TWO - SECANT FORMULA

by

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We give here a proof of the well-known "secant-formula", (see [S-R], p. 214). There exist several proofs of this formula, in [H], [Kl], [LK], [P-S]. Our proof, based on the existence of a secant-scheme, as defined in [La], shows that the formula actually gives the number of 2-secants of X in \( P = \mathbb{P}^{2 \dim X + 1}_k \) through a general point of P, multiplicities counted. Furthermore, one would hope that it prepares the way for a similar study of r-secants, \( r \geq 3 \).

I. STRATIFICATION OF THE SECANT-SCHEMЕ.

Let \( X \) be a closed algebraic subvariety of a projective space \( P = \mathbb{P}^N_k \) over an algebraically closed field \( k \).

Definition: Let \( r \) be a positive integer. A line \( l \) in \( P \) is said to be an \( r \)-secant of \( X \) if \( X \times l \) (the intersection of \( l \) and \( X \) in \( P \)) is a 0-dimensional scheme of degree \( r \).

Let us denote by \( G \) the Grassmann scheme parametrizing the rank 2 quotients of \( k^{N+1}_G \) (or equivalently lines in \( P \)) and by \( Q \) the universal rank 2 quotient bundle of \( k^{N+1}_G \) on \( G \). In [La], Laudal defined a locally closed subscheme \( \text{Sec}_r(X) \) of \( G \), the \( r \)-secant scheme of \( X \), parametrizing the \( r \)-secants of \( X \). Let us recall briefly his definition.
Consider the following fiber square, where \( L = \mathbb{P}_G(Q) \) denotes the tautological line bundle on \( G \), naturally embedded in \( P \times G \), and \( Z = L \times (X \times G) \) is the product of these spaces.

\[
\begin{array}{ccc}
Z & \rightarrow & L \\
\downarrow & t & \downarrow \\
X \times G (i, 1) & \rightarrow & P \times G \overset{p_2}{\rightarrow} G
\end{array}
\]

\((p_i, i=1,2, \text{ will always denote the projection of a product on the first and second factor respectively}).

As a coherent sheaf on \( P \times G \), the structure sheaf \( O_Z \) of \( Z \) defines a "flattening stratification of \( G"([M], \text{ lecture 8}), i.e. a disjoint family of locally-closed subschemes \( Y_i \) of \( G \) such that \( G = \bigcup_{i=1}^{n} Y_i \) and \( O_{Z \times Y_i} \) is flat over \( Y_i \). Let \( \text{Sec}_r(X) \), the \( r \)-secant scheme of \( X \), denote the union of all \( Y_i \) such that the fibers of \( p_2 \cdot t \) over \( Y_i \) are 0-dimensional of degree \( r \).

(The \( Y_i \) being indexed by the Hilbert-polynomial of \( O_{Z \times Y_i} \) on \( P \times Y_i \).

\( \text{Sec}_r(X) \) is a locally closed subscheme of \( G \), the \( k \)-points of which actually correspond to the \( r \)-secants of \( X \).

\( \text{Sec}_r(X) \) represents a subfunctor \( \underline{\text{Sec}}_r(X) \) of the Grassmann functor \( G \). One may prove (see [La]) that the fiber functors of \( \underline{\text{Sec}}_r(X) \) are deformation functors, the hull of which \( \hat{O}_{\text{Sec}_r(X)} \), \( l \) (the completed local ring of \( \text{Sec}_r(X) \) at \( 1 \)) can be computed. This provides local information on \( \text{Sec}_r(X) \). One result says that if an \( r \)-secant \( 1 \) is intersecting \( X \) in \( r \) distinct non singular points of \( X \), \( 1 \) is a non singular point of \( \text{Sec}_r(X) \) if and only if the embedding-dimension of \( O_{\text{Sec}_r(X),\hat{1}} \) is equal to \( 2(N-1) - r(N-\dim X-1) \). For \( r \leq 2 \), \( \text{Sec}_r(X) \) is always non singular at \( 1 \).
II. TWO-SECANT FORMULA

Let \( X \overset{i}{\hookrightarrow} P \) be a closed embedding as above, \( X \) being non singular. Writing \( S_2 = \text{Sec}_2(X) \), let \( L_2 \) be the restriction to \( S_2 \) of the tautological bundle \( L \) on \( G \), let \( s_2 : L_2 \overset{\pi}{\to} P \times S_2 \) be the canonical embedding and \( \varphi_2 = p_1 \circ s_2 : L_2 \overset{\pi}{\to} P \).

