On Lifemonitoring and Conditional Lifemonitoring of System Components with Application to Preventive System Maintenance

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Abstract

Consider a binary, monotone system of n independent components having absolutely continuous lifetime distributions. In Meilijson (1994) lifemonitoring of some components and conditional lifemonitoring of some others is considered. In the present paper the corresponding complete likelihood functions for the parameter vector, $\underline{\theta}$, of the general lifetime distributions of the components are arrived at, also covering the situation where the socalled autopsy data are not observed due to censoring. A marked point process framework is applied inspired by Arjas (1989). The construction of appropriate inspection strategies linked to the conditional lifemonitoring is considered in detail. Furthermore, preventive system maintenance is considered where components are replaced according to a specific strategy. Based on the likelihood functions a fully Bayesian approach to estimation of $\underline{\theta}$ is possible. For the case of exponentially distributed component lifetimes it is shown that the weighted sum of products of generalized gamma distributions, as introduced in Gåsemyr & Natvig (1998), is the natural conjugate prior for $\underline{\theta}$.

Key words: marked point process, likelihood function, censoring, autopsy data, inspection function, component replacement, Bayesian estimation, weighted sum of products of generalized gamma distributions.

1. Introduction

Consider a binary, monotone system (E, ϕ), where $E = \{1, \ldots, n\}$ is the set of components and ϕ is the structure function describing the state of the system in terms of the binary states of the components. The system may be a technological one, or a human being. We assume the components to be independent with absolutely continuous lifetime distributions. Denote the lifetime of the system by T and the lifetime of the ith component by T_i, with distribution function $F_i(t)$, survival function $\overline{F}_i(t) = 1 - F_i(t)$, p.d.f. $f_i(t)$ and failure rate $\lambda_i(t) = f_i(t)/\overline{F}_i(t)$, $i \in E$. Introduce $\underline{\overline{F}}(t) = (\overline{F}_1(t), \ldots, \overline{F}_n(t))$.

The state of the ith component at time t is denoted $X_i(t)$ and we have $X_i(t)=I(T_i>t)$, $i \in E$. Let $\underline{X}(t) = (X_1(t), \dots, X_n(t))$. We then have $\phi(\underline{X}(t)) = I(T > t)$. The reliability function, $h(\underline{\overline{F}}(t))$, of the system is given by $h(\underline{\overline{F}}(t)) = E\phi(\underline{X}(t)) = P(T > t)$. A path set, P, for the system is a set of components which ensures the functioning of the system if all components in P are functioning. Hence, $\phi(\underline{1}_P, \underline{0}_{P^c}) = 1$. The set P is called a minimal path set if no proper subset of P is a path set.

Furthermore, let

$$D = \text{the set of failed components} = \{i \in E | T_i \leq T\}.$$

A is a fatal set if and only if $P(D = A) > 0$. Introduce

$$\mathcal{A} = \{\text{fatal sets}\} = \{A \subset E | P(D = A) > 0\} = \{A_1, \dots, A_m\}.$$

Assume as a start that the system is observed until it fails. At this instant, the set of failed components, D, and the failure time of the system, T, are noted. The failure times of the components are not known. (T, D) are the socalled autopsy data of the system. Meilijson (1981), Nowik (1990), Antoine et al. (1993) and Gåsemyr (1998) discuss the corresponding identifiability problem; i.e. whether the distributions of T_i, $i \in E$ can be determined from the distribution of the autopsy data (T, D).

Following these papers let

$$\begin{aligned} C_A &= \text{the critical set corresponding to the fatal set } A \\ &= \{i \in A | P(T_i = T | D = A) > 0\} = \{i \in A | A^c \cup \{i\} \text{ is a path set}\}. \end{aligned}$$

This set consists of those components of the fatal set A which may have failed when the system failed at T and thus may have caused the failure of the system. The distribution of the autopsy data (T, D) is given by

$$G_A(t) = P(T \leq t,\ D = A)$$

with density function

$$g_A(t) = \frac{d}{dt} G_A(t).$$

The latter can be considered as a likelihood function on the space $R^+ \times \{1, 2, \dots, m\}$ with respect to the measure

$$\mu = \text{Lebesgue measure} \times \text{counting measure}.$$

The following result, essentially given in Meilijson (1981), is straightforward

$$g_A(t) = \sum_{i \in C_A} \lambda_i(t) \prod_{\ell \in A - \{i\}} F_\ell(t) \prod_{\ell \in A^c \cup \{i\}} \overline{F}_\ell(t). \tag{1.1}$$

In Gåsemyr & Natvig (1998) (1.1) is generalized to the case where components are dependent through the possible occurrence of independent common shocks, i.e. shocks that destroy several components at once.

In the present paper (1.1) is generalized in another direction. Assuming a model where autopsy data is known to be enough for identifiability, Meilijson (1994) goes beyond the identifiability question and into maximum likelihood estimation of the parameters of the component lifetime distributions based on empirical autopsy data from a sample of several systems. A corresponding Bayesian approach is indicated in Gåsemyr & Natvig (1998) for the mentioned shock model. Meilijson (1994) and Gåsemyr (1998) also considers lifemonitoring of some components and conditional lifemonitoring of some others. In Section 2 of the present paper a complete likelihood function for the parameter vector, $\underline{\theta}$, of the general lifetime distributions of the components is arrived at in the case where some components are lifemonitored, also covering the situation where autopsy data are not observed due to censoring. A marked point process framework is applied inspired by Arjas (1989). In Section 3 the corresponding likelihood function for the case where in addition some other components are conditionally lifemonitored, is given. These likelihood functions are generalizations of (1.1). The construction of appropriate inspection strategies linked to the conditional lifemonitoring is considered in detail at the end of Section 3.

In Section 4 we consider preventive system maintenance where components are replaced according to a specific strategy. Based on the likelihood functions a fully Bayesian approach to estimation of $\underline{\theta}$ is possible. For the case of exponentially distributed component lifetimes it is shown in Section 5 that the sum of products of generalized gamma distributions, as introduced in Gåsemyr & Natvig (1998), is the natural conjugate prior for $\underline{\theta}$. This section is concluded by considering a specific example.

2. Lifemonitored components

To know the autopsy data (T, D) means to know T and to know which component lifetimes are at most T and which are above T. The order of failure of the components and the failure times are indeed unknown. In actual practice, often some of the components are lifemonitored until system failure. Let

$$\begin{aligned} M &= \text{the set of lifemonitored components} \\ &= \{1, \cdots, p\} \subset E, \ 1 \leq p \leq n. \end{aligned}$$

This means that for $i \in M$ and $T_i \leq T, T_i$ is known. In this section a complete likelihood function for this case is arrived at for general lifetime distributions of the components, also covering the situation where the autopsy data are not observed due to censoring.

Let $Z_0^* = 0$ and

$$(Z_1^*, \cdots, Z_p^*) =$$

the order statistics of the lifetimes of the lifemonitored components.

$$\begin{aligned} Z_k &= Z_k^* \wedge T, \qquad k = 0, \cdots, p \\ Z_{p+1} &= T \end{aligned}$$

Z_k, $k = 1, \cdots, p+1$ are the points of time where either a component or system failure (or both) is observed. The number, K, of different such time points until system failure is at most $p+1$. We obviously have

$$K = \max\{k \in \{1, 2, \ldots\} | Z_k \neq Z_{k-1}\}.$$

Suppose $V > 0$ is a censoring time, either fixed in advance or being a random variable, being independent of T_i, $i = 1, \ldots, n$, and not depending on our parameter vector $\underline{\theta}$. The number, L, of different time points until system failure or censoring is given by

$$L = \max\{k \leq K | Z_k < V\}.$$

Introduce for $k = 1, \cdots, p+1(p)$

$I_k(I_k^*) = i$	if the ith lifemonitored component fails at time $Z_k(Z_k^*)$ (at which time the system may fail), $i \in M$
$I_k = 0$	if the system fails at time Z_k due to the failure of a non lifemonitored component
$J_k = j$	if the system fails at time Z_k with fatal set A_j, $j \in \{1, \cdots, m\}$
$J_k = 0$	if the system does not fail at time Z_k
$J_K = J$	
$R_0 = M$	

$R_k = M - \{I_1^*, \cdots, I_k^*\}$ = the set of lifemonitored components being at risk just after Z_k^*.

Now let $R \subset M$ be a set of lifemonitored components at risk and $R^c = M - R$ the corresponding set of failed components. Define

$$\begin{aligned} F(R, i) &= \{j \in \{1, \cdots, m\} | R^c \subset A_j,\ R \subset A_j^c,\ i \in C_{A_j} \cap R^c\} \qquad (2.1) \\ &= \text{the set of possible fatal sets, for which we also know that the} \\ &\quad\ \text{lifemonitored component } i \text{ is a member of the corresponding critical set.} \end{aligned}$$

For any i such that $j \in F(R, i)$, we then introduce

$$\begin{aligned} P_j(t) &= P[(J = j) \cap (T = t) | \left(\bigcap_{\ell \in R^c - \{i\}} (T_\ell \leq t) \right) \bigcap \left(\bigcap_{\ell \in R} (T_\ell > t) \right) \bigcap (T_i = t)] \\ &= \prod_{\ell \in A_j - M} F_\ell(t) \prod_{\ell \in A_j^c - M} \overline{F}_\ell(t). \qquad (2.2) \end{aligned}$$

Define

$$F(R) = \{j \in \{1, \cdots, m\} | R^c \subset A_j,\ R \subset A_j^c,\ C_{A_j} - M \neq \emptyset\}. \qquad (2.3)$$

For any $j \in F(R)$, we then introduce

$$P_j(s,t) = P[(J=j) \cap (s < T < t) | \left(\bigcap_{\ell \in R^c} (T_\ell \le s) \right) \bigcap \left(\bigcap_{\ell \in R} (T_\ell > t) \right)]$$
$$= \int_s^t \sum_{i \in C_{A_j} - M} \lambda_i(u) \prod_{\ell \in (A_j - M) - \{i\}} F_\ell(u) \prod_{\ell \in (A_j^c - M) \cup \{i\}} \overline{F}_\ell(u) du \,. \qquad (2.4)$$

For $k = 0, \cdots, K-1,\ t \ge Z_k^*$, introduce

$$R_k(t) = \{T_i > t,\ i \in R_k\} =$$

the event that all lifemonitored components at risk just after Z_k^*, are still at risk just after t.

$$\begin{aligned}
\mathcal{B}_0 &= \emptyset \\
\mathcal{B}_k &= \{Z_1, I_1, J_1, \cdots, Z_k, I_k, J_k\} \\
&= \text{the available information just after } Z_k \\
\mathcal{E}_0 &= \emptyset \\
\mathcal{E}_k &= \{Z_1^*, I_1^*, \ldots, Z_k^*, I_k^*\} \\
&= \text{the available information just after } Z_k^* \text{ on the lifemonitored components.}
\end{aligned}$$

For $k = 0, \ldots, K-1$ we have

$$\text{Information in } \{\mathcal{B}_k\} = \text{Information in } \{\mathcal{E}_k \cap (T > Z_k^*)\}. \qquad (2.5)$$

The fundamental theorem in this section is the following.

