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THE 1-SECTION OF A COUNTABLE FUNCTIONAL

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The continuous or countable functionals were independantly defined by Kleene [9] and Kreisel [10]. They were intended as a suitable basis for constructive mathematics, and thus it is interesting to investigate various notions of recursion on the countable functionals.

There have been two main streams in this investigation, the study of countable recursion and the study of computability or Kleene-recursion.

Countable recursion is the theory of recursion on the associates. Gandy-Hyland [3] and Hyland [7] are good sources for the recent development of countable recursion.

This paper will mostly be concerned with Kleene-recursion on the countable functionals as defined in Kleene [8] and [9]. We assume some familiarity with the countable functionals and associates, as presented in Kleene [9], Bergstra [1] or any other paper on the subject.

Pioneering work with recursion in nonnormal objects was done by Grilliot [4], who proved that a functional F of type 2 is normal if and only if its 1-section (that is the set of functions recursive in F) is closed under ordinary jump, and if and only if F is continuous on 1-section (F).

Hinman [6] constructed a countable functional that is not re-

cursively equivalent to a function, and thereby showed that recursion in non-normal functionals is an extension of ordinary recursion in functions.

In [6], Hinman asked if there are functionals with topless 1-sections, i.e. with no maximal elements in the Semi-lattice of degrees. This was answered in the affirmative by Bergstra [1], using a spoiling construction. Thus the class of 1-sections of functionals extends the class of 1-sections of functions.

As a general approach to recursion in functionals of type 2, Wainer [16] constructed hierarchies for the 1-sections of all functionals of type 2. This had also been done by Moschovakis [12] and Hinman [5], but Wainer's hierarchy is based on primitive recursive diagonalization and primitive recursive limit's, so one never needs to ask 2E whether or not a given subcomputation $\{e\}^\alpha$ is defined - it always will be.

The hierarchy is modelled on Shoenfield's [15] for normal functionals. With each F^2 is associated a set of notations 0^F , a well-founded ordering $<_F$ on 0^F and functions $\{f_a : a \in 0^F\}$ such that:

f is recursive in F iff f is primitive recursive in some f_a . If $a <_F b$ then f_a is primitive recursive in f_b . 0^F , $<_F$ and $\{f_a : a \in 0^F\}$ are all $\Pi_1^1(F)$.

This hierarchy generalizes the hierarchy of Shoenfield [15] which works only for normal functionals, but it lacks some of the hierarchical properties of Shoenfield's hierarchy such as the uniqueness-property. The uniqueness-property says that if $\|a\|_{0^F} < \|b\|_{0^F}$, then f_a is recursive in f_b . As we will see (corollary 1.2), this

uniqueness-property cannot hold for the hierarchy for a general non-normal functional.

Let F be a functional. We say that Wainer's hierarchy collapses for F if there is a notation a such that for all $f \in 1\text{-section}(F)$ there is a b of norm less than the norm of a such that f is recursive in f_b . It is clear that if F is normal, the hierarchy will not collapse, and if F is recursively equivalent to a function, the hierarchy will collapse. Wainer then asked if there are continuous functionals with non-collapsing hierarchies, and Harrington noticed that it will be equivalent to ask for a Π_1^1 1-section that is not Δ_1^1 .

The first serious approach to a solution of Wainer's problem was made in Bergstra-Wainer [2], and following ideas from that paper, Normann [13] constructed a functional recursive in $0'$ such that its 1-section is Π_1^1 but not Δ_1^1 .

As a basis for the construction, certain functionals F_a^b of Bergstra [1] were used. They are improvements of some non-reducible functionals of Hyland. One of our main results is that this construction is general when we are only concerned with 1-sections. (Theorem 3)

A Construction of a 1-section

Let T be Kleene's T -predicate with the following properties:

Each r.e. set is on the form $W_a = \{p: \exists q T(a, p, q)\}$.

For any p, a there is at most one q such that $T(a, p, q)$, and if $T(a, p, q)$ holds, then $q \geq 1$.

A computation-function α for W_a is a function which satisfies the following:

$$p \in W_a \Leftrightarrow \exists q < \alpha(p) T(a, p, q).$$

W_a will be uniformly primitive recursive in any computation-function for W_a .

Definition (Bergstra [1])

a Let σ be a sequence number.

$$R_a(\sigma) \Leftrightarrow \exists p, q (1 \leq p, q \leq \text{lh}(\sigma) \wedge T(a, p, q) \wedge \sigma(p) < q)$$

$$\underline{b} \quad F_a^b(\alpha) = \begin{cases} \mu t [T(b, \alpha(0), t) \wedge \neg R_a(\bar{\alpha}(t))] & \text{if such } t \text{ exists} \\ 0 & \text{otherwise.} \end{cases}$$

Remarks. $R_a(\sigma)$ says that σ is not the beginning of a computation-function for W_a and that this can be checked by regarding $W_a^{\text{lh}(\sigma)}$.

The idea of F_a^b is that if α is a computation-function for W_a , then $F_a^b(\alpha)$ gives the value at $\alpha(0)$ for a computation-function for W_b . If α is not a computation-function for W_a , $\bar{\alpha}(t)$ is in R_a for some t , and then $F_a^b(\alpha)$ gives the computation-function for W_b^t .

From the considerations above, the following lemma is easy to prove.

Lemma 1 (Bergstra [1])

$$\underline{a} \quad \forall \alpha, n [R_a(\bar{\alpha}(n)) \Rightarrow R_a(\bar{\alpha}(n+1))].$$

$$\underline{b} \quad \text{If } W_a \text{ is not primitive recursive in } \alpha, \text{ then } \exists n R_a(\bar{\alpha}(n)).$$

$$\underline{c} \quad \text{There exists } \alpha \text{ uniformly recursive in } W_a \text{ such that} \\ \forall n \neg R_a(\bar{\alpha}(n)).$$

$$\underline{d} \quad W_b \text{ is uniformly recursive in } W_a, F_a^b, a \text{ and } b.$$

$$\underline{e} \quad F_a^b \text{ is uniformly recursive in } W_b, a \text{ and thus continuous.}$$

f Let H_a^b be the partial recursive function defined by the following algorithm:

To compute H_a^b find the least t_0 such that $R_a(\bar{\alpha}(t_0))$, and then, if there is a $t < t_0$ such that $T(b, \alpha(0), t)$, let $H_a^b(\alpha)$ be the one such t , otherwise let $H_a^b(\alpha) = 0$. H_a^b is a subfunction of F_a^b defined at all α in which W_a is not primitive recursive.

