroid. Then we have:

$$d(P) = (-1)^{|E| - \rho(M)} D(P)$$

Proof: We observe that if  $(E, P)$  is noncoherent, then, by Corollary 2.6.,  $d(P) = D(P) = 0$ . Hence, the Lemma is trivially true in this case.

Assume then that  $(E, P)$  is coherent. In this case  $(F, M)$  is uniquely determined by  $(E, P)$  (by Proposition 6.15.) so  $\rho(M)$  is a unique number.

The proof is now by induction on  $|E|$ .

It is easy to verify that the lemma is true when  $|E| = 1$ .

Assume then that the lemma is true when  $|E| < n$  and assume that  $|E| = n$ .

We now choose  $e \in E$ .

By Proposition 6.16.  $(E-e, P_{+e})$  and  $(E-e, P_{-e})$  are both regular, with corresponding matroids:  $(F-e, M_{+e})$  and  $(F-e, M_{-e})$  respectively.

Furthermore, by Corollary 6.10.  $(F, M)$  is connected since  $(E, P)$  is coherent.

Hence, by Proposition 6.12.,  $\rho(M_{+e}) = \rho(M) - 1$  and  $\rho(M_{-e}) = \rho(M)$ .

We now apply Theorem 4.3. and the induction hypothesis and get:

$$\begin{aligned}
 d(P) &= d(P_{+e}) - d(P_{-e}) \\
 &= (-1)^{|E-e|-\rho(M_{+e})} D(P_{+e}) - (-1)^{|E-e|-\rho(M_{-e})} D(P_{-e}) \\
 &= (-1)^{(|E|-1)-(\rho(M)-1)} D(P_{+e}) - (-1)^{(|E|-1)-\rho(M)} D(P_{-e}) \\
 &= (-1)^{|E|-\rho(M)} (D(P_{+e}) + D(P_{-e})).
 \end{aligned}$$

Hence, by taking the absolute value on both sides we get:

$$|d(P)| = D(P) = |D(P_{+e}) + D(P_{-e})| = D(P_{+e}) + D(P_{-e})$$

Thus, by substituting  $(D(P_{+e}) + D(P_{-e}))$  by  $D(P)$  in the expression above, we get:

$$d(P) = (-1)^{|E|-\rho(M)} D(P) \quad \text{as stated.} \quad \square$$

As a direct consequence we now get the main result of this section:

Theorem 6.18. (The domination theorem)

Let  $(E, P)$  be a regular clutter where  $|E| > 2$  and let  $e \in E$  be a relevant component. Then we have:

$$D(P) = D(P_{+e}) + D(P_{-e}).$$

Proof: If  $(E, P)$  is noncoherent, it is easy to see that  $(E-e, P_{+e})$  and  $(E-e, P_{-e})$  are noncoherent as well since  $e$  is relevant.

Hence, by Corollary 2.6.,  $D(P) = D(P_{+e}) = D(P_{-e}) = 0$ , so the theorem is trivially true in this case.

If  $(E, P)$  is coherent, the theorem follows by the proof of Lemma 6.17.  $\square$

Corollary 6.19. Let  $(E, P)$  be a regular coherent structure. Then  $D(P) > 0$ . Moreover,  $D(P) = 1$  if and only if  $(E, P)$  is an s-p-structure.

Proof: The proof is by induction on  $|E|$ .

If  $|E| = 1$ , then obviously  $(E, P)$  is an s-p-structure, and  $D(P) = 1$ , so the result is trivially true.

Assume now that the result is true if  $|E| < n$ , and let  $|E| = n$ .

If  $(E, P)$  is an s-p-structure, then by Corollary 4.9.  $D(P) = 1$ .

Assume conversely that  $(E, P)$  is not an s-p-structure, and let  $(E^r, P^r)$  be the clutter obtained from  $(E, P)$  by performing all possible s-p-reductions.

Thus,  $(E^r, P^r)$  is obviously a minor of  $(E, P)$  and hence, by Proposition 6.16.,  $(E^r, P^r)$  is regular as well.

Moreover, by Corollary 3.7.,  $(E^r, P^r)$  is coherent.

Finally, since we have assumed that  $(E, P)$  is not an s-p-structure, then obviously  $(E^r, P^r)$  cannot be an s-p-structure.

If  $E^r \subsetneq E$ , i.e.  $|E^r| < n$ , we may apply the induction hypothesis and Corollary 4.8. and get:

$$D(P) = D(P^r) > 0.$$

Moreover, since  $(E^r, P^r)$  is not an s-p-structure, by the induction hypothesis we get:

$$D(P) = D(P^r) \neq 1, \text{ i.e. } D(P) > 1, \text{ as stated.}$$

If  $E^r = E$ , then  $(E, P)$  must be complex.

Hence, by Theorem 3.13. there exists a component  $j \in E$  such that both  $(E-j, P_{+j})$  and  $(E-j, P_{-j})$  are coherent.

Moreover,  $(E-j, P_{+j})$  and  $(E-j, P_{-j})$  are regular by Proposition 6.16., and  $|E-j| = n-1 < n$ .