Theorem 2.1 *Let*

$$S = V \wedge T$$
$$\Delta_i = I(T_i < S), \qquad i \in M,$$

where M is the set of lifemonitored components. Then the complete likelihood function for our parameter vector, $\underline{\theta}$, is given by

$$\begin{aligned}
L(\underline{\theta}) = & \prod_{i \in M} (\lambda_i(T_i))^{\Delta_i} \overline{F}_i(T_i \wedge S) \\
& \{I(V > S)[I(I_L \ne 0)\lambda_{I_L}(S) \prod_{\ell \in A_J - M} F_\ell(S) \prod_{\ell \in A_J^c - M} \overline{F}_\ell(S) \\
& + I(I_L = 0) \sum_{i \in C_{A_J - M}} \lambda_i(S) \prod_{\ell \in A_J - M - \{i\}} F_\ell(S) \prod_{\ell \in (A_J^c - M) \cup \{i\}} \overline{F}_\ell(S)] \\
& + I(V = S) h(\underline{1}_{R_L}, \underline{0}_{R_L^c}, \underline{\overline{F}}(S))\}.
\end{aligned}$$

Note that the product

$$\prod_{i \in M} (\lambda_i(T_i))^{\Delta_i} \overline{F}_i(T_i \wedge S)$$

represents the full likelihood function for the lifemonitored components up till just before system failure or censoring. The factor multiplied by $I(V > S)I(I_L \neq 0)$ is the intuitively obvious contribution to the likelihood function from a system failure due to the failure of a lifemonitored component. Similarly, the factor multiplied by $I(V > S)I(I_L = 0)$ is the intuitively obvious contribution from a system failure due to the failure of a non lifemonitored component. Finally, the factor multiplied by $I(V = S)$ is the intuitively obvious conditional survival probability of the system up till censoring. By setting $M = \emptyset$, $V = \infty$, $S = t$, noting that we then always have $I_L = 0$, $L(\underline{\theta})$ reduces to (1.1).

To prove the theorem we consider the process (Z_k, I_k, J_k) as a marked point process with (I_k, J_k) as the mark at Z_k, $k = 1, 2, \ldots$. In the following lemma we compute the intensities associated with this marked point process.

Lemma 2.2 *For $k = 0, \ldots, K-1$, $i \in \{0, 1, \ldots, n\}$, $j \in \{0, 1, \ldots, m\}$, $t \geq Z_k = Z_k^*$ define*

$$\begin{aligned} &\rho_{ij}(t; Z_1, I_1, \ldots, Z_k, I_k) \\ &\quad = \lim_{dt \to 0} P[(t < Z_{k+1} \leq t + dt) \cap (I_{k+1} = i) \cap (J_{k+1} = j) | \mathcal{B}_k \cap (Z_{k+1} > t)]/dt. \end{aligned}$$

We then have

$$\rho_{ij}(t; Z_1, I_1, \ldots, Z_k, I_k) = \frac{\gamma_{ij}(t; Z_1, I_1, \ldots, Z_k, I_k)}{h(\underline{1}_{R_k}, \underline{0}_{R_k^c}, \underline{\overline{F}}(t))}, \tag{2.6}$$

where

i) *for $i \in R_k$, $j \in F(R_k - \{i\}, i)$*

$$\gamma_{ij}(t; Z_1, I_1, \ldots, Z_k, I_k) = \lambda_i(t) P_j(t),$$

ii) *for $i \in R_k$*

$$\gamma_{i0}(t; Z_1, I_1, \ldots, Z_k, I_k) = \lambda_i(t) h(\underline{1}_{R_k - \{i\}}, \underline{0}_{R_k^c \cup \{i\}}, \underline{\overline{F}}(t)),$$

iii) *for $j \in F(R_k)$*

$$\gamma_{0j}(t; Z_1, I_1, \ldots, Z_k, I_k) = \frac{d}{dt} P_j(s, t),$$

where the right hand side is just the integrand of (2.4) with $u = t$, s being arbitrary,

iv) *otherwise*

$$\gamma_{ij}(t; Z_1, I_1, \ldots, Z_k, I_k) = 0.$$

Proof. For $k = 0, \ldots, K-1$, $i \in \{0, 1, \ldots, n\}$, $j \in \{0, 1, \ldots, m\}$, $t \geq Z_k = Z_k^*$, we have using (2.5) and the fact that $T \geq Z_{k+1}$

$$\begin{aligned}
&\rho_{ij}(t; Z_1, I_1, \ldots, Z_k, I_k) \\
&\quad = \lim_{dt \to 0} P[(t < Z_{k+1} \leq t + dt) \cap (I_{k+1} = i) \cap (J_{k+1} = j) | \mathcal{E}_k \cap (T > Z_k^*) \cap R_k(t) \cap (T > t)]/dt \\
&\quad = \lim_{dt \to 0} P[(t < Z_{k+1} \leq t + dt) \cap (I_{k+1} = i) \cap (J_{k+1} = j) | \mathcal{E}_k \cap R_k(t) \cap (T > t)]/dt \\
&\quad = \frac{\lim_{dt \to 0} P[(t < Z_{k+1} \leq t + dt) \cap (I_{k+1} = i) \cap (J_{k+1} = j) | \mathcal{E}_k \cap R_k(t)]/dt}{\mathcal{P}(T > t | \mathcal{E}_k \cap R_k(t))} .
\end{aligned}$$

Hence (2.6) follows by defining

$$\begin{aligned}
&\gamma_{ij}(t; Z_1, I_1, \ldots, Z_k, I_k) \\
&\quad = \lim_{dt \to 0} P[(t < Z_{k+1} \leq t + dt) \cap (I_{k+1} = i) \cap (J_{k+1} = j) | \mathcal{E}_k \cap R_k(t)]/dt,
\end{aligned}$$

and noting that

$$P(T > t | \mathcal{E}_k \cap R_k(t)) = P[\phi(\underline{1}_{R_k}, \underline{0}_{R_k^c}, \underline{X}(t)) = 1] = h(\underline{1}_{R_k}, \underline{0}_{R_k^c}, \underline{\overline{F}}(t)).$$

i) For $i \in R_k$, $j \in F(R_k - \{i\}, i)$

$$\begin{aligned}
&\gamma_{ij}(t; Z_1, I_1, \ldots, Z_k, I_k) \\
&= \lim_{dt \to 0} P\left[(t < T_i \leq t + dt) \cap \left(\bigcap_{\ell \in A_j - \{i\}} (T_\ell \leq T_i) \right) \cap \left(\bigcap_{\ell \in A_j^c} (T_\ell > T_i) \right) \middle| \mathcal{E}_k \cap R_k(t)\right]/dt \\
&= \lim_{dt \to 0} P\left[(t < T_i \leq t + dt) \cap \left(\bigcap_{\ell \in A_j - M} (T_\ell \leq T_i) \right) \cap \left(\bigcap_{\ell \in (A_j^c - M) \cup (R_k - \{i\})} (T_\ell > T_i) \right) \middle| \mathcal{E}_k \cap R_k(t)\right]/dt \\
&= \lim_{dt \to 0} P\left[(t < T_i \leq t + dt) \cap \left(\bigcap_{\ell \in A_j - M} (T_\ell \leq t) \right) \cap \left(\bigcap_{\ell \in (A_j^c - M) \cup (R_k - \{i\})} (T_\ell > t) \right) \middle| \mathcal{E}_k \cap R_k(t)\right]/dt \\
&= \lim_{dt \to 0} P\left[(t < T_i \leq t + dt) \cap \left(\bigcap_{\ell \in A_j - M} (T_\ell \leq t) \right) \cap \left(\bigcap_{\ell \in A_j^c - M} (T_\ell > t) \right) \middle| \mathcal{E}_k \cap R_k(t)\right]/dt \\
&= \lim_{dt \to 0} \{P[t < T_i \leq t + dt | T_i > t]/dt\} P\left[\left(\bigcap_{\ell \in A_j - M} (T_\ell \leq t) \right) \cap \left(\bigcap_{\ell \in A_j^c - M} (T_\ell > t) \right)\right] \\
&= \lambda_i(t) P_j(t)
\end{aligned}$$