THEOREM: Suppose \( X \overset{i}{\hookrightarrow} P \) is such that through a generic point of \( P \) there passes a finite number \( \neq 0 \) of 2-secants. Then

i) \( N = 2 \dim X + 1 \)
\[ \dim S_2 = 2 \dim X \quad \text{and} \quad \operatorname{im} \varphi_2 = P. \]

ii) Moreover \( S_2 \) is non singular and \( \varphi_2 \) is generically quasi-finite of degree \( \delta_2 \), the number of 2-secants through the generic point of \( P \).

iii) \( \delta_2 \) is given by the following two-secant formula:
\[ 2 \delta_2 = \deg^2(X) - \sum_{i=0}^{n} \binom{2n+1}{n+i} \deg s_i(X) \]
where \( s_i(X) \) is the \( i \)th Segre-class of \( X \) as an element of the Chow-ring \( A^*(X) \) and \( \deg(\cdot) \) denotes the degree in \( P \) of a subscheme or its corresponding class in \( A^*(P) \).

Let us prove the theorem.

1. Proof of i) and ii).

As an easy consequence of [La!] one deduces that \( S_2 \), if non empty, is non singular of dimension \( 2 \dim X \). In fact, if \( 1 \in S_2 \) is not a tangent, \( S_2 \) is non singular at \( 1 \), of dimension \( 2 \dim X \). This follows from \( \dim_k A^1 = 2 \dim X = 2(N-1) - 2(N-d-1) + 1 \) and \( \dim_k A^2 = \dim Q = 0 \) (see the discussion preceding the "trisecant lemma", §3 loc. cit.)

If \( 1 \in S_2 \) is a tangent, we still find \( \dim_k A^1 = 2 \dim X \), \( A^2 = 0 \) (see the forthcoming thesis of Tore Wentzel-Larsen), so that \( S_2 \)
is non singular at 1. In fact (see §3 loc. cit.)

\[ A^1 = \ker (1, m-n) \]
\[ A^2 = \text{Coker } (1, m-n) \]

where 1, m and n may be computed as follows. Suppose X is defined by the homogeneous ideal \( a \) of \( k[X_0, \ldots, X_N] \) and 1 by the ideal \( (X_2, \ldots, X_N) \). Let \( x \) be the point in which 1 cuts X.

We may assume \( \alpha X, x = (f_{d+1}, \ldots, f_N) \), where \( d = \dim X, x = (1, 0, \ldots, 0) \). Let \( Y_i = \frac{X_i}{X_0} \), \( i = 1, \ldots, N \). The morphisms 1, m, n of the diagram of (Laurent §3) is:

\[
\begin{align*}
H^0(X, \mathcal{O}_X) & \xrightarrow{\text{Hom}_k(\mathcal{E}/\mathcal{A}, k[\mathcal{E}])} \text{Hom}_{k[\mathcal{E}]}(\mathcal{A}/\mathcal{A}^1, k[\mathcal{E}]) \xrightarrow{\text{Hom}_{k[\mathcal{E}]}(\mathcal{A}/\mathcal{A}^1, k[\mathcal{E}])} \text{Hom}_{k[\mathcal{E}]}(\mathcal{A}_1^1, k[\mathcal{E}]) \xrightarrow{\text{Hom}_{k[\mathcal{E}]}(\mathcal{A}_1^1, k[\mathcal{E}])} \text{Hom}_{k[\mathcal{E}]}(\mathcal{A}_1^1, k[\mathcal{E}])
\end{align*}
\]