Thus, by the induction hypothesis we get:

$$D(P_{+j}) > 0 \text{ and } D(P_{-j}) > 0.$$

Hence, by Theorem 6.18. we get:

$$D(P) = D(P_{+j}) + D(P_{-j}) > 1 + 1 = 2 > 1, \text{ as stated. } \square$$

Corollary 6.20. Let  $(E, P)$  be an MSKT system. Then  $(E, P)$  is regular if and only if  $(E, P)$  is an s-p-structure.

Proof: It is not difficult to prove that every s-p-structure is regular.

Assume conversely that  $(E, P)$  is regular.

Hence, by Corollary 6.19.,  $D(P) \geq 1$ . Moreover, since  $(E, P)$  is an MSKT system, by Theorem 5.10., we get that  $D(P) \leq 1$ .

Thus, we must have  $D(P) = 1$ , which by Corollary 6.19. implies that  $(E, P)$  is an s-p-structure, as stated.  $\square$

In the next section we shall see how the domination theorem can be used to deduce properties of the factoring algorithm. Since the domination theorem is restricted to regular clutters, it is important to study this class further.

Lehman, A. (1964) provides an algorithm for checking whether a clutter is regular or not. However, this algorithm assumes that the family of minimal path sets are given. In order to find simpler conditions, we shall present some general results on regularity.

Theorem 6.21. The dual of a regular clutter is regular. That is, the class of regular clutters is closed under duality.

Proof: This result is a direct consequence of Lemma 18 and 19 in Lehman, A. (1964), so we omit the proof.  $\square$

Theorem 6.22. If a clutter can be decomposed into modules, the clutter is regular if and only if each module is regular and the organizing structure is regular.

Proof: The proof is straightforward but rather technical and thus omitted here.  $\square$

Theorem 6.23. Any  $k$ -out-of- $n$  system is regular.

Proof: Let  $(E, P)$  be a  $k$ -out-of- $n$  system. That is,  $|E|=n$  and  $P = \{P \subseteq E : |P| = k\}$ . We then introduce:

$$\tilde{P} = \{P \cup x : P \in P\} \text{ where } x \notin E \text{ is an artificial component.}$$

We shall extend  $\tilde{P}$  to a family  $M$  such that  $(E \cup x, M)$  is a matroid and  $\tilde{P} = M_x$ .

This is done by defining  $M$  as follows:

$$M = \{M \subseteq E \cup x : |M| = k+1\}.$$

Since, all sets in  $M$  have the same cardinality, they are obviously incomparable.

Furthermore, if  $M \in M$  and  $x \in M$ , then  $(M-x) \subseteq E$  and  $|M-x| = k$ , i.e.  $(M-x) \in P$ . So  $M_x \subseteq \tilde{P}$ .

Hence, since obviously  $\tilde{P} \subseteq M_x$ , we conclude that  $\tilde{P} = M_x$ .

It remains to prove that if  $A, B \in M$ ,  $A \neq B$  and  $j \in A \cap B$ , then there exists  $C \in M$  such that  $C \subseteq (A \cup B) - j$ .

Thus, we choose  $A, B \in M$ ,  $A \neq B$  such that  $j \in A \cap B$ .

Since  $A \neq B$ , obviously  $A$  and  $B$  are incomparable. Hence, there exists  $e \in E \cup x$  such that  $e \in A$  and  $e \notin B$ . Thus, since  $|A| = |B| = k+1$ , we must have:

$$|A \cup B| > k+2, \text{ i.e. } |(A \cup B) - j| > k+1.$$

Hence, there exists  $C \subseteq (A \cup B) - j$  such that  $|C| = k+1$ , i.e.  $C \in \mathcal{P}$ , so we conclude that  $(E, \mathcal{P})$  is a matroid. Thus, we have proved that  $(E, \mathcal{P})$  is regular as stated.  $\square$

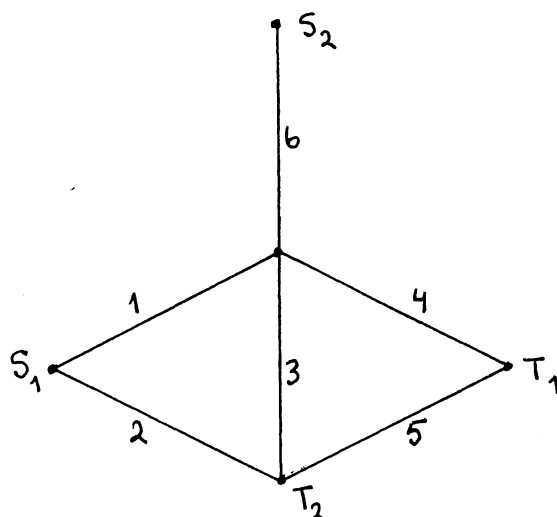
The next result we shall present, provides another large class of regular clutters. In Huseby (1983) it is shown that this class and the class of  $k$ -out-of- $n$  systems only have series and parallel systems in common.

Theorem 6.24. Let  $(E, \mathcal{P})$  be a clutter and let  $\mathcal{C}$  be the family of minimal cut sets. If  $|P \cap C|$  is odd for all  $P \in \mathcal{P}$  and  $C \in \mathcal{C}$ , then  $(E, \mathcal{P})$  is regular.

