

NEW LIGHT ON MEASURES OF IMPORTANCE OF SYSTEM COMPONENTS

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Abstract

In this paper the Natvig (1979) measure of the importance of a component in a binary coherent system is revisited. This measure is for the case of components not undergoing repair proportional to the expected reduction in remaining system lifetime due to the failure of the component. We now show that this expected reduction equals the expected increase in system lifetime by replacing the life distribution of the component by the corresponding one where exactly one minimal repair is allowed. It is also shown that our measure, as the Barlow-Proschian (1975) measure, is a weighted average of the Birnbaum (1969) measure and that it can easily be determined even for fairly large systems on a computer. Hence the foundation for numerical comparisons with other measures is established.

Some new results on our measure and on the corresponding one with total repair are also given, in addition to results on a measure suggested in Natvig (1982). A preliminary comparison seems to indicate that our measure is advantageous.

COHERENT STRUCTURE; COMPONENT IMPORTANCE; LIFETIME INCREASE

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1. INTRODUCTION AND MAIN IDEAS

Consider a system consisting of n components. In this paper, we shall restrict our attention to the case where the components, and hence the system, cannot be repaired. Let $(i=1, \dots, n)$

$$X_i(t) = \begin{cases} 1 & \text{if } i\text{th component functions at time } t, \\ 0 & \text{if } i\text{th component is failed at time } t. \end{cases}$$

Assume also that the stochastic processes $\{X_i(t), t \geq 0\}$, $i=1, \dots, n$ are mutually independent. Introduce $\underline{X}(t) = (X_1(t), \dots, X_n(t))$ and let

$$\phi(\underline{X}(t)) = \begin{cases} 1 & \text{if system functions at time } t, \\ 0 & \text{if system is failed at time } t. \end{cases}$$

Now let the i th component have an absolutely continuous life distribution $F_i(t)$ with density $f_i(t)$. Then the reliability of this component at time t is given by

$$P(X_i(t)=1) = 1 - F_i(t) \triangleq \bar{F}_i(t).$$

Introduce $\bar{\underline{F}}(t) = (\bar{F}_1(t), \dots, \bar{F}_n(t))$. Then the reliability of the system at time t is given by $P(\phi(\underline{X}(t))=1) = h(\bar{\underline{F}}(t))$, where h is the system's reliability function. The following notation will be used:

$$(\cdot_i, x) = (x_1, \dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_n).$$

We also assume the structure function ϕ to be coherent. For an excellent introduction to coherent structure theory, we refer to Barlow and Proschan (1981).

Now introduce the random variable

Z_i = reduction in remaining system lifetime due to the failure of the i th component.

In Natvig (1979) we suggested the following new measure of the importance of the i th component:

$$(1.1) \quad I_{N_1}^{(i)} = EZ_i / \sum_{j=1}^n EZ_j,$$

tacitly assuming $EZ_i < \infty$, $i=1, \dots, n$. Furthermore, we proved that

$$(1.2) \quad EZ_i = \int_0^{\infty} \sum_{(\cdot, \underline{x})} \prod_{j \neq i} F_j(t)^{1-x_j} \bar{F}_j(t)^{x_j} \\ \times \int_0^{\infty} [h(\bar{H}_{-t}^{(1, \underline{x})}(u)) - h(\bar{H}_{-t}^{(0, \underline{x})}(u))] du f_i(t) dt$$

where

$$\bar{H}_{-t}^{\underline{x}}(u) = (\bar{H}_{1,t}^{x_1}(u), \dots, \bar{H}_{n,t}^{x_n}(u))$$

and

$$\bar{H}_{i,t}^1(u) = \bar{F}_i(t+u)/\bar{F}_i(t), \quad \bar{H}_{i,t}^0(u) = 0.$$

In Natvig (1982) we introduced the following random variables

Y_i^1 = remaining system lifetime just after the failure of the i th component, which, however, immediately undergoes a minimal repair; i.e. it is repaired to have the same distribution of remaining lifetime as it had just before failing.

Y_i^0 = remaining system lifetime just after the failure of the i th component.

We then gave Z_i the following interpretation

$$Z_i = Y_i^1 - Y_i^0.$$

Denote that EZ_i may perhaps best be interpreted as the reduction in expected remaining system lifetime due to the failure of the i th component.

