BAYESIAN ESTIMATION OF COMPONENT LIFETIMES BASED ON AUTOPSY DATA

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Consider a binary system of n independent components having absolutely continuous lifetime distributions. The system is observed until it fails. At this instant, the set of failed components and the failure time of the system are noted. The failure times of the components are not known. These are the socalled autopsy data of the system. Meilijson (1981), Nowik (1990) and Antoine et al. (1993) discuss the corresponding identifiability problem; i.e. whether the component life distributions can be determined from the distribution of the observed data. Assuming a model where autopsy data is known to be enough for identifiability, Meilijson (1992) goes beyond the identifiability question and into maximum likelihood estimation of the parameters of the component lifetime distributions based on empirical autopsy data from a sample of several systems. He also considers lifemonitoring of some components and conditional lifemonitoring of some other. In the present paper a corresponding Bayesian approach is presented. Due to prior information one advantage is that the identifiability problem represents no obstacle.

Key words. Fatal set; critical set; natural conjugate prior; mixture of products of gamma distributions; lifemonitoring of components.

1. Introduction

Consider a binary system of n independent, binary components having absolutely continuous lifetime distributions. Denote the lifetime of the system by T and the lifetime of the ith component by T_i, with distribution function $F_i(t)$, survival function $\overline{F}_i(t) = 1 - F_i(t)$ and p.d.f. $f_i(t), i\varepsilon E = \{1, \cdots, n\}$. Let

$$\begin{aligned} D &= \text{the fatal set} = \{i\varepsilon E | T_i \leq T\} \\ \mathcal{A} &= \{\text{possible fatal sets}\} = \{A \subset E | P(D = A) > 0\} \\ &= \{A_1, \cdots, A_m\} \end{aligned}$$

The system is observed until it fails. At this instant, the fatal set, D, and the failure time of the system, T, are noted. The failure times of the components are not known. (T, D) are the socalled autopsy data of the system. Meilijson (1981), Nowik (1990) and Antoine et al. (1993) discuss the corresponding identifiability problem; i.e. whether the distributions of T_i, $i\varepsilon E$ can be determined from the distribution of the autopsy data (T, D). For a very readable presentation of these efforts we recommend to start with the latter paper.

Following these papers let

$$\begin{aligned} C_A &= \text{the critical set corresponding to the fatal set } A \\ &= \{i\varepsilon A | P(T_i = T | D = A) > 0\} \end{aligned}$$

This set consists of those components of the fatal set A which may have failed when the system failed at T and thus may have caused the failure of the system. The distribution of the autopsy data (T, D) is given by

$$G_i(t) = P(T \le t, D = A_i)$$

with density function

$$g_i(t) = \frac{d}{dt} G_i(t)$$

The latter can be considered as a likelihood function on the space $R^+ \times \{1, 2, \cdots, m\}$ with respect to the measure

$$\mu = \text{Lebesgue measure} \times \text{ counting measure.}$$

This follows since

$$\int\limits_{[0,t] \times \{i\}} g_j(s) d\mu(s, j) = \int\limits_{[0,t]} g_i(s) ds = G_i(t)$$

Because the system fails when the last component of the critical set corresponding to the fatal set fails, we have

$$\begin{aligned} g_i(t) &= \prod_{l \varepsilon A_i - C_{A_i}} F_l(t) \frac{d}{dt} \prod_{l \varepsilon C_{A_i}} F_l(t) \prod_{l \varepsilon A_i^c} \overline{F}_l(t) \\ &= \sum_{j \varepsilon C_{A_i}} f_j(t) \prod_{l \varepsilon A_i - \{j\}} F_l(t) \prod_{l \varepsilon A_i^c} \overline{F}_l(t) \end{aligned}$$

Introduce the failure rate $\lambda_i(t)$ and the cumulative failure rate

$$\Lambda_i(t) = \int_0^t \lambda_i(s) ds$$

corresponding to $F_i(t), i \varepsilon E$. We then get

$$\begin{aligned} g_i(t) &= \sum_{j \varepsilon C_{A_i}} \lambda_j(t) \prod_{l \varepsilon A_i - \{j\}} (1 - e^{-\Lambda_l(t)}) \prod_{l \varepsilon A_i^c \cup \{j\}} e^{-\Lambda_l(t)} \\ &= \sum_{j \varepsilon C_{A_i}} \lambda_j(t) \sum_{A_i^c \cup \{j\} \subset B} (-1)^{|B| - |A_i^c| - 1} \prod_{l \varepsilon B} e^{-\Lambda_l(t)} \end{aligned} \tag{1.1}$$

The λ_i's depend on a parameter vector $\underline{\theta}$; i.e.

$$\lambda_i(t) = \lambda_i(t; \underline{\theta})$$

Assuming a model where autopsy data is known to be enough for identifiability, Meilijson (1992) goes beyond the identifiability question and into maximum likelihood estimation of the parameter vector $\underline{\theta}$ based on empirical autopsy data from a sample of several systems. In the present paper a corresponding Bayesian approach is presented. Let the prior distribution of $\underline{\theta}$ be $\pi(\underline{\theta})$. Then the posterior distribution of $\underline{\theta}$ given the autopsy data $(T = t, D = A_i)$ is obviously

$$\pi(\underline{\theta} | T = t, D = A_i) \propto g_i(t) \pi(\underline{\theta}) \tag{1.2}$$

By the Bayesian approach one avoids the asymptotics of Meilijson (1992), which is of less help in reliability where data often are scarce.

In Section 2 we consider, as Meilijson (1992), a model where the T_i's are exponentially distributed. Due to prior information a further advantage of the Bayesian approach is that the identifiability problem represents no obstacle. To illustrate this in detail we treat in Section 3 a parallel system of two components. Meilijson (1992) also considers lifemonitoring of some components and conditional lifemonitoring of some other components. In Sections 4 and 5 the complete likelihood function for these cases are arrived at for general lifetime distributions of the components making a fully Bayesian approach possible. Our present work is an attempt to get as much information as possible from a failing system. This parallels and should be combined with Gåsemyr and Natvig (1994) which is concerned with the combination of expert opinions about the lifetimes of components having a multivariate exponential distribution of the Marshall-Olkin type. Finally, it should be stated that the general idea of Sections 4 and 5 is much inspired by Arjas (1989) although we have found it inconvenient to use his framework of marked point processes.

2. Exponentially distributed component lifetimes

Assume now

$$\lambda_i(t) = \theta_i, \ i\varepsilon E;$$

i.e. the component lifetimes are exponentially distributed. Assume furthermore the prior distributions of θ_i, to be independent and gamma with shape parameter a_i and scale parameter $b_i, i\varepsilon E$. Denote the corresponding p.d.f. $g(\theta_i; a_i, b_i)$. Then from (1.1) and (1.2) we get the following posterior distribution of $\underline{\theta}$ given the autopsy data $(T = t, D = A_i)$

$$\begin{aligned}
&\pi(\underline{\theta}|T = t, D = A_i) \propto \sum_{j\varepsilon C_{A_i}} \theta_j \sum_{A_i^c \cup \{j\} \subset B} (-1)^{|B|-|A_i^c|-1} \\
&\times \prod_{l=1}^{n} \frac{b_l^{a_l}}{\Gamma(a_l)} \theta_l^{a_l-1} e^{-\theta_l(b_l+I(l\varepsilon B)t)} \\
&= \sum_{j\varepsilon C_{A_i}} \sum_{A_i^c \cup \{j\} \subset B} (-1)^{|B|-|A_i^c|-1} \frac{a_j}{b_j+t} \prod_{l\varepsilon B} (\frac{b_l}{b_l+t})^{a_l} \\
&\times \prod_{l=1}^{n} g(\theta_l; a_l + \delta_{jl}, b_l + I(l\varepsilon B)t)
\end{aligned} \tag{2.1}$$

This is a mixture of products of gamma distributions. Concerning the number of addends in (2.1) we see that the minimum of 1 is obtained for a series system whereas the maximum of $n2^{n-1}$ is obtained for a parallel system.

To deal with autopsy data from a sample of several systems we show how the updating works when autopsy data from a single system arrives. Assume the prior distribution of $\underline{\theta}$ to be given as a mixture of products of gamma distributions; i.e.

$$\pi(\underline{\theta}) = \sum_{k=1}^{K} w_k \prod_{l=1}^{n} g(\theta_l; a_{kl}, b_{kl}), \tag{2.2}$$

where some of the w_k's may be negative. Then parallel to (2.1) we get the following posterior distribution of $\underline{\theta}$ given the autopsy data $(T = t, D = A_i)$

$$\begin{aligned} \pi(\underline{\theta}|T = t, D = A_i) \propto \sum_{k=1}^{K} w_k \sum_{j \epsilon C_{A_i}} \sum_{A_i^c \cup \{j\} \subset B} (-1)^{|B|-|A_i^c|-1} \\ \times \frac{a_{kj}}{b_{kj} + t} \prod_{l \epsilon B} (\frac{b_{kl}}{b_{kl} + t})^{a_{kl}} \prod_{l=1}^{n} g(\theta_l; a_{kl} + \delta_{jl}, b_{kl} + I(l \epsilon B)t) \end{aligned} \tag{2.3}$$

This is again a mixture of products of gamma distributions. Hence this distribution is the natural conjugate prior for $\underline{\theta}$ with respect to our exponential autopsy model. This seems to be a completely new generalization of the fact that the gamma distribution is the natural conjugate prior for the failure rate in an exponential model.