Proof: See Theorem 47 in Lehman, A. (1964).  $\square$

In Huseby (1983) a clutter satisfying the condition given in Theorem 6.24, is called a linear clutter. It can be shown that if  $(E, \mathcal{P})$  is a linear clutter, then the corresponding matroid can be represented by a matrix over the field of order 2,  $GF(2)$ . (See Lehman, A. (1964).) In Huseby (1983) it is shown that every 2-terminal undirected network system is linear. Hence, we obtain especially that every 2-terminal undirected network system is regular, (as indicated in Example 6.14). However, the class of linear clutters is in fact much larger. Thus, for a general linear clutter it is suggested to use the matrix over  $GF(2)$  as a representation of the system. The following example provides a linear system not being a 2-terminal undirected network system.

Example 6.25. Consider the undirected network system  $(E, \mathcal{P})$  shown in Figure 6.4.



Figur 6.4.

The system is said to be functioning if  $S_1$  and  $T_1$  can communicate with each other or  $S_2$  and  $T_2$  can communicate with each other.

In this case we get that  $E = \{1, 2, \dots, 6\}$  and  $P = \{P_1, \dots, P_7\}$  where:

$$P_1 = \{1, 4\}, P_2 = \{2, 5\}, P_3 = \{1, 3, 5\}, P_4 = \{2, 3, 4\},$$

$$P_5 = \{1, 2, 6\}, P_6 = \{3, 6\} \text{ and } P_7 = \{4, 5, 6\}.$$

The family of minimal cut sets of  $(E, P)$ ,  $C$  is given by:

$C = \{C_1, \dots, C_4\}$  where:

$$C_1 = \{1, 2, 6\}, C_2 = \{1, 3, 5\}, C_3 = \{2, 3, 4\} \text{ and } C_4 = \{4, 5, 6\}.$$

Hence, it is easy to check that  $|P \cap C|$  is odd for all  $P \in P$  and  $C \in C$ , and thus  $(E, P)$  is linear and hence regular by Theorem 6.24.  $\square$

We close this section by proving that every  $K$ -terminal undirected network system is regular. However, it turns out that this result is a special case of a much more general result, so we start by presenting the general case. We shall need the following lemma:



Lemma 6.26. Let  $(F, M)$  be a matroid, and assume that  $A$  and  $B$  are two distinct sets in  $M$  such that  $j \in A \cap B$ . Then for any  $k \in A - B$  there exists a set  $C \in M$  such that  $k \in C$  and  $C \subseteq (A \cup B) - j$ .

Proof: See Theorem 2., p. 24 in Welsh (1976).  $\square$

We also need the following notation:

Let  $(E, P_1), \dots, (E, P_m)$  be clutters. Then  $P = P_1 \cdot P_2 \cdots P_m =$  The family of minimal sets  $P$  of the form  $P = S_1 \cup \dots \cup S_m$  where  $S_r \in P_r$ ,  $r = 1, \dots, m$ . We observe that  $(E, P)$  is the series connection of  $(E, P_1), \dots, (E, P_m)$ . That is,  $(E, P)$  is functioning if and only if  $(E, P_1), \dots, (E, P_m)$  are functioning. Moreover, as usually, if  $P$  is a family of sets, then  $\tilde{P}$  denotes the family defined by  $\tilde{P} = \{P \cup x : P \in P\}$ , where  $x$  is some fixed artificial component.

Theorem 6.27. Let  $(E, P_1), \dots, (E, P_m)$  be regular clutters and let  $(E \cup x, N_1), \dots, (E \cup x, N_m)$  be the corresponding matroids respectively. Assume that  $N_j = \tilde{P}_j \cup M$ , where  $M$  is the family of sets  $M \in N_j$  such that  $x \notin M$ ,  $j = 1, \dots, m$  (i.e.  $M$  is independent of  $j$ ). Then  $(E, P) = (E, P_1 \cdots P_m)$  is regular, and  $(E \cup x, \tilde{P} \cup M)$  is the corresponding matroid. ( $\tilde{P} = \{P \cup x : P \in P\} = \{P \cup x : P \in P_1 \cdots P_m\}$ .)

Proof: It is sufficient to prove that  $(E \cup x, \tilde{P} \cup M)$  is a matroid since this implies that  $(E, P)$  is regular. Hence, we must prove:

- (i)  $\tilde{P} \cup M$  is a family of incomparable subsets of  $E \cup x$ .
- (ii) If  $A, B \in (\tilde{P} \cup M)$ ,  $A \neq B$  and  $j \in A \cap B$ , then there exists a set  $C \in (\tilde{P} \cup M)$  such that  $C \subseteq (A \cup B) - j$ .

We start by proving (i).

Obviously,  $\tilde{P}$  and  $M$  are both families of incomparable subsets of  $E \cup x$ .

Moreover, if  $P \in \tilde{\mathcal{P}}$  and  $M \in \mathcal{M}$ , then we cannot have  $P \subseteq M$  since

$(\tilde{\mathcal{P}}_1 \cup M), \dots, (\tilde{\mathcal{P}}_m \cup M)$  are families of incomparable subsets of  $E \cup x$ , and  $P \in \tilde{\mathcal{P}}$  implies  $P$  being a union of sets in  $\tilde{\mathcal{P}}_1, \dots, \tilde{\mathcal{P}}_m$ .

So, it remains only to prove that if  $P \in \tilde{\mathcal{P}}$  and  $M \in \mathcal{M}$ , then  $M \not\subseteq P$ .