Let now T be the lifetime of a new system, and T_i the lifetime of a new system where the life distribution of the i th component is replaced by the corresponding one where exactly one minimal repair of the component is allowed; i.e. $\bar{F}_i(t)$ is replaced by

$$\bar{F}_i(t) + \int_0^t f_i(t-u) \frac{\bar{F}_i(t-u+u)}{\bar{F}_i(t-u)} du = \bar{F}_i(t)(1 - \ln \bar{F}_i(t)).$$

Let furthermore S_i be the lifetime of the i th component (until the minimal repair). Consider now two cases. In case 1 we assume

$S_i < T$. Then

$$Z_i = S_i + Y_i^1 - (S_i + Y_i^0) = T_i - T.$$

In case 2 we assume $S_i > T$. Then

$$Z_i = Y_i^1 - Y_i^0 = 0 - 0 = 0 = T_i - T.$$

Hence we always have the relation

$$(1.3) \quad Z_i = T_i - T.$$

This is also mentioned in Bergman (1984). Hence

$$(1.4) \quad EZ_i = \int_0^{\infty} h(\bar{F}_i(t)(1 - \ln \bar{F}_i(t)), \bar{F}(t)) dt - \int_0^{\infty} h(\bar{F}(t)) dt,$$

assuming the integrals exist. By performing a pivotal decomposition on the i th component in (1.4) we get

$$(1.5) \quad EZ_i = \int_0^{\infty} \bar{F}_i(t)(-\ln \bar{F}_i(t)) I_B^{(i)}(t) dt, \quad \text{where}$$

$$(1.6) \quad I_B^{(i)}(t) = h(1_i, \bar{F}(t)) - h(0_i, \bar{F}(t)).$$

$I_B^{(i)}(t)$ is the Birnbaum (1969) measure of the importance of the i th component at time t , which is obviously the probability of the component being critical for system functioning at t . Hence $I_{N_1}^{(i)}$ is a weighted average of the latter measure as is true for the Barlow-Proshan (1975) measure given by

$$(1.7) \quad I_{B-P}^{(i)} = \int_0^{\infty} f_i(t) I_B^{(i)}(t) dt.$$

This is then the probability of the component "causing" system failure.

One might ask why the component in the $I_{N_1}^{(i)}$ measure undergoes a minimal repair. If instead a total repair of the i th component is allowed, i.e. the component is repaired to have the same distribution of remaining lifetime as originally, the expected increase in system lifetime is given by

$$(1.8) \quad EU_i = \int_0^{\infty} \int_0^t f_i(t-u) \bar{F}_i(u) du I_B^{(i)}(t) dt.$$

Note that for exponentially distributed lifetimes a minimal repair and a total repair are just the same and $EZ_i = EU_i$. Finally, the expected increase in system lifetime by replacing the i th component by a perfect one, i.e. $\bar{F}_i(t)$ is replaced by 1, is given by

$$(1.9) \quad EV_i = \int_0^{\infty} \bar{F}_i(t) I_B^{(i)}(t) dt.$$

Now let the components have proportional hazards, i.e.,

$$\bar{F}_i(t) = \exp(-\lambda_i R(t)), \quad \lambda_i > 0, t > 0, i=1, \dots, n,$$

where λ_i , $i=1, \dots, n$ are the proportional hazard rates.

In Natvig (1982) the following measure is suggested

$$(1.10) \quad I_{N_2}^{(i)} = \frac{\partial ET}{\partial \lambda_i^{-1}} / \sum_{j=1}^n \frac{\partial ET}{\partial \lambda_j^{-1}},$$

again tacitly assuming $\partial ET / \partial \lambda_i^{-1} < \infty$, $i=1, \dots, n$. Assuming we are allowed to reverse the order of differentiation and integration, we get

$$(1.11) \quad \frac{\partial ET}{\partial \lambda_i^{-1}} = \int_0^{\infty} \frac{\partial}{\partial \lambda_i^{-1}} h(\bar{F}(t)) dt = \int_0^{\infty} \frac{\partial h(\bar{F}(t))}{\partial \bar{F}_i(t)} \frac{\partial \bar{F}_i(t)}{\partial \lambda_i^{-1}} dt$$

$$\int_0^{\infty} \lambda_i^2 R(t) \exp(-\lambda_i R(t)) I_B^{(i)}(t) dt = \lambda_i E Z_i.$$

This relation was shown to be true just for a series and parallel system in Natvig (1982).

We now define the measures

$$I_{N_3}^{(i)} = EU_i / \sum_{j=1}^n EU_j, \quad I_{N_4}^{(i)} = EV_i / \sum_{j=1}^n EV_j,$$

again assuming $EU_i < \infty$, $EV_i < \infty$, $i=1, \dots, n$. The latter measure is also suggested in Bergman (1984). Hence we see that all measures $I_{N_k}^{(i)}$, $k=1, 2, 3, 4$ and $I_{B-P}^{(i)}$ are weighted averages of the Birnbaum (1969) measure. Especially, due to the independence of $\{X_i(t), t \geq 0\}$, $i=1, \dots, n$, this implies that all measures give zero importance to irrelevant components. In Section 4 we will have a closer look at the different weight functions. A preliminary comparison seems to indicate that the $I_{N_1}^{(i)}$ measure is advantageous. A closer numerical comparison of the measures is planned using the fault tree analysis program SAW (1984) to establish $I_B^{(i)}(t)$ for suitable $t \geq 0$. This program is constructed to deal efficiently with fault trees of replicated events.