ii) For $i \in R_k$

$$
\begin{aligned}
&\gamma_{i0}(t; Z_1, I_1, \ldots, Z_k, I_k) \\
&\quad = \lim_{dt \to 0} P[(t < T_i \le t + dt) \cap R_k(T_i) \cap (T > Z^*_{k+1}) | \mathcal{E}_k \cap R_k(t)]/dt \\
&\quad = \lim_{dt \to 0} \{P[R_k(T_i) \cap (T > T_i) | \mathcal{E}_k \cap R_k(t) \cap (t < T_i \le t + dt)] P[t < T_i \le t + dt | T_i > t]/dt\} \\
&\quad = \lambda_i(t) \lim_{dt \to 0} \Big\{P[T > T_i | \mathcal{E}_k \cap R_k(T_i) \cap (t < T_i \le t + dt)] \\
&\qquad\qquad \times P[R_k(T_i) | \mathcal{E}_k \cap R_k(t) \cap (t < T_i \le t + dt)]\Big\} \\
&\quad = \lambda_i(t) \lim_{dt \to 0} P[\phi(\underline{1}_{R_k - \{i\}}, \underline{0}_{R_k^c \cup \{i\}}, \underline{X}(T_i)) = 1 | t < T_i \le t + dt] \\
&\quad = \lambda_i(t) h(\underline{1}_{R_k - \{i\}}, \underline{0}_{R_k^c \cup \{i\}}, \underline{\overline{F}}(t))
\end{aligned}
$$

iii) For $j \in F(R_k)$

$$
\begin{aligned}
&\gamma_{0j}(t; Z_1, I_1, \ldots, Z_k, I_k) \\
&\quad = \lim_{dt \to 0} P\left[\bigcup_{i \in C_{A_j} - M} \left\{(t < T_i \le t + dt) \cap \left(\bigcap_{\ell \in A_j - \{i\}} (T_\ell \le T_i)\right)\right.\right. \\
&\qquad\qquad \left.\left. \cap \left(\bigcap_{\ell \in A_j^c} (T_\ell > T_i)\right)\right\} | \mathcal{E}_k \cap R_k(t)\right] /dt \\
&\quad = \sum_{i \in C_{A_j} - M} \lim_{dt \to 0} P[(t < T_i \le t + dt) \cap \left(\bigcap_{\ell \in (A_j - M) - \{i\}} (T_\ell \le t)\right) \cap \left(\bigcap_{\ell \in A_j^c - M} (T_\ell > t)\right) | T_i > t]/dt \\
&\quad = \frac{d}{dt} P_j(s, t)
\end{aligned}
$$

iv) This is obvious.

Proof of Theorem 2.1. Introducing $\Delta = \sum_{i=1}^{n} \Delta_i$, we may write the likelihood, $L(\underline{\theta})$, as

$$
\begin{aligned}
L(\underline{\theta}) &= \left\{\prod_{k=0}^{\Delta - 1} [P(Z_{k+1} > t | \mathcal{B}_k) \rho_{I_{k+1} J_{k+1}}(t; Z_1, I_1, \ldots, Z_k, I_k)]_{t = Z_{k+1}}\right\} \\
&\quad \times \{P(Z_{k+1} > t | \mathcal{B}_k)[I(V > S) \rho_{I_{\Delta+1} J}(t; Z_1, I_1, \ldots, Z_\Delta, I_\Delta) + I(V = S)]\}_{(t,k) = (S, \Delta)}. \quad (2.7)
\end{aligned}
$$

By applying (2.5) we have since $t \geq Z_k^*$

$$\begin{aligned} &P(Z_{k+1} > t|\mathcal{B}_k) = P(R_k(t) \cap (T > t)|\mathcal{B}_k) \\ &= P(R_k(t)|\mathcal{B}_k)P(T > t|\mathcal{E}_k \cap (T > Z_k^*) \cap R_k(t)) \\ &= \prod_{i \in R_k} (\overline{F}_i(t)/\overline{F}_i(Z_k))P(T > t|\mathcal{E}_k \cap R_k(t))/P(T > Z_k^*|\mathcal{E}_k) \\ &= \prod_{i \in R_k} (\overline{F}_i(t)/\overline{F}_i(Z_k))h(\underline{1}_{R_k}, \underline{0}_{R_k^c}, \underline{\overline{F}}(t))/h(\underline{1}_{R_k}, \underline{0}_{R_k^c}, \underline{\overline{F}}(Z_k^*)). \end{aligned}$$

Inserting this into (2.7), applying (2.6), we get

$$\begin{aligned} L(\underline{\theta}) = &\prod_{k=0}^{\Delta-1} \left[\prod_{i \in R_k} (\overline{F}_i(Z_{k+1})/\overline{F}_i(Z_k)) \right] \prod_{i \in R_\Delta} (\overline{F}_i(S)/\overline{F}_i(Z_\Delta)) \\ &\times \left\{ \left[\prod_{k=0}^{\Delta-1} \gamma_{I_{k+1}0}(Z_{k+1}; Z_1, I_1, \ldots, Z_k, I_k)/h(\underline{1}_{R_k}, \underline{0}_{R_k^c}, \underline{\overline{F}}(Z_k^*)) \right] / h(\underline{1}_{R_\Delta}, \underline{0}_{R_\Delta^c}, \underline{\overline{F}}(Z_\Delta^*)) \right\} \\ &\times \{ I(V > S)\gamma_{I_{\Delta+1}J}(S; Z_1, I_1, \ldots, Z_\Delta, I_\Delta) \\ &+ I(V = S)h(\underline{1}_{R_\Delta}, \underline{0}_{R_\Delta^c}, \underline{\overline{F}}(S))\}. \end{aligned}$$

We now apply Lemma 2.2, noting that $(Z_{k+1}, I_{k+1}) = (Z_{k+1}^*, I_{k+1}^*)$ for $k = 0, \ldots, \Delta - 1$. Since $h(\underline{1}_{R_0}, \underline{0}_{R_0^c}, \underline{\overline{F}}(Z_0^*)) = h(\underline{1}) = 1$ and $R_k - \{I_{k+1}^*\} = R_{k+1}$, we get

$$\begin{aligned} L(\underline{\theta}) = &\prod_{k=0}^{\Delta-1} \left[\prod_{i \in R_k} (\overline{F}_i(Z_{k+1})/\overline{F}_i(Z_k)) \right] \prod_{i \in R_\Delta} (\overline{F}_i(S)/\overline{F}_i(Z_\Delta)) \prod_{k=0}^{\Delta-1} \lambda_{I_{k+1}}(Z_{k+1}) \\ &\{ I(V > S)[I(I_{\Delta+1} \neq 0)\lambda_{I_{\Delta+1}}(S)P_J(S) + I(I_{\Delta+1} = 0)\frac{d}{dt}P_J(s,t)|_{t=S}] \\ &+ I(V = S)h(\underline{1}_{R_\Delta}, \underline{0}_{R_\Delta^c}, \underline{\overline{F}}(S))\}. \end{aligned} \tag{2.8}$$

By applying (2.2) and (2.4) our proof is now completed by noting that $L = \Delta + 1$ when $V > S$, whereas $L = \Delta$ when $V = S$.

3. Lifemonitored and conditionally lifemonitored components

In this section we will extend the model of Section 2 and also allow for conditional lifemonitoring of some components. Let

$$\begin{aligned} C &= \text{the set of conditionally lifemonitored components} \\ &= \{p+1, \cdots, p+q\} \subset E \,, \; 1 \leq p < p+q \leq n. \end{aligned}$$

For $i \in C$ there exists some arbitrary stopping time (inspection time), τ_i, such that the ith component is monitored from τ_i onwards until system failure. This means that if

$i \in C$ and $\tau_i < T_i \leq T$, then T_i is known and the ith component is, after τ_i, dealt with as a lifemonitored component. If on the other hand, $T_i \leq \tau_i \leq T$, only this inequality becomes known. The inspection times are assumed to occur immediately after the failure of a component that is currently being monitored. Furthermore, the set of components being inspected each time is chosen according to a specific strategy determined in advance. The idea behind the model is that lifemonitoring of components is expensive and special equipment might be needed. Hence for some components this is started only when we know that the system is in a serious state.

The quantities Z_k^*, Z_k, I_k^*, I_k and R_k are obvious modifications of the corresponding ones in Section 2, whereas the definitions of J_k, J, $R_k(t)$, K and L are exactly the same. The role of R_k^c is now played by Q_k. The new feature is the set H_k of components that are inspected immediately after Z_k^*. This set splits into $H_{k,0}$ and $H_{k,1}$, the sets of components in H_k that are found to have respectively failed or not on inspection.

Formal definitions are given inductively in the following due to the sequential nature of the set up.

$$Z_0^* = Z_0 = 0, \qquad R_0 = M, \qquad Q_0 = \emptyset$$

Assume $R_{k-1} \neq \emptyset$.

$Z_k^* = \min\{T_i | i \in R_{k-1}\}$

$Z_k = Z_k^* \wedge T$

$I_k^* = i \quad \text{if } Z_k^* = T_i$

$$I_k = \begin{cases} I_k^* & \text{if } Z_k = Z_k^* \\ 0 & \text{otherwise} \end{cases}$$

$H_k =$ $\{i \in C | \tau_i = Z_k^*\}$ = a subset of conditionally monitored components in $C - R_{k-1} \cup Q_{k-1}$ being monitored from Z_k^* onwards, determined according to a specific strategy on the basis of information that is or becomes available at Z_k^* about components in $M \cup C$, provided $Z_k^* < T$

$H_{k,0} = \{i \in H_k | X_i(Z_k^*) = 0\}$

$H_{k,1} = \{i \in H_k | X_i(Z_k^*) = 1\}$

$N_k = \bigcup_{\ell=1}^{k} H_\ell$

$N_{k,0} =$ $\bigcup_{\ell=1}^{k} H_{\ell,0}$ = the set of conditionally lifemonitored components being failed on inspections, not after Z_k^*

$N_{k,1} =$ $\bigcup_{\ell=1}^{k} H_{\ell,1}$ = the set of conditionally lifemonitored components being functioning on inspections, not after Z_k^*

$R_k =$ $(R_{k-1} - I_k^*) \cup H_{k,1}$ = the set of lifemonitored and conditionally lifemonitored components being at risk just after Z_k^*

$Q_k =$ $(Q_{k-1} \cup I_k^*) \cup H_{k,0}$ = the set of lifemonitored and conditionally lifemonitored components having failed not after Z_k^*

It should be noted that if $R_k = \emptyset$, there are no more lifemonitored and conditionally lifemonitored components at risk just after $Z_k^* = Z_k$, and we define $Z_{k+1} = T$.