Assume conversely that  $P \in \tilde{\mathcal{P}}$ ,  $M \in \mathcal{M}$  and  $M \subseteq P$ , and let

$P = S_1 \cup \dots \cup S_m$  where  $S_r \in \tilde{\mathcal{P}}_r$ ,  $r = 1, \dots, m$ .

Now, we choose an arbitrary element  $j \in M$ , and let  $J = \{r : j \in S_r, 1 \leq r \leq m\}$ .

Since  $M \subseteq P = S_1 \cup \dots \cup S_m$ ,  $J$  must be nonempty, and we have  $j \in M \cap S_r$  for all  $r \in J$ . Furthermore, since  $S_r \in \tilde{\mathcal{P}}_r$ ,  $r = 1, \dots, m$  and  $M \in \mathcal{M}$ , we especially have that  $x \in S_r - M$  for all  $r \in J$ . (In fact, this is true for all  $r$ .)

Hence, since  $(E \cup x, \tilde{\mathcal{P}}_r \cup M)$  is a matroid for all  $r \in J$ , by Lemma 6.26 there exists a set  $S'_r \in \tilde{\mathcal{P}}_r \cup M$  such that  $x \in S'_r$ , i.e.  $S'_r \in \tilde{\mathcal{P}}_r$ , and  $S'_r \subseteq (S_r \cup M) - j$  for all  $r \in J$ .

However, this implies that:

$$\begin{aligned} P' &= \left[ \bigcup_{r \in J} S'_r \right] \cup \left[ \bigcup_{r \notin J} S_r \right] \subseteq \left[ \bigcup_{r \in J} (S_r \cup M) - j \right] \cup \left[ \bigcup_{r \notin J} S_r \right] \\ &= \left[ \bigcup_{r=1}^m S_r \right] - j \subseteq \left[ \bigcup_{r=1}^m S_r \right] = P \end{aligned}$$

We have assumed that  $P \in \tilde{\mathcal{P}}$ . That is,  $P$  is a minimal union of sets  $S_r \in \tilde{\mathcal{P}}_r$ . Thus, we have arrived at a contradiction.

Hence, we conclude that (i) is true.

We now turn to the proof of (ii). Assume first that  $A, B \in \mathcal{M}$ , and  $j \in A \cap B$ . Since  $(E \cup x, \tilde{\mathcal{P}}_r \cup M)$  ( $1 \leq r \leq m$ ) is a matroid, by Proposition 6.11,  $((E \cup x) - x, (\tilde{\mathcal{P}}_r \cup M) - x) = (E, M)$  is a matroid. Hence, there exists a set  $C \in \mathcal{M}$  such that  $C \subseteq (A \cup B) - j$ , so we conclude that (ii) is true.

Next, assume that  $A \in \tilde{\mathcal{P}}$ ,  $B \in M$  and  $j \in A \cap B$ , and let  $A = S_1 \cup \dots \cup S_m$  where  $S_r \in \tilde{\mathcal{P}}_r$ ,  $r = 1, \dots, m$ . Furthermore, let  $J = \{r: j \in S_r, 1 \leq r \leq m\}$ .

We observe that we have  $j \in S_r \cap B$  and  $x \in S_r - B$  for all  $r \in J$ .

Hence, since  $(Eux, \tilde{\mathcal{P}}_r \cup M)$  is a matroid for all  $r \in J$ , by Lemma 6.26, there exists a set  $S'_r \in \tilde{\mathcal{P}}_r \cup M$  such that  $x \in S'_r$ , i.e.  $S'_r \in \tilde{\mathcal{P}}_r$ , and  $S'_r \subseteq (S_r \cup B) - j$  for all  $r \in J$ .

Now, let  $D = \left[ \bigcup_{r \in J} S'_r \right] \cup \left[ \bigcup_{r \notin J} S_r \right]$ . Thus, since  $\tilde{\mathcal{P}}$  is the family of minimal such unions, there must exist a set  $C \in \tilde{\mathcal{P}}$  such that

$C \subseteq D$ . Moreover, we get that:  $D \subseteq \left[ \bigcup_{r \in J} (S_r \cup B) - j \right] \cup \left[ \bigcup_{r \notin J} S_r \right] =$

$\left[ \left( \bigcup_{r=1}^m S_r \right) \cup B \right] - j = (A \cup B) - j$ . So, we conclude that (ii) is true.

Finally, assume that  $A, B \in \tilde{\mathcal{P}}$ , and  $j \in A \cap B$ , and let  $A = S_1 \cup \dots \cup S_m$  and  $B = T_1 \cup \dots \cup T_m$ , where  $S_r, T_r \in \tilde{\mathcal{P}}_r$ ,  $r = 1, \dots, m$ .

Since we assume  $A \neq B$ , there must exist at least one  $s$ ,  $1 \leq s \leq m$ , such that  $S_s \neq T_s$ . Let  $J = \{r: j \in T_r, 1 \leq r \leq m\}$ . We now consider three possible cases:

Case 1:  $j \notin (S_s \cup T_s)$ .