In Section 2 we establish (1.5) directly from (1.2) and give some generalizations and simplifications of results in Natvig (1979, 1982) on $I_{N_1}^{(i)}$. In the latter paper we are arriving at the distribution of Z_i and especially $P(Z_i=0)$. It should be noted that these expressions are not simplified by the discovery of (1.5). This is, however, true for the distribution of

Z_{K_k} = reduction in remaining system lifetime due to the failure of the k th minimal cut set.

In Section 3 we give some results on the measures $I_{N_k}^{(i)}$, $k=2,3,4$. Finally, the measures by Birnbaum (1969), Barlow and Proschan (1975) and Natvig (1979), being developed for binary systems are generalized in Natvig (1984) to the multistate case.

2. NEW RESULTS ON $I_{N_1}^{(i)}$

We start by establishing (1.5), the idea being that the conditioning on the states of the other components at the point of time when the i th component fails, is unnecessary. From (1.2) we get

$$\begin{aligned} EZ_i &= \int_0^\infty \int_0^\infty \left[h\left(\frac{\bar{F}_i(t+u)}{\bar{F}_i(t)} \middle| i, \bar{F}(t+u)\right) - h(0_i, \bar{F}(t+u)) \right] du f_i(t) dt \\ &= \int_0^\infty \int_0^\infty \left[h(1_i, \bar{F}(t+u)) - h(0_i, \bar{F}(t+u)) \right] \frac{\bar{F}_i(t+u)}{\bar{F}_i(t)} du f_i(t) dt \\ &= \int_0^\infty \bar{F}_i(v) (-\ln \bar{F}_i(v)) I_B^{(i)}(v) dv. \end{aligned}$$

Hence we have proved (1.5).

Completely similarly we get the following simplified versions of Theorem 3.8 of Natvig (1979) and Theorem 2.7 of Natvig (1982) treating Z_{K_k} defined in Section 1.

Theorem 2.1

$$\begin{aligned} P(Z_{K_k} \leq z) &= 1 - \sum_{i \in K_k} \int_0^\infty \prod_{j \in K_k - \{i\}} F_j(t) h\left(\frac{\bar{F}_i(t+z)}{\bar{F}_i(t)} \middle| i, \underline{0}^{K_k - \{i\}}, \bar{F}(t+z)\right) f_i(t) dt \\ EZ_{K_k} &= \sum_{i \in K_k} \int_0^\infty \prod_{j \in K_k - \{i\}} F_j(t) \int_0^\infty h\left(\frac{\bar{F}_i(t+z)}{\bar{F}_i(t)} \middle| i, \underline{0}^{K_k - \{i\}}, \bar{F}(t+z)\right) dz f_i(t) dt \end{aligned}$$

Note that the component whose failure coincides with the failure of the minimal cut set, must be the last to fail within this set. Hence Z_{K_k} cannot be interpreted as an increase in system lifetime by improving the life distributions of some components. We also observe that EZ_{K_k} cannot be interpreted as the reduction in expected remaining system lifetime due to the failure of the minimal cut set.

Let s be the number of minimal cut sets. In Natvig (1979) the following measure of importance of the k th minimal cut set is suggested

$$I_{N_1}^{(K_k)} = EZ_{K_k} / \sum_{j=1}^s EZ_{K_j},$$

This is now numerically feasible.

Similarly, when giving an expression for the expected reduction in remaining system lifetime due to the failure of a module, in Theorem 3.7 of Natvig (1979), conditioning on the states of the components outside the module at the time of failing of the module, is unnecessary.

Note that the expressions for $I_{N_1}^{(i)}$ for a series and parallel system given in Theorem 3.2 of Natvig (1979) are straightforward from (1.5). We now generalize Theorem 3.3 of the same paper. We also prove corresponding results to the ones given in Lemmas 3.6, 3.7 of Barlow and Proschan (1975) for their measure.

Lemma 2.2

Let the i th component be in series (parallel) with the rest of the system. Assuming $\bar{F}_r(t) > 0$ for $t \in [0, \infty)$, $r \neq i$, EZ_i is strictly increasing in $\bar{F}_i(t)(-\ln \bar{F}_i(t))$. Assuming $0 < \bar{F}_i < 1$ for $t \in (0, \infty)$, EZ_i is strictly increasing (decreasing) in $\bar{F}_j(t)$, $j \neq i$, $0 < \bar{F}_j(t) < 1$.