Joining various F_a^b 's together, we may construct functionals with interesting 1-sections. This method was used in Bergstra [1], Bergstra-Wainer [2] and Normann [13].

We will now use it to prove the following.

Theorem 1

Let $A \subseteq \mathbb{N}$ be a Π_1^1 -set. Then there is a recursive 1-1-function ρ and a functional F of type 2 recursive in $0'$ such that for all n

$$W_{\rho(n)} \in 1\text{-section}(F) \iff n \in A$$

Proof. Let $B \subseteq \mathbb{N} \times \mathbb{N}$ be semi-recursive such that for no n

B_n is recursive in B_{-n} , where

$$B_n = \{m; (n, m) \in B\}$$

$$B_{-n} = \{(k, m) \in B; k \neq n\}$$

B is constructed by a standard priority construction.

Since A is Π_1^1 , there is a recursive family $\langle C_n \rangle_{n \in \mathbb{N}}$ of linear orderings such that

$$n \in A \iff C_n \text{ is a well-ordering}$$

If k field (C_n) , let

$$X_k^n = \{(\langle n, e \rangle, a); e <_{C_n} k \text{ and } (\langle n, e \rangle, a) \in B\}$$

$$Y_k^n = \{(\langle n, e \rangle, a); e \leq_{C_n} k \text{ and } (\langle n, e \rangle, a) \in B\}$$

$$Z^n = \{(\langle n, e \rangle, a); e \text{ field } (C_n) \text{ and } (\langle n, e \rangle, a) \in B\}$$

Let τ_1, τ_2 and ρ be recursive such that

$$X_k^n = W_{\tau_1(n,k)}, Y_k^n = W_{\tau_2(n,k)} \text{ and } Z^n = W_{\rho(n)}$$

Let $F(n, k, \alpha) = F_{\tau_1(n,k)}^{\tau_2(n,k)}(\alpha)$ whenever $k \in \text{field } (C_n)$

$$F(n, k, \alpha) = 0 \text{ otherwise.}$$

It follows by lemma 1 and the recursion theorem that if C_n is a well-ordering all X_k^n such that $k \in \text{field } (C_n)$ will be uniformly recursive in F , and thus $Z^n = W_{\rho(n)} \in 1\text{-section } (F)$. So now assume that C_n is not a well-ordering, and let $\{k_i\}_{i \in \mathbb{N}}$ be a C_n -decreasing sequence. By induction on the computations in F , we show that they are all computations in the partial functionals

$$K_i(m, k, \alpha) = \begin{cases} F(m, k, \alpha) & \text{if } m \neq n \text{ or } m = n \text{ and } \neg(k_i \leq k) \\ H(m, k, \alpha) & \text{if } m = n \text{ and } k_i \leq k \end{cases}$$

where $H(n, k, \alpha) = H_{\tau_1(n,k)}^{\tau_2(n,k)}(\alpha)$.

K_i is recursive in $B_{\langle n, k_i \rangle}$, so $B_{\langle n, k_i \rangle}$ is not recursive in K_i .

The only non-trivial part of the induction is when S8 is used,

$$\{e\}^F(\vec{a}) = F(\lambda x \{e_1\}^F(x, \vec{a})).$$

Let $\alpha = \lambda x \{e_1\}^F(x, \vec{a})$.

By the induction hypothesis

$$\alpha = \lambda x \{e_1\}^{K_{i+1}}(x, \vec{a})$$

so α is recursive in $B_{\langle n, k_{i+1} \rangle}$. It follows that $B_{\langle n, k_{i+1} \rangle}$ is not recursive in α , and for $k_{i+1} <_{C_n} k$ we have that $W_{\tau_1(n, k)}$ is not recursive in α . Then $K_i(\alpha)$ is defined and equals $F(\alpha)$, so

$$\{e\}^F(\vec{a}) = \{e\}^{K_i}(\vec{a}).$$

We may conclude that 1-section $(F) = 1\text{-section } K_i$, so 1-section $(F) \subseteq 1\text{-section } (B_{\langle n, k_i \rangle})$. Then $Z^n = W_{\rho(n)}$ cannot be an element of 1-section (F) . This ends the proof of the theorem.

Remarks.- The same type of argument is carried through in some more details in Normann [13].

In Theorem 5 we see that functionals of higher type with interesting 1-sections are easier to find.

Corollary 1.1 (Normann [13])

There is a functional F recursive in $0'$ such that

$$1\text{-section } (F) \in \Pi_1^1 \setminus \Delta_1^1.$$

Proof. Let $A \in \Pi_1^1 \setminus \Delta_1^1$ in Theorem 1.

Corollary 1.2

There is a functional F recursive in $0'$ such that for no Π_1^1 -linearly ordered set C of Δ_2^0 -degrees, 1-section (F) is generated by C .

Proof. There are Δ_1^1 -sets D_1, D_2 and D_3 , mutually disjoint, such that $\mathbb{N} = D_1 \cup D_2 \cup D_3$, D_2 is infinite and all infinite arithmetic sets intersects both D_1 and D_3 .

Let $A \in \Pi_1^1 \setminus \Delta_1^1$ be such that $D_1 \subseteq A$ and $D_3 \cap A^c \neq \emptyset$.
 Let F be as in Theorem 1, and in order to obtain a contradiction, assume that C is a linear Π_1^1 -subset of Δ_2^0 -degrees.
 1-section (F) . Now $\{W_{p(m)}; m \in D_1\}$ is Δ_1^1 , and by the principle, there is a $\alpha \in C$ such that $W_{p(m)}$ is recursive in α for all $m \in D_1$.

Now $\{m; W_{p(m)} \text{ is recursive in } \alpha\}$ is arithmetic, thus infinite and intersects D_3 . But if $m \in D_3$, $W_{p(m)} \notin 1\text{-section}(F)$, so $W_{p(m)}$ cannot be recursive in $\alpha \in C \subseteq 1\text{-section}(F)$. This gives the contradiction.

Remark. Let G be any countable functional. If $1\text{-section}(G) = 1\text{-section}(\beta)$ for some function β such that $1\text{-section}(G) = 1\text{-section}(\beta)$, then the natural hierarchy for $1\text{-section}(\beta)$ will be a hierarchy for $1\text{-section}(G)$ with the uniqueness-property. On the other hand, if $1\text{-section}(G) \in \Delta_1^1$ and there is no hierarchy for $1\text{-section}(G)$ with the uniqueness-property, then by the principle there will be a function β in this hierarchy such that every element of $1\text{-section}(G)$ is recursive in β . If $1\text{-section}(G) = 1\text{-section}(\beta)$.

