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In [1] Alfsen, Shultz, and Størmer have studied a class of normed Jordan algebras over the reals, named JB-algebras. These algebras generalize the Jordan algebras of self-adjoint elements of a C^* -algebra, and also the more general Jordan operator algebras (JC-algebras), studied by Topping [6]. In [1] a structure theory for JB-algebras is given, generalizing the corresponding theory for finite dimensional formally real Jordan algebras by Jordan, von Neumann and Wigner.[3]. It is also shown that the JB-algebras share with C^* -algebras many of the basic properties needed for applications to physics, e.g. the spectral theory and the fact that the idempotents form a complete orthomodular lattice. However, we miss one important concept: the tensor product. Thus, it is of some interest to try to define a "good" notion of tensor product for some class of JB-algebras properly containing the self-adjoint parts of C^* -algebras.

In this note we show that this is impossible. In fact, assume that A is a JB-algebra such that the tensor product of the real linear spaces A and $M_2(\mathbb{C})_{sa}$ (the self-adjoint 2×2 -matrices over the complex numbers) can be equipped with a Jordan structure satisfying certain minimal requirements of "good behavior" relating it to the given Jordan products in the two factors, then we show that A itself is the self-adjoint part of a C^* -algebra. This is derived from a general result on Jordan algebras over a field of characteristic different from two, which may be of some inde-

pendent interest.

All algebras in this note will be assumed to have an identity, denoted by 1. In a Jordan algebra \mathcal{A} (see [2] for definition and basic properties), the product of two elements x, y is denoted by $x \circ y$. The mapping $y \mapsto x \circ y$ of \mathcal{A} into itself is denoted by T_x . Two elements x, y are said to operator commute if $T_x T_y = T_y T_x$. It follows from the definition of the triple product [2;p.36], that if $x, y \in \mathcal{A}$, then

$$(1.1) \quad \{xyx\} = 2x \circ (x \circ y) - x^2 \circ y ;$$

thus, defining the map $U_x : y \mapsto \{xyx\}$ we have

$$(1.2) \quad U_x = 2T_x^2 - T_{x^2} .$$

An associative algebra \mathcal{A} will also be considered as a Jordan algebra under the product $a \circ b = \frac{1}{2}(ab + ba)$. The algebra of 2×2 -matrices over a field K will be denoted by $M_2(K)$, or simply M_2 when K is understood. M_2 has a basis of matrix units ϵ_{ij} , $i, j = 1, 2$ such that

$$(1.3) \quad \epsilon_{11} + \epsilon_{22} = 1, \quad \epsilon_{ij} \epsilon_{kl} = \delta_{jk} \epsilon_{il} .$$

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2. The algebraic result.

All algebras in this section will be over a fixed field K of characteristic different from 2.

First, let us assume that \mathcal{A} is an associative algebra. Then $\mathcal{A} \otimes M_2$ is an associative algebra under the product defined by $(x \otimes \alpha)(y \otimes \beta) = xy \otimes \alpha\beta$. Considering the induced Jordan products on \mathcal{A}, M_2 and $\mathcal{A} \otimes M_2$ respectively, we easily verify the formulas:

$$(2.1) \quad (x \otimes 1) \circ (y \otimes \beta) = (x \circ y) \otimes \beta ,$$

$$(2.2) \quad (1 \otimes \alpha) \circ (y \otimes \beta) = y \otimes (\alpha \circ \beta) .$$