Proof

Let the i th component be in series with the rest of the system. Then from (1.5)

$$EZ_i = \int_0^\infty \bar{F}_i(t)(-\ln \bar{F}_i(t))h(1_i, \bar{F}(t))dt.$$

Hence since $h(1_i, \bar{F}(t)) > 0$, $t \in [0, \infty)$, EZ_i is strictly increasing in $\bar{F}_i(t)(-\ln \bar{F}_i(t))$. Furthermore, since the i th component is in series with the rest of the system, $\phi(1_i, \underline{x})$ is a coherent system. Hence $h(1_i, \bar{F}(t))$ is strictly increasing in $\bar{F}_j(t)$, $j \neq i$ for $0 < \bar{F}_j < 1$. The same is true for EZ_i since $\bar{F}_i(t)(-\ln \bar{F}_i(t)) > 0$, $t \in (0, \infty)$.

The proof is completely similar when the i th component is in parallel with the rest of the system. \square

Due to the normalization in (1.1) corresponding results for $I_{N_1}^{(i)}$ do not follow. Hence we do not easily generalize Theorem 3.5 of Natvig (1979).

Lemma 2.3

Let the i th component be in series (parallel) with the rest of the system and let the i th and j th component have common life distribution F . Then

$$I_{N_1}^{(i)} > I_{N_1}^{(j)}, \quad i \neq j.$$

Proof

Let the i th component be in series with the rest of the system. Then from (1.5)

$$\begin{aligned} EZ_i &= \int_0^\infty \bar{F}(t) (-\ln \bar{F}(t)) h(1_i, \bar{F}(t)) dt \\ &= \int_0^\infty \bar{F}(t) (-\ln \bar{F}(t)) [\bar{F}(t) h(1_i, 1_j, \bar{F}(t)) + F(t) h(1_i, 0_j, \bar{F}(t))] dt \\ EZ_j &= \int_0^\infty \bar{F}(t) (-\ln \bar{F}(t)) \bar{F}(t) [h(1_i, 1_j, \bar{F}(t)) - h(1_i, 0_j, \bar{F}(t))] dt \\ &= EZ_i - \int_0^\infty \bar{F}(t) (-\ln \bar{F}(t)) h(1_i, 0_j, \bar{F}(t)) dt < EZ_i. \end{aligned}$$

By normalizing, the inequality is established. The proof is completely similar when the i th component is in parallel with the rest of the system. \square

Finally we give the following generalization of Theorem 3.3 in Natvig (1979), the result now being closer to the one given in Theorem 3.8 of Barlow and Proschan (1975) for their measure. Note that we are not benefitting from Lemmas 2.2, 2.3 in the proof.

Theorem 2.4

Let the i th and j th component be in series (parallel) with the rest of the system. Let for $j \neq i$ $F_i(t) > F_j(t)$ ($\bar{F}_i(t) > \bar{F}_j(t)$) for all $t > 0$. Then

$$I_{N_1}^{(i)} > I_{N_1}^{(j)}.$$

Proof

We give the proof for the case where the i th and j th component are in parallel with the rest of the system. Then

$$\begin{aligned}
 EZ_i - EZ_j &= \int_0^{\infty} \{ \bar{F}_i(t) (-\ln \bar{F}_i(t)) F_j(t) [1 - h(0_i, 0_j, \bar{F}(t))] \\
 &\quad - \bar{F}_j(t) (-\ln \bar{F}_j(t)) F_i(t) [1 - h(0_i, 0_j, \bar{F}(t))] \} dt \\
 &= \int_0^{\infty} [1 - h(0_i, 0_j, \bar{F}(t))] F_i(t) F_j(t) \\
 &\quad \times [\bar{F}_i(t) (-\ln \bar{F}_i(t)) / F_i(t) - \bar{F}_j(t) (-\ln \bar{F}_j(t)) / F_j(t)] dt.
 \end{aligned}$$

The latter expression is nonnegative since $\bar{F}_i(t) > \bar{F}_j(t)$ and $\bar{F}_i(t) (-\ln \bar{F}_i(t)) / (1 - \bar{F}_i(t))$ is an increasing function of $\bar{F}_i(t)$. The proof is hence completed. \square

3. NEW RESULTS ON $I_{N_k}^{(i)}$, $k=2,3,4$

We now have a look at the measures $I_{N_k}^{(i)}$, $k=2,3,4$ considering them in reversed order.

It is immediately checked that Lemmas 2.2, 2.3 and Theorem 2.4 always hold for the case where the i th component is in series with the rest of the system, when considering the $I_{N_4}^{(i)}$ measure. When the i th component is in parallel with the rest of the system

$$\begin{aligned}
 EV_i &= \int_0^{\infty} h(1_i, \bar{F}(t)) dt - \int_0^{\infty} h(\bar{F}(t)) dt = \\
 &= \int_0^{\infty} 1 dt - ET = \infty - ET,
 \end{aligned}$$

so we have a degenerate case where the pivotal decomposition leading to (1.9) is not allowed. If this relation is formally applied, the results in Lemmas 2.2, 2.3 hold. In Theorem 2.4 we get

$$I_{N_4}^{(i)} = I_{N_4}^{(j)}, \quad i \neq j,$$

when the i th and j th component both are in parallel with the rest of the system, irrespective of their life distributions! As a conclusion the bad behaviour of the $I_{N_4}^{(i)}$ measure for components being in parallel with the rest of the system, and hence for pure parallel systems, seems to be a sufficient reason for dismissing it.