Since  $S_s, T_s \in \tilde{\mathcal{P}}_s$ , we know that  $x \in S_s \cap T_s$ . Thus, since  $(Eux, \tilde{\mathcal{P}}_s \cup M)$  is a matroid, and  $S_s \neq T_s$ , there exists  $C \in (\tilde{\mathcal{P}}_s \cup M)$  such that  $C \subseteq (S_s \cup T_s) - x$ . So we have that  $C \in M$ . Hence, especially  $C \in (\tilde{\mathcal{P}} \cup M)$ . Moreover, since  $j \notin (S_s \cup T_s)$ , we get:

$$C \subseteq (S_s \cup T_s) - x \subseteq (S_s \cup T_s) = (S_s \cup T_s) - j \subseteq \bigcup_{r=1}^m [(S_r \cup T_r) - j] = (A \cup B) - j.$$

Hence, we conclude that (ii) is true in this case.

Case 2:  $j \in S_s - T_s$  (or  $j \in T_s - S_s$ ).

Since  $S_s, T_s \in \tilde{\mathcal{P}}_s$ , we know that  $x \in S_s \cap T_s$ . Thus, since  $(Eux, \tilde{\mathcal{P}}_s \cup M)$  is a matroid, and  $S_s \neq T_s$ , by Lemma 6.26 there exists a set  $M \in (\tilde{\mathcal{P}}_s \cup M)$  such that  $j \in M$  and  $M \subseteq (S_s \cup T_s) - x$ , i.e.  $M \in M$ .

Furthermore, since  $(EUX, \tilde{p}_r UM)$  is a matroid and since  $x \in T_r - M$  for all  $r \in J$ , by Lemma 6.26 there exists a set  $T'_r \in (\tilde{p}_r UM)$  such that  $x \in T'_r$ , i.e.  $T'_r \in \tilde{p}_r$  and  $T'_r \subseteq (T_r UM) - j$ , for all  $r \in J$ . Now, let  $D = [\bigcup_{r \in J} T'_r] \cup [\bigcup_{r \notin J} T_r]$ . Since  $\tilde{p}$  is the family of minimal such unions, there must exist a set  $C \in \tilde{p}$  such that  $C \subseteq D$ . Moreover, we get that:  $D \subseteq [\bigcup_{r \in J} (T_r UM) - j] \cup [\bigcup_{r \notin J} T_r] = [(\bigcup_{r=1}^m T_r) UM] - j = (BUM) - j \subseteq (AUB) - j$ . So, we conclude that (ii) is true in this case.

Case 3:  $j \in S_s \cap T_s$ .

Since  $(EUX, \tilde{p}_s UM)$  is a matroid, and  $S_s \neq T_s$ , there exists a set  $M \in (\tilde{p}_s UM)$  such that  $M \subseteq (S_s \cup T_s) - j$ . If  $x \notin M$ , then  $M \in \mathcal{M}$ . Hence, by taking  $C = M$  we get:  $C \subseteq (S_s \cup T_s) - j \subseteq \bigcup_{r=1}^m [(S_r \cup T_r) - j] = (AUB) - j$ , and thus (ii) is true. Assume, conversely that  $x \in M$ , i.e.  $M \in \tilde{p}_s$ . Hence, since  $S_s \in \tilde{p}_s$ , we know that  $x \in S_s \cap M$  and  $j \in S_s - M$ . So, obviously  $M \neq S_s$ . Thus, since  $(EUX, \tilde{p}_s UM)$  is a matroid, by Lemma 6.26 there exists a set  $M' \in (\tilde{p}_s UM)$  such that  $j \in M'$  and  $M' \subseteq (S_s UM) - x$ , i.e.  $M' \in \mathcal{M}$ . Furthermore, since  $(EUX, \tilde{p}_r UM)$  is a matroid and since  $x \in T_r - M'$  for all  $r \in J$ , by Lemma 6.26 there exists a set  $T'_r \in (\tilde{p}_r UM)$  such that  $x \in T'_r$ , i.e.  $T'_r \in \tilde{p}_r$  and  $T'_r \subseteq (T_r UM') - j$ , for all  $r \in J$ .

Now let  $D = [\bigcup_{r \in J} T'_r] \cup [\bigcup_{r \notin J} T_r]$ .

Since  $\tilde{p}$  is the family of minimal such unions, there must exist a set  $C \in \tilde{p}$  such that  $C \subseteq D$ . Moreover, we get that:  $D \subseteq [\bigcup_{r \in J} (T_r UM') - j] \cup [\bigcup_{r \notin J} T_r] = [(\bigcup_{r=1}^m T_r) UM'] - j = (BUM') - j \subseteq (BUS_s UM) - j \subseteq (BUS_s \cup T_s) - j \subseteq (AUB) - j$ . So, we conclude that (ii) is true in this case.

Thus (ii) is proved to be true in all possible cases, and hence the theorem is true.  $\square$

Now, it is easy to prove the following result:

Theorem 6.28. Let  $G$  be an undirected graph with edge set  $E$  and let  $\{S_1, T_1\}, \dots, \{S_m, T_m\}$  be  $m$  pairs of nodes (not necessarily disjoint) in  $G$ . Furthermore, let  $(E, P_r)$  be the system which is functioning if  $S_r$  and  $T_r$  can communicate through  $G$ ,  $r = 1, \dots, m$ . Then  $(E, P) = (E, P_1 \cdots P_m)$  is regular.