The information obtained by the inspections immediately after Z_k^* can be summarized as a vector $\underline{Y}_k^* = (Y_{k,p+1}^*, \ldots, Y_{k,p+q}^*) \in \{-1, 0, 1\}^q$ defined by

$$\underline{Y}_k^* = (\underline{1}_{H_{k,1}}, \underline{0}_{H_{k,0}}, -\underline{1}).$$

H_k, $H_{k,0}$ and $H_{k,1}$ can be recovered from $\underline{Y}_k^*$ by means of the functions g, g_0 and g_1 from $\{-1, 0, 1\}^q$ into C defined by

$$\begin{aligned} g(\underline{y}) &= \{i \in C | y_i \neq -1\} \\ g_0(\underline{y}) &= \{i \in C | y_i = 0\} \\ g_1(\underline{y}) &= \{i \in C | y_i = 1\}. \end{aligned}$$

Also define

$$\underline{Y}_k = \begin{cases} \underline{Y}_k^* & \text{if } k < K \\ -\underline{1} & \text{if } k = K, \end{cases}$$

reflecting that there is no additional inspection after the failure of the system since by that time the autopsy data (T, D) are known.

Now let $R \subset M \cup C$ be a set of lifemonitored and conditionally lifemonitored components known to be at risk and $Q \subset M \cup C$ a corresponding set of components known to have failed. We have $R \cap Q = \emptyset$. However, since we might lack information on some of the conditionally lifemonitored components, we do not have $Q = M \cup C - R$.

We then define $F(R, Q, i)$ and $F(R, Q)$ by replacing R^c by Q in respectively (2.1) and (2.3). Similarly, we define $P_j^{N_k}(t)$ and $P_j^{N_k}(s, t)$ by replacing M by $M \cup N_k$ in respectively (2.2) and (2.4).

Furthermore, we introduce

$\mathcal{B}_0 = \emptyset$

$\mathcal{B}_k = \{Z_1, I_1, \underline{Y}_1, J_1, \ldots, Z_k, I_k, \underline{Y}_k, J_k\}$

$\mathcal{E}_0 = \emptyset$

$\mathcal{E}_k = \{Z_1^*, I_1^*, \underline{Y}_1^*, \ldots, Z_k^*, I_k^*, \underline{Y}_k^*\} =$

the available information just after Z_k^* on the lifemonitored and conditionally lifemonitored components.

With these definitions (2.5) is still valid for $k = 0, \ldots, K - 1$.

Finally, the inspection strategy is defined in such a way that the following condition is satisfied

$$\begin{aligned} \lim_{dt \to 0} P[\underline{Y}_k^* &= \underline{y} | \mathcal{E}_{k-1} \cap (t < Z_k^* \leq t + dt) \cap (I_k^* = i)] \\ &= \prod_{\ell \in g_0(\underline{y})} F_\ell(t) \prod_{\ell \in g_1(\underline{y})} \overline{F}_\ell(t), \end{aligned} \tag{3.1}$$

for $k = 0, 1, \dots$ and for all $\underline{y} \in \{-1, 0, 1\}^q$ such that the left hand side is positive. Denote these sets by $G_{t,i}^{\mathcal{E}_{k-1}}$, $k = 0, 1, \dots$ We will return to the construction of inspection strategies satisfying (3.1).

The generalization of Theorem 2.1 is the following

Theorem 3.1 *Let*

$$\begin{aligned} S &= V \wedge T \\ \Delta_i &= I(T_i < S), \quad i \in M \\ \Delta_i &= I(\tau_i < T_i < S), \quad i \in C, \end{aligned}$$

where M and C respectively are the sets of lifemonitored and conditionally lifemonitored components.

$$\Delta = \sum_{i=1}^{n} \Delta_i$$

Then the complete likelihood function for our parameter vector, $\underline{\theta}$, is given by

$$\begin{aligned} L(\underline{\theta}) = & \prod_{i \in M \cup C} (\lambda_i(T_i))^{\Delta_i} \prod_{i \in M \cup N_{\Delta,1}} \overline{F}_i(T_i \wedge S) \prod_{i \in N_{\Delta,0}} F_i(\tau_i) \\ & \times \{I(V > S)[I(I_L \neq 0)\lambda_{I_L}(S) \prod_{\ell \in A_J - M - N_{L-1}} F_\ell(S) \prod_{\ell \in A_J^c - M - N_{L-1}} \overline{F}_\ell(S) \\ & + I(I_L = 0) \sum_{i \in C_{A_J} - M - N_{L-1}} \lambda_i(S) \prod_{\ell \in A_J - M - N_{L-1} - \{i\}} F_\ell(S) \prod_{\ell \in (A_J^c - M - N_{L-1}) \cup \{i\}} \overline{F}_\ell(S)] \\ & + I(V = S) h(\underline{1}_{R_L}, \underline{0}_{Q_L}, \underline{\overline{F}}(S))\} . \end{aligned}$$

To prove the theorem we consider the process $(Z_k, I_k, \underline{Y}_k, J_k)$ as a marked point process with $(I_k, \underline{Y}_k, J_k)$ as the mark at Z_k, $k = 1, 2, \dots$ In the following generalization of Lemma 2.2 we compute the intensities associated with this marked point process.

Lemma 3.2 *For $k = 0, \dots, K-1$, $i \in \{0, 1, \dots, n\}$, $j \in \{0, 1, \dots, m\}$, $\underline{y} \in \{-1, 0, 1\}^q$, $t \geq Z_k = Z_k^*$ define*

$$\begin{aligned} & \rho_{i\underline{y}j}(t; Z_1, I_1, \underline{Y}_1, \dots, Z_k, I_k, \underline{Y}_k) \\ & = \lim_{dt \to 0} P[(t < Z_{k+1} \leq t + dt) \cap (I_{k+1} = i) \cap (\underline{Y}_{k+1} = \underline{y}) \cap (J_{k+1} = j) | \mathcal{B}_k \cap (Z_{k+1} > t)]/dt \end{aligned}$$

We then have

$$\rho_{i\underline{y}j}(t; Z_1, I_1, \underline{Y}_1, \dots, Z_k, I_k, \underline{Y}_k) = \frac{\gamma_{i\underline{y}j}(t; Z_1, I_1, \underline{Y}_1, \dots, Z_k, I_k, \underline{Y}_k)}{h(\underline{1}_{R_k}, \underline{0}_{Q_k}, \underline{\overline{F}}(t))}, \tag{3.2}$$

where

i) *for* $i \in R_k$, $j \in F((R_k - \{i\}), (Q_k \cup \{i\}), i)$

$$\gamma_{i(-\underline{1})j}(t; Z_1, I_1, \underline{Y}_1, \ldots, Z_k, I_k, \underline{Y}_k) = \lambda_i(t) P_j^{N_k}(t),$$

ii) *for* $i \in R_k$, $\underline{y} \in G_{t,i}^{\mathcal{E}_k}$

$$\begin{aligned}\gamma_{i\underline{y}0}&(t; Z_1, I_1, \underline{Y}_1, \ldots, Z_k, I_k, \underline{Y}_k) \\ &= \lambda_i(t) \prod_{\ell \in g_0(\underline{y})} F_\ell(t) \prod_{\ell \in g_1(\underline{y})} \overline{F}_\ell(t) h(\underline{1}_{(R_k - \{i\}) \cup g_1(\underline{y})}, \underline{0}_{(Q_k \cup \{i\}) \cup g_0(\underline{y})}, \underline{F}(t)),\end{aligned}$$

iii) *for* $j \in F(R_k, Q_k)$

$$\gamma_{0(-\underline{1})j}(t; Z_1, I_1, \underline{Y}_1, \ldots, Z_k, I_k, \underline{Y}_k) = \frac{d}{dt} P_j^{N_k}(s, t),$$

iv) *otherwise*

$$\gamma_{i\underline{y}j}(t; Z_1, I_1, \underline{Y}_1, \ldots, Z_k, I_k, \underline{Y}_k) = 0.$$

Proof: Except for ii) the proof is completely parallel to the one of Lemma 2.2. Hence we only prove ii).