Now turning to the $I_{N_3}^{(i)}$ measure, we immediately prove a corresponding version of Lemma 2.2.

Lemma 3.1

Let the i th component be in series (parallel) with the rest of the system. Assuming $\bar{F}_r(t) > 0$ for $t \in [0, \infty)$, $r \neq i$, EU_i is strictly increasing in $\int_0^t f_i(t-u)\bar{F}_i(u)du$. Assuming $\int_0^t f_i(t-u)\bar{F}_i(u)du > 0$, $t \in (0, \infty)$, EU_i is strictly increasing (decreasing) in $\bar{F}_j(t)$, $j \neq i$, $0 < \bar{F}_j(t) < 1$.

Furthermore, for this measure Lemma 2.3 easily follows, whereas it is not easy to establish a theorem corresponding to Theorem 2.4. Our only contribution so far is the following.

Theorem 3.2

Let the i th and j th component be in series with the rest of the system. (Consider a parallel system.) Assume

$$\bar{F}_r(t) = \sum_{k=0}^{m-1} \frac{(\lambda_r t)^k}{k!} e^{-\lambda_r t}, \quad \lambda_r > 0, t > 0, r = i, j (r = 1, \dots, n),$$

where $m > 1$ is an integer; i.e. the life lengths of the i th and j th component (all components) are gamma distributed. Let for $j \neq i$ $F_i(t) > F_j(t)$ ($\bar{F}_i(t) > \bar{F}_j(t)$) for all $t > 0$. Then

$$I_{N_3}^{(i)} > I_{N_3}^{(j)}.$$

Proof

After some algebra or from a direct argument it follows that

$$\int_0^t f_i(t-u)\bar{F}_i(u)du = \sum_{k=m}^{2m-1} \frac{(\lambda_i t)^k}{k!} e^{-\lambda_i t}$$

Note that $F_i(t) > F_j(t)$ ($\bar{F}_i(t) > \bar{F}_j(t)$) for all $t > 0$ is just equivalent to $\lambda_i > \lambda_j$ ($\lambda_i < \lambda_j$). Assume without loss of generality that $i = 1$ and $j = 2$. Consider first the series case and hence $\lambda_1 > \lambda_2$. Then from (1.8)

$$EU_1 - EU_2 = \sum_{j_1=m}^{2m-1} \sum_{j_2=0}^{m-1} \int_0^{\infty} h(1_i, 1_j, \bar{F}(t)) t^{j_1+j_2} e^{-(\lambda_1+\lambda_2)t} dt \\ \times \frac{(\lambda_1 \lambda_2)^{j_2}}{j_1! j_2!} (\lambda_1^{j_1-j_2} - \lambda_2^{j_1-j_2}) > 0,$$

and this part of the proof is completed.

Now consider the parallel system and hence $\lambda_1 < \lambda_2$. Then we get

$$EU_1 = \sum_{j_1=m}^{2m-1} \sum_{j_2=m}^{\infty} \dots \sum_{j_n=m}^{\infty} \frac{(j_1+\dots+j_n)!}{(\lambda_1+\dots+\lambda_n)^{j_1+\dots+j_n+1}} \prod_{r=1}^n \frac{(\lambda_r)^{j_r}}{j_r!} \\ = \sum_{j_1=m}^{2m-1} \frac{1}{\lambda_1} \sum_{j_2=m}^{\infty} \dots \sum_{j_n=m}^{\infty} \frac{(j_1+\dots+j_n)!}{j_1! \dots j_n!} p_1^{j_1+1} p_2^{j_2} \dots p_n^{j_n},$$

where $p_i = \lambda_i / (\lambda_1 + \dots + \lambda_n)$ $i=1, \dots, n$ can be interpreted as probabilities for multinomial events E_1, \dots, E_n respectively.

Then

$$\sum_{j_2=m}^{\infty} \dots \sum_{j_n=m}^{\infty} \frac{(j_1+\dots+j_n)!}{j_1! \dots j_n!} p_1^{j_1+1} p_2^{j_2} \dots p_n^{j_n}$$

is the probability of E_r happening at least m times, for all $r=2, \dots, n$, before E_1 is happening j_1+1 times. EU_2 is obtained from EU_1 by just interchanging λ_1 and λ_2 . The associated multinomial probability is now the one discussed above with E_1 and E_2 interchanged. Since $p_1 < p_2$, this new probability is the smaller. Furthermore, $\lambda_1^{-1} > \lambda_2^{-1}$. Hence $EU_1 > EU_2$ and the proof is completed. \square

Now introduce the random variable

W_i = reduction in remaining system lifetime due to the wear and the failure of the i th component.