Proof: Since  $(E, P_1), \dots, (E, P_m)$  are 2-terminal undirected network systems, by Example 6.14, we know that  $(E, P_1), \dots, (E, P_m)$  are regular clutters and that the corresponding matroids can be written  $(E \cup x, \tilde{P}_1 \cup M), \dots, (E \cup x, \tilde{P}_m \cup M)$  respectively, where  $M$  is the family of minimal circuits in  $G$ , (i.e.  $M$  is independent of the choice of terminals) and  $x$  is some fixed artificial component. (Since we consider several pairs of terminals, we cannot interpret  $x$  as a component between the terminals in this case.) Hence, by Theorem 6.27,  $(E, P)$  is regular.  $\square$

Corollary 6.29. Every  $K$ -terminal undirected network system is regular.

Proof. Let  $T_1, \dots, T_k$  be  $k$  terminals in an undirected network  $G$ , and let  $(E, P)$  be the system which is functioning if  $T_1, \dots, T_k$  can communicate through  $G$ . Moreover, let  $(E, P_r)$  be the system which is functioning if  $T_r$  and  $T_{r+1}$  can communicate through  $G$ ,  $r = 1, \dots, k-1$ . Then obviously  $(E, P) = (E, P_1 \cdots P_{k-1})$ . Hence  $(E, P)$  is regular by Theorem 6.28.  $\square$

## 7. The factoring algorithm.

In Section 5 we saw how the domination concept was used in order to improve the Inclusion-Exclusion method in reliability computations. Now, we shall see how domination can be used in order to improve another method, namely the factoring algorithm. This was first done in Satyanarayana and Chang (1983) in the case of K-terminal undirected network systems. However, having established the generalized domination theorem, it is easy to extend the earlier results to the case of regular clutters. However, before we do so, let us review the method.

In this section we only consider systems with statistically independent components. (The method may be extended to the case of dependent components as well. However, this requires information concerning the conditional reliabilities of the components.)

Definition 7.1. Let  $(E, P)$  be a clutter where  $E = \{1, \dots, n\}$  and let  $\phi$  be the structure function of the system. Then introduce  $X_i$ , the random variable denoting the state of the  $i$ -th component at a fixed point of time,  $i = 1, \dots, n$ ,  $\underline{X} = (X_1, \dots, X_n)$ , and define:

The reliability of the  $i$ -th component  $= P(X_i = 1) = p_i$ ,  
 $i = 1, \dots, n$ , and:

The reliability of the system  $= P(\phi(\underline{X}) = 1) = h_\phi$ .

Under the assumption of independent components, we have:

$$h_\phi = h_\phi(\underline{p}) \quad \text{where } \underline{p} = (p_1, \dots, p_n).$$

Now, the factoring algorithm is based on the following two results:

Proposition 7.2. Let  $(E, P)$  be a clutter with structure function  $\phi$ , and let  $e \in E$ . Then we have (assuming independence):

$$h_{\phi}(p) = p_e h_{\phi_{+e}}(p) + (1-p_e) h_{\phi_{-e}}(p)$$

where  $\phi_{+e}$  and  $\phi_{-e}$  are the structure functions of  $(E-e, P_{+e})$  and  $(E-e, P_{-e})$  respectively. This reduction method is called pivotal decompositions.

Proof: Obvious, see Barlow and Proschan (1981).  $\square$

Proposition 7.3. Let  $(E, P)$  be a clutter with structure function  $\phi$ , and let  $i, j \in E$ .

(i) If  $i$  and  $j$  are in series, then  $h_{\phi}$  will depend on  $p_i$  and  $p_j$  only through  $p_i \cdot p_j$ . Hence, replacing  $i$  and  $j$  by the single component  $i$  (i.e. a series reduction) with updated reliability  $p_i' = p_i \cdot p_j$  does not alter  $h_{\phi}$ .

(ii) If  $i$  and  $j$  are in parallel, then  $h_{\phi}$  will depend on  $p_i$  and  $p_j$  only through  $(p_i + p_j - p_i p_j)$ . Hence, replacing  $i$  and  $j$  by the single component  $i$  (i.e. a parallel reduction) with updated reliability  $p = p_i + p_j - p_i p_j$  does not alter  $h_{\phi}$ .

Proof: The proof is straightforward and thus omitted.  $\square$

Now, the algorithm can be formulated as follows:

Algorithm 7.4. Let  $(E, P)$  be a clutter with structure function  $\phi$ .

If  $(E, P)$  contains irrelevant components only, then  $\phi$  is simply a constant not depending on the component states. Hence, in this case we have:

$$h_{\phi} = \phi.$$

Assume now that  $(E, P)$  has at least one relevant component. We now proceed as follows:

Step 1. Perform all possible s-p-reductions and update the reliability of the components as indicated in Proposition 7.3.

Step 2. Let  $(E^r, P^r)$  be the resulting clutter after Step 1, and  $\phi^r$  the corresponding structure function. since  $(E, P)$  contains at least one relevant component, by Corollary 3.7,  $(E^r, P^r)$  must contain at least one as well.

Case 1.  $(E^r, P^r)$  contains exactly one relevant component, say component  $e$  with updated reliability  $p_e$ .

Then  $h_{\phi^r} = P(X_e=1) = p_e$

Case 2.  $(E^r, P^r)$  contains more than one relevant component; i.e.  $(E^r, P^r)$  is complex.