where \( A = (Y_2^2, Y_2, \ldots, Y_N) \) is the ideal of \( z = X \cap 1 \) in the affine piece where \( X_0 \neq 0 \) and \( k[\mathcal{E}] = k[Y_1, \ldots, Y_N]/A \). Consider first \( A^1 \); clearly \( n \) is an isomorphism, thus \( A^1 = \ker l \). Since \( X \) is non singular, \( \text{rank}_{k[\mathcal{E}]}((f_{d+1}, \ldots, f_N)/A(f_{d+1}, \ldots, f_N), k[\mathcal{E}]) = N - d \). By a proper choice of bases in the diagram above, one gets \( A^1 = \ker M \), where \( M \) is the matrix

\[
\begin{pmatrix}
M_1 & 0 \\
M_2 & M_1
\end{pmatrix}
\]

where \( M_1 = \begin{pmatrix}
1 & \frac{\partial^2 f_1(x)}{\partial Y_1^2} & \frac{\partial f_1(x)}{\partial Y_1} & \ldots & \frac{\partial f_i(x)}{\partial Y_1} \\
\frac{\partial Y_1}{\partial Y_2} & \frac{\partial Y_1}{\partial Y_2} & \ldots & \frac{\partial Y_1}{\partial Y_N} \\
\vdots & \vdots & \ddots & \vdots \\
\end{pmatrix}
\]

and \( M_2 \) contains higher derivatives of the \( f_1 \)'s. By non singularity of \( X \), \( \text{rank } M = 2\text{rank } M_1 = 2(N - d) \), therefore \( \text{coker } (1, m-n) = \text{coker } l = 0 \), i.e. \( A^2 = 0 \).
Since $S_2$ is non singular of dimension $2\dim X$, we have $\dim L_2 = 2\dim X + 1$. By assumption, the generic fiber of $\varphi_2$ is finite and non empty, which implies that $\dim L_2 = \dim P$, that is, $N = 2 \dim X + 1$ and $\Im \varphi_2 = P$. $\varphi_2$ is generically quasi-finite of degree $\delta_2$, the number of 2-secants through the generic point of $P$. In order to compute $\delta_2$, we need a useful morphism $\sigma: X^{\sim}X \to G$, where $X^{\sim}X$ denotes the blowing up of $X \times X$ along the diagonal.

2. The morphism $\sigma: X^{\sim}X \to G$.

Let $P^{\sim}P$ denote the blowing-up of $P \times P$ along the diagonal, and $\varepsilon_P$ (resp.$\varepsilon_X$) the exceptional locus in $P^{\sim}P$ (resp. $X^{\sim}X$). Then there exists a morphism $\gamma: P^{\sim}P \to G$ (see [Kl.], V.B). Let us recall the definition.

a) If $\Omega_P^1$ denotes the cotangent sheaf on $P$, we have the following exact sequence on $P$:

$$0 \to \Omega_P^1(1) \to O_P^{N+1} \to O_P(1) \to 0.$$ 

Let $R$ denote the projective bundle $\mathbb{P}(\Omega_P^1(1))$ and $p$ the structure morphism $R \to P$. Let $K_R$ be the kernel of the natural surjective homomorphism $p^* \Omega_P^1(1) \to O_R(1) \to 0$, and $E$ the locally free rank 2 $O_R$-module defined to make the following diagram exact and commutative:

$$
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
K_R & \to & K_R \\
\downarrow & & \downarrow \\
0 & \to & p^* \Omega_P^1(1) \to O_R^{N+1} \to p^* O_P(1) \to 0. \\
\downarrow & & \downarrow \\
0 & \to & O_R(1) \to E \to p^* O_P(1) \to 0. \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}
$$

$E$, as a rank 2 quotient of $O_R^{N+1}$, provides a morphism
\[ \alpha : R \to G \] such that \( \alpha^* Q = E \). The fiber of \( p \) over a point \( x \in P \) consists, as a set, of all lines in \( P \) through \( x \).