For $i \in R_k$, $\underline{y} \in G_{t,i}^{\mathcal{E}_k}$

$$\begin{aligned}
&\gamma_{i\underline{y}0}(t; Z_1, I_1, \underline{Y}_1, \ldots, Z_k, I_k, \underline{Y}_k) \\
&\quad = \lim_{dt \to 0} P[(t < T_i \le t + dt) \cap R_k(T_i) \cap (\underline{Y}_{k+1} = \underline{y}) \cap (T > Z_{k+1}^*) | \mathcal{E}_k \cap R_k(t)]/dt \\
&\quad = \lim_{dt \to 0} \{P[R_k(T_i) \cap (\underline{Y}_{k+1} = \underline{y}) \cap (T > T_i) | \mathcal{E}_k \cap R_k(t) \cap (t < T_i \le t + dt)] \\
&\qquad\qquad \times P(t < T_i \le t + dt | T_i > t)/dt\} \\
&\quad = \lambda_i(t) \lim_{dt \to 0} \{P[(T > T_i) \cap (\underline{Y}_{k+1} = \underline{y}) | \mathcal{E}_k \cap R_k(T_i) \cap (t < T_i \le t + dt)] \\
&\qquad\qquad \times P[R_k(T_i) | \mathcal{E}_k \cap R_k(t) \cap (t < T_i \le t + dt)]\} \\
&\quad = \lambda_i(t) \lim_{dt \to 0} P[(T > T_i) \cap (Y_{k+1}^* = \underline{y}) | \mathcal{E}_k \cap R_k(T_i) \cap (t < T_i \le t + dt)] \\
&\quad = \lambda_i(t) \lim_{dt \to 0} \{P[Y_{k+1}^* = \underline{y} | \mathcal{E}_k \cap (t < Z_{k+1}^* \le t + dt) \cap (I_{k+1}^* = i)] \\
&\qquad\quad \times P[T > T_i | \mathcal{E}_k \cap R_k(T_i) \cap (t < T_i \le t + dt) \cap (Y_{k+1}^* = \underline{y})]\} \\
&\quad = \lambda_i(t) \prod_{\ell \in g_0(\underline{y})} F_\ell(t) \prod_{\ell \in g_1(\underline{y})} \overline{F}_\ell(t) \\
&\qquad\qquad \times \lim_{dt \to 0} P[\phi(\underline{1}_{(R_k - \{i\}) \cup g_1(\underline{y})}, \underline{0}_{(Q_k \cup \{i\}) \cup g_0(\underline{y})}, \underline{X}(T_i)) = 1 | t < T_i \le t + dt] \\
&\quad = \lambda_i(t) \prod_{\ell \in g_0(\underline{y})} F_\ell(t) \prod_{\ell \in g_1(\underline{y})} \overline{F}_\ell(t) h(\underline{1}_{(R_k - \{i\}) \cup g_1(\underline{y})}, \underline{0}_{(Q_k \cup \{i\}) \cup g_0(\underline{y})}, \underline{F}(t)),
\end{aligned}$$

having applied (3.1).

Proof of Theorem 3.1: Parallelling the proof of Theorem 2.1 all the way we end up with the following counterpart to (2.8), noting that $g_1(\underline{Y}_{k+1}) = H_{k+1,1}$ and $g_0(\underline{Y}_{k+1}) = H_{k+1,0}$

$$
\begin{aligned}
L(\underline{\theta}) = & \prod_{k=0}^{\Delta-1}[\prod_{\ell \in H_{k+1,1}} \overline{F}_\ell(Z_{k+1}) \prod_{i \in R_k} (\overline{F}_i(Z_{k+1})/\overline{F}_i(Z_k))] \prod_{i \in R_\Delta}(\overline{F}_i(S)/\overline{F}_i(Z_\Delta)) \\
& \times \prod_{k=0}^{\Delta-1} \prod_{\ell \in H_{k+1,0}} F_\ell(Z_{k+1}) \prod_{k=0}^{\Delta-1} \lambda_{I_{k+1}}(Z_{k+1}) \\
& \times \{I(V > S)[I(I_{\Delta+1} \neq 0)\lambda_{I_{\Delta+1}}(S) P_J^{N_\Delta}(S) \qquad (3.3) \\
& + I(I_{\Delta+1} = 0)\frac{d}{dt} P_J^{N_\Delta}(s,t)|_{t=S}] + I(V = S)h(\underline{1}_{R_\Delta}, \underline{0}_{Q_\Delta}, \underline{F}(S))\} \; .
\end{aligned}
$$

Our proof is now completed parallel to the one of Theorem 2.1 by noting that the factor displayed in the first line of (3.3) can be written as

$$
\begin{aligned}
& \prod_{k=0}^{\Delta-1} \left[\prod_{\ell \in R_{k+1} \cup \{I_{k+1}\}} \overline{F}_\ell(Z_{k+1}) / \prod_{\ell \in R_k} \overline{F}_\ell(Z_k) \right] \prod_{i \in R_\Delta} (\overline{F}_i(S)/\overline{F}_i(Z_\Delta)) \\
& = \prod_{k=0}^{\Delta-1} \overline{F}_{I_{k+1}}(Z_{k+1}) \prod_{i \in R_\Delta} \overline{F}_i(S) = \prod_{i \in M \cup N_{\Delta,1}} \overline{F}_i(T_i \wedge S).
\end{aligned}
$$

We now return to the construction of inspection strategies satisfying (3.1). The simplest way this can be done is to let H_k be determined by a function $H_{t,i}^{\mathcal{E}_{k-1}}$ into the set of subsets of $C - (R_{k-1} - \{i\}) \cup (Q_{k-1} \cup \{i\}) = C - R_{k-1} \cup Q_{k-1}$, being piecewise constant and right continuous in $t > Z_{k-1}^*$, and with $i \in R_{k-1}$. We then determine H_k by

$$
H_k = H_{Z_k^*, I_k^*}^{\mathcal{E}_{k-1}}.
$$

To see that (3.1) is satisfied, note that there exists $dt > 0$ such that $H_{s,i}^{\mathcal{E}_{k-1}}$ is constant for $s \in [t, t + dt)$. To ensure the left hand side of (3.1) to be positive, $\underline{y} \in \{-1, 0, 1\}^q$ must be such that $g(\underline{y})$ is equal to this constant value. Hence, for such $\underline{y}$ this left hand side equals

$$
\lim_{dt \to 0} P \left[\bigcap_{\ell \in g(\underline{y})} (X_\ell(T_i) = y_\ell) | t < T_i \leq t + dt \right] ,
$$

which again is equal to the right hand side of (3.1).

The assumption of piecewise constancy of $H_{t,i}^{\mathcal{E}_{k-1}}$ ensures some stability in the inspection strategy, and hence seems reasonable.

The specification of the inspection strategy requires the specification of the functions $H_{t,i}^{\mathcal{E}_{k-1}}$ for each possible history $\mathcal{E}_{k-1}$. When these functions are constant in t and depend on the history only through the set of components at risk and the set of components known to have failed, i.e. $H_{t,i}^{\mathcal{E}_{k-1}}$ is of the form $H(R_{k-1} - \{i\}, Q_{k-1} \cup \{i\})$, we obtain an inspection plan of type 1 as defined in Gåsemyr (1998). In analogy, we may call the inspection strategy discussed above a history dependent inspection plan of type 1.

A more flexible strategy can be defined by allowing H_k to depend more extensively on information that becomes available at Z_k^*. We then let H_k be determined through an iterative procedure involving a function $H_{t,i}^{\mathcal{E}_{k-1}}(R, Q)$ into $\{C - R \cup Q\} \cup \{\emptyset\}$, being piecewise constant and right continuous in $t > Z_{k-1}^*$. Here $i \in R_{k-1}$, whereas R and Q are disjoint subsets of C with $R_{k-1} - \{i\} \subseteq R$ and $Q_{k-1} \cup \{i\} \subseteq Q$.

H_k is then the result of repeated applications of $H_{Z_k^*, I_k^*}^{\mathcal{E}_{k-1}}(R, Q)$ with $(R, Q) =$ $(R_{k-1} - I_k^*, Q_{k-1} \cup I_k^*)$ as initial values. For each iteration, a new component is selected for inspection and afterwards the pair (R, Q) is updated by adding the component to R if it is functioning and to Q if not. This procedure is stopped when for the first time $H_{Z_k^*, I_k^*}^{\mathcal{E}_{k-1}}(R, Q) = \emptyset$. We assume the procedure takes zero operational time.

To see that (3.1) is satisfied, note that there exists $dt > 0$ such that the function $H_{s,i}^{\mathcal{E}_{k-1}}(R, Q)$ is constant for $s \in [t, t + dt)$ for each of the finitely many possible pairs (R, Q). Hence, for each iteration, where a single component is selected, (3.1) is true. Due to the independence of components it follows that (3.1) is satisfied for the whole inspection.

If $H_{t,i}^{\mathcal{E}_{k-1}}(R, Q) = H(R, Q)$ depends on the history only through the pair (R, Q) of components known to respectively be at risk and to have failed, H is called an inspection function and the strategy is an inspection plan of type 2 according to the terminology in Gåsemyr (1998). In that paper, the sets $H(R, Q)$ are allowed to contain more than one component. This generality could be allowed in our history dependent framework as well, giving rise to a history dependent inspection function and a history dependent inspection plan of type 2. Again (3.1) will be satisfied.

As a special case let us consider a socalled cause-controlling inspection plan as introduced in Gåsemyr (1998). An inspection plan of type 2 is said to be cause-controlling if for $k = 1, 2, \ldots$, the risk set R_k, arising from the whole inspection immediately after Z_k^*, is always a path set for the system if possible. This can be achieved by using an inspection function H satisfying $H(R, Q) \neq \emptyset$ if R is not a path set. Suppose in addition that $M \cup C = E$ and that $R_0 = M$ is a path set. It is then easy to see that the system failure time T must coincide with the failure time of a component that is currently being monitored. Hence, the identity of the component causing system failure becomes known.

This leads to the following corollary of Theorem 3.1.

Corollary 3.3 *For the case of a cause-controlling inspection plan the complete likelihood function for our parameter vector, $\underline{\theta}$, reduces to*

$$L(\underline{\theta}) = \prod_{i \in M \cup C} (\lambda_i(T_i))^{\Delta_i} \prod_{i \in M \cup N_{\Delta,1}} \overline{F}_i(T_i \wedge S) \prod_{i \in N_{\Delta,0}} F_i(\tau_i)$$
$$\times \{I(V > S)\lambda_{I_L}(S) \prod_{\ell \in A_J - M - N_{L-1}} F_\ell(S) \prod_{\ell \in A_J^c - M - N_{L-1}} \overline{F}_\ell(S) + I(V = S)\}$$

Proof: The corollary follows immediately from Theorem 3.1 since for a cause-controlling inspection plan we always have $I_L \neq 0$. In addition, the contribution $h(\underline{1}_{R_L}, \underline{0}_{Q_L}, \underline{\overline{F}}(S)) = 1$ since R_L is here a path set.