Then choose one  $e \in E^r$  and compute  $h_{\phi}$  by applying Proposition 7.2. That is, we get:

$$h_{\phi^r} = p_e h_{\phi_{+e}^r} + (1-p_e) h_{\phi_e^r}$$

where  $\phi_{+e}$  and  $\phi_{-e}$  are the structure functions of  $(E-e, P_{+e})$  and  $(E-e, P_{-e})$  respectively, and where  $h_{\phi_{+e}^r}$  and  $h_{\phi_{-e}^r}$  are computed by applying Algorithm 7.4. once more.

The recursion process defined in Algorithm 7.4. can be illustrated by a tree. (See Figure 7.1.)

In this tree the nodes represent the minors obtained during the process, while the edges denote s-p-reductions and pivotal decompositions. The leaves (the nodes at the bottom of the tree) will be minors having at most one relevant component.



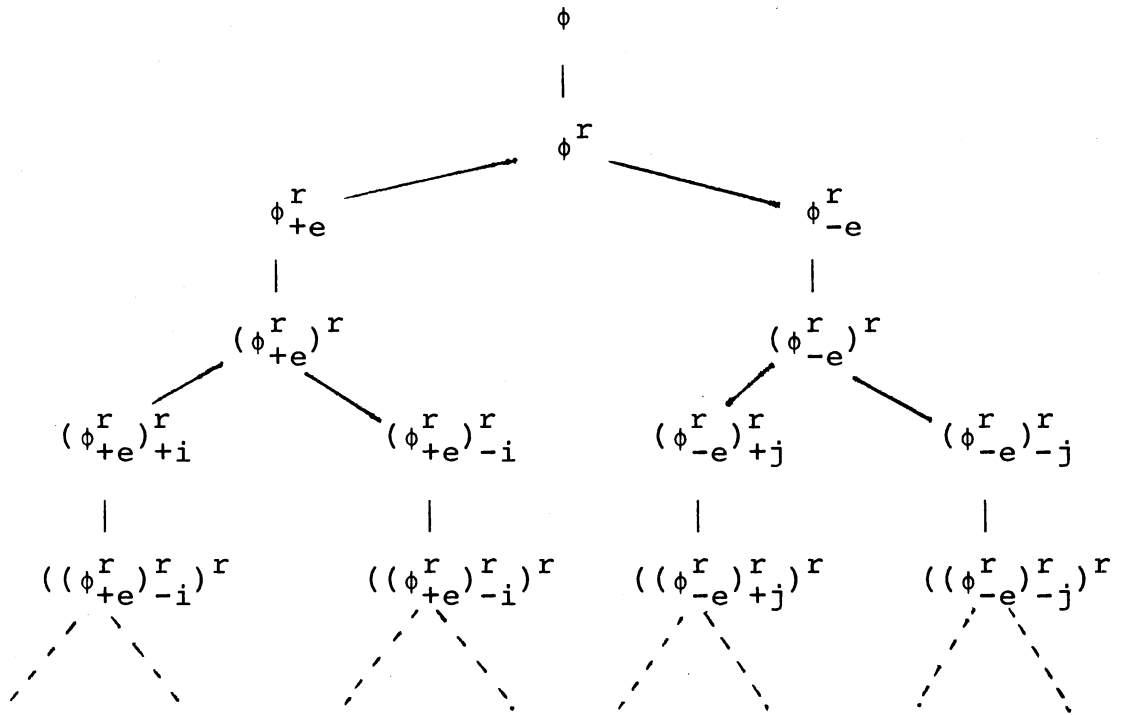


Figure 7.1.

In order to reduce the running time of the algorithm, it is obviously important that the number of leaves in the tree is as small as possible. However, it is easy to see that this number in general depends on the choice of the component  $e$  in Step 2. in the algorithm. We shall now see how the problem of choosing the right  $e$  can be solved using the domination concept.

First, we introduce the following notation:

Let  $(E, P)$  be a clutter. Then we define:

$t(P)$  = The minimal number of leaves in a tree  
obtained by applying Algorithm 7.4 to  $(E, P)$ .

The following proposition is a direct consequence of this definition:

Proposition 7.5. Let  $(E, P)$  be a clutter, and let  $(E^r, P^r)$  be the structure obtained from  $(E, P)$  by performing all possible s-p-reductions.

Then we have:  $t(P^r) = t(P)$ .

Moreover if  $(E, P)$  is complex, (i.e.  $(E^r, P^r) = (E, P)$ ), then  $t(P) = \min_{e \in E} (t(P_{+e}) + t(P_{-e}))$ .

Hence,  $e$  is an optimal choice if and only if  $t(P) = t(P_{+e}) + t(P_{-e})$ .

Proof:  $t(P^r) = t(P)$  follows by considering Figure 7.1. Now, assume that  $(E, P)$  is complex. That is, we have arrived at Step 2. Case 2 in Algorithm 7.4. Hence, we must choose a component  $e \in E$ , and then apply Proposition 7.2. When  $e$  is chosen, the smallest possible tree will have  $t(P_{+e}) + t(P_{-e})$  leaves. Hence, in order to minimize the number of leaves, we must find  $e$  such that  $t(P_{+e}) + t(P_{-e})$  is as small as possible.

