ISBN 82-553-0351-0

Mathematics
No 9 - August 1978

MULTITRANSVERSE BOARDMAN MAPPINGS ARE GENERIC

by

Hans Brodersen

Oslo

PREPRINT SERIES - Matematisk institutt, Universitetet i Oslo

Introduction

Let $f : N \to P$ be a smooth mapping between smooth manifolds and S^k a stratification of the jet space $J^k(N,P)$. Recall that f is multitransverse to S^k if its jet extension j^kf is transverse to S^k and the restrictions of f to the strata of $(j^kf)^{-1}S^k$ form a family in general position ([1] p.127). We call f a multi Boardman mapping if $S^k = \{\Sigma^I\}_I$ is the Boardman stratification of $J^k(N,P)$ and f is multitransverse to S^k for each k. The purpose of this paper is to prove there is a dense open subset of multi Boardman maps in $C^\infty(N,P)$, when N is compact.

Such a result can be obtained from Mather's method for producing a dense open set of topologically stable maps,[6] . In fact Mather in his lectures at the Nordic Summer School in Oslo 1976 demonstrated how to get a dense open set of topologically stable mappings multitransverse to a prescribed invariant jet stratification. The present proof is based on Mather's techniques but adapted to our specific purpose, and is accordingly brief.

§1 Stratifications § S

In the sequal we will use the terminology of [1]. We refer to [1] ch.I for basic results on stratifications in general, and to [7] for particulars about the Boardman stratifications.

Let S be a Whitney stratification of a closed subset A of a smooth manifold P. It is a well known fact that the set $\{f \in C^\infty(N,P) : f \pitchfork S\}$ form an open dense subset of $C^\infty(N,P)$. If S is Whitney stratification of a closed subset of the jet-space $J^k(N,P)$ then the set $\Omega = \{f \in C^\infty(N,P) : j^kf \pitchfork S\}$ is likewise open and dense.

Let $A \subset \mathbb{R}^p$ be a semialgebraic subset. Let T be a finite partition of A into semialgebraic subsets. A slight modification of the proof of 2.7 [1] chapter I gives us

Theorem 1.1. There exists a finite Whitney stratification S of A refining T where the strata of S are smooth submanifolds and semialgebraic subset of $\mathbb{R}^p$.

We want to state the analogue of 2.1 for certain subset of smooth manifolds. Let P be a smooth manifold and for each $y \in P$ let (O_y, ϕ_y) be a chart at y. A subset $A \subset P$ is a locally semi-algebraic set with respect to $\{(O_y, \phi_y)\}$ if $\phi_y(O_y \cap A) \subset \mathbb{R}^p$ is semi-algebraic for each y.

Theorem 1.2. Let $A \subset P$ be a locally semialgebraic subset, and T a locally finite partition of A in locally semialgebraic subset all with respect to a given covering of local charts. Then we can find a Whitney stratification of A which refines T.

The proof of 1.1 is almost identical to the proof of 1.2. We construct S by inductively removing points where Whitney condition b) or the condition of being a smooth submanifold are not satisfied. (See the proof of 2.7 [1] page 20.) This is possible, because it follows from 2.5 and 2.6 [1] chapter I, that the singular subsets are semialgebraic in the given charts, hence locally semialgebraic subsets of P.

Let $J^k(n,p)$ be the space of k jets of smooth mappings $(R^n,0) \to (R^p,0)$. Let $\{\Sigma^I\}_{I=(i_1,\dots,i_k)}$ be the stratification of $J^k(n,p)$ in the Boardman varieties. Let N,P be two smooth manifolds of dimension n and p respectively. We get induced a stratification $\{\Sigma^I(N,P)\}_{I=(i_1,\dots,i_k)}$ from the stratification of the fiber $J^k(n,p)$. We will call this the Boardman stratification of the jetspace $J^k(N,P)$. Since the Boardman varieties in the fiber $J^k(n,p)$ are semialgebraic, we can use 1.1 to refine $\{\Sigma^I\}_{I=(i_1,\dots,i_k)}$ to a Whitney stratification. Since the Boardman varieties are invariant with respect to the structure group $L^k(n) \times L^k(p)$, the obtained Whitney stratification is also invariant. Hence we get a Whitney stratification of $J^k(N,P)$ which refines the Boardman stratification. Let $f \in C^\infty(N,P)$. Call f a Boardman mapping if the jet extension $J^kf : N \to J^k(N,P)$ is transverse to the Boardman stratification for all k. It follows from the codimension formula for the Boardman varieties that f is a Boardman mapping provided $J^{n+1}f: N \to J^{n+1}(N,P)$ is transverse to the Boardman stratification. Since this stratification can be refined to a Whitney stratification, the Boardman mappings contain an open dense subset of $C^\infty(N,P)$.