3. A parallel system of two components

Note that with the Bayesian approach the identifiability problem represents no obstacle. To illustrate this in detail we now consider a parallel system of two components. From the references given in Section 1 it is well known that the lifetime distributions of the two components are unidentifiable. This is obvious since, under the autopsy model, one in effect observes only the system failure time, which has the distribution function $F_1(t)F_2(t)$, from which it is impossible to single out $F_1(t)$ and $F_2(t)$.

Assume as in Section 2 that component lifetimes are exponentially distributed, but with, as a start, a general prior distribution $\pi(\theta_1, \theta_2)$. Obviously $T = \max(T_1, T_2)$ and the only fatal set is $A_1 = \{1, 2\}$. Introduce

$$\begin{aligned} B_i(t) &= \{T_i = t, T_j \le t\} \quad i = 1, 2; j \ne i \\ p_i(t) &= P(B_i(t)|T = t) \quad i = 1, 2 \end{aligned}$$

Then

$$\{T = t\} = B_1(t) \cup B_2(t),$$

where the events $B_1(t)$ and $B_2(t)$ are a.s. disjoint. The posterior distribution of (θ_1, θ_2) given the autopsy data, $T = t$, can now be written as

$$\pi(\theta_1, \theta_2|T = t) = \sum_{i=1}^{2} p_i(t)\pi(\theta_1, \theta_2|B_i(t)) \tag{3.1}$$

Assuming θ_1 and θ_2 to be prior independent we will show that they are posterior negatively correlated. From (3.1) we have

$$\begin{aligned} \mathrm{Cov}(\theta_1, \theta_2|T = t) &= \sum_{\substack{i=1 \\ j \ne i}}^{2} p_i(t) \int_0^\infty \int_0^\infty [\theta_1 - E(\theta_1|T = t)][\theta_2 - E(\theta_2|T = t)] \\ &\times \pi(\theta_i|T_i = t)\pi(\theta_j|T_j \le t) d\theta_1 d\theta_2 \\ &= \sum_{\substack{i=1 \\ j \ne i}}^{2} p_i(t)[E[\theta_i|T_i = t) - E(\theta_i|T = t)][E(\theta_j|T_j \le t) - E(\theta_j|T = t)] \end{aligned}$$

By noting that $(i = 1, 2; j \neq i)$

$$E(\theta_i|T = t) = p_i(t)E(\theta_i|T_i = t) + p_j(t)E(\theta_i|T_i \leq t),$$

we get since $p_1(t) + p_2(t) = 1$

$$\mathrm{Cov}(\theta_1, \theta_2|T = t) = -p_1(t)p_2(t)$$

$$\times [E(\theta_1|T_1 = t) - E(\theta_1|T_1 \leq t)][E(\theta_2|T_2 = t) - E(\theta_2|T_2 \leq t)]$$

From Barlow and Proschan (1985) we have "under mild regularity conditions" that $E(\theta_i|T_i)$ is nonincreasing in $T_i, i = 1, 2$. Hence $\mathrm{Cov}(\theta_1, \theta_2|T = t) \leq 0$.

Assume furthermore the prior distribution of θ_1 and θ_2 to be gamma. Then from (2.1) and (3.1) $(i = 1, 2; j \neq i)$

$$\begin{aligned}
&\pi(\theta_1, \theta_2|T = t) \propto [\theta_1 e^{-\theta_1 t}(1 - e^{-\theta_2 t}) + \theta_2 e^{-\theta_2 t}(1 - e^{-\theta_1 t})] \\
&\times g(\theta_1; a_1, b_1) \cdot g(\theta_2; a_2, b_2) \\
&\pi(\theta_1, \theta_2|B_i(t)) = g(\theta_i; a_i + 1, b_i + t) \\
&\times \frac{g(\theta_j; a_j, b_j) - (b_j/(b_j + t))^{a_j} g(\theta_j; a_j, b_j + t)}{1 - (b_j/(b_j + t))^{a_j}} \\
&\qquad p_i(t) = \alpha_i(t)/(\alpha_1(t) + \alpha_2(t)), \quad \text{where} \\
&\qquad \alpha_i(t) = \frac{a_i}{b_i + t}\left(\frac{b_i}{b_i + t}\right)^{a_i}\left(1 - \left(\frac{b_j}{b_j + t}\right)^{a_j}\right) \\
&\qquad p_1(t)/p_2(t) = \alpha_1(t)/\alpha_2(t)
\end{aligned} \tag{3.2}$$

Now by L'Hôpital's rule

$$\lim_{t \to 0} p_1(t)/p_2(t) = \frac{a_1/b_1}{a_2/b_2} \lim_{t \to 0} \frac{a_2(b_2/b_2 + t))^{a_2 - 1} b_2(b_2 + t)^{-2}}{a_1(b_1/b_1 + t))^{a_1 - 1} b_1(b_1 + t)^{-2}} = 1$$

Hence $p_1(0) = p_2(0) = 1/2$; i.e. equal weight is allocated to the two components when system lifetime approaches zero, irrespective of prior assessments.

Now again by L'Hôpital's rule $(i = 1, 2; j \neq i)$

$$\begin{aligned}
&\lim_{t \to 0} \pi(\theta_1, \theta_2|B_i(t)) = g(\theta_i; a_i + 1, b_i)(-b_j/a_j) \\
&\times [\lim_{t \to 0} \frac{d}{dt} g(\theta_j; a_j, b_j + t) - (a_j/b_j)g(\theta_j; a_j, b_j)] \\
&= g(\theta_1; a_1 + 1, b_1)g(\theta_2; a_2 + 1, b_2)
\end{aligned}$$

From (3.1) we now have

$$\lim_{t \to 0} \pi(\theta_1, \theta_2|T = t) = g(\theta_1; a_1 + 1, b_1)g(\theta_2; a_2 + 1, b_2)$$

This is intuitively obvious since the shape parameters are each added by 1 corresponding to failures of both components whereas the scale parameters are unchanged due to zero time at test.

Now assume $a_2 > a_1$. Then

$$\lim_{t\to\infty} p_1(t)/p_2(t) = \lim_{t\to\infty} \frac{a_1 b_1^{a_1}(b_2+t)^{a_2+1}}{a_2 b_2^{a_2}(b_1+t)^{a_1+1}} = \infty$$

Hence $p_1(\infty) = 1$ and $p_2(\infty) = 0$; i.e. all weight is allocated to the component with the smallest shape parameter when system lifetime approaches infinity. Now by (3.1) and the expression for $\pi(\theta_1, \theta_2|B_1(t))$ in (3.2) the probability measure corresponding to $\pi(\theta_1, \theta_2|T = t)$ converges weakly when $t \to \infty$ to the product measure of the Dirac measure at $0, \delta_0(\theta_1)$, and the measure corresponding to $g(\theta_2; a_2, b_2)$. For the second component this is intuitively obvious since we just know that its lifetime is less than infinity and hence our prior assessment is unchanged.

Now let $a_2 > a_1, b_2 < b_1$. Here the prior mean, a_1/b_1, of θ_1 is less than the prior mean, a_2/b_2, of θ_2. Consider a vector, $\underline{T}$, of system lifetimes. We shall show that

$$\pi(\theta_1 \geq \theta_0|\underline{T}) < \pi(\theta_2 \geq \theta_0|\underline{T});$$

i.e. θ_2 is posterior stochastically larger than θ_1. Hence the first component is still the better one.