Analogous to (1.5) it is easy to prove that

$$EW_i = EU_i.$$

Moreover, the distribution of W_i and especially $P(W_i=0)$ can be established as for Z_i in Natvig (1982). Note that wear comes into the distribution of Z_i just indirectly, as opposed to W_i , through $f_i(t)$. As a conclusion the $I_{N_3}^{(i)}$ measure is mathematically rather inconvenient, but its behaviour revealed so far gives no reason for dismissing it. (i)

We finally turn to the I_{N_2} measure. At least for the special case where components are exponentially distributed this measure is easily motivated since λ_i^{-1} is the expected lifetime of the i th component.

Remembering the relation (1.11) it is immediately checked that Lemmas 2.2, 2.3 hold for the $I_{N_2}^{(i)}$ measure. However, when considering Theorem 2.4 in this case, we end up with the following results.

Theorem 3.3.

Let the i th and j th component be in series (parallel) with the system. (Let $R(t)/R'(t)$ be increasing for all $t > 0$, covering the case where the life lengths of the components are Weibull distributed with the same shape parameter.) Let for $j \neq i$ $F_i(t) > F_j(t)$ ($\bar{F}_i(t) > \bar{F}_j(t)$) for all $t > 0$. Then

$$I_{N_2}^{(i)} > I_{N_2}^{(j)}.$$

Proof.

Again note that $F_i(t) > F_j(t)$ ($\bar{F}_i(t) > \bar{F}_j(t)$) for all $t > 0$, is just equivalent to $\lambda_i > \lambda_j$ ($\lambda_i < \lambda_j$). For the series case we get from (1.11)

$$\frac{\partial ET}{\partial \lambda_i^{-1}} = \lambda_i^2 \int_0^{\infty} R(t) \exp(-(\lambda_i + \lambda_j)R(t)) h(1_i, 1_j, \bar{F}(t)) dt,$$

which is obviously no less than $\partial ET / \partial \lambda_j^{-1}$ and this part of the proof is completed.

If the i th and j th component are in parallel with the rest of the system, we get from (1.11) and the corresponding part of the proof of Theorem 2.4

$$\begin{aligned} \frac{\partial ET}{\partial \lambda_i^{-1}} - \frac{\partial ET}{\partial \lambda_j^{-1}} &= \int_0^{\infty} [1-h(0_i, 0_j, \bar{F}(t))] F_i(t) F_j(t) / R(t) \\ &\times [\bar{F}_i(t) (-\ln \bar{F}_i(t))^2 / F_i(t) - \bar{F}_j(t) (-\ln \bar{F}_j(t))^2 / F_j(t)] dt \end{aligned}$$

Introduce the functions

$$g(x) = x(\ln x)^2 / (1-x), \quad 0 < x < 1,$$

and

$$\phi(t) = g(\bar{F}_i(t)) - g(\bar{F}_j(t)), \quad t > 0.$$

Now it is easy to see that $g(x)$ obtains a single maximum in $(0,1)$ and that $g(0) = g(1) = 0$. Since by assumption $\bar{F}_i(t) > \bar{F}_j(t)$ for all $t > 0$, and both functions are decreasing in t , there exists $t_0 > 0$ such that

$$\phi(t) < 0 \quad \text{for } 0 < t < t_0, \quad \phi(t) > 0 \quad \text{for } t > t_0.$$

Hence since $(1-h(0_i, 0_j, \bar{F}(t))R(t)/R'(t))$ is increasing in t :

$$\begin{aligned} \frac{\partial ET}{\partial \lambda_i^{-1}} - \frac{\partial ET}{\partial \lambda_j^{-1}} &> \int_0^{t_0} [1-h(0_i, 0_j, \bar{F}(t_0))] (R(t_0)/R'(t_0)) \\ &\times (F_i(t) F_j(t)) R'(t) / (R(t))^2 \phi(t) dt \\ &+ \int_{t_0}^{\infty} [1-h(0_i, 0_j, \bar{F}(t_0))] (R(t_0)/R'(t_0)) \\ &\times (F_i(t) F_j(t)) R'(t) / (R(t))^2 \phi(t) dt \\ &= [1-h(0_i, 0_j, \bar{F}(t_0))] (R(t_0)/R'(t_0)) \\ &\times \int_0^{\infty} (F_i(t) F_j(t)) R'(t) / (R(t))^2 \phi(t) dt \end{aligned}$$

Note that $R'(t) > 0$, $t > 0$. By substituting $u = R(t)$ in the integral, this is reduced to zero, and the proof is completed. \square

Theorem 3.3 leads to the speculation of what might happen if $R(t)/R'(t)$ is strictly decreasing or constant in the parallel case. The following theorem answers this question.