The functional we constructed in corollary 1 is not a collapsing hierarchy, but no hierarchy with the uniqueness-property. We conjecture that there is no non-normal functional with a topless 1-section and a hierarchy with the uniqueness-property.

Computations on Countable Functionals

In this section we investigate the construction of a Kleene computation over the type structure of functionals. Kleene [9] gave one such interpretation of the Kleene computation over the type structure of functionals.

reducing defined computations $\{e\}(\varphi)$ to "countable recursions" $\{e'\}(\alpha_\varphi)$ on associates of φ . There are however, more countable recursions than there are computations - the fan functional is countably recursive but not computable. We will approach the problem more directly by analyzing the computations themselves, assuming that higher-type objects φ as "given".

A constructive object x of type $k+1$ must in some way be determined as the (pointwise) limit of a sequence $\langle x_n \rangle_{n < \omega}$ of clearly calculable (eg. primitive recursive) approximations. The complexity of x is then reflected by its modulus function M_x such that for all φ of type k , $M_x(\varphi) = \mu m (\forall n > m) (x_n(\varphi) = x_m(\varphi))$. Thus we would like to associate with each defined computation $x = \{e\}(\varphi)$, a sequence $\langle x_n \rangle$ uniformly primitive recursive in x , such that $x = \lim x_n$ and the modulus M_x is computable uniformly in x . This is clearly possible for φ of type ≤ 1 , taking x_n to be the result (if any) after n steps in the computation. It is also possible for φ of type-2, as was shown in Wainer [17], but a much more detailed analysis of computations is required in this case. A direct result of this analysis is that the 1-section of every non-normal type-2 object is generated by its "r.e." elements, and so each such type-2 object can be viewed as an "r.e. set construction". This line is developed further in the next section. We now generalize [17] to all finite types.

Let $\varphi = \langle \varphi_1, \dots, \varphi_r \rangle$ denote any list of countable objects of types $\leq k$, encoded as a single type- k object. With each possible computation $\{e\}(\varphi)$ associate the sequence $\lambda n. h(e, \varphi, n)$ of approximations as follows:

If $n = 0$ or if e is not of the correct form for an index, set $h(e, \varphi, n) = 0$. Otherwise:

If $\{e\}(\varphi)$ is defined by an outright computation S1, S2, S3 or S7 then for every $n > 0$ set $h(e, \varphi, n) = \{e\}(\varphi)$.

If $\{e\}(\varphi) \simeq \{e_1\}(\{e_2\}(\varphi), \varphi)$ by S4, set

$$h(e, \varphi, n) = h(e_1, \langle h(e_2, \varphi, n-1), \varphi \rangle, n-1).$$

If $\{e\}(\varphi, \kappa)$ is defined by primitive recursion S5 from $\{e_1\}$ and $\{e_2\}$, set

$$h(e, \langle \varphi, \kappa \rangle, n) = \begin{cases} h(e_1, \varphi, n-1) & \text{if } \kappa = 0 \\ h(e_2, \langle h(e, \langle \varphi, \kappa-1 \rangle, n-1), \varphi, \kappa \rangle, n-1) & \text{ow.} \end{cases}$$

If $\{e\}(\varphi) \simeq \{e_1\}(\varphi')$ by S6, where φ' is a permutation of φ , set $h(e, \varphi, n) = h(e_1, \varphi', n-1)$.

If $\{e\}(\varphi) \simeq \varphi_i(\lambda \beta. \{e_1\}(\varphi, \beta))$ by S8, set

$$h(e, \varphi, n) = \varphi_i(\lambda \beta. h(e_1, \langle \varphi, \beta \rangle, n-1)).$$

Finally, if $\{e\}(z, \varphi) \simeq \{z\}(\varphi)$ by S9, set

$$h(e, \langle z, \varphi \rangle, n) = h(z, \varphi, n-1).$$

Since h is defined by simple induction on n , it is total and primitive recursive.

Theorem 2

- (i) If $\{e\}(\varphi)$ is defined, then $\lim_n h(e, \varphi, n) = \{e\}(\varphi)$.
- (ii) There is a partial recursive functional M such that if $\{e\}(\varphi)$ is defined, then

$$M(e, \varphi) = \mu m (\forall n \geq m) (h(e, \varphi, n) = \{e\}(\varphi)).$$

Proof. We must prove (i) and (ii) together, by induction over computations $\{e\}(\varphi)$, using the Recursion Theorem to define M .

All cases except S8 are straightforward. For example suppose

$\{e\}(\varphi) \simeq \{e_1\}(\{e_2\}(\varphi), \varphi)$ by S4, so that

$h(e, \varphi, n) = h(e_1, \langle h(e_2, \varphi, n-1), \varphi \rangle, n-1)$. Inductively, we can compute

$m_2 = M(e_2, \varphi)$ such that for all $n > m_2$ $h(e_2, \varphi, n-1) = \{e_2\}(\varphi)$,

and then compute $m_1 = M(e_1, \{e_2\}(\varphi), \varphi)$ such that for all $n > m_1$,

$h(e_1, \langle \{e_2\}(\varphi), \varphi \rangle, n-1) = \{e\}(\varphi)$. Then for all $n > \max(m_1, m_2)$,

$h(e, \varphi, n) = \{e\}(\varphi)$, and $M(e, \varphi) = \mu m (\forall n) [m \leq n \leq \max(m_1, m_2) + 1 \rightarrow$

$h(e, \varphi, n) = h(e, \varphi, m)]$.

Now suppose $\{e\}(\varphi) \simeq \varphi_i(\lambda \beta. \{e_1\}(\varphi, \beta))$ by S8, so that

$h(e, \varphi, n) = \varphi_i(\lambda \beta. h(e_1, \langle \varphi, \beta \rangle, n-1))$. Inductively, we can assume

that for every β and every $n > M(e_1, \varphi, \beta)$

$$h(e_1, \langle \varphi, \beta \rangle, n-1) = \{e_1\}(\varphi, \beta).$$

Let α_φ be a fixed list of associates for φ , and for each β

let α_β be any associate. By Kleene [9], since $\lambda \beta. M(e_1, \varphi, \beta)$ and $\lambda \beta. h(e_1, \langle \varphi, \beta \rangle, M(e_1, \varphi, \beta)) = \lambda \beta. \{e_1\}(\varphi, \beta)$ are computable in φ , they

have associates A_M and A_{hoM} , recursive in α_φ . Clearly A_M

and A_{hoM} can be chosen so that for every α_β there is an x such

that for every finite sequence σ extending $\bar{\alpha}_\beta(x)$,

$$A_M(\sigma) = M(e_1, \varphi, \beta) + 1$$

and $A_{hoM}(\sigma) = h(e_1, \langle \varphi, \beta \rangle, M(e_1, \varphi, \beta)) + 1 = \{e_1\}(\varphi, \beta) + 1$.