Denote the likelihood function by $L(\theta_1, \theta_2; \underline{T})$. This is obviously symmetric in θ_1 and θ_2. Define $(i = 1, 2)$

$$I(\theta_i) = I(\theta_i \geq \theta_0)$$

We now have

$$\begin{aligned}
&\pi(\theta_1 \geq \theta_0|\underline{T}) - \pi(\theta_2 \geq \theta_0|\underline{T}) \propto \\
&\int_{0<\theta_2<\theta_1<\infty} L(\theta_1, \theta_2; \underline{T})[I(\theta_1) - I(\theta_2)]\pi(\theta_1, \theta_2) d\theta_1 d\theta_2 \\
&+ \int_{0<\theta_1<\theta_2<\infty} L(\theta_1, \theta_2; \underline{T})[I(\theta_1) - I(\theta_2)]\pi(\theta_1, \theta_2) d\theta_1 d\theta_2
\end{aligned}$$

By interchanging θ_1 and θ_2 in the last integral using the symmetry of $L(\theta_1, \theta_2; \underline{T})$ this equals

$$\begin{aligned}
&= \int_{0<\theta_2<\theta_1<\infty} L(\theta_1, \theta_2; \underline{T})[I(\theta_1) - I(\theta_2)][\pi(\theta_1, \theta_2) - \pi(\theta_2, \theta_1)] d\theta_1 d\theta_2 \\
&\propto \int_{0<\theta_2<\theta_1<\infty} L(\theta_1, \theta_2; \underline{T})[I(\theta_1) - I(\theta_2)](\theta_1\theta_2)^{a_1-1} e^{-(\theta_1+\theta_2)b_2} \\
&\times [\theta_2^{a_2-a_1} e^{-(b_1-b_2)\theta_1} - \theta_1^{a_2-a_1} e^{-(b_1-b_2)\theta_2}] d\theta_1 d\theta_2 < 0
\end{aligned}$$

Finally, we have in Figure 3.1 tabulated $\pi(\theta_1, \theta_2)$ and in Figures 3.2, 3.3, 3.4 tabulated $\pi(\theta_1, \theta_2|T = t)$ for $t = 1,\ 5,\ 50$, for the more complex case $a_1 = 4,\ b_1 = 1;\ a_2 = 6,\ b_2 = 3$. Here the prior mean of θ_2 is less than the one of θ_1. Note that $\pi(\theta_1, \theta_2|T = 1)$ is not that far from $\pi(\theta_1, \theta_2)$. $\pi(\theta_1, \theta_2|T = 5)$ indicates that the second component is still the better one, whereas $\pi(\theta_1, \theta_2|T = 50)$ reveals, as in the case above where $t \to \infty$, that the first component is the better one.

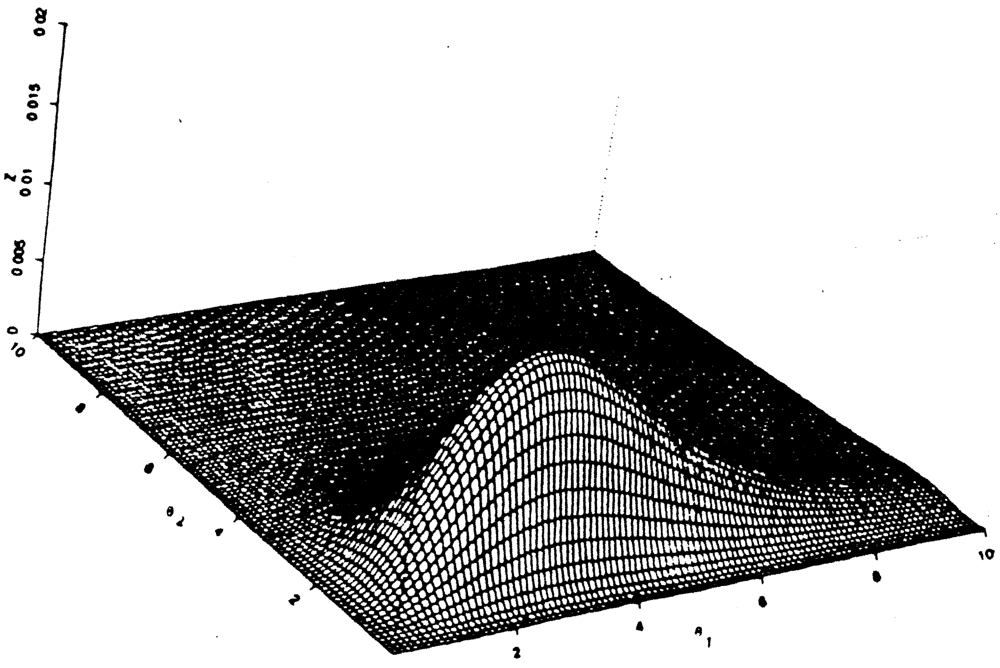

Figure 3.1 $\pi(\theta_1, \theta_2)$ for $a_1 = 4, b_1 = 1; a_2 = 6, b_2 = 3$.

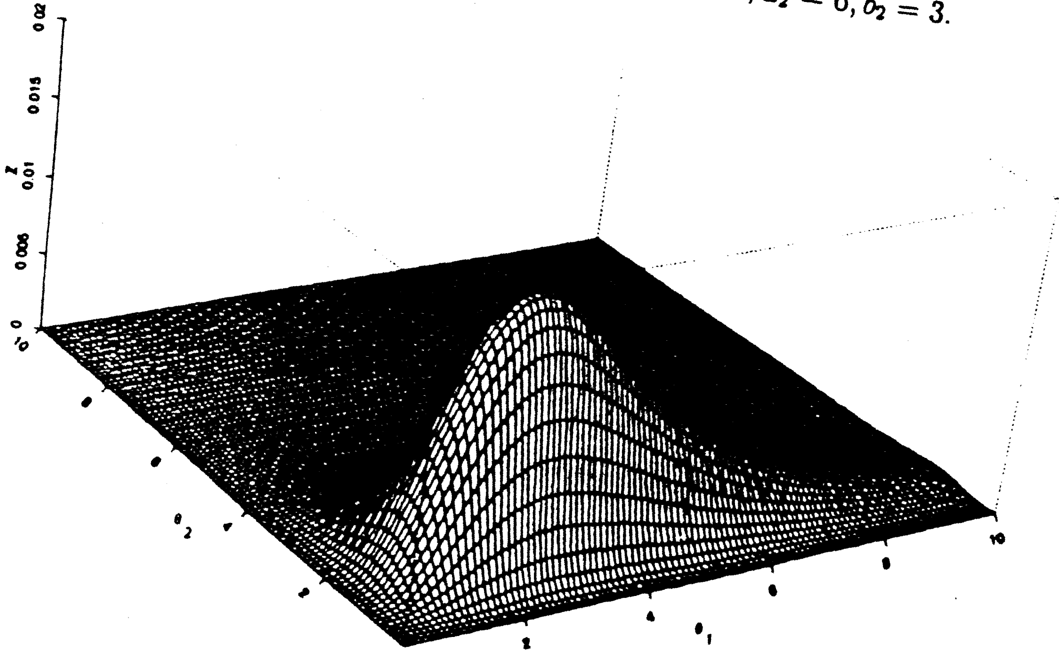

Figure 3.2 $\pi(\theta_1, \theta_2 | T = 1)$ for $a_1 = 4, b_1 = 1; a_2 = 6, b_2 = 3$.

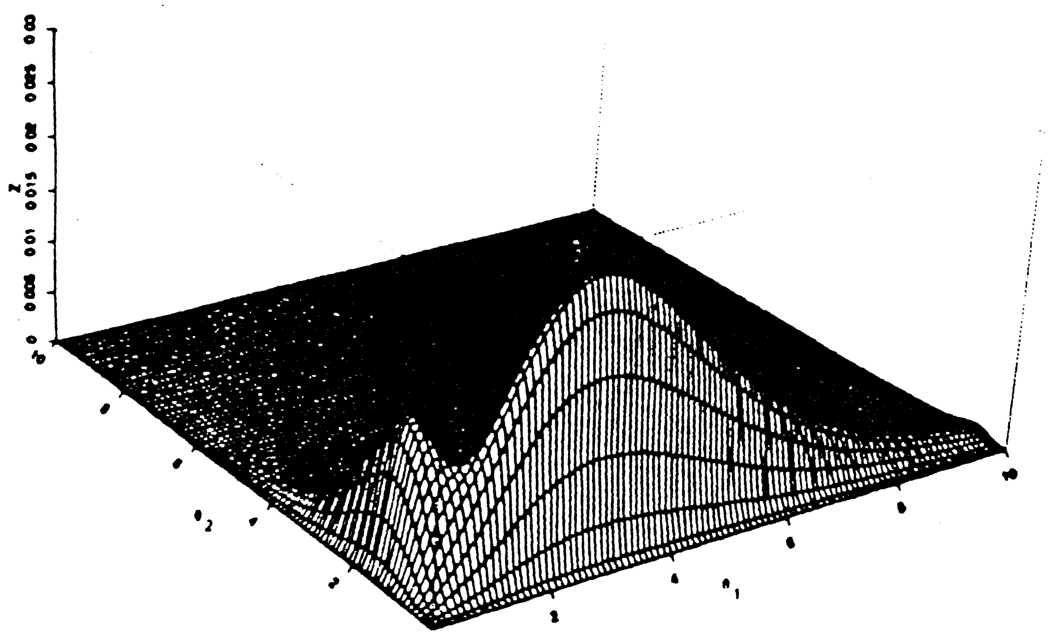

Figure 3.3 $\pi(\theta_1, \theta_2|T = 5)$ for $a_1 = 4, b_1 = 1; a_2 = 6, b_2 = 3$.
Note the scale on the Z-axis.