Assume now only that \mathcal{A} is a Jordan algebra and that $\mathcal{A} \otimes M_2$ is equipped with some Jordan product. Then the two formulas above are equivalent to the following four:

$$(2.3) \quad (x \otimes 1) \circ (y \otimes 1) = (x \circ y) \otimes 1 ,$$

$$(2.4) \quad (1 \otimes \alpha) \circ (1 \otimes \beta) = 1 \otimes (\alpha \circ \beta) ,$$

$$(2.5) \quad (1 \otimes \alpha) \circ (x \otimes 1) = x \otimes \alpha ,$$

$$(2.6) \quad [T_{1 \otimes \alpha}, T_{x \otimes 1}] = 0 .$$

Indeed, (2.3)-(2.5) are immediate consequences of (2.1) and (2.2). So is (2.6) when (2.1) is rewritten as $T_{x \otimes 1}(y \otimes \beta) = (x \circ y) \otimes \beta$, and similarly for (2.2). We prove the converse implication as easily, first using (2.5) to write $y \otimes \beta = T_{y \otimes 1}(1 \otimes \beta) = T_{1 \otimes \beta}(y \otimes 1)$.

Note that (2.3) states that the natural embedding $x \mapsto x \otimes 1$ of \mathcal{A} into $\mathcal{A} \otimes M_2$ is a Jordan homomorphism, while (2.4) is the same statement for M_2 .

Finally, (2.6) states that any element of $1 \otimes M_2$ operator commutes with any element of $\mathcal{A} \otimes 1$. Thus it seems natural to

claim that (2.3)-(2.6), or equivalently, (2.1), (2.2), should be satisfied for a "good" tensor product of Jordan algebras.

Theorem 2.1. Let \mathcal{A} be a Jordan algebra with identity over a field K of characteristic different from two. Assume that there exists a Jordan product in $\mathcal{A} \otimes M_2(K)$ satisfying (2.1) and (2.2). Then there exists a unique associative product in \mathcal{A} inducing the given Jordan products both in \mathcal{A} and in $\mathcal{A} \otimes M_2(K)$.

The remainder of this section is devoted to the proof of Theorem 2.1. Thus, the assumptions of the Theorem will be kept throughout. As noted above, the formulas (2.3)-(2.6) are also valid in the present setting.

We will make extensive use of the "linearised Jordan identity" [2; p.34]:

$$(2.7) \quad [T_{a \circ b}, T_c] + [T_{b \circ c}, T_a] + [T_{c \circ a}, T_b] = 0 ,$$

which is valid for a, b, c in any Jordan algebra. Putting $a = c = 1 \otimes \alpha$ and $b = x \otimes 1$ in (2.7) and using (2.6) yields:

$$(2.8) \quad [T_{x \otimes \alpha}, T_{1 \otimes \alpha}] = 0 ,$$

Using this together with (2.1) and (2.2) we may compute:

$$(x \otimes \alpha) \circ (y \otimes \alpha) = T_{x \otimes \alpha} T_{1 \otimes \alpha} (y \otimes 1) = T_{1 \otimes \alpha} T_{x \otimes \alpha} (y \otimes 1) = (x \circ y) \otimes \alpha^2 ,$$

which applied to $\alpha = \epsilon_{ij}$ yields the formulas:

$$(2.9) \quad (x \otimes \epsilon_{11}) \circ (y \otimes \epsilon_{11}) = (x \circ y) \otimes \epsilon_{11} \quad \text{and similarly for } \epsilon_{22},$$

$$(2.10) \quad (x \otimes \epsilon_{21}) \circ (y \otimes \epsilon_{21}) = 0 \quad \text{and similarly for } \epsilon_{12},$$

$$(2.11) \quad (x \otimes \epsilon_{11}) \circ (y \otimes \epsilon_{22}) = 0 .$$

The last formula comes from (2.9), (2.1) and $\epsilon_{22} = 1 - \epsilon_{11}$.

The following lemma is the main step in the proof of the theorem.

Lemma 2.2. If $x, y \in \mathcal{A}$, there exists $z \in \mathcal{A}$ such that
 $(x \otimes \epsilon_{21}) \circ (y \otimes \epsilon_{11}) = z \otimes \epsilon_{21}$.

Proof: Writing

$$(2.12) \quad (x \otimes \epsilon_{21}) \circ (y \otimes \epsilon_{11}) = \sum_{i,j=1}^2 z_{ij} \otimes \epsilon_{ij},$$

we have to show $z_{11} = z_{12} = z_{22} = 0$.