Theorem 3.4

Consider a parallel system of two components and let $R(t)/R'(t)$ be strictly decreasing (constant) for all $t > 0$. Let $\bar{F}_1(t) > \bar{F}_2(t)$ for all $t > 0$. Then:

$$I_{N_2}^{(1)} < I_{N_2}^{(2)} \quad (I_{N_2}^{(1)} = I_{N_2}^{(2)})$$

Proof

The proof is completely parallel to the one above, just noting that since $n = 2$, $1 - h(0_i, 0_j, \bar{F}(t)) = 1$.

Hence we have found an example where according to the $I_{N_2}^{(i)}$ measure, the poorest component in a parallel system is the most important. This is intuitively unacceptable. An example where $R(t)/R'(t)$ is strictly decreasing (constant) is $R(t) = e^{t^2}$ ($R(t) = e^t$). Hence the $I_{N_2}^{(i)}$ measure is unacceptable for systems of components wearing out rapidly.

4. COMPARISON OF THE WEIGHT FUNCTIONS

The discussion in this section is just meant to be preliminary, postponing closer numerical comparisons on real life systems to a later paper.

First of all there is a basic difference between the $I_{B-P}^{(i)}$ measure and the $I_{N_k}^{(i)}$, $k=1,2,3,4$ measures, which we illustrate by looking at the $I_{N_1}^{(i)}$ measure. From (1.1) and (1.5)

$$\begin{aligned} I_{N_1}^{(i)} &= \int_0^\infty \bar{F}_i(t) (-\ln \bar{F}_i(t)) I_B^{(i)}(t) dt / \sum_{j=1}^n \int_0^\infty \bar{F}_j(u) (-\ln \bar{F}_j(u)) I_B^{(j)}(u) du \\ &= \int_0^\infty w_{N_1}^{(i)}(t) I_B^{(i)}(t) dt, \end{aligned}$$

where

$$w_{N_1}^{(i)}(t) = \bar{F}_i(t) (-\ln \bar{F}_i(t)) / \sum_{j=1}^n \int_0^\infty \bar{F}_j(u) (-\ln \bar{F}_j(u)) I_B^{(j)}(u) du.$$

Hence the weight function $w_{N_1}^{(i)}(t)$ depends on $I_B^{(i)}(t)$, which is somewhat awkward. This is the prize for ensuring that the measures of importance of the components add up to one. Secondly,

$$\int_0^{\infty} w_{N_1}^{(i)}(t) dt = k_i ,$$

depends on i , as is easily checked in the exponential case. For $I_{B-P}^{(i)}$ we have $w_{B-P}^{(i)}(t) = f_i(t)$, integrating up to one. This might suggest introducing a modified weight function,

$$w_{N_1}^{(i)*}(t) = w_{N_1}^{(i)}(t)/k_i ,$$

with corresponding measure

$$I_{N_1}^{(i)*} = \int_0^{\infty} w_{N_1}^{(i)*}(t) I_B^{(i)}(t) dt.$$

Since $w_{N_1}^{(i)*}(t)$ may be interpreted as a modified probability density of the lifetime of the i th component, these measures add up to one. The drawback of this measure is that a lot of intuition is lost.

One objection against the $I_B^{(i)}(t)$ measure is that it gives the importance of the components at fixed points of time leaving for the analyst to determine these points. We now have several theories, corresponding to the different weight functions, to determine which time points are important for a time independent measure. Obviously, points of time where $I_B^{(i)}(t)$ is large must be taken into account as well; i.e. points where the probability of the i th component being critical for system functioning, is large.

To compare these weight functions assume $F_i(t)$ $i=1, \dots, n$ fixed. For the $I_{N_4}^{(i)}$ measure the weight function is proportional to $F_i(t)$, which always gives most weight to large t . This seems unreasonable. Consider the $I_{B-P}^{(i)}$ measure with gamma distributed lifetimes of the components; i.e.

$$w_{B-P}^{(i)}(t) = \lambda_i^\alpha t^{\alpha-1} e^{-\lambda_i t} / \Gamma(\alpha).$$

This function is unimodal with maximum in $t_{B-P}^{(i)} = \max(0, (\alpha-1)/\lambda_i)$, which is less than the expected value α/λ_i . Note that for $\alpha < 1$, including the exponential case ($\alpha=1$), small values of t give most weight, which again seems unreasonable.