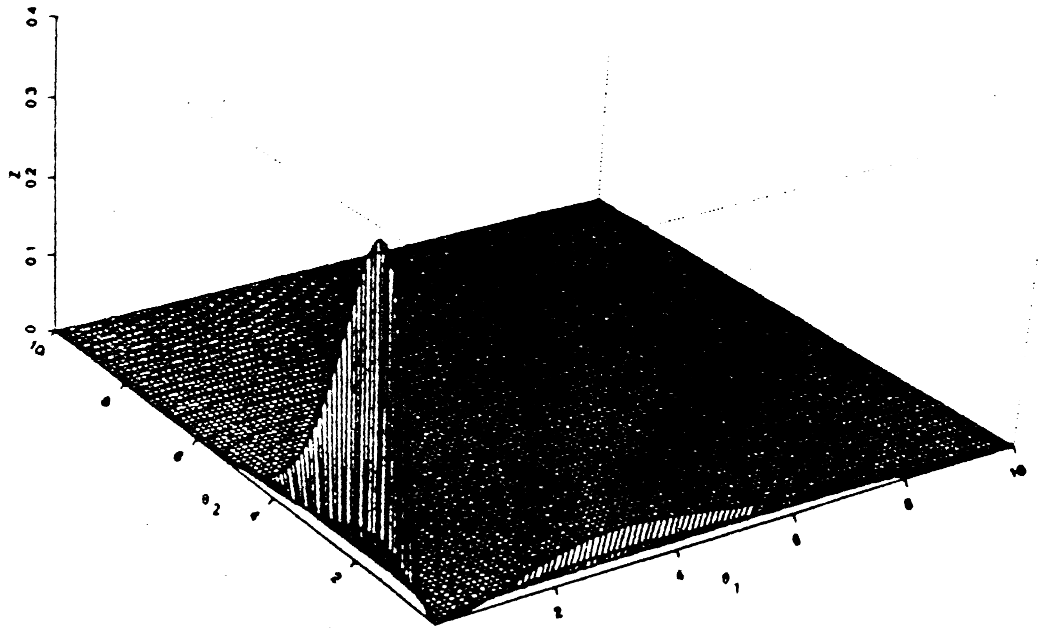

Figure 3.4 $\pi(\theta_1, \theta_2|T = 50)$ for $a_1 = 4, b_1 = 1; a_2 = 6, b_2 = 3$.
Note the scale on the Z-axis.

4. Lifemonitored components

To know the autopsy data (T, D) means to know T and to know which component lifetimes are at most T and which are above T. The order of failure of the components and the failure times are indeed unknown. In actual practice, often some of the components are lifemonitored until system failure. Let

$$\begin{aligned} M &= \text{the set of lifemonitored components} \\ &= \{1, \cdots, p\} \subset E,\ 1 \leq p \leq n \end{aligned}$$

This means that for $i \varepsilon M$ and $T_i \leq T, T_i$ is known. In this section a complete likelihood function for this case is arrived at for general lifetime distributions of the components making a fully Bayesian approach possible.

We are now back at the notation of Section 1. Let furthermore $Z_0^* = 0$ and

$$(Z_1^*, \cdots, Z_p^*) =$$

the order statistics of the lifetimes of the lifemonitored components

$$\begin{aligned} Z_i &= Z_i^* \wedge T\ ,\ i = 0, \cdots, p \\ Z_{p+1} &= T \end{aligned}$$

$Z_i,\ i = 1, \cdots, p+1$ are the points of time where either a component or system failure (or both) is observed. The number, K, of different such time points (until system failure) is at most $p+1$. We obviously have

$$K = \min\{k | Z_{k+1} = Z_k\} \wedge p + 1$$

Introduce $(k = 1, \cdots, K)$

$I_k(I_k^*) = i$ if the ith lifemonitored component fails at time $Z_k(Z_k^*)$ (at which time the system may fail), $i \varepsilon M$

$I_k = 0$ if the system fails at time Z_k due to the failure of a non lifemonitored component

$J_k = i$ if the system fails at time Z_k with fatal set $A_i,\ i \varepsilon \{1, \cdots, m\}$

$J_k = 0$ if the system does not fail at time Z_k

$J_K = J$

$R_0 = M$

$R_k = M - \{I_1, \cdots, I_k\}$ = the set of lifemonitored components being at risk just after Z_k

R_k^* is defined correspondingly by replacing $I_1, \cdots, I_k$ by $I_1^*, \cdots, I_k^*$ and Z_k by Z_k^*.

$$\begin{aligned}\mathcal{B}_0 &= \emptyset \\ \mathcal{B}_k &= \{Z_1, I_1, J_1, \cdots, Z_k, I_k, J_k\} \\ &= \text{the available information just after } Z_k\end{aligned}$$

Our likelihood function will be a density function on the space

$$\Omega = (R^+ \times \{0, 1, \cdots, p\} \times \{0, 1, \cdots, m\})^{p+1}$$

with respect to the measure

$$\mu = (\text{Lebesgue measure } \times \text{ counting measure } \times \text{ counting measure})^{p+1}$$

Now let $R \subset M$ be a set of lifemonitored components at risk and $R^c = M - R$ the corresponding set of failed components. Introduce

$$P(t, i|R) = \prod_{l \varepsilon A_i - R^c} F_l(t) \prod_{l \varepsilon A_i^c - R} \overline{F}_l(t) \tag{4.1}$$

= the probability that the system has failed at time t with fatal set A_i, given the set R of lifemonitored components at risk just after t.

The associated set of possible fatal sets, for which we also know that component I is a member of the corresponding critical set, is given by

$$F(R, I) = \{i \varepsilon \{1, \cdots, m\} | R^c \subset A_i,\ R \subset A_i^c,\ I \varepsilon C_{A_i}\} \tag{4.2}$$

$$\begin{aligned} &G(s, t, i|R) \\ &= \int_s^t \prod_{l \varepsilon A_i - C_{A_i} - R^c} F_l(u) \frac{d}{du}\Big(\prod_{l \varepsilon C_{A_i} - R^c} F_l(u)\Big) \prod_{l \varepsilon A_i^c - R} \overline{F}_l(u) du \end{aligned} \tag{4.3}$$

= the probability that the system fails in $(s, t]$ with fatal set A_i given the set R of lifemonitored components at risk in this time interval.

The associated set of possible fatal sets is given by

$$F(R) = \{i \varepsilon \{1, \cdots, m\} | R^c \subset A_i,\ R \subset A_i^c,\ C_{A_i} - R^c \neq \emptyset\} \tag{4.4}$$

For $k = 0, \cdots, K - 1,\ t \geq Z_k$, introduce

$$R_k(t) = \{T_i > t,\ i \varepsilon R_k\} =$$

the event that all lifemonitored components at risk just after Z_k, are still at risk just after t.

$R_k^*(t)$ is defined correspondingly by replacing R_k by R_k^*.

$$\begin{aligned} E_0 &= \emptyset \\ E_k &= \{Z_1^*, I_1^*, \cdots, Z_k^*, I_k^*\}\end{aligned}$$

For $k = 0, \cdots, K-1$ we have

$$\mathcal{B}_k = E_k \cap \{T > Z_k\} \tag{4.5}$$

We will start by proving two lemmas.

Lemma 4.1

For $k = 0, \cdots, K-1$ and $t \geq Z_k$, we have

$$\begin{aligned}
&P(T > t | E_k \cap R_k(t)) \\
&= 1 - \sum_{j=1}^{k} \sum_{i \varepsilon F(R_j, I_j)} P(Z_j, i | R_j) \\
&- \sum_{j=1}^{k} \sum_{i \varepsilon F(R_{j-1})} G(Z_{j-1}, Z_j, i | R_{j-1}) - \sum_{i \varepsilon F(R_k)} G(Z_k, t, i | R_k)
\end{aligned}$$

Proof. Introduce the events

$$A_i(t) = \{J = i \cap T \leq t\} \quad , \; i \varepsilon \{1, \cdots, m\}$$

Then obviously for $k = 0, \cdots, K-1, t \geq Z_k^*$ we have

$$\begin{aligned}
&P(T > t | E_k \cap R_k^*(t)) \\
&= P((\bigcup_{i=1}^{m} A_i(t))^c | E_k \cap R_k^*(t)) \\
&= 1 - \sum_{i=1}^{m} P(A_i(t) | E_k \cap R_k^*(t)) \\
&= 1 - \sum_{j=1}^{k} \sum_{i \varepsilon F(R_j^*, I_j^*)} P(Z_j^*, i | R_j^*) \\
&- \sum_{j=1}^{k} \sum_{i \varepsilon F(R_{j-1}^*)} G(Z_{j-1}^*, Z_j^*, \; i | R_{j-1}^*) - \sum_{i \varepsilon F(R_k^*)} G(Z_k^*, t, i | R_k^*)
\end{aligned}$$

The first double sum is the contribution from the lifemonitored components causing system failure. The second double sum is the contribution from non lifemonitored components causing system failure before Z_k^* and the last sum the corresponding contribution after Z_k^*. Note that the lifetime distributions of the non lifemonitored components are unaffected by $E_k \cap R_k^*(t)$ since this just contains independent information on the lifemonitored components. Since $k < K, (Z_j^*, R_j^*)$ can be replaced by $(Z_j, R_j), \; j = 0, \cdots, k$ and the proof is completed.