Thus we can define an associate A_{e_1} for $\lambda \beta. \{e_1\}(\varphi, \beta)$ by

$$A_{e_1}(\sigma) = \begin{cases} A_{hoM}(\sigma) & \text{if } A_M(\sigma) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

The value of $\{e\}(\varphi) = \varphi_i(\lambda \beta. \{e_1\}(\varphi, \beta))$ is therefore determined

(with respect to α_{φ_i}) by some finite initial segment of A_{e_1} , say

$\bar{A}_{e_1}(m)$. Let $N = \max\{A_M(\sigma) \mid \sigma \leq m\}$.

Now for any $n \geq N$ and every $\bar{\alpha}_\beta(x) \leq m$ such that $A_{e_1}(\bar{\alpha}_\beta(x)) > 0$

we have $n \geq A_M(\bar{\alpha}_\beta(x)) = M(e_1, \varphi, \beta) + 1$, and so

$h(e_1, \langle \varphi, \beta \rangle, n-1) = \{e_1\}(\varphi, \beta) = A_{e_1}(\bar{\alpha}_\beta(x)) - 1$. Therefore there is an associate of $\lambda\beta.h(e_1, \langle \varphi, \beta \rangle, n-1)$ which extends $\bar{A}_{e_1}(m)$, and hence

$$h(e, \varphi, n) = \varphi_i(\lambda\beta.h(e_1, \langle \varphi, \beta \rangle, n-1)) = \varphi_i(\lambda\beta.\{e_1\}(\varphi, \beta)) = \{e\}(\varphi).$$

Thus $h(e, \varphi, n) = \{e\}(\varphi)$ for every $n \geq N$, and (i) is proved.

For part (ii) we must show how to compute

$$M(e, \varphi) = \mu m (\forall n > m) (h(e, \varphi, n) = h(e, \varphi, m)).$$

Clearly it will be sufficient to show how to decide for each n , whether or not the following holds:

$$\exists m > n (h(e, \varphi, m) \neq h(e, \varphi, n))$$

$$\text{i.e. } \exists m > n (\varphi_i(\lambda\beta.h(e_1, \langle \varphi, \beta \rangle, m-1)) \neq \varphi_i(\lambda\beta.h(e_1, \langle \varphi, \beta \rangle, n-1))).$$

To do this, first define a functional γ_n as follows:

$$\gamma_n(\beta) = \begin{cases} h(e_1, \langle \varphi, \beta \rangle, m_0-1) & \text{if } m_0 = \mu m (n < m \leq M(e_1, \varphi, \beta) \wedge h(e, \varphi, m) \neq h(e, \varphi, n)) \\ h(e_1, \langle \varphi, \beta \rangle, M(e_1, \varphi, \beta)) & \text{if there is no such } m_0. \end{cases}$$

Clearly, γ_n is uniformly recursive in n, φ , and it is easy to see that

$$\gamma_n = \begin{cases} \lambda\beta.h(e_1, \langle \varphi, \beta \rangle, m_0-1) & \text{if } m_0 = \mu m > n (h(e, \varphi, m) \neq h(e, \varphi, n)) \\ \lambda\beta.\{e_1\}(\varphi, \beta) & \text{if } \forall m > n (h(e, \varphi, m) = h(e, \varphi, n)). \end{cases}$$

Therefore $\exists m > n (h(e, \varphi, m) \neq h(e, \varphi, n))$ if and only if $h(e, \varphi, n) \neq \varphi_i(\gamma_n)$.

This completes the proof of (ii).

Corollary 2.1

For every computable type $k+1$ functional $\lambda\varphi.\{e\}(\varphi)$ over the type structure of countable functionals, there is a computable Σ_1^0 set

$$D_e^{k+1} = \{\langle \varphi, n \rangle \mid \exists m > n (h(e, \varphi, m) \neq h(e, \varphi, n))\}$$

such that for every φ ,

$$\{e\}(\varphi) = h(e, \varphi, \mu m (\langle \varphi, m \rangle \in D_e^{k+1}))$$

Corollary 2.2

For each countable functional φ of type $> k$ define

$$h_{\varphi}^k(e, \beta, n) = h(e, \langle \varphi, \beta \rangle, n)$$

and $D_{e, \varphi}^k = \{\langle \beta, n \rangle \mid \exists m > n (h_{\varphi}^k(e, \beta, m) \neq h_{\varphi}^k(e, \beta, n))\}$.

Then for every $\lambda \beta. \{e\}(\varphi, \beta) \in k\text{-sc}(\varphi)$ we have

(i) $D_{e, \varphi}^k \in k\text{-sc}(\varphi)$, since $\langle \beta, n \rangle \in D_{e, \varphi}^k$ if $n < M(e, \varphi, \beta)$.

(ii) $\lambda \beta. \{e\}(\varphi, \beta)$ is μ -recursive in h_{φ}^k and $D_{e, \varphi}^k$.

Hence $k\text{-sc}(\varphi)$ is generated by its $\Sigma_1^0(h_{\varphi}^k)$ elements.

Corollary 2.3

For every countable functional φ of type $k+2$, $1\text{-sc}(\varphi)$ is a $\Pi_k^1(h_{\varphi}^1)$ ($\Pi_1^1(h_{\varphi}^1)$ if $k=0$) set of reals generated by its r.e.-in- h_{φ}^1 elements.

More precisely, there is a $\Pi_{(k+1)+1}^1(h_{\varphi}^1)$ set B of r.e.-in- h_{φ}^1 indices such that

$$1\text{-sc}(\varphi) = \{f \mid f \leq_T W_i^{h_{\varphi}^1} \text{ for some } i \in B\}.$$

Proof. By Corollary 2.2, we need only to take

$$B = \{i \mid \exists e (W_i^{h_{\varphi}^1} = h_{\varphi}^1 \cup D_{e, \varphi}^1 \wedge \lambda x. \{e\}(\varphi, x) \text{ is total})\}.$$

By results of Bergstra [1] on $2\text{-en}(\varphi)$, $\{e \mid \lambda x. \{e\}(\varphi, x) \text{ is total}\}$ is $\Pi_{(k+1)+1}^1$ in h_{φ}^1 , and therefore so is B .