4. An application to component replacement

In this section we consider preventive system maintenance where components are replaced according to a specific strategy. We have to take into account that it is costly to intervene in system operation. Hence, it is desirable to postpone replacement of failed components as long as possible in order to replace several components at a time. On the other hand, it is obviously important to avoid a system failure. As a compromise we assume that components are replaced as soon as system weakening has reached a certain level; i.e. when $\psi(\underline{X}(t))$ jumps to zero, where ψ is a binary, monotone structure function such that $\psi(\underline{X}(t)) \leq \phi((\underline{X}(t))$. At this time a total inspection of the components is carried through and all failed components are replaced, while the others are not affected. We assume this procedure takes zero operational time. Afterwards, the replaced components are assumed to have the same lifetime distributions as the initial ones.

It is natural to choose ψ such that when $\psi(\underline{X}(t))$ jumps to zero, at least one additional component must fail for $\phi(\underline{X}(t))$ to jump to zero. If for instance ϕ is a k-out-of-n system, we can choose ψ as a $(k+1)$-out-of-n system.

We denote by $T_{i,\ell}$ and $S_{i,\ell}$ respectively the time for the ℓth failure and replacement of the ith component, $i \in E$, $\ell = 1, 2, \ldots$ The successive times of preventive system maintenance are denoted $T^1, T^2, \ldots$, and the interval $(T^{r-1}, T^r]$ is called the rth operational period of the system, $r = 1, 2, \ldots$ Here $T^0 = 0$. Formally, these variables are related as follows. Let

$$
\begin{aligned}
& X_{i,1}(t) = I(T_{i,1} > t), \qquad i \in E, \; t > 0 \\
& \underline{X}_1(t) = (X_{1,1}(t), \ldots, X_{n,1}(t)) \\
& T^1 = \inf\{t > 0 | \psi(\underline{X}_1(t)) = 0\} \\
& X_{i,2}(t) = I(T_{i,1} > t) + I(T_{i,1} \leq T^1) I(T_{i,2} > t), \qquad i \in E, \; t > T^1 \\
& \underline{X}_2(t) = (X_{1,2}(t), \ldots, X_{n,2}(t)) \\
& T^2 = \inf\{t > T^1 | \psi(\underline{X}_2(t)) = 0\} \\
& \quad \vdots \\
& X_{i,r}(t) = I(T_{i,1} > t) + I(T_{i,1} \leq T^{r-1}) I(T_{i,2} > t) + \cdots \\
& \qquad + I(T_{i,r-1} \leq T^{r-1}) I(T_{i,r} > t), \qquad i \in E, \; t > T^{r-1} \\
& \underline{X}_r(t) = (X_{1,r}(t), \ldots, X_{n,r}(t)) \\
& T^r = \inf\{t > T^{r-1} | \psi(\underline{X}_r(t)) = 0\} \\
& S_{i,0} = 0 \\
& S_{i,\ell} = \min\{T^r, r \in \{1, 2, \ldots\} | T_{i,\ell} \leq T^r\}, \qquad \ell = 1, 2, \ldots
\end{aligned}
$$

The component states at time t are given by

$$\underline{X}(t) = \sum_{r=1}^{\infty} I(T^{r-1} < t \leq T^r)\underline{X}_r(t).$$

Our distributional assumptions can formally be stated by introducing

$$V_{i,\ell} = T_{i,\ell} - S_{i,\ell-1}, \qquad i \in E, \ \ell = 1, 2, \ldots$$

Then the variables $V_{i,\ell}$ are independent and $V_{i,\ell}$ has distribution function $F_i(t)$.

In order to immediately register the successive times, $T^1, T^2, \ldots$, of preventive system maintenance, a history independent inspection plan of type 2, which is cause-controlling with respect to ψ, is followed. At the consecutive failure times, Z_k^*, of the currently monitored components, the risk set is updated by $H_{k,1}$, the set of conditionally lifemonitored components being functioning on inspection, $k = 1, 2, \ldots$ If $Z_k^* = T^r$ for some $r = 1, 2, \ldots$, first all failed components are replaced, while the others are not affected. The risk set, R_k, can then be constructed by for instance starting out with the inspection function $H(R_{k-1}, \emptyset)$ or simply start all over again with $R_k = R_0 = M$.

The inspection strategy leads to inspection times $\tau_{i,\ell}$ defined as follows. Introduce

$$\tau_{i,\ell}^* = \min\{Z_k^* | (Z_k^* \geq S_{i,\ell-1}) \cap (i \in H_k)\}, \qquad i \in E, \ \ell = 1, 2, \ldots$$

The possibilities $\tau_{i,\ell}^* = S_{i,\ell-1}$ and $\tau_{i,\ell}^* = +\infty$ are not excluded. Let

$$\tau_{i,\ell} = \min\{\tau_{i,\ell}^*, S_{i,\ell}\}, \qquad i \in E, \ \ell = 1, 2, \ldots$$

Hence, we define exactly one inspection time $\tau_{i,\ell}$ for the ith component in its ℓth renewal cycle $[S_{i,\ell-1}, S_{i,\ell}]$.

We want to calculate the likelihood function, $L(\underline{\theta})$, based on data from observing the system components according to the scheme described above on the interal $[0, t_0]$, where t_0 is either a fixed time point or the result of random censoring. Define

$\widetilde{T}_{i,0} = 0$

$\widetilde{T}_{i,\ell} =$ $\max\{T_{i,\ell}, \tau_{i\ell}\}$ = the time when the ℓth failure of the ith component is known

$L_i = \max\{\ell \in \{0, 1, \ldots\} | \widetilde{T}_{i,\ell} \leq t_0\}$

$=$ the number of known failures for the ith component before t_0

$\Delta_{i,\ell} = I(\tau_{i,\ell} \leq T_{i,\ell})$

$R_{i,\ell} = \max\{r \in \{1, 2, \ldots\} | T^r < S_{i,\ell}\}$

$=$ the number of operational periods of the system before the one that ends with the ℓth replacement of the ith component

$R = \max\{r \in \{1, 2, \ldots\} | T^r \leq t_0\}$

$=$ the number of operational periods of the system completed before t_0

$R(t_0) =$ the risk set at t_0.

For $r = 0, \dots, R$ define k_r by
$$Z^*_{k_r} = T^r.$$
Note that since $Z^*_0 = T^0 = 0$, we have $k_0 = 0$. Finally, let

$$\begin{aligned} G(R) &= E - \bigcup_{\{k \geq k_R+1 | Z^*_k < t_0\}} (H_{k,0} \cup \{I^*_k\}) - R(t_0) \\ &= \text{the set of components for which no information is acquired in the interval } (T^R, t_0]. \end{aligned}$$

We can now prove the following theorem.

Theorem 4.1 *The complete likelihood function for our parameter vector, $\underline{\theta}$, in the component replacement model, where all failed components are replaced, is given by*

$$\begin{aligned} L(\underline{\theta}) = & \prod_{i \in E} \prod_{\ell=1}^{L_i} [\lambda_i(T_{i,\ell} - S_{i,\ell-1}) \overline{F}_i(T_{i,\ell} - S_{i,\ell-1})]^{\Delta_{i,\ell}} \\ & \times [F_i(\tau_{i,\ell} - S_{i,\ell-1}) - F_i(T^{R_{i,\ell}} - S_{i,\ell-1})]^{1-\Delta_{i,\ell}} \\ & \times \prod_{i \in R(t_0)} \overline{F}_i(t_0 - S_{i,L_i}) \prod_{i \in G(R)} \overline{F}_i(T^R - S_{i,L_i}). \end{aligned}$$

Proof: The likelihood is found by linking contributions from the consecutive operational periods of the system. With obvious notation we then have

$$L(\underline{\theta}) = L(\underline{\theta}|(0, t_0]) = \prod_{r=0}^{R-1} [L(\underline{\theta}|(T^r, T^{r+1}])] L(\underline{\theta}|(T^R, t_0]). \tag{4.1}$$

The available information at the beginning of the $(r+1)$th operational period is $\mathcal{B}_{k_r}$. Introduce the corresponding conditional distributions $(r = 0, 1, \dots, R-1)$

$$G^r_i(t) = P(X_{i,r+1}(t) = 0 | \mathcal{B}_{k_r}), \qquad t > T^r,$$

with corresponding p.d.f. $g^r_i(t)$ and failure rate $\gamma^r_i(t) = g^r_i(t)/\overline{G}^r_i(t)$, $i \in E$.

Furthermore, introduce

$$\begin{aligned} N^r_0 &= \bigcup_{k_r+1 \leq k \leq k_{r+1}} H_{k,0} \\ &= \text{the set of components being failed on inspections in the } (r+1)\text{th operational period} \end{aligned}$$

$$\begin{aligned} N^R_0 &= \bigcup_{\{k \geq k_R+1 | Z^*_k < t_0\}} H_{k,0} \\ &= \text{the set of components being failed on inspections in } (T^R, t_0). \end{aligned}$$

$$\begin{aligned} Q^r &= \bigcup_{k_r+1 \leq k \leq k_{r+1}} \{I^*_k\} \\ &= \text{the set of components observed to fail in the } (r+1)\text{th operational period} \end{aligned}$$

$$
\begin{aligned}
Q^R &= \bigcup_{\{k \geq k_R+1 | Z_k^* < t_0\}} \{I_k^*\} \\
&= \text{the set of components observed to fail in } (T^R, t_0) \\
A_{J^{r+1}} &= \{i \in E | X_{i,r+1}(T^{r+1}) = 0\} \\
&= \text{the fatal set corresponding to the } (r+1)\text{th jump of } \psi(\underline{X}(t)) \text{ to zero} \\
L_i^r &= \min\{\ell \in \{1,2,\ldots\} | S_{i,\ell} > T^r\} \\
&= \text{the number of renewal cycles for the } i\text{th component needed to just exceed } T^r
\end{aligned}
$$