\( \alpha \) takes a point of \( R \), that is, a line \( l \) through \( x \in P \), to the corresponding point \( l \) in \( G \). The factorization \( \alpha^* Q = E \to p^* O_p(1) \), where all the homomorphisms are surjective, gives a factorization of \((p, \alpha)\) through \( \mathbb{P}_G(Q) \):

\[
\begin{array}{ccc}
(p, \alpha) & \rightarrow & P \times G \\
\downarrow & & \downarrow \\
\mathbb{P}_G(Q) & &
\end{array}
\]

and it is easy to prove that the morphism \( R \to \mathbb{P}_G(Q) \) defined by \( E \to p^* O_p(1) \to 0 \) is in fact an isomorphism, by which we can identify \( R \) and \( \mathbb{P}_G(Q) \) as \((P \times G)\)-schemes.

b) There is on the other hand the key morphism \( \lambda : P \times P \to R \), defined in \([Kl.]\). (See also \([H], [Lk]\)). \( \lambda \) can be described as follows. Outside of \( \varepsilon_p \), \( \lambda \) is equal to the projection of \( P \times P \) from the diagonal \( \mathbb{P}_p(0_p(1)) \) on \( R = \mathbb{P}_p \left( \frac{1}{p}(1) \right) \). \( \lambda \) takes a point \((x, y)\), not in \( \varepsilon_p \) to \((x, xy) \in R \subset P \times G \), where \( xy \) is the line through \( x \) and \( y \) as an element of \( G \), while the restriction of \( \lambda \) to \( \varepsilon_p = \mathbb{P}_p(\Omega^1_p) \) is the natural \( P \)-isomorphism \( \mathbb{P}_p(\Omega^1_p) \cong \mathbb{P}_p(\Omega^1_p(1)) \). Consider now the projective bundle \( \mathbb{P}_R(E) \) over \( R \). It is a subbundle of \( P \times R \) and a \((P \times P)\)-scheme by the composed morphism

\[
\begin{array}{ccc}
\mathbb{P}_R(E) & \rightarrow & P \times P \\
\downarrow & & \downarrow (P \times P) \\
\mathbb{P}_R(E) & &
\end{array}
\]

\( P \times P \) can be identified with \( \mathbb{P}_R(E) \) both as \( R \)-scheme (via \( \lambda \)) and a \( P \times P \)-scheme (via the blowing-up morphism).
c) Consider the morphism \( \gamma = \alpha \ast \lambda : P \times P = \mathbb{P}_R(Q_L) \to R = \mathbb{P}_G(Q) = L \to G \). By the identification \( \mathbb{P}_R(\alpha \ast Q) \simeq \mathbb{P}_G(Q) \times \mathbb{P}_G(Q) = G \times G \), the subscheme \( \mathbb{P}_R(\alpha_\ast Q(1)) \) of \( \mathbb{P}_R(Q_L) \) is identified with the diagonal of \( L \times L \), the morphism \( \lambda \) with the first \( G \)-projection, and \( \gamma \) with the structure morphism \( L \times L \to G \). By the inclusion \( L \subset P \times G \), \( L \times L \) has a natural structure of \( G \)-\( P \times P \)-scheme and the identification with \( P \times \alpha \ast Q \), whence with \( P \times P \), is a \( P \times P \)-isomorphism. The morphism \( \gamma : P \times P \to G \) is "natural" in the following sense: it takes a point of \( P \times P = L \times L \), that is, a triple \( (x,y,1) \) where \( 1 \) is a line through \( x, y \in P \), to the corresponding point \( 1 \in G \).

d) \( X \times X \) can be seen as the proper transform of \( X \times X \) under the blowing up \( P \times P \). We define \( \sigma : X \times X \to G \) to be the restriction of \( \gamma \) to \( X \times X \).