The Enumeration Operator of a Non-Normal Type-2 Object

Let F be an arbitrary fixed non-normal type-2 object. Since $1\text{-sc}(F)$ is generated entirely "from within", and since F is continuous on its 1-section, all the results of the previous section carry through unchanged for recursions in F (alternatively see Wainer [17]). Thus we have a total primitive-recursive-in- F function

$$h_F(e, a, n) = h(e, \langle F, a \rangle, n)$$

and a partial-recursive-in- F function

$$M_F(e, a) = M(e, F, a)$$

such that for every e and every list $a = \langle a_1 \dots a_r \rangle$ of integers:

$$\{e\}^F(a) \text{ defined} \Rightarrow M_F(e, a) \text{ defined and equal to} \\ \mu m \forall n \geq m [h_F(e, a, n) = \{e\}^F(a)].$$

Therefore with each $\{e\}^F \in 1\text{-sc}(F)$ is associated a r.e.-in- h_F set

$$D_{e,F} = \{ \langle a, n \rangle \mid \exists m > n (h_F(e, a, m) \neq h_F(e, a, n)) \}$$

such that

- (i) $D_{e,F} \in 1\text{-sc}(F)$, since $\langle a, n \rangle \in D_{e,F}$ iff $n < M_F(e, a)$.
- (ii) $\{e\}^F \leq_T h_F, D_{e,F}$.

Henceforth we will usually omit the subscripts F .

Now there is a primitive recursive function $\lambda e.e'$ such that $\{e'\}^F(a) \simeq F(\lambda x. \{e\}^F(a, x))$. Thus the action of F can be regarded as a "jump" from the h-r.e. set D_e to the h-r.e. set $D_{e'}$. As was remarked in [17], this begs the question whether it might be possible to replace F , in the generation of its 1-section, by a continuous-everywhere Bergstra-type functional $F_e^{e'}(\alpha)$. We now show that this is indeed the case. Thus from the point of view of 1-sections, the Bergstra-type functionals are the only ones.

Definition

(a) For finite sequences σ let

$$\text{Mod}(e, a, \sigma) \iff \forall x, j < \text{lh}(\sigma) [j > \sigma_x \rightarrow h(e, \langle a, x \rangle, j) = h(e, \langle a, x \rangle, \sigma_x)].$$

(b) Then associate with F the enumeration operator

$$J_F(\alpha) = \{ \langle e, a, n \rangle \mid \exists m > n (h(e', a, m) \neq h(e', a, n) \wedge \text{Mod}(e, a, \bar{\alpha}(m))) \}$$

Note (cf. Lemma 1)

(i) To compute $J_F(\alpha)(\langle e, a, n \rangle)$ from h first see if

$$n \in D_{e', a} = \{ n \mid \exists m > n (h(e', a, m) \neq h(e', a, n)) \}.$$

If so, find the first stage m which witnesses its membership and then for each α give value 1 if $\text{Mod}(e, a, \bar{\alpha}(m))$.

Otherwise give value 0.

Hence J_F is continuous and of Kalmar rank $\leq \omega$.

(ii) $\text{Mod}(e, a, \bar{\alpha}(m))$ says that up to stage m , α looks like a correct modulus function for the sequence $\lambda j x. h(e, \langle a, x \rangle, j)$ approximating $\lambda x. \{e\}(a, x)$. Thus

$$\forall m \text{Mod}(e, a, \bar{\alpha}(m)) \iff \forall x (M(e, a, x) \leq \alpha(x)).$$

Therefore if $\forall m \text{Mod}(e, a, \bar{\alpha}(m))$ then the set

$$D_{e, a} = \{ \langle x, n \rangle \mid \exists m > n (h(e, \langle a, x \rangle, m) \neq h(e, \langle a, x \rangle, n)) \}$$

is primitive recursive in α .

Conversely if $\lambda x. \{e\}^F(a, x)$ is total, then from $D_{e, a}$ we can compute $M(e, a, x) = \mu n (\langle x, n \rangle \notin D_{e, a})$, so that

$$n \in D_{e', a} \iff \langle e, a, n \rangle \in J_F(\lambda x. M(e, a, x)).$$

Thus $J_F(\alpha)(\langle e, a, n \rangle) \approx F_{e, a}^{e', a}(n * \alpha)$.

Theorem 3

$$1\text{-sc}(F) = 1\text{-sc}(h_F, J_F) .$$

Corollary 3.1

For every non-normal type-2 object there is a continuous type-2 object, of Kalmar rank $\leq \omega$, with the same 1-section.

Theorem 3 is proved by the two following lemmas.

Lemma 2

There is a recursive function d such that

$$\{e\}^F(a) \text{ defined} \Rightarrow D_{e,a} = \{d(e,a)\}^{h,J} .$$

Hence if $\{e\}^F$ is total, then $\{e\}^F \leq_T h, D_e \in 1\text{-sc}(h, J)$.

Proof. By induction over computations $\{e\}^F(a)$, using the Recursion Theorem to define d .

For example suppose $\{e\}^F(a) \simeq \{e_1\}^F(\{e_2\}^F(a), a)$ by S4 .

Inductively we can use $d(e_2, a)$ to compute $k_2 = \mu k \notin D_{e_2, a}$, and then $k_1 = \mu k \notin D_{e_1, u, a}$ where $u = h(e_2, a, k_2) = \{e_2\}^F(a)$. Then $D_{e,a} = \{n \mid \exists m > n (h(e, a, m) \neq h(e, a, n))\}$ where, for every $m > \max(k_1, k_2)$,

$$h(e, a, m) = h(e_1, \langle h(e_2, a, m-1), a \rangle, m-1) = h(e_1, \langle u, a \rangle, m-1) = \{e\}^F(a) .$$

So $D_{e,a} = \{n \mid \exists m (n < m \leq \max(k_1, k_2) \wedge h(e, a, n) \neq h(e, a, m))\}$, and its index $d(e, a)$ is clearly given by a primitive recursive function of $d(e_2, a)$ and $d(e_1, \langle u, a \rangle)$.