Lemma 4.2

For $k = 0, \cdots, K-1, \; t > Z_k$ and $i \varepsilon \{1, \cdots, m\}$ define

$$\begin{aligned}
&\rho_i(t | E_k \cap R_k(t) \cap T \geq t) \\
&= \lim_{dt \to 0} P[(J = i) \cap (t \leq T \leq t + dt) | E_k \cap R_k(t) \cap T \geq t] / dt
\end{aligned}$$

Then

$$\rho_i(t|E_k \cap R_k(t) \cap T \geq t)$$
$$= [\sum_{j\varepsilon C_{A_i}-R_k^c} \lambda_j(t) \prod_{l\varepsilon A_i - R_k^c - \{j\}} F_l(t) \prod_{l\varepsilon (A_i^c - R_k)\cup\{j\}} \overline{F}_l(t)]$$
$$/P(T > t|E_k \cap R_k(t)),$$

where the denominator is given by Lemma 4.1.

Proof. By introducing the events $A_i(t)$, $i\varepsilon\{1, \cdots, m\}$ as in Lemma 4.1 we have (since $t > Z_k$)

$$\rho_i(t|E_k \cap R_k(t) \cap T \geq t)$$

$$= \lim_{dt\to 0} P[A_i(t + dt) - A_i(t)|E_k \cap R_k(t) \cap T \geq t]/dt$$

$$= \lim_{dt\to 0} \frac{P[A_i(t + dt) - A_i(t)|E_k \cap R_k(t)]/dt}{P(T \geq t|E_k \cap R_k(t))}$$

$$= \lim_{dt\to 0} \frac{G(t, t + dt, i|R_k)/dt}{P(T \geq t|E_k \cap R_k(t))}$$

The proof is completed by using (4.3) and noting that $P(T \geq t|E_k \cap R_k(t)) =$ $P(T > t|E_k \cap R_k(t))$ since $t > Z_k$.

Theorem 4.3

Let

$$\Delta_i = I(T_i \leq T), \quad i\varepsilon M$$

The complete likelihood function of our parameter vector $\underline{\theta}$ is, for the case where M is the set of lifemonitored components, given by

$$L(\underline{\theta}) = \prod_{i\varepsilon M} (\lambda_i(T_i))^{\Delta_i} \overline{F}_i(T_i \wedge T)$$

$$\times \prod_{k=0}^{K-1} P(T > Z_{k+1}|E_k \cap R_k(Z_{k+1}))/P(T > Z_k|E_k)$$

$$\times \prod_{k=0}^{K-2} (1 - \sum_{i\varepsilon F(R_{k+1}, I_{k+1})} P(Z_{k+1}, i|R_{k+1}))$$

$$\times [I(I_K \neq 0)P(Z_K, J|R_K)$$

$$+ I(I_K = 0)\rho_J(Z_K|E_{K-1} \cap R_{K-1}(Z_K) \cap T \geq Z_K)]$$

An explicit expression is arrived by applying (4.1)-(4.4) and Lemmas 4.1 and 4.2. Se also our subsequent (4.12).

Proof. With obvious notation we can write $L(\underline{\theta})$ in the following form

$$\begin{aligned} L(\underline{\theta}) = \prod_{k=0}^{K-1} & L(\underline{\theta}; Z_{k+1}|\mathcal{B}_k) L(\underline{\theta}; I_{k+1}|\mathcal{B}_k, Z_{k+1}) \\ & \times L(\underline{\theta}; J_{k+1}|\mathcal{B}_k, Z_{k+1}, I_{k+1}) \end{aligned} \tag{4.6}$$

We start by establishing $L(\underline{\theta}; Z_{k+1}|\mathcal{B}_k)$. Assume $t > Z_k$. Then for $k = 0, \cdots, K-1$ using (4.5) we have

$$P(Z_{k+1} > t|\mathcal{B}_k) = P(\bigcap_{i\epsilon R_k} (T_i > t) \cap T > t|\mathcal{B}_k)$$

$$= P(R_k(t) \cap T > t|\mathcal{B}_k)$$

$$= P(R_k(t)|\mathcal{B}_k) P(T > t|\mathcal{B}_k \cap R_k(t))$$

$$= \prod_{i\epsilon R_k} (\overline{F}_i(t)/\overline{F}_i(Z_k)) P(T > t|E_k \cap T > Z_k \cap R_k(t))$$

$$= \prod_{i\epsilon R_k} (\overline{F}_i(t)/\overline{F}_i(Z_k)) P(T > t|E_k \cap R_k(t))/P(T > Z_k|E_k),$$

since

$$P(T > Z_k|E_k \cap R_k(t)) = P(T > Z_k|E_k \cap R_k(Z_k)) = P(T > Z_k|E_k)$$

Hence

$$-\frac{d}{dt} P(Z_{k+1} > t|\mathcal{B}_k) = [-\frac{d}{dt} P(T > t|E_k \cap R_k(t))$$

$$+ \sum_{j\epsilon R_k} \lambda_j(t) P(T > t|E_k \cap R_k(t))] \prod_{i\epsilon R_k} (\overline{F}_i(t)/\overline{F}_i(Z_k))$$

$$/P(T > Z_k|E_k)$$

$$= [P(T > t|E_k \cap R_k(t))/P(T > Z_k|E_k)] \prod_{i\epsilon R_k} \overline{F}_i(t)/\overline{F}_i(Z_k)$$

$$\times [\sum_{j\epsilon R_k} \lambda_j(t) + \sum_{j\epsilon F(R_k)} \rho_j(t|E_k \cap R_k(t) \cap T \geq t)],$$

having applied the definition of $\rho_i(t|E_k \cap R_k(t) \cap T \geq t)$ given in Lemma 4.2 and the fact that $t > Z_k$, and also (4.4). By inserting $t = Z_{k+1}$ in the expression above we get for

$k = 0, \cdots, K-1$

$$L(\underline{\theta}; Z_{k+1}|\mathcal{B}_k) = [P(T > Z_{k+1}|E_k \cap R_k(Z_{k+1}))/P(T > Z_k|E_k)]$$

$$\times \prod_{i\varepsilon R_k} (\overline{F}_i(Z_{k+1})/\overline{F}_i(Z_k))[\sum_{j\varepsilon R_k} \lambda_j(Z_{k+1}) + \sum_{j\varepsilon F(R_k)} \rho_j(Z_{k+1}|E_k \cap R_k(Z_{k+1}) \cap T \geq Z_{k+1})] \tag{4.7}$$

By a competing risk argument we get for $k = 0, \cdots, K-2$ by using (4.5)

$$L(\underline{\theta}; I_{k+1}|\mathcal{B}_k, Z_{k+1}) = L(\underline{\theta}; I_{k+1}|E_k, R_k(Z_{k+1}), T \geq Z_{k+1})$$

$$= \lambda_{I_{k+1}}(Z_{k+1})/[\sum_{j\varepsilon R_k} \lambda_j(Z_{k+1}) + \sum_{j\varepsilon F(R_k)} \rho_j(Z_{k+1}|E_k \cap R_k(Z_{k+1}) \cap T \geq Z_{k+1})] \tag{4.8}$$

whereas

$$L(\underline{\theta}; I_K|\mathcal{B}_{K-1}, Z_K) =$$

$$[\lambda_{I_K}(Z_K)I(I_K \neq 0) + \sum_{j\varepsilon F(R_{K-1})} \rho_j(Z_K|E_{K-1} \cap R_{K-1}(Z_K) \cap T \geq Z_K)I(I_K = 0)] \tag{4.9}$$

$$/[\sum_{j\varepsilon R_{K-1}} \lambda_j(Z_K) + \sum_{j\varepsilon F(R_{K-1})} \rho_j(Z_K|E_{K-1} \cap R_{K-1}(Z_K) \cap T \geq Z_K)]$$

For $k = 0, \cdots, K-2$, we get by applying (4.2)

$$L(\underline{\theta}; J_{k+1}|\mathcal{B}_k, Z_{k+1}, I_{k+1}) = L(\underline{\theta}; 0|\mathcal{B}_k, Z_{k+1}, I_{k+1})$$

$$= 1 - \sum_{i\varepsilon F(R_{k+1}, I_{k+1})} P(Z_{k+1}, i|R_{k+1}) \tag{4.10}$$

Finally, we get by using (4.5)

$$L(\underline{\theta}; J_K|\mathcal{B}_{K-1}, Z_K, I_K)$$

$$= I(I_K \neq 0)P(Z_K, J|R_K)$$

$$+ I(I_K = 0)\rho_J(Z_K|E_{K-1} \cap R_{K-1}(Z_K) \cap T \geq Z_K) \tag{4.11}$$

$$/[\sum_{j\varepsilon F(R_{K-1})} \rho_j(Z_K|E_{K-1} \cap R_{K-1}(Z_K) \cap T \geq Z_K)]$$

By inserting (4.7)-(4.11) into (4.6) our proof is completed.