If $f \in C^\infty(N,P)$ is a Boardman mapping we can define $\{\Sigma^I(f)\}_{I=(i_1,\dots,i_k)}$ by $\Sigma^I(f) = (J^kf)^{-1}(\Sigma^I(N,P))$. This gives the repeted rank stratification as defined by Thom (see [7] page 246). The stratifications $\{\Sigma^I(f)\}_{I=(i_1,\dots,i_k)}$ for different k will refine each other, but one can prove that we get the finest stratification when $k = n$. We call $\{\Sigma^I(f)\}_{I=(i_1,\dots,i_n)}$

the Thom-Boardman stratification of N associated to f and the strata in the stratification Thom-Boardman strata. Note that if $\Sigma^I(f)_I$ $(I=(i_1,\dots,i_n))$ is a Thom-Boardman stratum, then $f|\Sigma^I(f)$ is either a submersion or an immersion. If $\Sigma(f) = \{x \in N: df(x)$ is not surjectiv$\}$, then $\Sigma(f)$ is a union of Thom-Boardman strata.

§2 Properties about stable unfoldings

Assume N is compact. Mather defines in [6] a certain open dense subset of $C^\infty(N,P)$ consisting of mappings of finite singularity type. If f is a mapping of finite singularity type then there exists a stable unfolding of f. To be precise there is a commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{i} & N' \\ \downarrow f & & \downarrow F \\ P & \xrightarrow[j]{} & P' \end{array}$$

where i,j are closed embeddings, F a proper and infinitesimally stable mapping transverse to $j(P)$, and $\operatorname{codim}(i(N),N') = \operatorname{codim}(j(P),P')$. (In this definition we will drop the claim $i(N) = F^{-1}j(P)$ since we only need to know that $i(N)$ is a union of components of $F^{-1}(j(P))$). In fact Mather gives an explicit construction of the unfolding of the following type

$$\begin{array}{ccc} N & \xrightarrow{i} & N \times U \\ f\downarrow & & \downarrow F \\ P & \xrightarrow[j]{} & P \times U \end{array}$$

where U is an open neighbourhood of the origin in $\mathbb{R}^k$, F is of the form $(x,t) \to (F(x,t),t)$ where $(x,t) \in N \times U$, $(F(x,t),t) \in P \times U$ and i,j are the embeddings $x \to (x,0), y \to (y,0)$.

Proposition 2.1. *Let* $f : N \to P$ *be a mapping of finite singularity type. Let*

$$\begin{array}{ccc} N & \xrightarrow{i} & N' \\ \downarrow f & & \downarrow F \\ P & \xrightarrow[j]{} & P' \end{array}$$

be a stable unfolding of f. *Then there is a neighbourhood* W *of* f *in* $C^\infty(N,P)$ *and continuous mappings* $H : W \to C^\infty(N,N')$, $K : W \to C^\infty(P,P')$ with $H(f) = i, K(f) = j$ *such that*

$$\begin{array}{ccc} N & \xrightarrow{H(g)} & N' \\ \downarrow g & & \downarrow F \\ P & \xrightarrow[K(g)]{} & P' \end{array}$$

is a stable unfolding of g, *for every* $g \in W$

Proof. Let $k : P' \to \mathbb{R}^l$ be a closed embedding and let (U,π) be a tubular neighbourhood of $k(P')$. Using i we will consider N as a submanifold of N' and using j and k we will consider P as a submanifold of $\mathbb{R}^l$. Since $N \subset N'$ is compact there is a finite number of charts $(U_j,\psi_j)j = 1,\ldots,m$ in N' with $N \subset \bigcup_{j=1}^{m} U_j$ such that in the chart (U,ψ_j) we have coordinates (x,t) where $(x,0)$ are coordinates in $U_j \cap N$. Further we will choose $\overline{U}_j$ compact. Let V be an open set in N' with $\overline{V} \subset \bigcup^{m} U_j$ and $N \subset V$. Let $U_{m+1} = N' - \overline{V}$. Then $\{\{U_j\}_{j=1}^{m+1}$ is an open covering of N'. Let $\{\phi_j\}_{j=1}^{m+1}$ be a partition of unity subordinate to $\{U_j\}$. Let $g \in C^\infty(N,P)$ and define $G_j(g) : U_j \to \mathbb{R}^l$ $j=1,\ldots,m$ by $G_j(g)(x,t) = \phi_j(x,t)(g(x,0) - f(x,0))$ Since supp $\phi_j \subset U_j$ we can extend $G_j(g)$ to N' setting