3. Proof of the secant-formula iii).

Let \( S \) denote the scheme-theoretical image by \( \sigma \) of \( X \times X \) in \( G \). Since \( X \) is reduced, irreducible, so are \( X \times X \) and \( S \). We have clearly \( S_2 \subset S \). Since \( S_2 \) is non singular and \( \dim S_2 = \dim X \times X = 2 \dim X \), \( S \) is the closure in \( G \) of \( S_2 \).

Let \( L_s \) denote the restriction of \( L \) to \( S \) and \( \varphi_S \) the composed morphism of the natural embedding \( L_s \to P \times S \) with \( P_1 \)

\[
\begin{array}{ccc}
L_s & \rightarrow & P \times S \\
\varphi_S & \downarrow & P_1 \\
& \varphi & \\
\end{array}
\]

We have clearly \( S = S_2 \), \( L = L_2 \), \( \text{im } \varphi_S = \overline{\text{im } \varphi_2} \) and \( \varphi_S \) is generically finite of degree \( \delta_2 \).

Let \( B = X \times X \), let \( Q_B \) denote the pull-back of \( Q \) to \( B \), with \( L_B = \mathbb{P}_B(Q_B) \) and let \( \sigma_L \) be the induced morphism \( L_B \to L_S \)
in the cartesian diagram

\[
\begin{array}{c}
\sigma_L \downarrow & \sigma \downarrow & \phi_S \downarrow \\
L_B \rightarrow L_S \rightarrow P \times S & \rightarrow & P_1 \\
B \rightarrow S & \rightarrow & P
\end{array}
\]

\(\sigma,\) and hence \(\sigma_L,\) are generically finite of degree 2.

Put \(\varphi = \phi_S \cdot \sigma_L.\) Then \(\varphi\) is generically of finite degree:

\[\deg \varphi = 2 \deg \phi_S = 2 \delta_2.\]

Let us compute \(\deg \varphi.\)

We denote by \(\tilde{\sigma}_i.\) \(P \times P + P, i=1,2,\) the composed morphism of the blowing up \(P \times P \rightarrow P \times P\) and the projection \(p_i, i=1,2,\) as well as its restriction to \(X \times X\) if no confusion is possible. Consider the following commutative diagram:

\[
\begin{array}{c}
B \xrightarrow{j} L_B \\
\tilde{\sigma}_1 \xrightarrow{i} P \times B \\
\xrightarrow{r} (i,1) \\
\xrightarrow{t} X \times B \\
\xrightarrow{i} P \\
\xrightarrow{\varphi} P_1 \setminus P_2 \\
X \xrightarrow{i} P \xrightarrow{t_1} B
\end{array}
\]

where \(r = (\tilde{\sigma}_1,1),\) \(t = (i,1) \cdot r,\) and \(s\) is the natural embedding \(L_B \rightarrow P \times B.\) The image by \(t\) of an element \(b \in B\) lying over \((x,y) \in X \times X\) is \((x,b).\) We have seen that \(b\) corresponds to a line through \(x\) and \(y;\) we have therefore \((x,b) \in L_B\) and \(t\) factorizes through \(L_B.\) Let \(j\) be such that \(t = s \cdot j.\)

All the schemes considered are quasi-projective, non singular. We can therefore identify their Chow cohomology and homology groups of cycles modulo rational equivalence.
To the proper morphism \( \varphi \) corresponds a homomorphism \( \varphi_\ast \) of Chow cohomology groups, \( \varphi_\ast : A^\ast (L_B) \rightarrow A^\ast (P) = \mathbb{Z}[T]/T^{n+1} \) where \( T \) is the class in \( A^1(P) \) of an hyperplane. \( L_B \) is reduced, irreducible, and \( \varphi \) has generic degree \( 2\delta_2 \), so we have

\[ \varphi_\ast (1) = 2\delta_2. \]

a) Let us compute first \( s_\ast (1) \), the class of \( L_B \) in \( A^\ast (P \times B) \).