First, we apply $T_{1 \otimes \epsilon_{11}}$ to both sides of (2.12). On the left-hand side we find, using (2.8) and (2.2):

$$\begin{aligned} T_{1 \otimes \epsilon_{11}}((x \otimes \epsilon_{21}) \circ (y \otimes \epsilon_{11})) &= T_{1 \otimes \epsilon_{11}} T_{y \otimes \epsilon_{11}}(x \otimes \epsilon_{21}) = \\ &= T_{y \otimes \epsilon_{11}} T_{1 \otimes \epsilon_{11}}(x \otimes \epsilon_{21}) = \\ &= T_{y \otimes \epsilon_{11}}(\frac{1}{2}x \otimes \epsilon_{21}) = \\ &= \frac{1}{2}(x \otimes \epsilon_{21}) \circ (y \otimes \epsilon_{11}). \end{aligned}$$

On the righthand side we find

$$T_{1 \otimes \epsilon_{11}}(\sum_{i,j}^2 z_{ij} \otimes \epsilon_{ij}) = z_{11} \otimes \epsilon_{11} + \frac{1}{2}(z_{12} \otimes \epsilon_{12} + z_{21} \otimes \epsilon_{21}).$$

Since the ϵ_{ij} 's are linearly independent, $z_{11} = z_{22} = 0$ follows.

Next, applying $T_{1 \otimes \epsilon_{21}}$ and using the same technique, we find $z_{12} = 0$; thus proving the lemma. \square

Note that Lemma 2.2 is also valid when ϵ_{22} is substituted for ϵ_{11} . Thus, we may define maps $R_x : A \rightarrow A$ and $L_x : A \rightarrow A$ for given $x \in A$ by

$$(2.13) \quad R_x(y) \otimes \epsilon_{21} = 2(x \otimes \epsilon_{11}) \circ (y \otimes \epsilon_{21}),$$

$$(2.14) \quad L_x(y) \otimes \epsilon_{21} = 2(x \otimes \epsilon_{22}) \circ (y \otimes \epsilon_{21}).$$

Lemma 2.3. $[R_y, L_x] = 0$ for all $x, y \in \mathcal{A}$.

Proof: This will follow from the identity $[T_{y \otimes \epsilon_{11}}, T_{x \otimes \epsilon_{22}}] = 0$, which is proved as follows: Let $a = y \otimes \epsilon_{11}$, $b = x \otimes \epsilon_{22}$, $c = 1 \otimes \epsilon_{22}$. Then $a \circ b = a \circ c = 0$ and $b \circ c = x \otimes \epsilon_{22}$. Applying (2.7) yields the desired equality. \square

Corollary 2.4. There exists an associative product in \mathcal{A} with 1 as an identity and such that $xy = L_x(y) = R_y(x)$ and $x \circ y = \frac{1}{2}(xy + yx)$ for all $x, y \in \mathcal{A}$.

Proof: By (2.2), (2.13) and (2.14), $L_x(1) = R_x(1) = x$. Lemma 2.3 now guarantees existence and associativity of the product defined by $xy = L_x(y) = R_y(x)$. Also, $T_{x \otimes \epsilon_{11}} + T_{x \otimes \epsilon_{22}} = T_{x \otimes 1}$, which together with (2.1) proves that $x \circ y = \frac{1}{2}(R_x + L_x)(y) = \frac{1}{2}(yx + xy)$. \square

Concluding proof of Theorem 2.1. Now, \mathcal{A} is shown to be ^{an} associative algebra, and so is $\mathcal{A} \otimes M_2$. What remains to prove is that $a \circ b = \frac{1}{2}(ab + ba)$ for all a, b in $\mathcal{A} \otimes M_2$. First, rewrite (2.13) and (2.14) as follows:

$$(2.15) \quad (x \otimes \epsilon_{11}) \circ (y \otimes \epsilon_{21}) = \frac{1}{2}yx \otimes \epsilon_{21},$$

$$(2.16) \quad (x \otimes \epsilon_{22}) \circ (y \otimes \epsilon_{21}) = \frac{1}{2}xy \otimes \epsilon_{21}.$$