$\tilde{G}_j(g) \equiv 0$ outside U_j. Hence we get a mapping $g \to \tilde{G}_j(g)$, $C^\infty(N,P) \to C^\infty(N',\mathbb{R}^1)$. Since the partial derivatives of ϕ_j are bounded on the compact supp ϕ_j and ϕ_j vanish outside supp ϕ_j it is clear that the mapping $g \to \tilde{G}_j(g)$ is continuous in the Whitney C^∞ topology. Define $\tilde{G}(g) = \sum_{j=1}^{m} \tilde{G}_j(g) + F$ by 3.6 and 3.8 of [2] chapter II, $g \to \tilde{G}(g)$ is a continuous mapping. $C^\infty(N,P) \to C^\infty(N', \mathbb{R}^1)$. From the definition of $\tilde{G}(g)$ follows that $\tilde{G}(g)|N \equiv g$ By continuity $\tilde{G}(g)(N') \subset U$ for g sufficiently close to f. Hence in a neighbourhood of f we can define $G(g) = \pi o \tilde{G}(g)$. By 3.5 [2] chapter II, this is a continuous mapping $G : C^\infty(N,P) \to C^\infty(N',P')$. Notice that $G(g)|N \equiv g$ and that $G(f) = F$. Hence if g is sufficiently close to f, $G(g) \pitchfork j(P)$, since $j(P)$ is a closed submanifold of P', and since F is stable, $G(g)$ is smoothly equivalent to F. Then by [3] it is possible to find $\tilde{H}(g) \in \text{Diff } N'$, $\tilde{K}(g) \in \text{Diff } P'$ continuously dependent of g such that $G(g) = \tilde{K}(g)F\tilde{H}(g)$ where $\tilde{H}(f) = id_{N'}$, $\tilde{K}(f) = id_{P'}$. Define $H(g) = \tilde{H}(g)oi$ and $K(g) = \tilde{K}(g)^{-1}oj$. $H(g)$, $K(g)$ are continuously dependent of g and by construction $H(f) = i$ and $K(f) = j$. Since $G(g)oi = jog$ and $G(g) = \tilde{K}(g)F\,\tilde{H}(g)$ the diagram

$$\begin{array}{ccc} N & \xrightarrow{H(g)} & N' \\ g\downarrow & & \downarrow F \\ P & \xrightarrow[K(g)]{} & P' \end{array}$$

will commute. Since $G(g) \pitchfork j(P)$ and $F = \tilde{K}(g)^{-1}G(g)\tilde{H}(g)^{-1}$, $F \pitchfork \tilde{K}(g)^{-1}j(P) = K(g)(P)$. It is clear that $H(g)$ and $K(g)$ are closed embeddings hence

$$\begin{array}{ccc} N & \xrightarrow{H(g)} & N' \\ \downarrow g & & \downarrow F \\ P & \xrightarrow[K(g)]{} & P' \end{array}$$

is a stable unfolding of g which varies continuously with g.

§ 3. A stratification associated to a proper infinitesimally stable mapping.

3.1. The property of the stratification.

Let $f: N \to P$ be a proper and infinitesimally stable mapping. Since the Boardman strata of the jet spaces $J^k(N,P)$ are unions of orbits under the group action of Diff $N \times$ Diff P, it follows that f is a Boardman mapping and even a multi Boardman mapping,[5]. Let $\{\Sigma^I(f)\}_I$ be the Thom-Boardman stratification of N as explained in § 1.