Since  $t(P)$  is the minimal number of leaves, we must have:  $t(P) = \min_{e \in E} (t(P_{+e}) + t(P_{-e}))$ . Hence,  $e$  is an optimal choice if and only if  $t(P) = t(P_{+e}) + t(P_{-e})$ .  $\square$

Now, we can prove the main result of this section:

Theorem 7.6. Let  $(E, P)$  be a regular clutter. Then we have:

$$D(P) \leq t(P).$$

Especially,  $D(P) = t(P)$  if  $(E, P)$  is coherent.

Furthermore, if  $(E, P)$  is a complex coherent structure, then  $e$  is an optimal choice for the pivotal decomposition if  $(E - e, P_{+e})$  and  $(E - e, P_{-e})$  both are coherent.

Proof: We start by proving that  $D(P) \leq t(P)$  and that  $D(P) = t(P)$  if  $(E, P)$  is coherent. Since, by Corollary 2.6.,  $D(P) = 0$  if  $(E, P)$  is noncoherent, we may assume that  $(E, P)$  is coherent. The proof then is by induction on  $n = |E|$ .

It is easily seen that the theorem is true if  $n = 1$ .

Assume now that the theorem is true for all  $m \leq n_0$  and assume that  $n = n_0 + 1$  ( $n \geq 2$ ).

Let  $(E^r, P^r)$  be the structure obtained from  $(E, P)$  by performing all possible s-p-reductions.

If  $(E, P)$  is not complex, then  $E^r \subset E$ , and hence  $|E^r| \leq n_0$ . Furthermore, by Corollary 3.7.  $(E^r, P^r)$  is regular. Hence, by Corollary 4.8., the induction hypothesis and Proposition 7.5, we get:

$$D(P) = D(P^r) = t(P^r) = t(P) \quad \text{as stated.}$$

Assume on the other hand that  $(E, P)$  is complex.

By Proposition 6.16.  $(E-e, P_{+e})$  and  $(E-e, P_{-e})$  are both regular for all  $e \in E$ .

Hence, by Theorem 6.18. and the induction hypothesis we get:

$$D(P) = D(P_{+e}) + D(P_{-e}) \leq t(P_{+e}) + t(P_{-e}) \quad \text{for all } e \in E.$$

Thus especially by Proposition 7.5. we get:

$$D(P) \leq \min_{e \in E} (t(P_{+e}) + t(P_{-e})) = t(P).$$

However, since  $(E, P)$  is a complex coherent structure, by

Theorem 3.13. there exists a component  $e_0 \in E$  such that

$(E-e_0, P_{+e_0})$  and  $(E-e_0, P_{-e_0})$  both are coherent. Hence, by the induction hypothesis we have  $D(P_{e_0}) = t(P_{+e_0})$  and  $D(P_{-e_0}) = t(P_{-e_0})$ , and thus we get:

$$t(P_{+e_0}) + t(P_{-e_0}) = D(P) \leq t(P) \leq t(P_{+e_0}) + t(P_{-e_0}).$$

Hence, we conclude that:

$$D(P) = t(P) \quad \text{as stated.}$$

Moreover,  $t(P) = t(P_{+e_0}) + t(P_{-e_0})$ . That is,  $e_0$  is an optimal choice by Proposition 7.5. and hence the theorem is proved.  $\square$

## 8. Final remarks.

In the present report we have tried to develop a theoretical basis for efficient reliability calculations. This has been done by studying the concepts of domination and signed domination, and it has led to several new results generalizing earlier works.

However, there are still lots of ends to tie up and unanswered questions left. In these final remarks we shall list some of the most interesting problems:

- 1) Theorem 5.10. provides a formula for computing the signed domination of an MSKT system. As we mentioned in Section 5, for the time being the algorithm of Satyanarayana and Prabhakar (1978) has not been generalized to MSKT systems. Hence, an interesting, and probably easy, problem would be to carry out this generalization.
- 2) In Theorem 7.6. we characterized an optimal factoring strategy valid for all regular clutters. However, we still need an efficient algorithm selecting the right edges in the pivoting process. In Section 3 we characterized such edges by a certain partial ordering, but in order to apply this condition we need to know the minimal path and cut sets of the system. Hence, the problem of designing an optimal factoring algorithm, valid for all regular clutters, is still unsolved.
- 3) Instead of considering the case of general regular clutters, it is perhaps possible to obtain some new results by studying special structures. Especially, it would be interesting to generalize the work of Satyanarayana and Chang (1983) to the class of systems described in Theorem 6.28.
- 4) We have proved that regularity is a sufficient condition for the domination theorem to be true. However, it is easy to see

that this is not a necessary condition. There exist nonregular clutters for which the domination theorem holds. As an example, consider the 2-terminal directed bridge structure shown in Figure 5.2. Since this structure is an SKT-system which is not an s-p-structure, by Corollary 6.20 the structure is nonregular. It is easy, however, to verify that the domination theorem is valid for this structure. We still believe that regularity is a key concept in the study of the factoring algorithm. Especially, we conjecture that regularity is necessary for Theorem 7.6 to be true. In relation to this we recall that Theorem 7.6 is proved by induction on the number of components in the structure. Hence, it is not sufficient that the domination theorem is valid for the structure itself. In the proof it is implicitly assumed that the domination theorem is valid for all minors of the structure as well. We hope to return to this problem in a forthcoming paper.

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