Note that the product

$$\prod_{i\varepsilon M} (\lambda_i(T_i))^{\Delta_i} \overline{F}_i(T_i \wedge T)$$

represents the full likelihood function for the lifemonitored components. Secondly, note that by applying Lemma 4.1 we have

$$P(T > Z_{k+1}|E_k \cap R_k(Z_{k+1}))/P(T > Z_k|E_k)$$

$$= 1 - \sum_{i\varepsilon F(R_k)} G(Z_k, Z_{k+1}, i|R_k)/[1 - \sum_{j=1}^{k} \sum_{i\varepsilon F(R_j, I_j)} P(Z_j, i|R_j) \tag{4.12}$$

$$- \sum_{j=1}^{k} \sum_{i\varepsilon F(R_{j-1})} G(Z_{j-1}, Z_j, i|R_{j-1})]$$

5. Lifemonitored and conditionally lifemonitored components

In this section we will extend the model of Section 4 and also allow for conditional lifemonitoring of some components. Let

$$\begin{aligned} C = &\text{ the set of conditionally lifemonitored components} \\ = &\ \{p+1, \cdots, p+q\} \subset E\ ,\ 1 \leq p < p+q \leq n. \end{aligned}$$

For $i \varepsilon C$ there exists some arbitrary stopping time (inspection time), τ_i, in terms of the monitored components such that the ith component is monitored from τ_i onwards until system failure. This means that if $i \varepsilon C$ and $\tau_i < T_i \leq T$, then T_i is known and the ith component is, after τ_i, dealt with as a lifemonitored component. If on the other hand, $T_i \leq \tau_i \leq T$, only this inequality becomes known. The idea behind the model is that lifemonitoring of components is expensive and special equipment might be needed. Hence for some components this is started only when we know that the system is in a serious state. Note that the system may well be a human being at a hospital.

Introduce

$$X_i(t) = I(T_i > t),\ Y_i = X_i(\tau_i),\ i \varepsilon C$$

For each $\tau_i \leq T,\ i \varepsilon C$, we assume that $\tau_i = Z_k$ for some $k = 1, \cdots, K$, where now

$$K = \min\{k | Z_{k+1} = Z_k\} \wedge p + q + 1,$$

implying that inspections take zero (operational) time.

Let for $k = 1, \cdots, K$

$$H_0 = \emptyset$$

$H_k = \{i \varepsilon C | \tau_i = Z_k\}$ = the set of conditionally lifemonitored components being monitored from Z_k onwards

$$H_{k,0} = \{i \varepsilon H_k | Y_i = 0\}$$

$$H_{k,1} = \{i \varepsilon H_k | Y_i = 1\}$$

$N_{k,0} = \bigcup_{j=1}^{k} H_{j,0}$ = the set of conditionally lifemonitored components being failed on inspections, not after Z_k

$N_{k,1} = \bigcup_{j=1}^{k} H_{j,1}$ = the set of conditionally lifemonitored components being functioning on inspections, not after Z_k

$$R_0 = M$$

$R_k = M \cup N_{k,1} - \{I_1, \cdots, I_k\}$ = the set of lifemonitored and conditionally lifemonitored components being at risk just after Z_k

$$Q_0 = \emptyset$$

$Q_k = N_{k,0} \cup \{I_1, \cdots, I_k\}$ = the set of lifemonitored and conditionally lifemonitored components having failed not after Z_k

$\mathcal{B}_0 = \emptyset$

$\mathcal{B}_k = \{Z_1, I_1, J_1, Y_i, i\varepsilon H_1, \cdots, Z_k, I_k, J_k, Y_i, i\varepsilon H_k\}$

Our likelihood function will be a density function on the space

$$\Omega = (R^+ \times \{0, 1, \cdots, p+q\} \times \{0, 1, \cdots, m\})^{p+q+1} \times \{0, 1\}^q$$

with respect to the measure

$$\mu = (\text{Lebesgue measure } \times \text{ counting measure } \times \text{ counting measure})^{p+q+1} \times (\text{counting measure})^q$$

Now let N_0 be a set of conditionally lifemonitored components being failed on inspections. Compared with the deductions in Section 4 we have for $l\varepsilon N_0$ and $0 \leq t \leq \tau_l$ to replace $F_l(t)$ by $F_l(t)/F_l(\tau_l)$ and hence $\overline{F}_l(t)$ by $(\overline{F}_l(t) - \overline{F}_l(\tau_l))/F_l(\tau_l)$. Introduce

$$\phi_{l,N_0}(t) = \overline{F}_l(t) - I(l\varepsilon N_0)\overline{F}_l(\tau_l) \qquad 0 \leq t \leq \tau_l \tag{5.1}$$

Let furthermore $R \subset M \cup C$ be a set of lifemonitored and conditionally lifemonitored components known to be at risk and $Q \subset M \cup C$ a corresponding set of components known to be failed. We have $R \cap Q = \emptyset$. However, since we might lack information on some of the conditionally lifemonitored components, we do not have $Q = M \cup C - R$.

Parallel to (4.1)-(4.4) we now introduce

$$P(t, i|Q, R, N_0) = \prod_{l\varepsilon A_i - Q} F_l(t) \prod_{l\varepsilon A_i^c - R} \phi_{l,N_0}(t) / \prod_{l\varepsilon[(A_i-Q)\cup(A_i^c-R)]\cap N_0} F_l(\tau_l) \tag{5.2}$$

$$F(Q, R, I) = \{i\varepsilon\{1, \cdots, m\} | Q \subset A_i, R \subset A_i^c, I\varepsilon C_{A_i}\} \tag{5.3}$$

$$\begin{aligned} &G(s, t, i|Q, R, N_0) \\ &= \int_s^t \prod_{l\varepsilon A_i - C_{A_i} - Q} F_l(u) \frac{d}{du}\Big(\prod_{l\varepsilon C_{A_i} - Q} F_l(u)\Big) \prod_{l\varepsilon A_i^c - R} \phi_{l,N_0}(u) du / \prod_{l\varepsilon[(A_i-Q)\cup(A_i^c-R)]\cap N_0} F_l(\tau_l) \end{aligned} \tag{5.4}$$

$$F(Q, R) = \{i\varepsilon\{1, \cdots, m\} | Q \subset A_i, R \subset A_i^c, C_{A_i} - Q \neq \emptyset\} \tag{5.5}$$

Introduce

$$E_0 = \emptyset$$
$$E_k = \{Z_1, I_1, Y_i, i\varepsilon H_1, \cdots, Z_{k-1}, I_{k-1}, Y_i, i\varepsilon H_{k-1}, Z_k, I_k\}$$

For $k = 0, \cdots, K-1$ we have

$$\mathcal{B}_k = \{E_k, Y_i, i\varepsilon H_k\} \cap \{T > Z_k\}. \tag{5.6}$$

Let $(0 \leq j \leq k)$

$R^k_j = M \cup N_{k-1,1} - \{I_1, \cdots, I_j\}$ = the set of lifemonitored and conditionally lifemonitored components being at risk just after Z_j according to E_k.

Note that $R^k_{k-1} = R_{k-1}$. The R^k_j's are determined recursively from the relations.

$$\begin{aligned} &R^0_0 = M \quad , \; H_{0,1} = \emptyset \\ &R^k_0 = R^{k-1}_0 \cup H_{k-1,1} \quad , \; 1 \leq k \\ &R^k_j = R^k_{j-1} - I_j \quad , \; 1 \leq j \leq k \end{aligned}$$

For $k = 0, \cdots, K-1, t \geq Z_k$ define $R_k(t)$ from R_k as in Section 4. The proofs of the two following lemmas are very parallel to the ones of Lemmas 4.1 and 4.2 and are left to the reader.