Next, put $a = x \otimes \epsilon_{21}$, $b = y \otimes \epsilon_{11}$, $c = 1 \otimes \epsilon_{11}$, so that $a \circ b = \frac{1}{2}xy \otimes \epsilon_{21}$, $b \circ c = y \otimes \epsilon_{11}$, $a \circ c = \frac{1}{2}x \otimes \epsilon_{21}$. Applying (2.7) we get $[T_{xy \otimes \epsilon_{21}}, T_{1 \otimes \epsilon_{11}}] + [T_{y \otimes \epsilon_{11}}, T_{x \otimes \epsilon_{21}}] = 0$. Applying this operator identity to $1 \otimes \epsilon_{12}$ and computing by means of previous formulas, we find

$$(2.17) \quad (x \otimes \epsilon_{21}) \circ (y \otimes \epsilon_{12}) = \frac{1}{2}yx \otimes \epsilon_{11} + \frac{1}{2}xy \otimes \epsilon_{22}.$$

Now, repeat the foregoing discussion with the indices 1,2 interchanged. The result will be another associative product on \mathcal{A} such that the analogues of the previous formulas hold. But (2.17) is invariant under this reversal of indices, thus proving that the two products are equal. Now the formulas (2.9), (2.10), (2.11), (2.15), (2.16), (2.17) together with the "reversed" analogues of (2.15), (2.16), show that the formula $a \circ b = \frac{1}{2}(ab + ba)$ holds for all a, b of the form $x \otimes \epsilon_{ij}$, thus for all $a, b \in \mathcal{A} \otimes M_2$. This completes the proof of Theorem 2.1. \square

3. The JB-algebra result.

In this section the result of the previous section will be applied to JB-algebras. First we will prove that if a JB-algebra A is the self-adjoint part of a $*$ -algebra \mathcal{A} , then \mathcal{A} is in fact a C^* -algebra. This result and the proof that follows is due to F.W. Shultz [5].

Lemma 3.1. Let \mathcal{A} be a complex $*$ -algebra whose self-adjoint part A is a JB-algebra under the product $a \circ b = \frac{1}{2}(ab + ba)$ and some norm. If $x \in \mathcal{A}$ then x^*x is a positive element of A .

Proof. An element a of A is easily seen to be invertible in the associative algebra \mathcal{A} iff it is invertible in the Jordan algebra A . It follows that the real part of its spectrum, defined with respect to \mathcal{A} , coincides with its spectrum $\sigma(a)$ with respect to A . Thus, by the usual argument $\sigma(x^*x) \cup \{0\} = \sigma(xx^*) \cup \{0\}$. (See e.g. [4, Prop.1.1.8]). Also, the norm-closed real algebra $C(x^*x) \subseteq A$ generated by x^*x and 1 is isomorphic to $C_{\mathbb{R}}(X)$ for

some compact Hausdorff space X . Now the proof of [4, Thm.1.4.4] will complete the proof of the lemma. \square

Proposition 3.2. [5] Let \mathcal{A} be a complex $*$ -algebra whose self-adjoint part A is a JB-algebra under the product $a \circ b = \frac{1}{2}(ab + ba)$ and some norm. Then \mathcal{A} is a C^* -algebra under the norm defined by $\|x\| = \|x^*x\|^{\frac{1}{2}}$, where the latter norm is the JB-algebra norm on A .

Proof: Obviously, on A , the new norm coincides with the JB-algebra norm. Let K be the state space of A . By Lemma 3.1, we have $p(x^*x) \geq 0$ for $x \in \mathcal{A}$, $p \in K$, so the Schwarz inequality

$$|p(y^*x)| \leq p(y^*y)^{\frac{1}{2}} p(x^*x)^{\frac{1}{2}}$$

is valid. As a special case, $|p(x)| \leq p(x^*x)^{\frac{1}{2}}$, so that if $x^*x = 0$ then $x = 0$; that is, if $\|x\| = 0$ then $x = 0$.