If $\{e'\}^F(a) \simeq F(\lambda x. \{e\}^F(a, x))$ by S8, then inductively we have, for each x , $D_{e,a,x} = \{d(e, \langle a, x \rangle)\}^{h,J}$. Therefore $D_{e,a} = \{\langle x, n \rangle \mid n \in D_{e,a,x}\}$ is recursive in h, J with index given by a primitive recursive function of e, a , and an index for d . Note (ii) above then shows how to obtain the required J -index for $D_{e',a}$.

Lemma 3

There is a partial recursive ψ such that if F is continuous on $1\text{-sc}(\alpha, h)$, then for all e, a, n ,

$$J_F(\alpha)(\langle e, a, n \rangle) = \psi(F, \alpha, e, a, n).$$

Hence $1\text{-sc}(h, J) \subseteq 1\text{-sc}(F)$.

Proof. The parameters a will be deleted from the following argument, since they remain inactive throughout. Recalling the definition of $J_F(\alpha)$ and of $h(e', m)$, we simply have to decide (recursively in F) the following:

$$\exists m > n (F(\lambda x. h(e, x, m-1)) \neq F(\lambda x. h(e, x, n-1)) \wedge \text{Mod}(e, \bar{\alpha}(m))).$$

The procedure is a refinement of that used for part (ii) of Theorem 2.

Define g_n recursively in h, α, e, n as follows: Given x , look for the least m_0 such that $n < m_0 \leq \alpha(x) \wedge h(e', n) \neq h(e', m_0) \wedge \text{Mod}(e, \bar{\alpha}(m_0))$. If such an m_0 is found, set $g_n(x) = h(e, x, m_0-1)$. If no such m_0 is found, set $g_n(x) = h(e, x, \alpha(x))$.

Suppose $\exists m > n (h(e', n) \neq h(e', m) \wedge \text{Mod}(e, \bar{\alpha}(m)))$ and let m_0 be the least such. Clearly if $m_0 \leq \alpha(x)$ then $g_n(x) = h(e, x, m_0-1)$. If $\alpha(x) < m_0$ then since $\text{Mod}(e, \bar{\alpha}(m_0))$ holds, we have (putting $j = m_0-1$)

$$g_n(x) = h(e, x, \alpha(x)) = h(e, x, m_0-1).$$

Therefore

$$g_n = \begin{cases} \lambda x. h(e, x, m_0-1) & \text{if } \langle e, n \rangle \in J_F(\alpha) \\ \lambda x. h(e, x, \alpha(x)) & \text{if } \langle e, n \rangle \notin J_F(\alpha). \end{cases}$$

With the aid of g_n we can now compute $J_F(\alpha)(\langle e, n \rangle)$ recursively in F as follows:

(1) First see if $h(e', n) = F(\lambda x. h(e, x, \alpha(x)))$. If so, then

$$\langle e, n \rangle \in J_F(\alpha) \iff h(e', n) \neq J(g_n).$$

because if $\langle e, n \rangle \in J_F(\alpha)$ then the m_0 above exists and $F(g_n) = F(\lambda x. h(e, x, m_0 - 1)) = h(e', m_0)$.

(2) Now suppose $h(e', n) \neq F(\lambda x. h(e, x, \alpha(x)))$. Define

$$\beta_m(x) = \begin{cases} h(e, x, \alpha(x)) & \text{if } x, \alpha(x) < m \\ h(e, x, m-1) & \text{otherwise.} \end{cases}$$

Then $\lambda m x. \beta_m(x) \in 1\text{-sc}(\alpha, h)$ and $\lambda x. h(e, x, \alpha(x)) = \lim \beta_m$.

Therefore $F(\lambda x. h(e, x, \alpha(x))) = \lim F(\beta_m)$ since F is continuous on $1\text{-sc}(\alpha, h)$, so we can compute

$$m_1 = \mu m > n (h(e', n) \neq F(\beta_m)).$$

Now if $\langle e, n \rangle \in J_F(\alpha)$ let

$$m_0 = \mu m > n (h(e', n) \neq h(e', m) \wedge \text{Mod}(e, \bar{\alpha}(m))).$$

Since $\text{Mod}(e, \bar{\alpha}(m_0))$ holds, we have for every $j \leq m$,

$$\beta_j = \lambda x. h(e, x, j-1)$$

and hence $F(\beta_j) = h(e', j)$.

Therefore $m_0 = m_1$ and so $\text{Mod}(e, \bar{\alpha}(m_1))$ holds.

Conversely if $\text{Mod}(e, \bar{\alpha}(m_1))$ holds, then for every $j \leq m$, $F(\beta_j) = h(e', j)$ and so m_1 is the first witness to the fact that $\langle e, n \rangle \in J_F(\alpha)$. Hence

$$\langle e, n \rangle \in J_F(\alpha) \iff \text{Mod}(e, \bar{\alpha}(m_1)).$$

This completes case (2) and the proof of Lemma 3.

Corollary 3.2.

If F is everywhere-continuous, then J_F is recursive in F .

A natural question to ask of J_F is whether or not it is a uniform enumeration operator (i.e. whether or not there is a recursive function $j(e)$ such that if α_1 is recursive in α_2 with index e then $J_F(\alpha_1)$ is recursive in $J_F(\alpha_2)$ with index $j(e)$). If this were the case, then one would hope to be able to give a notation-free degree-theoretic hierarchy for $1\text{-sc}(F)$.

Theorem 4

If $1\text{-sc}(F)$ is topless, then J_F is not uniform.

Proof. If $1\text{-sc}(F)$ is topless, then $h \cup J_F(h) <_T h'$, for otherwise $1\text{-sc}(F) = 1\text{-sc}(h')$. But then by Theorem 3 of Lachlan [11], $J_F(h \cup J_F(h)) \leq_T h \cup J_F(h)$. So if J_F were uniform, a straightforward induction on computations would give $1\text{-sc}(F) = 1\text{-sc}(h \cup J_F(h))$, again contradicting the fact that $1\text{-sc}(F)$ is topless.

Remark

Theorem 3 shows that every non-normal 1-section is an ideal in the degrees, generated from a real h by iterating a certain "r.e. set construction" J along (simultaneously generated) well-orderings.