Lemma 5.1

For $k = 0, \cdots, K-1$ we have

$$\begin{aligned} &P(T > Z_k | E_k) \\ &= 1 - \sum_{j=1}^{k-1} \sum_{i \varepsilon F(Q_j, R^k_j, I_j)} P(Z_j, i | Q_j, R^k_j, N_{k-1,0}) - \sum_{i \varepsilon F(Q_{k-1} \cup I_k, R^k_k, I_k)} P(Z_k, i | Q_{k-1} \cup I_k, R^k_k, N_{k-1,0}) \\ &- \sum_{j=1}^{k} \sum_{i \varepsilon F(Q_{j-1}, R^k_{j-1})} G(Z_{j-1}, Z_j, i | Q_{j-1}, R^k_{j-1}, N_{k-1,0}) \end{aligned} \tag{5.7}$$

Here

$$\begin{aligned} &[(A_i - Q_j) \cup (A^c_i - R^k_j)] \cap N_{k-1,0} = N_{k-1,0} - N_{j,0}, j = 0, \cdots, k-1 \\ &[(A_i - Q_{k-1} \cup I_k) \cup (A^c_i - R^k_k)] \cap N_{k-1,0} = \emptyset \end{aligned}$$

For $k = 0, \cdots, K-1$ and $t \geq Z_k$, we have

$$\begin{aligned} &P(T > t | \{E_k, Y_i, i \varepsilon H_k\} \cap R_k(t)) \\ &= 1 - \sum_{j=1}^{k} \sum_{i \varepsilon F(Q_j, R^{k+1}_j, I_j)} P(Z_j, i | Q_j, R^{k+1}_j, N_{k,0}) \\ &- \sum_{j=1}^{k} \sum_{i \varepsilon F(Q_{j-1}, R^{k+1}_{j-1})} G(Z_{j-1}, Z_j, i | Q_{j-1}, R^{k+1}_{j-1}, N_{k,0}) - \sum_{i \varepsilon F(Q_k, R_k)} G(Z_k, t, i | Q_k, R_k, N_{k,0}) \end{aligned} \tag{5.8}$$

Here, in addition

$$[(A_i - Q_k) \cup (A^c_i - R_k)] \cap N_{k,0} = \emptyset$$

Lemma 5.2

For $k = 0, \cdots, K-1, t > Z_k$ and $i \varepsilon \{1, \cdots, m\}$ define

$$\rho_i(t | \{E_k, Y_i, i\varepsilon H_k\} \cap R_k(t) \cap T \geq t)$$

$$= \lim_{dt \to 0} P[(J = i) \cap (t \leq T \leq t + dt) | \{E_k, Y_i, i\varepsilon H_k\} \cap R_k(t) \cap T \geq t]/dt$$

Then

$$\rho_i(t | \{E_k, Y_i, i\varepsilon H_k\} \cap R_k(t) \cap T \geq t)$$

$$= [\sum_{j\varepsilon C_{A_i} - Q_k} \lambda_j(t) \prod_{l\varepsilon A_i - Q_k - \{j\}} F_l(t) \prod_{l\varepsilon(A_i^c - R_k)\cup\{j\}} \phi_{l,N_{k,0}}(t)]$$

$$/P(T > t | \{E_k, Y_i, i\varepsilon H_k\} \cap R_k(t)),$$

where the denominator is given by Lemma 5.1.

Theorem 5.3

Let

$$\Delta_i = I(T_i \leq T) \quad , i\varepsilon M$$
$$\Delta_i = I(\tau_i < T_i \leq T), \ i\varepsilon C$$

The complete likelihood function of our parameter vector $\underline{\theta}$ is, for the case where M is the set of lifemonitored components and C the set of conditionally lifemonitored components, given by

$$L(\underline{\theta}) = \prod_{i\varepsilon M \cup N_{K-1,1}} (\lambda_i(T_i))^{\Delta_i} \overline{F}_i(T_i \wedge T) \times \prod_{i\varepsilon N_{K-1,0}} F_i(\tau_i)$$

$$\times \prod_{k=0}^{K-1} P(T > Z_{k+1} | \{E_k, Y_i, i\varepsilon H_k\} \cap R_k(Z_{k+1}))/P(T > Z_k | E_k)$$

$$\times \prod_{k=0}^{K-2} (1 - \sum_{i\varepsilon F(Q_k \cup I_{k+1}, R_{k+1}^{k+1}, I_{k+1})} P(Z_{k+1}, i | Q_k \cup I_{k+1}, R_{k+1}^{k+1}, N_{k,0})$$

$$\times [I(I_K \neq 0) P(Z_K, J | Q_{K-1} \cup I_K, R_K^K, N_{K-1,0})$$

$$+ I(I_K = 0) \rho_J(Z_K | \{E_{K-1}, Y_i, i\varepsilon H_{K-1}\} \cap R_{K-1}(Z_K) \cap T \geq Z_K)]$$

An explicit expression is arrived at by applying (5.2)-(5.5) and Lemmas 5.1 and 5.2.

Proof.

With obvious notation we can write $L(\underline{\theta})$ in the following form

$$L(\underline{\theta}) = \prod_{k=0}^{K-1} L(\underline{\theta}; Z_{k+1}|\mathcal{B}_k)L(\underline{\theta}; I_{k+1}|\mathcal{B}_k, Z_{k+1})$$
$$\times L(\underline{\theta}; J_{k+1}|\mathcal{B}_k, Z_{k+1}, I_{k+1})L(\underline{\theta}; Y_i, i\varepsilon H_{k+1}|\mathcal{B}_k, Z_{k+1}, I_{k+1}, J_{k+1}) \tag{5.9}$$

By an argument parallel to the one leading to (4.7) replacing (4.5) by (5.6) we get for $k = 0, \cdots, K-1$

$$L(\underline{\theta}; Z_{k+1}|\mathcal{B}_k) = [P(T > Z_{k+1}|\{E_k, Y_i, i\varepsilon H_k\} \cap R_k(Z_{k+1}))$$
$$/P(T > Z_k|\{E_k, Y_i, i\varepsilon H_k\})]$$
$$\times \prod_{i\varepsilon R_k} (\overline{F}_i(Z_{k+1})/\overline{F}_i(Z_k))[\sum_{j\varepsilon R_k} \lambda_j(Z_{k+1}) + \sum_{j\varepsilon F(Q_k,R_k)} \rho_j(Z_{k+1}|\{E_k, Y_i, i\varepsilon H_k\}$$
$$\cap R_k(Z_{k+1}) \cap T \geq Z_{k+1})] \tag{5.10}$$

Parallel to (4.8) we get for $k = 0, \cdots, K-2$

$$L(\underline{\theta}; I_{k+1}|\mathcal{B}_k, Z_{k+1}) = L(\underline{\theta}; I_{k+1}|\{E_k, Y_i, i\varepsilon H_k\}, R_k(Z_{k+1}), T \geq Z_{k+1})$$
$$= \lambda_{I_{k+1}}(Z_{k+1})/[\sum_{j\varepsilon R_k} \lambda_j(Z_{k+1}) + \sum_{j\varepsilon F(Q_k,R_k)} \rho_j(Z_{k+1}|\{E_k, Y_i, i\varepsilon H_k\} \cap R_k(Z_{k+1}) \cap T \geq Z_{k+1})], \tag{5.11}$$

whereas

$$L(\underline{\theta}; I_K|\mathcal{B}_{K-1}, Z_K) =$$
$$[\lambda_{I_K}(Z_K)I(I_K \neq 0) + \sum_{j\varepsilon F(Q_{K-1},R_{K-1})} \rho_j(Z_K|\{E_{K-1}, Y_i, i\varepsilon H_{K-1}\} \cap R_{K-1}(Z_K) \cap T \geq Z_K)$$
$$I(I_K = 0)]/[\sum_{j\varepsilon R_{K-1}} \lambda_j(Z_K) + \sum_{j\varepsilon F(Q_{K-1},R_{K-1})} \rho_j(Z_K|\{E_{K-1}, Y_i, i\varepsilon H_{K-1}\} \cap R_{K-1}(Z_K) \cap$$
$$T \geq Z_K)] \tag{5.12}$$

For $k = 0, \cdots, K-2$, we get by applying (5.3)

$$L(\underline{\theta}; J_{k+1}|\mathcal{B}_k, Z_{k+1}, I_{k+1}) = L(\underline{\theta}; 0|\mathcal{B}_k, Z_{k+1}, I_{k+1})$$

$$= 1 - \sum_{i\varepsilon F(Q_k \cup I_{k+1}, R_{k+1}^{k+1}, I_{k+1})} P(Z_{k+1}, i|Q_k \cup I_{k+1}, R_{k+1}^{k+1}, N_{k,0}) \tag{5.13}$$

By using (5.6) we get

$$L(\underline{\theta}, J_K|\mathcal{B}_{K-1}, Z_K, I_K)$$

$$= I(I_K \neq 0) P(Z_K, J|Q_{K-1} \cup I_K, R_K^K, N_{K-1,0})$$

$$+ I(I_K = 0)\rho_J(Z_K|\{E_{K-1}, Y_i, i\varepsilon H_{K-1}\} \cap R_{K-1}(Z_K) \cap T \geq Z_K) \tag{5.14}$$

$$/[\sum_{j\varepsilon F(Q_{K-1}, R_{K-1})} \rho_j(Z_K|\{E_{K-1}, Y_i, i\varepsilon H_{K-1}\} \cap R_{K-1}(Z_K) \cap T \geq Z_K)]$$