By applying Corollary 3.3 for the case $V > S$ with respect to ψ we get

$$
\begin{aligned}
L(\underline{\theta}|(T^r, T^{r+1}]) &= \prod_{i \in Q^r} \gamma_i^r(T_{i,L_i^r}) \prod_{i \in Q^r} \overline{G}_i^r(T_{i,L_i^r}) \\
&\times \prod_{i \in N_0^r} G_i^r(\tau_{i,L_i^r}) \prod_{i \in A_{J^{r+1}} - Q^r \cup N_0^r} G_i^r(T^{r+1}) \prod_{i \in A^c_{J^{r+1}}} \overline{G}_i^r(T^{r+1}) \, .
\end{aligned} \tag{4.2}
$$

Now we have

$$
\overline{G}_i^r(t) = \overline{F}_i(t - S_{i,L_i^r-1}) / \overline{F}_i(T^r - S_{i,L_i^r-1}), \qquad t > T^r
$$

Furthermore,

$$
\begin{aligned}
\tau_{i,L_i^r} &= T^{r+1}, \qquad i \in A_{J^{r+1}} - Q^r \cup N_0^r. \\
S_{i,L_i^r-1} &= S_{i,L_i^{r+1}-1}, \qquad i \in A^c_{J^{r+1}}
\end{aligned}
$$

Hence, we get from (4.2), noting that for $i \in A_{J^r}$, $S_{i,L_i^r-1} = T^r$,

$$
\begin{aligned}
L(\underline{\theta}|(T^r, T^{r+1}]) &= \prod_{i \in A^c_{J^r}} [\overline{F}_i(T^r - S_{i,L_i^r-1})]^{-1} \\
&\times \prod_{i \in Q^r} \lambda_i(T_{i,L_i^r} - S_{i,L_i^r-1}) \prod_{i \in Q^r} \overline{F}_i(T_{i,L_i^r} - S_{i,L_i^r-1}) \\
&\times \prod_{i \in A_{J^{r+1}} - Q^r} [F_i(\tau_{i,L_i^r} - S_{i,L_i^r-1}) - F_i(T^r - S_{i,L_i^r-1})] \\
&\times \prod_{i \in A^c_{J^{r+1}}} \overline{F}_i(T^{r+1} - S_{i,L_i^{r+1}-1}) \, .
\end{aligned} \tag{4.3}
$$

Correspondingly, by applying Corollary 3.3 for the case $V = S$ we get

$$
\begin{aligned}
L(\underline{\theta}|(T^R, t_0]) &= \prod_{i \in A^c_{J^R}} [\overline{F}_i(T^R - S_{i,L_i^R-1})]^{-1} \\
&\times \prod_{i \in Q^R} \lambda_i(T_{i,L_i^R} - S_{i,L_i^R-1}) \prod_{i \in Q^R} \overline{F}_i(T_{i,L_i^R} - S_{i,L_i^R-1}) \\
&\times \prod_{i \in N_0^R} [F_i(\tau_{i,L_i^R} - S_{i,L_i^R-1}) - F_i(T^R - S_{i,L_i^R-1})] \\
&\times \prod_{i \in R(t_0)} \overline{F}_i(t_0 - S_{i,L_i^R-1}) \prod_{i \in E - Q^R \cup N_0^R \cup R(t_0)} \overline{F}_i(T^R - S_{i,L_i^R-1}) \, .
\end{aligned} \tag{4.4}
$$

Inserting (4.3) and (4.4) into (4.1), changing the order of multiplication, we get

$$
\begin{aligned}
L(\underline{\theta}) = \prod_{i\in E} \Bigg\{ & \prod_{r=0}^{R} [\lambda_i(T_{i,L_i^r} - S_{i,L_i^r-1})\overline{F}_i(T_{i,L_i^r} - S_{i,L_i^r-1})]^{I(i\in Q^r)} \\
& \times \prod_{r=0}^{R-1} \left[F_i(\tau_{i,L_i^r} - S_{i,L_i^r-1}) - F_i(T^r - S_{i,L_i^r-1})\right]^{I(i\in A_{J^{r+1}} - Q^r)} \\
& \times \left[F_i(\tau_{i,L_i^R} - S_{i,L_i^R-1}) - F_i(T^R - S_{i,L_i^R-1})\right]^{I(i\in N_0^R)} \Bigg\} \\
& \times \prod_{i\in R(t_0)} \overline{F}_i(t_0 - S_{i,L_i^R-1}) \prod_{i\in E-Q^R\cup N_0^R \cup R(t_0)} \overline{F}_i(T^R - S_{i,L_i^R-1}) \\
= \prod_{i\in E} \Bigg\{ & \prod_{\ell=1}^{L_i} \left[\lambda_i(T_{i,\ell} - S_{i,\ell-1})\overline{F}_i(T_{i,\ell} - S_{i,\ell-1})\right]^{I(i\in Q^{R_{i,\ell}})} \\
& \times \left[F_i(\tau_{i,\ell} - S_{i,\ell-1}) - F_i(T^{R_{i,\ell}} - S_{i,\ell-1})\right]^{I(i\in A_{J^{R_{i,\ell}+1}} - Q^{R_{i,\ell}})I(R^{i,\ell}<R) + I(i\in N_0^R)I(R^{i,\ell}=R)} \Bigg\} \\
& \times \prod_{i\in R(t_0)} \overline{F}_i(t_0 - S_{i,L_i^R-1}) \prod_{i\in E-Q^R\cup N_0^R\cup R(t_0)} \overline{F}_i(T^R - S_{i,L_i^R-1}),
\end{aligned}
$$

having used the fact that $L_i^{R_{i,\ell}} = \ell$ and that $L_i = L_i^R$ for $i \in Q^R \cup N_0^R$. By noting that $I(i \in Q^{R_{i,\ell}}) = \Delta_{i,\ell}$ and $I(i \in A_{J^{R_{i,\ell}+1}} - Q^{R_{i,\ell}})I(R^{i,\ell} < R) + I(i \in N_0^R)I(R^{i,\ell} = R) = 1 - \Delta_{i,\ell}$ and finally that $L_i = L_i^R - 1$ for $i \in E - Q^R \cup N_0^R$, our proof is completed.

An alternative component replacement model is obtained if no total inspection of the components is carried through and hence only components known to have failed are replaced. We now define inductively

$$
\begin{aligned}
& S_{i,0} = 0 \\
& \tau_{i,\ell} = \min\{Z_k^* | (Z_k^* \geq S_{i,\ell-1}) \cap (i \in H_k)\} \\
& S_{i,\ell} = \min\{T^r, r \in \{1,2,\ldots\} | \tilde{T}_{i,\ell} \leq T^r\}.
\end{aligned}
$$

Parallel to Theorem 4.1 we now get

Theorem 4.2 *The complete likelihood function for our parameter vector, $\underline{\theta}$, in the component replacement model, where only components known to have failed are replaced, is given by*

$$
\begin{aligned}
L(\underline{\theta}) = & \prod_{i\in M\cup C} \prod_{\ell=1}^{L_i} [\lambda_i(T_{i,\ell} - S_{i,\ell-1})\overline{F}_i(T_{i,\ell} - S_{i,\ell-1})]^{\Delta_{i,\ell}} \\
& \times [F_i(\tau_{i,\ell} - S_{i,\ell-1})]^{1-\Delta_{i,\ell}} \prod_{i\in R(t_0)} \overline{F}_i(t_0 - S_{i,L_i}).
\end{aligned}
$$

Proof: The proof is very similar to the one of Theorem 4.1. We now have

$$\overline{G}_i^r(t) = \begin{cases} \overline{F}_i(t - S_{i,L_i^r-1})/\overline{F}_i(T^r - S_{i,L_i^r-1}), & i \in R_{k_r}, \quad t > T^r \\ \overline{F}_i(t - S_{i,L_i^r-1}), & i \in M \cup C - R_{k_r}, \quad t > T^r \end{cases}$$

Hence, we get, noting that $N_0^r \subseteq M \cup C - R_{k_r}$,

$$L(\underline{\theta}|(T^r, T^{r+1}]) = \prod_{i \in R_{k_r}} [\overline{F}_i(T^r - S_{i,L_i^r-1})]^{-1} \prod_{i \in Q^r} \lambda_i(T_{i,L_i^r} - S_{i,L_i^r-1}) \prod_{i \in Q^r} \overline{F}_i(T_{i,L_i^r} - S_{i,L_i^r-1})$$
$$\times \prod_{i \in N_0^r} F_i(\tau_{i,L_i^r} - S_{i,L_i^r-1}) \prod_{i \in R_{k_{r+1}}} \overline{F}_i(T^{r+1} - S_{i,L_i^{r+1}-1})$$
$$L(\underline{\theta}|(T^R, t_0]) = \prod_{i \in R_{k_R}} [\overline{F}_i(T^R - S_{i,L_i^R-1})]^{-1} \prod_{i \in Q^R} \lambda_i(T_{i,L_i^R} - S_{i,L_i^R-1}) \prod_{i \in Q^R} \overline{F}_i(T_{i,L_i^R} - S_{i,L_i^R-1})$$
$$\times \prod_{i \in N_0^R} F_i(\tau_{i,L_i^R} - S_{i,L_i^R-1}) \prod_{i \in R(t_0)} \overline{F}_i(t_0 - S_{i,L_i^R-1}),$$

leading to the completion of the proof.

5. Bayesian estimation of component parameters

Having complete likelihood functions for our parameter vector, $\underline{\theta}$, a Bayesian approach to estimation of $\underline{\theta}$ is possible for our models. Taking prior knowledge into account this approach is especially suitable in reliability where data often are scarce and asymptotic properties of estimators are of less help. Let the prior distribution of $\underline{\theta}$ be $\pi(\underline{\theta})$. Then the posterior distribution given the data $\underline{D}$ is by Bayes theorem

$$\pi(\underline{\theta}|\underline{D}) \propto L(\underline{\theta})\pi(\underline{\theta}). \tag{5.1}$$

The posterior distribution of $\underline{\theta}$ gives through (5.1) the basis for Bayesian inference on component lifetimes. A specific parameter may for instance be estimated by the expectation in its posterior marginal distribution. When exact methods for calculating the expectation are not available, one may use Markov Chain Monte Carlo simulation to obtain approximate values. See for instance Smith & Roberts (1993).