By an "r.e. set construction" we mean a procedure of the following kind:

An arbitrary Δ_2^0 -set A ($<_T \underline{0}'$) is presented in the form of a pair (e, α) where e is the index of a primitive recursive sequence $\lambda s, x. [e](x, s)$ such that

$$A(x) = i \iff \forall s \leq \alpha(x) ([e](x, s) = i) \quad i = 0 \text{ or } 1.$$

Then a (recursive) sequence $\lambda m. h(e, a, m)$ is defined so that for each a , $h(e, a, m) = 0$ unless at some stage m it is decided, on the basis of the finite set $\{[e](x, s) \mid x, s < m\}$ of approximations

to A "available" at stage m , to enumerate a into the r.e. set being constructed, in which case $h(e,a,m) = 1$. For the decision to have been a vital one, it is therefore only necessary that, up to m , $[e]$ looks like a correct sequence for A . This can be expressed by the relation

$$\text{mod}(e, \bar{\alpha}(m)) \neq \forall x, s < m(s) \rightarrow \alpha(x) \rightarrow [e](x, s) = [e](x, \alpha(x)).$$

Thus the construction can be regarded as an enumeration operator J of the following familiar sort:

$$J(\alpha) = \{ \langle e, a, n \rangle \mid \exists m > n (h(e, a, m) \neq h(e, a, n) \wedge \text{mod}(e, \bar{\alpha}(m))) \},$$

and the set constructed from A as above is then

$$A_1 = \{ a \mid \exists m (h(e, a, m) = 1) \} = \{ a \mid \langle e, a, 0 \rangle \in J(\alpha) \}.$$

From e and $J(\alpha)$ we can then compute a presentation (e_1, α_1) of A_1 , and repeat the construction in order to obtain a new r.e. set

$$A_2 = \{ a \mid \langle e_1, a, 0 \rangle \in J(\alpha_1) \},$$

and so on.

It therefore makes perfectly good sense to talk about the 1-section of an r.e. set construction, i.e. $1\text{-sc}(h, J)$, and it is to be hoped that many interesting 1-sections will be generated directly by appropriate combinations of priority constructions.

The 1-section of a type $k+2$ functional ($k \geq 1$)

In the previous section we described a standard procedure for creating 1-sections of countable type-2 functionals. In this section we will give a general 1-section construction for higher type functionals. We will show that the necessary conditions for a class of sets to be a 1-section given in corollary 2.3 will in fact

be sufficient.

In Normann [14], a method of constructing higher type functionals with interesting 1-sections were developed. We will show that all 1-sections of functionals of type >2 may be obtained using that method. We need the following lemma:

Lemma 4 (Normann [14])

There is a primitive recursive list φ_n of primitive recursive functionals of type k such that:

For all Π_k^1 -sets $B \subseteq \omega$ there is a recursive relation R such that

$$m \in B \iff \exists \Psi \in Ct(k+1) \forall n R(m, \langle \Psi(\varphi_0), \dots, \Psi(\varphi_{n-1}) \rangle).$$

Moreover, if $m \in B$, we may choose Ψ recursive uniformly in m .

Now, given any Π_k^1 -set B of indices, we define

$$\Phi(e, k, \Psi) = \begin{cases} 1 & \text{if } \exists s (T(e, k, s) \ \& \ \forall n \leq s \ R(e, \langle \Psi(\varphi_0), \dots, \Psi(\varphi_{n-1}) \rangle)) \\ 0 & \text{otherwise} \end{cases}$$

If $e \in B$ and $\forall n R(e, \langle \Psi(\varphi_0), \dots, \Psi(\varphi_{n-1}) \rangle)$ we see that $\lambda k \Phi(e, k, \Psi)$ is the characteristic function of W_e , so $\{W_e; e \in B\}$ will be a subset of 1-section (Φ) .

It is also easily seen that Φ is recursive in $0'$.

We prove the unrelativized version of the theorem:

Theorem 5 ($k \geq 1$)

Let $A \subseteq \mathcal{P}(\omega)$ be Π_k^1 , closed under recursion in finite lists and recursively generated by it's r.e. elements.

Then there is a continuous functional Φ of type $k+2$ recursive in $0'$ such that

$$A = 1\text{-section}(\Phi).$$

Proof. We prove the theorem in detail for $k=1$. With some modifications, the same argument works for the general case.

Let $C \in \Delta_1^1$ be a subset of ω such that both C and its complement intersects all infinite arithmetic sets.

Let $B_0 = \{e; W_e \in A\}$, $B = B_0 \cap C$. B will be Π_1^1 .

If W is an r.e. set, $\{e; W = W_e\}$ is an infinite arithmetic set, so each r.e. set in A has an index in C , and thus in B . On the other hand all arithmetic subsets of B are finite. So, if we construct Φ as above, the first property gives us that

$$A \subseteq 1\text{-section}(\Phi).$$

We will see that the other property gives us equality.

Definition.

Let $\alpha_{e,k}$ be the canonical associate for $\lambda \Psi(e,k,\Psi)$.

Let $\alpha = \langle \alpha_{e,k} \rangle_{e,k \in \omega}$.

Remark

$\alpha_{e,k}$ will be uniformly recursive in W_e and k . Moreover, if e_1 is an index for a Kleene-computation, \vec{f} a list of functions and $\{e_1\}(\Phi, \vec{f}) \downarrow$, the value of this computation is uniformly recursive in α, \vec{f} .

Now assume that $\lambda x \{e_1\}(\Phi, x)$ is a total function. We call a subcomputation $\{e_2\}(\Phi, \vec{f})$ of $\{e_1\}(\Phi, x)$ essential if the list \vec{f} is from $\omega \cup \{\varphi_n; n \in \omega\}$.

Claim. The set of essential subcomputations of $\lambda x \{e_1\}(\Phi, x)$ is arithmetic with an arithmetic enumeration.

Proof. The essential subcomputations may be defined by a $\Sigma_1^0(\alpha)$ -positive inductive definition. We give two of the clauses:

- iv If $\{e_2\}(\phi, \vec{f}) = \{e_3\}(\phi, \{e_4\}(\phi, \vec{f}), \vec{f})$ is an essential subcomputation, then $\{e_4\}(\phi, \vec{f}) = k$ is an essential subcomputation, and $\{e_3\}(\phi, k, \vec{f})$ is an essential subcomputation. k is found using α .
- viii If $\{e_2\}(\phi, e, k, \vec{f}) = \phi(e, k, \lambda g\{e_3\}(\phi, g, \vec{f}))$ is an essential subcomputation, then $\{e_3\}(\phi, \varphi_n, \vec{f})$ are essential subcomputations for all n .