Finally, by using (5.6) the new contribution is for $k = 0, \cdots, K-2$ given by

$$L(\underline{\theta}; Y_i, i\varepsilon H_{k+1}|\mathcal{B}_k, Z_{k+1}, I_{k+1}, J_{k+1})$$

$$= L(\underline{\theta}; Y_i, i\varepsilon H_{k+1}|\{E_k, Y_i, i\varepsilon H_k\}, Z_{k+1}, I_{k+1}, T > Z_{k+1})$$

$$= P(Y_i, i\varepsilon H_{k+1} \cap T > Z_{k+1}|E_{k+1})/P(T > Z_{k+1}|E_{k+1})$$

$$= P(T > Z_{k+1}|E_{k+1}, Y_i, i\varepsilon H_{k+1}) P(Y_i, i\varepsilon H_{k+1}|E_{k+1})/P(T > Z_{k+1}|E_{k+1})$$

$$= [P(T > Z_{k+1}|E_{k+1}, Y_i, i\varepsilon H_{k+1})/P(T > Z_{k+1}|E_{k+1})] \prod_{i\varepsilon H_{k+1,0}} F_i(Z_{k+1}) \prod_{i\varepsilon H_{k+1,1}} \overline{F}_i(Z_{k+1}), \tag{5.15}$$

whereas

$$L(\underline{\theta}; Y_i, i\varepsilon H_K|\mathcal{B}_{K-1}, Z_K, I_K, J) = 1, \tag{5.16}$$

since at Z_K the system fails and we know which components (A_J) that have failed and which (A_J^c) that have survived.

We now insert (5.10)-(5.16) into (5.9). Note that

$$P(T > Z_0|\{E_0, Y_i, i\varepsilon H_0\}) = P(T > Z_0|E_0) = P(T > 0) = 1.$$

Furthermore, we have by letting $\tau_i = 0, i\varepsilon M$

$$\begin{aligned}
&\prod_{k=1}^{K-1}\prod_{i\varepsilon H_{k,1}} \overline{F}_i(Z_k) \times \prod_{k=0}^{K-1}\prod_{i\varepsilon R_k} \overline{F}_i(Z_{k+1})/\overline{F}_i(Z_k) \\
&= \prod_{i\varepsilon N_{K-1,1}} \overline{F}_i(\tau_i) \times \prod_{i\varepsilon M\cup N_{K-1,1}} \prod_{k=0}^{K-1} (\overline{F}_i(Z_{k+1})/\overline{F}_i(Z_k))^{I(\tau_i\leq Z_k<T_i\wedge T)} \\
&= \prod_{i\varepsilon M\cup N_{K-1,1}} \overline{F}_i(\tau_i) \times \prod_{i\varepsilon M\cup N_{K-1,1}} \overline{F}_i(T_i\wedge T)/\overline{F}_i(\tau_i) \\
&= \prod_{i\varepsilon M\cup N_{K-1,1}} \overline{F}_i(T_i\wedge T)
\end{aligned}$$

Hence our proof is completed.

Let $R \subset M \cup C$ still be a set of lifemonitored and conditionally lifemonitored components known to be at risk and $Q \subset M\cup C$ a corresponding set of components known to be failed. Assume that at each inspection time point, a set of conditionally lifemonitored components is chosen iteratively according to the set function $H(Q,R)$. This is done to improve the control of the inspection procedure. Obviously

$$H(Q,R) \subset C - (Q\cup R)$$

In general, at each $Z_k, k = 1,\cdots,K-1$, inspection is stopped when $H(Q_k, R_k) = \emptyset$. By a slightly generalized argument it is seen that Theorem 5.3 is valid also in this case.

Assume now that one is allowed and can afford always to inspect components such that all the time at least one of the lifemonitored and conditionally lifemonitored components known to be at risk, must fail for the system to fail. Then the likelihood function of Theorem 5.3 becomes especially simple as is seen from the following intuitive obvious corollary.

Corollary 5.4

Let $H(Q,R)$ be chosen such that

$$(H(Q,R)\cup R)\cap A_i \neq \emptyset \quad \text{for all } i\varepsilon\{1,\cdots,m\},$$

implying that at $Z_k, k = 1,\cdots,K-1$

$$R_k \cap A_i \neq \emptyset \quad \text{for all } i\varepsilon\{1,\cdots,m\}.$$

Then

$$\begin{aligned}
L(\underline{\theta}) = &\prod_{i\varepsilon M\cup N_{K-1,1}} (\lambda_i(T_i))^{\Delta_i}\overline{F}_i(T_i\wedge T) \times \prod_{i\varepsilon N_{K-1,0}} F_i(\tau_i) \\
&\times \prod_{i\varepsilon\{E-M-N_{K-1,1}-N_{K-1,0}|Y_i(T)=0\}} F_i(T) \times \prod_{i\varepsilon\{E-M-N_{K-1,1}-N_{K-1,0}|Y_i(T)=1\}} \overline{F}_i(T)
\end{aligned}$$

Proof.

By assumption

$$R_k \not\subset A_i^c \qquad k = 1, \cdots, K-1 \quad \text{for all } i \varepsilon \{1, \cdots, m\}$$

Hence from (5.3) and (5.5) we have

$$F(Q_k, R_k, I_k) = F(Q_k, R_k) = \emptyset \quad k = 1, \cdots, K-1$$

Furthermore, this implies

$$F(Q_j, R_j^k, I_j) = F(Q_j, R_j^k) = \emptyset \quad j < k = 1, \cdots, K$$

Hence, all the double sums in (5.7) and all sums in (5.8) disappear and we get for $k = 0, \cdots, K-1$

$$P(T > Z_k | E_k) = 1 - \sum_{i \varepsilon F(Q_{k-1} \cup I_k, R_k^k, I_k)} P(Z_k, i | Q_{k-1} \cup I_k, R_k^k, N_{k-1,0})$$

$$P(T > Z_{k+1} | \{E_k, Y_i, i \varepsilon H_k\} \cap R_k(Z_{k+1})) = 1 \tag{5.17}$$

Since $F(Q_{K-1}, R_{K-1}) = \emptyset$, it follows from (5.12) that $I(I_K = 0) = 0$. Finally from (5.1) and (5.2), noting that $[(A_J - Q_{K-1} \cup I_K) \cup (A_J^c - R_K^K)] \cap N_{K-1,0} = \emptyset$, we have

$$P(Z_K, J | Q_{K-1} \cup I_K, R_K^K, N_{K-1,0})$$

$$= \prod_{l \varepsilon A_J - Q_{K-1} \cup I_K} F_l(Z_K) \prod_{l \varepsilon A_J^c - R_K^K} \phi_{l, N_{K-1,0}}(Z_K) \tag{5.18}$$

$$= \prod_{i \varepsilon \{E - M - N_{K-1,1} - N_{K-1,0} | Y_i(T) = 0\}} F_i(T) \prod_{i \varepsilon \{E - M - N_{K-1,1} - N_{K-1,0} | Y_i(T) = 1\}} \overline{F}_i(T)$$

By inserting (5.17) and (5.18) into the likelihood function of Theorem 5.3, remembering that $I(I_K = 0) = 0$, our proof is completed.

Since inspections are costly and should be reduced to a minimum it is reasonable to stop inspection at $Z_k, k = 1, \cdots, K-1$ when for the first time $R_k \cap A_i \neq \emptyset$ for all $i \varepsilon \{1, \cdots, m\}$.

Note that the likelihood function of Corollary 5.4 is of the same simple form as the one given by (1.1). Hence by assuming exponentially distributed component lifetimes, the posterior distribution of $\underline{\theta}$ is arrived at as in Section 2. Remember that the maximum number of addends in (2.1) was obtained for a parallel system and the minimum for a series system. In contrast, in the set up of Corollary 5.4 one just has to monitor one component at a time for the parallel system whereas for the series system all components have to be monitored. For a k-out-of n system, where at least k components must function for the system to function, obviously a minimum of k components must be monitored at a time. Note also that the fatal sets $A_1, \cdots, A_m$ in Corollary 5.4 can be replaced by the minimal cut sets of the system.

It should also be stressed that the inspection procedure of Corollary 5.4, in spite of giving simple mathematics and a simple likelihood function, does not need to be a realistic one in a specific application. Hence, the general result of Theorem 5.3 is indeed of importance.

Although this paper is concerned with autopsy data, it is finally worth noting that if these data are not observed, due to interrupted operation of the system, our results can easily be adapted to cover this case as well. Suppose $V > 0$ is a censoring time, either fixed in advance or being a random variable, with an absolutely continuous distribution, being independent of $T_i, i = 1, \cdots, n$. The corresponding intensity is assumed not to depend on $\underline{\theta}$. We now just have to replace T by min (T, V). Then the likelihood functions for $\underline{\theta}$ given in Theorems 4.3 and 5.3 are just modified by replacing the last factor $[\cdots]$, corresponding to a system failure at Z_K, by $[\cdots]I(J \neq 0) + I(J = 0)$. Here $J = 0$ corresponds to a censoring at Z_K.

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