Now assume that the lifetime of the ith component, T_i, is exponentially distributed with failure rate θ_i, $i \in E$. We have $\underline{\theta} = (\theta_1, \ldots, \theta_n)$.

The following definition of the generalized gamma distribution is given in Gåsemyr & Natvig (1998).

Definition 5.1 *For positive real numbers $a, b, t_1, \ldots, t_r$ define the functions*

$$\begin{aligned} f(\theta; a, b, \underline{t}) &= \theta^{a-1} e^{-b\theta} \prod_{i=1}^{r} (1 - e^{-\theta t_i}), \quad \theta \geq 0 \\ f(\theta; a, b) &= \theta^{a-1} e^{-b\theta}, \quad \theta \geq 0, \end{aligned} \tag{5.2}$$

where $\underline{t} = (t_1, \ldots, t_r)$. *Define the normalizing constant* $\gamma(a, b, \underline{t})$ *by*

$$(\gamma(a,b,\underline{t}))^{-1} = \int_0^\infty f(\theta; a, b, \underline{t})d\theta = \sum_{B\subset\{1,\ldots,r\}} (-1)^{|B|} \int_0^\infty f(\theta; a, b + \sum_{i=1}^r I_B(i)t_i)d\theta$$

$$= \sum_{\underline{d}\in\{0,1\}^r} (-1)^{|\underline{d}|} \int_0^\infty f(\theta; a, b + \underline{d}\cdot\underline{t})d\theta = \Gamma(a) \sum_{\underline{d}\in\{0,1\}^r} (-1)^{|\underline{d}|}(b + \underline{d}\cdot\underline{t})^{-a}, \qquad (5.3)$$

where $|B|$ *denotes the number of elements in* B, $d_i = I_B(i)$ *and* $|\underline{d}|$ *denotes* $d_1 + \cdots + d_r$. *The generalized gamma distribution with parameters* a, b *and* $\underline{t}$ *is then defined as the probability distribution on* $[0, \infty)$ *with density function given by*

$$g(\theta; a, b, \underline{t}) = \gamma(a, b, \underline{t}) f(\theta; a, b, \underline{t}), \qquad \theta \geq 0. \qquad (5.4)$$

The ordinary gamma distribution, $g(\theta; a, b)$, *is a special case corresponding to* $r = 0$.

Note that conditionally on $\underline{\theta}$, the reliability function of the system can be written as

$$h(\underline{\overline{F}}(t)) = \sum_{\underline{x}\in\{0,1\}^n} \phi(\underline{x}) \prod_{i=1}^n (e^{-\theta_i t})^{x_i}(1 - e^{-\theta_i t})^{1-x_i}, \qquad (5.5)$$

by total state enumeration. This method is computationally inefficient, but will serve the purpose to prove Theorem 5.3 to follow. In applications more efficient approaches are needed such as for instance the technique of recursive disjoint products, see Abraham (1979), Ball & Provan (1988) and Locks (1980, 1982). For network systems the factoring algorithm can be very efficient, see Satyanarayana & Chang (1983).

The following lemma is obtained immediately from Theorem 3.1 using (5.5).

Lemma 5.2 *For the case of exponentially distributed component lifetimes the likelihoood function in Theorem 3.1 can be written in the form*

$$L(\underline{\theta}) = \sum_{k=1}^K \prod_{i\in B_k} f(\theta_i; 1, 0, t_{k,i}) \prod_{i\in C_k} f(\theta_i; 1, t_{k,i}) \prod_{i\in D_k} f(\theta_i; 2, t_{k,i}), \qquad (5.6)$$

where B_k, C_k, D_k *are disjoint subsets of* E *for each* $k = 1, \ldots, K$.

Our main result in our Bayesian approach is the following theorem being completely parallel to Theorem 2.2 in Gåsemyr & Natvig (1998).

Theorem 5.3 a) *Suppose that the failure rates* θ_i, $i \in E$ *for the components of a binary, monotone system* (E, ϕ) *have a joint prior distribution of the form*

$$\pi(\underline{\theta}) \propto \sum_{j=1}^J \prod_{i=1}^n f(\theta_i; a_{j,i}, b_{j,i}, \underline{t}_{j,i}) = \sum_{j=1}^J \prod_{i=1}^n \gamma(a_{j,i}, b_{j,i}, \underline{t}_{j,i})^{-1} g(\theta_i; a_{j,i}, b_{j,i}, \underline{t}_{j,i}). \qquad (5.7)$$

Then the posterior distribution of $\underline{\theta}$ associated with the likelihood function given by (5.6) is of the form

$$\pi(\underline{\theta}|\underline{D}) \propto \sum_{j=1}^{J}\sum_{k=1}^{K}\Big[\prod_{i\in B_k} f(\theta_i; a_{j,i}, b_{j,i}, \underline{t}_{j,i}, t_{k,i})\Big]$$
$$\times\Big[\prod_{i\in C_k} f(\theta_i; a_{j,i}, b_{j,i} + t_{k,i}, \underline{t}_{j,i})\Big] \prod_{i\in D_k} f(\theta_i; a_{j,i} + 1, b_{j,i} + t_{k,i}, \underline{t}_{j,i})$$
$$\times\Big[\prod_{i\in E-B_k-C_k-D_k} f(\theta_i; a_{j,i}, b_{j,i}, \underline{t}_{j,i})\Big]. \tag{5.8}$$

b) *The class of distributions of the form (5.7) is a natural conjugate class of priors for our exponential model.*

c) *Suppose the prior distribution,*

$$\pi(\underline{\theta}) = \prod_{i=1}^{n} g(\theta_i; a_i, b_i), \tag{5.9}$$

for $\underline{\theta}$ is updated with data from r independent systems with likelihood functions of the form given by Theorem 3.1. Then the posterior distribution is of the form (5.7).

Proof: a) is a straightforward application of Bayes theorem. b) follows since (5.8) is of the same general form as (5.7). c) follows by repeated use of a).

Theorem 5.3 states that the weighted sum of products of generalized gamma distributions is the natural conjugate prior for $\underline{\theta}$ with respect to our exponential model. This seems to be a completely new generalization of the fact that the gamma distribution is the natural conjugate prior for the failure rate in an exponential model parallel to the generalization given in Gåsemyr & Natvig (1998). For further comments we refer to that paper and to Gåsemyr & Natvig (1996) where simulation procedures for parameter estimation and prediction in the model of Gåsemyr & Natvig (1998) are suggested.

By considering the likelihood functions given in Theorems 4.1 and 4.2 instead of the one given in Theorem 3.1, Theorem 5.3 is still valid. This follows since for the case of exponentially distributed component lifetimes we have

$$F_i(\tau_{i,\ell} - S_{i,\ell-1}) - F_i(T^{R_{i,\ell}} - S_{i,\ell-1}) =$$
$$\exp(-\theta_i(T^{R_{i,\ell}} - S_{i,\ell-1}))[1 - \exp(-\theta_i(\tau_{i,\ell} - T^{R_{i,\ell}}))].$$

We conclude this section by considering, for general lifetime distributions of the components, an example of a history dependent inspection plan of type 2. The example is motivated by a situation where due to scarcity of financial resources, technical equipment or personell, it is only possible to monitor a fixed number of components at a time. Whenever a currently monitored component fails, exactly one new component is added to the risk set. To decide which component to start monitoring at Z_k^*, we order the components in $C - R_{k-1} \cup Q_{k-1}$ according to the size of the quantity

$$M_i(k) = E[T|\mathcal{B}_k \cap (X_i(Z_k^*) = 1)] - E[T|\mathcal{B}_k \cap (X_i(Z_k^*) = 0)]. \tag{5.10}$$

The components are inspected in decreasing order of $M_i(k)$ until a functioning one is found. This component is then included in the risk set. The motivation is that a component with larger $M_i(k)$ carries more information about the expected system lifetime. This quantity is very much linked to the Natvig measure of importance of components, see Natvig (1985).

For $\delta = 0, 1$ we have

$$\begin{aligned} & P[T > t|\mathcal{B}_k \cap (X_i(Z_k^*) = \delta)] \\ = & \int\limits_{[0,\infty]^n} P[T > t|\mathcal{B}_k \cap (X_i(Z_k^*) = \delta), \underline{\theta}]\pi(\underline{\theta}|\mathcal{B}_k \cap (X_i(Z_k^*) = \delta))d\underline{\theta} \\ \propto & \int\limits_{[0,\infty]^n} P[T > t|\mathcal{B}_k \cap (X_i(Z_k^*) = \delta), \underline{\theta})L(\underline{\theta}|\mathcal{B}_k \cap (X_i(Z_k^*) = \delta))\pi(\underline{\theta})d\underline{\theta}, \end{aligned} \tag{5.11}$$

where $\pi(\underline{\theta}|\mathcal{B}_k \cap (X_i(Z_k^*) = \delta))$ is the posterior distribution of $\underline{\theta}$. $M_i(k)$ is determined by (5.11) since

$$M_i(k) = \int\limits_0^\infty \{P[T > t|\mathcal{B}_k \cap (X_i(Z_k^*) = 1)] - P[T > t|\mathcal{B}_k \cap (X_i(Z_k^*) = 0)]\}dt\,.$$

Hence we see that the current updating of the information on $\underline{\theta}$ is not just a consequence of our monitoring scheme, but is in fact an integrated part of the scheme. The computation of $M_i(k)$ must in most cases be carried through by simulation. One starts by generating a sample from the posterior distribution of $\underline{\theta}$ by for instance rejection sampling, and then simulates the process $\underline{X}(t)$ for given $\underline{\theta}$.

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