Considering the $I_{N_2}^{(i)}$ measure, defined in the proportional hazards case, we see from (1.11) that the weight function is proportional to $\bar{F}_i(t)(-\ln \bar{F}_i(t))$, leaving the discussion to the one of the $I_{N_1}^{(i)}$ measure. It is now easy to see that $w_{N_1}^{(i)}(t)$ starts out in 0 for $t=0$, obtains a single maximum for $t_{N_1}^{(i)} = F_i^{-1}(1-e^{-1})$, the 0.368 upper point in the life distribution of the component, before asymptotically approaching zero when $t \rightarrow \infty$. In the proportional hazards case

$$t_{N_1}^{(i)} = R^{-1}(\lambda_i^{-1}),$$

which reduce to $\lambda_i^{-1/\alpha}$ in the Weibull case. Note that the expected value is $\Gamma(\frac{\alpha+1}{\alpha})\lambda_i^{-1/\alpha}$ in the latter case. For the exponential case $t_{N_1}^{(i)}$ equals the expectation. In the gamma case

$$t_{N_1}^{(i)} = \frac{1}{2\lambda_i} \chi_{2\alpha, 1-e^{-1}}^2,$$

where $\chi_{2\alpha, 1-e^{-1}}^2$ is the lower $1-e^{-1}$ point in the χ^2 -distribution with 2α degrees of freedom.

Now finally we turn to the $I_{N_3}^{(i)}$ measure. Then the weight function is proportional to

$$\int_0^t f_i(t-u)\bar{F}_i(u)du = F_i(t) - \int_0^\infty f_i(u)F_i(t-u)du.$$

Hence $w_{N_3}^{(i)}(t)$ starts out in 0 for $t=0$ and approaches 0 when $t \rightarrow \infty$. It obtains extremal points for values of t satisfying

$$f_i(t) = \int_0^t f_i(u)f_i(t-u)du.$$

In the gamma case $w_{N_3}^{(i)}(t)$ obtains a single maximum for

$$t_{N_3}^{(i)} = (\Gamma(2\alpha)/\Gamma(\alpha))^{\alpha/\lambda_i}.$$

We immediately realize that for $\alpha > 1$

$$t_{N_3}^{(i)} > (\alpha^\alpha)^{1/\alpha} / \lambda_i = \alpha / \lambda_i ,$$

so as opposed to $t_{B-P}^{(i)}$, which is always less than the expected value in the gamma case, $t_{N_3}^{(i)}$ is always larger for $\alpha > 1$. To compare $t_{N_1}^{(i)}$, $t_{N_3}^{(i)}$, $t_{B-P}^{(i)}$ and the expectation in the gamma case, we have worked out the following table assuming for simplicity that $\lambda_i = 1$.

α	Expected value	$t_{N_1}^{(i)}$	$t_{N_3}^{(i)}$	$t_{B-P}^{(i)}$
$\frac{1}{2}$	$\frac{1}{2}$	≈ 0.42	0.32	0
1	1	1	1	0
2	2	≈ 2.15	2.45	1
4	4	≈ 4.3	5.38	3
10	10	≈ 10.7	14.21	9

Table 1. Table of $t_{N_1}^{(i)}$, $t_{N_3}^{(i)}$, $t_{B-P}^{(i)}$ for various gamma distributions.

We can obviously conclude that $t_{N_1}^{(i)}$ is the one which is always closest to the expected value. Note that in the gamma case, the failure rate is decreasing in time for $\alpha < 1$. Hence a minimal repair in this case is better than a total repair. For $\alpha > 1$ it is the other way round.

Remembering the discussion of the weight function for the $I_{N_4}^{(i)}$ measure, we may roughly say, for all measures, that the more the i th component's distribution is improved, the more weight is put on large values of time. This explains why $t_{N_1}^{(i)} > t_{N_3}^{(i)}$ for $\alpha = \frac{1}{2}$, whereas $t_{N_1}^{(i)} < t_{N_3}^{(i)}$ for $\alpha = 2, 4, 10$ in Table 1. At least to us a total repair in the case of life distributions with decreasing failure rate, or perhaps more realistically a "bathtub" shape failure rate, seems not very sensible. In any case a minimal repair

seems to be the right amount of improvement. The "repair" corresponding to the $I_{B-P}^{(i)}$ measure seems to be too small.

If one does agree with these points, this leaves us with the $I_{N_1}^{(i)}$ and $I_{N_2}^{(i)}$ measures. The latter is harder to motivate, it is restricted to the proportional hazards case and it is at least unacceptable for systems of components wearing out rapidly. Hence we end up with the $I_{N_1}^{(i)}$ measure.

ACKNOWLEDGEMENTS

We are thankful to Morten Sørum and Professor Bo Bergman for thoughtprovoking comments without which this paper had never been written, and to Arne Bang Huseby for clarifying discussions and very valuable comments on the manuscript. Theorem 3.3 was originally proved for a parallel system with Weibull distributed life lengths of the components. The proof in the more general case is due to him along with that of Theorem 3.4.

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