Let \( K \) denote the kernel of the quotient \( V_B \rightarrow Q_B \rightarrow 0 \) defined by \( \sigma \), with \( V = k^{N+1} \). By [Lk] (lemma 2 p.173), \( L_B = \mathbb{P}_C(Q_B) \) is the "scheme of zeroes" of the homomorphism \( \kappa \) obtained by composing the homomorphism \( \sigma \rightarrow L_B \rightarrow V_B \) with the surjective homomorphism \( V_B \rightarrow O_{P \times B}(1) \) on \( P \times B = \mathbb{P}_B(V) \).

\( P \times B \) is non singular and \( L_B \) has codimension \( N-1 \), equal to the rank of \( K_{P \times B} \). Let us denote by \( c_1(F) \) the \( i \)th Chern class in \( A^i(P \times B) \) of a locally free \( O_{P \times B} \)-module \( F \), and by \( s_i(F) \) the \( i \)th "inverse" Chern class of \( F \) in \( A^i(P \times B) \), defined by

\[ \left( \sum_{i \in \mathbb{N}} s_i(F) \right) \left( \prod_{i \in \mathbb{N}} c_i(F) \right) = 1. \]

Then by [G] (théorème 2) the class \( [L_B] \) of \( L_B \) in \( A^\ast (P \times B) \) is given by

\[ [L_B] = c_{N-1}(K_{P \times B} \otimes_{O_{P \times B}} O_{P \times B}(1)) \]

where \( K_{P \times B} \) is the dual \( O_{P \times B} \)-module of \( K_{P \times B} \).

b) Let us prove

\[ \psi_\ast (1) = P_1 \ast [L_B] = P_1 \ast (s_{N-1}(Q_{P \times B})). \]

By the exact sequence \( \sigma \rightarrow K_{P \times B} \rightarrow V_{P \times B} \rightarrow Q_{P \times B} \rightarrow 0 \), we get

\[ c_{N-1}(K_{P \times S} \otimes O_{P \times S}(1)) = \prod_{i \geq 0} (Q_{P \times B})_{T^{N-1-i}}. \]
But \( p_1^* (s_i(Q_{P \times B}) \cdot T^{N-i}) = T^{N-i} p_1^* (s_i(Q_{P \times B})) \) and since \( \dim B = N-1 \),
\( p_1^* (s_i(Q_{P \times B})) = 0 \) for all \( i < N-1 \). We note that, \((N-1)\) being even,
\( s_{N-1}(Q_{P \times B}) = s_{N-1}(Q_{P \times B})^\vee \).

Q.E.D.

c) Consider the diagram (*) above.

We have in \( A^*(P) \) the following formula:

\[(3) \quad p_1^* t^* (s_{N-1}Q_B) = \delta_2 T^N.\]

Proof of (3). The morphism \( p_2 \circ t \) is the identity morphism on \( B \).

Let us write
\[ t^* (1) = \sum_{i \geq 0} b_i T^{N-i}, \quad b_i \in A^i(B). \]

Then
\[ (p_2 \circ t)^* (1) = \text{id}_B^* (1) = 1 \]
and
\[ p_2^* (t^* (1)) = p_2^* (b_0 T^N) = b_0 \]
which implies \( b_0 = 1 \) in \( A^*(B) \).

By functoriality of the Chern classes, we have:
\[ s_{N-1}(Q_B) = t^* s_{N-1}(Q_{P \times B}). \]