Starting with $\{e_1\}(\phi, x)$, we will then get to all essential subcomputations, so this is a $\Sigma_1^0(\alpha)$ -class and by the effective enumeration of the φ_n 's, it is arithmetically enumerable.

We say that ϕ is used non-effectively at e if there is an essential subcomputation

$$\{e_2\}(\phi, e, k, \vec{f}) = \phi(e, k, \lambda g\{e_3\}(\phi, g, \vec{f}))$$

such that

$$\forall n R(e, \langle \{e_3\}(\phi, \varphi_0, \vec{f}), \dots, \{e_3\}(\phi, \varphi_{n-1}, \vec{f}) \rangle).$$

If ϕ is used non-effectively at e , then e must be in B , since $\psi = \lambda g\{e_3\}(\phi, g, \vec{f})$ is total and for $e \notin B$ there is an n such that

$$\neg R(e, \langle \psi(\varphi_0), \dots, \psi(\varphi_{n-1}) \rangle).$$

Moreover $D = \{e; \phi \text{ is used non-effectively at } e\}$ is arithmetic, so D is a finite subset of B .

Let $\alpha_D = \langle \alpha_{e,k} \rangle_{e \in D}$.

α_D will be recursive in $\langle W_e \rangle_{e \in D}$, so α_D is recursive in some element in A .

By induction on the length of the essential computations we

prove that they will be uniformly recursive in α_D . The only non-trivial case is

$$\{e_2\}(\phi, e, k, \vec{f}) = \phi(e, k, \lambda g\{e_3\}(\phi, g, \vec{f}))$$

where we split the instruction in two cases:

i $e \in D$ Then $k \notin W_e \Leftrightarrow \alpha_{e,k}(< >) = 1$. If $k \in W_e$, we give out value 0. If $k \in W_e$ find recursively the s such that $T(e, k, s)$. If

$$\forall n \leq s R(e, <\{e_3\}(\phi, \varphi_0, \vec{f}), \dots, \{e_3\}(\phi, \varphi_{n-1}, \vec{f})>)$$

we give out value 1, otherwise we give value 0.

By the induction hypothesis we may recursively in α_D decide the statement above.

ii $e \notin D$ Then for some least n

$$\neg R(e, <\{e_3\}(\phi, \varphi_0, \vec{f}), \dots, \{e_3\}(\phi, \varphi_{n-1}, \vec{f})>)$$

which we find recursively in α_D .

Then check if for some $s < n$, $T(e, k, s)$. If there is one, give out value 1, otherwise give out value 0.

It follows that $\lambda x\{e_1\}(\phi, x)$ is recursive in α_D , and thus recursive in some element of A . This means that 1-section $(\phi) \subseteq A$, and the theorem is established for $k = 1$.

For $k > 1$, we let C be Δ_K^1 such that both C and the complement of C intersect all infinite Σ_{k-1}^1 -sets. We construct ϕ in the analogue way of the construction above. Again $A \subseteq 1$ -section (ϕ) is trivial. Assume that $\lambda x\{e_1\}(\phi, x)$ is total. We let the essential subcomputations be all subcomputations where arguments of type- k are from the list $\{\varphi_n\}_{n \in \omega}$. Replacing functionals of type $< k$ by arbitrary associates for them, we see that

the set of essential subcomputations will be Σ_{k-1}^1 . Then $\{e; \phi$ is used non-effectively at $e\}$ will be a Σ_{k-1}^1 -subset of B , and thus finite. The rest of the argument goes exactly as above, replacing functionals of type $< k$ by arbitrary associates for them.

Corollary 5.1

Every 1-section of a countable functional is the 1-section of a 1-obtainable functional of the same type.

Remark. Corollary 5.1 is an analogue of corollary 3.1. The notion of Kalmar-Rank for higher-type functionals is meaningless, but for our constructed functional ϕ , when e and k are given, there will be a finite, fixed list of functionals $\phi_0, \dots, \phi_{n-1}$ such that $D(e, k, \Psi)$ is decided by $\Psi(\phi_0), \dots, \Psi(\phi_{n-1})$. So these functionals really have well-defined rank ω .

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References

1. J. Bergstra, Computability and continuity in finite types, Disertation, Utrecht 1976.
2. J. Berstra - S.S. Wainer, The "real" ordinal of the 1-section of a continuous functional, Paper submitted to Logic Colloquium '76, Oxford 1976. Abstract in J.S.L.
3. R.O. Gandy - J.M.E. Hyland, Computable and recursively countable functions of higher type, in R.O. Gandy and J.M.E. Hyland (eds.) Logic Colloquium '76 407-438, North Holland 1977.
4. T. Grilliot, On effectively discontinuous type-2 objects, J.S.L. 36 (1971) 245-248.

5. P. Hinman, Hierarchies of Effective Descriptive Set Theory, T.A.M.S. 142 (1969) 111-140.
6. P. Hinman, Degrees of Continuous Functionals, J.S.L. 38 (1973) 393-395.
7. J.M.E. Hyland, The intrinsic recursion theory on the countable or continuous functionals, in J.E. Fenstad, R.O. Gandy and G.E. Sacks (eds): Generalized recursion theory II, North Holland 1978.
8. S.C. Kleene, Recursive functionals and Quantifiers of finite types I, T.A.M.S. 91 (1959) 1-52 and II, 108 (1963) 106-142.
9. S.C. Kleene, Countable functionals, in A. Heyting (ed.) Constructivity in mathematics, 81-100, North Holland 1959.
10. G. Kreisel, Interpretation of analysis by means of functionals of finite type, in A. Heyting (ed.) Constructivity in mathematics, 100-128, North Holland 1959.
11. A.H. Lachlan, Uniform enumeration operations, J.S.L. 40 (1975) 401-409.
12. Y.N. Moschovakis, Hyperanalytic Predicates, T.A.M.S. 129 (1967) 249-282.
13. D. Normann, A Continuous type-2 functional with non-collapsing hierarchy, to appear in J.S.L.
14. D. Normann, Countable functionals and the analytic hierarchy, Oslo Preprint Series no 17 (1977).
15. J.R. Shoenfield, A hierarchy based on a type-2 object, T.A.M.S. 134 (1968) 103-108.
16. S.S. Wainer, A hierarchy for the 1-section of any type-2 object, J.S.L. 39 (1974) 88-94.
17. S.S. Wainer, The 1-section of a non-normal type-2 object, in J.E. Fenstad, R.O. Gandy and G.E. Sacks (eds.) Generalized recursion theory II, North Holland 1978.