To prove that $\|\cdot\|$ is subadditive, we calculate:

$$\begin{aligned} \|x+y\|^2 &= \|(x+y)^*(x+y)\| = \\ &= \sup_{p \in K} p(x^*x + y^*x + x^*y + y^*y) \leq \\ &\leq \sup_{p \in K} (p(x^*x) + 2p(x^*x)^{\frac{1}{2}} p(y^*y)^{\frac{1}{2}} + p(y^*y)) \leq \\ &\leq \|x^*x\| + 2\|x^*x\|^{\frac{1}{2}}\|y^*y\|^{\frac{1}{2}} + \|y^*y\| = \\ &= \|x\|^2 + 2\|x\| \cdot \|y\| + \|y\|^2 = (\|x\| + \|y\|)^2. \end{aligned}$$

The C^* -identity $\|x^*x\| = \|x\|^2$ follows by definition. To prove submultiplicativity, note that by Lemma 4.1, the map $x \mapsto y^*xy$ is positive. Thus

$$\|xy\|^2 = \|(xy)^*xy\| = \|y^*(x^*x)y\| \leq \|y^*(\|x^*x\| \cdot 1)y\| = \|x\|^2 \cdot \|y\|^2.$$

Finally, to prove that \mathcal{A} is complete, note that the involution on \mathcal{A} is isometric. Thus, if (x_n) is a Cauchy sequence in \mathcal{A} , then $(x_n + x_n^*)$ and $(i(x_n - x_n^*))$ are both Cauchy, and therefore convergent in A . The proposition is now proved. \square

Theorem 3.3. Let A be a JB-algebra and assume that $A \otimes M_2(\mathbb{C})_{sa}$ can be given a Jordan structure such that (2.1) and (2.2) hold. Then A is (isometrically isomorphic to) the self-adjoint part of a C^* -algebra.

Proof: Let $\mathcal{A} = A \otimes iA$ be the complexification of A . Then \mathcal{A} is a Jordan $*$ -algebra over \mathbb{C} . The involutions in \mathcal{A} and M_2 induce an involution in $\mathcal{A} \otimes M_2$ defined by $(x \otimes \alpha)^* = x^* \otimes \alpha^*$. Under this involution the self-adjoint part of $\mathcal{A} \otimes M_2$ is

$$(\mathcal{A} \otimes M_2)_{sa} = \mathcal{A}_{sa} \otimes (M_2)_{sa} = A \otimes (M_2)_{sa}.$$

Thus $\mathcal{A} \otimes M_2$ is the complexification of $A \otimes (M_2)_{sa}$, and so is a Jordan $*$ -algebra under a product satisfying (2.1) and (2.2). By Theorem 2.1, there exists an associative product in \mathcal{A} inducing the given Jordan products both in \mathcal{A} and in $\mathcal{A} \otimes M_2$. In particular, for $x, y \in \mathcal{A}$ we have:

$$\begin{aligned} (xy)^* \otimes \epsilon_{12} &= (xy)^* \otimes \epsilon_{21}^* = (xy \otimes \epsilon_{21})^* = 2((x \otimes \epsilon_{21}) \circ (y \otimes \epsilon_{11}))^* = \\ &= 2(x \otimes \epsilon_{21})^* \circ (y \otimes \epsilon_{11})^* = 2(x^* \otimes \epsilon_{12}) \circ (y^* \otimes \epsilon_{11}) = \\ &= y^* x^* \otimes \epsilon_{12}, \end{aligned}$$

so $(xy)^* = y^* x^*$, and \mathcal{A} is a $*$ -algebra. Its self-adjoint part is A , so by Proposition 3.2, \mathcal{A} is a C^* -algebra, and the proof is complete. \square

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