Then (2) implies (3), by:

\[ p_1^* t^* (s_{N-1}(Q_B)) = p_1^* (s_{N-1}(Q_{P \times B}) t^* (1)) \quad \text{(projection formula)} \]

\[ = p_1^* (s_{N-1}(Q_{P \times B}) \sum_{i=0}^{N-1} b_i T^{N-i}) = p_1^* (s_{N-1}(Q_{P \times B}) b_0 T^N) \]

\[ = T^N p_1^* (s_{N-1}(Q_{P \times B})) = \delta_2 T^N. \]

d) In the commutative diagram (*), we have:

\[ p_1 \circ t = i \circ p_1 \circ r = i \circ \tilde{p}_1, \]
so we can replace (3) by (3'):

\[ \delta_2 T^N = i_\ast \tilde{p}_1^*(s_{N-1}Q_B). \]
In order to compute \( s_{n-1}(Q_B) \), let us put
\[
\alpha_1 = \tilde{p}_1^* i^*(T) \\
\alpha_2 = \tilde{p}_2^* i^*(T)
\]
and let \( \tau \) denote the inclusion morphism \( \varepsilon_X \to B \) of the exceptional locus \( \varepsilon_X \) of \( B \).

Then we have in \( A^*(B) \):

**Lemma 2:** \( c_1(Q_B) = \alpha_1 + \alpha_2 - \tau_*(1) \)
\[
c_2(Q_B) = \alpha_1 + \alpha_2 - \tau_*(1).
\]

**Proof of lemma 2:** We have identified \( P \times P \) with \( L \times L \) as \( G \)-schemes and \( P \times P \)-schemes. Let \( \mathcal{I} \) denote the embedding
\[
X \times X \to L \times L = P \times P \tag{G}
\]
which corresponds to \( i : X \to P \) in a natural way. Let \( \Delta \) denote the diagonal in \( L \times L \) and
\[
\Delta_1 = p_1^* O_L(1) \\
\Delta_2 = p_2^* O_L(2).
\]
Look at the morphisms \( \lambda \) and \( \alpha : \)
\[
L \times L = \mathbb{P}_L(Q_L) \xrightarrow{\lambda} L = \mathbb{P}_G(Q) \xrightarrow{\alpha} G.
\]
By identifying \( L \times L \) and \( \mathbb{P}_L(Q_L) \) we also get:
\[
\Delta = \mathbb{P}_L(O_L(1)), \quad \Delta_1 = \lambda^* O_L(1),
\]
\[
\Delta_2 = \mathbb{P}_L(Q_L)(1).
\]

Using again \([Lk]\) (lemma 2), we see that the \( L \)-subbundle \( \Delta \) of \( L \times L \) is the "scheme of zeroes" of the composed morphism:
\[ \lambda^*H \to \mathcal{O}_{\mathbb{P}L}(Q_L) \to \mathcal{O}_{\mathbb{P}L}(Q_L)(1) \], where \( H \) is the kernel of \( Q_L \to O_{L}(1) \to 0 \) on \( L \), and by \([G]\) (théorème 2) we get

\[ \Delta = c_1(\lambda^*H \otimes \mathcal{O}_{\mathbb{P}L}(Q_L)(1)) \text{ in } \mathbb{A}^*(\mathcal{O}_{\mathbb{P}L}(Q_L)) \]

\[ = -c_1(\lambda^*H) + c_1(\mathcal{L}_2). \]

By the exact sequence \( 0 \to H \to Q_L \to O_L(1) \to 0 \) this becomes

\[ \Delta = -c_1(Q_{L \times L}) + c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2). \]

Since \( \mathcal{L}_1 \) is a quotient of \( Q \), we get

\[ c_1(Q_{L \times L}) = c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2) - [\Delta] \]

\[ c_2(Q_{L \times L}) = c_1(\mathcal{L}_1)(c_1(\mathcal{L}_2) - [\Delta]). \]

To prove the lemma, it is enough to see that \( \tilde{i}^*[\Delta] = \tau_*(1) \). By \( P \times P = L \times L \), \( \Delta \) is equal to the exceptional divisor \( \varepsilon_P \) and we know that the restriction of the invertible sheaf \( O_{\mathbb{P} \times P}(\varepsilon_P) \) to \( X \times X \) is \( O_{X \times X}(\varepsilon_X) \), which implies that

\[ \tilde{i}^*[\Delta] = [\varepsilon_X] = \tau_*(1). \]

Q.E.D.

It is now easy to compute \( s_{N-1}(Q_B) = s_{2n}(Q_B) \) where \( n = \text{dim } X \).

By lemma 2

\[ s_{2n}(Q_B) = \sum_{i=0}^{2n} a_1^i(a_2 - \tau_*(1))^{2n-i}. \]

Clearly \( a_1^i a_2^{2n-i} = 0 \) for all \( i \neq n \), so we get:

\[ s_{2n}(Q_B) = a_1^n a_2^n + \tau_*(1) \sum_{i=0}^{2n} a_1^{2n-i} \sum_{j=1}^{i} (-1)^j (i-j) a_2^j \tau_*(1)^{j-1}. \]
But \( \tau^* \tau_*(1) = c_1(N\varepsilon_X|B) \) (self intersection formula)

where \( N\varepsilon_X|B \) is the conormal sheaf of \( \varepsilon_X \) in \( B \) and we know that on \( \varepsilon_X = \mathbb{P}X(\Omega^1_X) \), \( N\varepsilon_X|B = \mathbb{P}X(\Omega^1_X)(1) \) (\( \Omega^1_X \) denotes the cotangent sheaf on \( X \)). Let us put \( \xi = \tau^* \tau_*(1) = c_1(\mathbb{P}X(\Omega^1_X)(1)) \).

Then, using the "projection formula", we get

\[
s_{2n}(Q_B) = \alpha_1^n \alpha_2^n + \tau^* \left( \sum_{i=1}^{2n} \tau^*(\alpha_1)^{2n-i} \right) \cdot \left( \sum_{j=1}^{i} (-1)^{2j-1}(\tau^*(\alpha_2)^{i-j}) \xi^{j-1} \right)
\]

Let \( e \) be the structure morphism \( \varepsilon_X \to X \), and diag the diagonal morphism in the following diagram:

\[
\begin{array}{ccc}
\varepsilon_X = \mathbb{P}X(\Omega^1_X) & \xrightarrow{\tau} & X \times X \\
| & & | \\
e & & (i,i) \\
X & \xrightarrow{\text{diag}} & X \times X
\end{array}
\]

\[
\xrightarrow{P \times P}
\]

\[
\begin{array}{ccc}
P_1 & \xrightarrow{p_1} & X \\
& & \\
& & \ddl{p_1}
\end{array}
\]

It is easily seen that

\[
\tau^* \alpha_1 = \tau^* \alpha_2 = e^* i^*(T).
\]

Consider formula (3').

\[
i_\sim P_1^* S_{2n}(Q_B) = i_\sim P_1^* (\alpha_1^n \alpha_2^n) - i^*(\sum_{i=1}^{2n} i e^* (\tau^* T^{2n-j})^{i-j} \xi^j)
\]

Clearly \( i_\sim P_1^* (\alpha_1^n \alpha_2^n) = P_1^* (i,i)^* (T^* T^n \otimes T^* T^n) \)

\[
= P_1^* (T^n i^*(1) \otimes T^n i^*(1)) = P_1^* (T^{2n+1} \deg X \otimes T^{2n+1} \deg X) = (\deg X)^2 T^{2n+1}, \text{ and}
\]
Let $s_i$ denote the $i^{th}$ Segre class of $X$.

By definition, $e_*(\xi^{i-1}) = s_{j-n}(X)$, and by the identity
\[
\sum_{i \geq j} \binom{i}{j} = \binom{2n+1}{j}
\]
we get
\[
i_*e_*s_{2n}(Q_B) = (\deg X)^2T^{2n+1} - \sum_{j=0}^{n} \binom{2n+1}{n+j} i_*s_j(X)T^{2n-j}.
\]

Finally, this implies the two-secant formula:
\[
2\delta_2 = (\deg X)^2 - \sum_{j=0}^{n} \binom{2n+1}{n+j} \deg(s_j(X)).
\]
REFERENCES