We want to construct a Whitney stratification S of P with the following property: Let $y \in P$ and $\{x_1,\dots,x_k\} \subset f^{-1}(y)$. Let $\Sigma^I(f)$ be Thom-Boardman stratum through x_i, $i=1,\dots,k$ and U the S stratum through y. Then locally at y we will have

$$U \subset \bigcap_{1=1}^{k} f(\Sigma^I(f)).$$

3.2. A locally semialgebraic partition of P.

Let $\Sigma(f) = \{x \in N \mid df(x) \text{ is not surjective}\}$. As mentioned in §1, $\Sigma(f)$ is a union of Thom-Boardman strata. We will define an equivalence relation on P by setting $y \sim y'$ if $\# f^{-1}(y) \cap \Sigma^I(f) = \# f^{-1}(y') \cap \Sigma^I(f)$ for all $\Sigma^I(f) \subset \Sigma(f)$. Let T be the induced partition of P in equivalence classes. Let $y \in P$ and $\Sigma^I(f)$ be a Thom-Boardman stratum. Define $n_{I(y)} = \# f^{-1}(y) \cap \Sigma^I(f)$ and $N_{k,I} = \{y \in P \mid n_{I(y)=k}\}$. Then it is clear, that y belongs to the element of T given by the expression $\bigcap_I N_{n_{I(y)},I}$, where the intersection is taken over indexes I with $\Sigma^I(f) \subset \Sigma(f)$.

3.3. Proposition. The elements of T are locally semialgebraic subsets of P in the sense of §1.

Proof: Let $y \in P$ and assume $y \notin f(\Sigma(f))$. Since $\Sigma(f)$ is closed and f is proper, $f(\Sigma(f))$ is closed. It is clear that we can find chart (V,η) around y with $V \cap f(\Sigma(f) = \emptyset$ and

$\eta(V) \subset \mathbb{R}^p$ semialgebraic. Since $y' \in V$ implies $y' \sim y$ it follows that at every $y \notin f(\Sigma(f))$ we can find charts which intersect T in semialgebraic sets. Now we have to treat the case $y \in P$, $f^{-1}(y) \cap \Sigma(f) \neq \emptyset$. Let us first make one remark.

Remark. Assume $f : U \to V$ is a polynomial mapping between open semialgebraic sets $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^p$. Let $\Sigma \subset U$ be semialgebraic. Let $y \in V$ and $n(y) = \# f^{-1}(y) \cap \Sigma$. Let $N_k \subset V$ be given as $N_k = \{y | n(y) = k\}$ where k is a non negative integer. Then N_k is semialgebraic. The proof here is an application of the Tarski-Seidenbergs theorem and can be found in [6] page 137.

Now assume that $y \in P$ and $f^{-1}(y) \cap \Sigma(f) = \{x_1,\ldots,x_m\}$ (Since f is stable f has regular intersections and $m \leq p$). From [4]it follows that we can find chart (U_i,ψ_i) at x_i, (V,η) chart at y with $\eta o f o \psi_i^{-1} : U_i \to V$ a polynomial and $\psi_i(U_i), \eta(V)$ semialgebraic subsets of euclidean spaces. Since f is proper, we can find $V' \subset V$ such that $f^{-1}(V') \cap \Sigma(f) \subset \bigcup_{i=1}^{m} U_i$ and $\eta(V')$ semialgebraic. Since f is a polynomial in local charts and the Boardman stratification is locally semialgebraic, $\psi_i(U_i \cap \Sigma^I(f))$ is semialgebraic for each Thom-Boardman stratum $\Sigma^I(f) \subset \Sigma(f)$. Define $N_{k,I_i} = \{y \in V | \# f^{-1}(y) \cap \Sigma^I(f) \cap U_i = k\}$. From the remark follows that $\eta(N_{k,I_i})$ is semialgebraic. The fact that $f^{-1}(V') \cap \Sigma(f) \subset \bigcup_{i=1}^{m} U_i$ will imply that $V' \cap N_{k,I} = (U(\bigcap_i N_{k_i,I_i})) \cap V'$ where the union is taken over the sets of m tupples $(k_1,\ldots,k_m)$ with $\Sigma k_i = k$. Everything in the expression above is semialgebraic in the chart (V,η), hence $\eta(V' \cap N_{k,I})$ is semialgebraic. Since the elements in T are intersections of such sets, we have showed that the elements in T are semialgebraic in the chart (V',η).

3.4 A stratification of P. From §1 follows that T can be refined to a Whitney stratification S. We will show that S has the property mentioned above. From §1 follows that if $\Sigma^I(f) \subset \Sigma(f)$ is a Thom-Boardman stratum, then $f|\Sigma^I(f)$ is an immersion. Let $y \in P$ and $f^{-1}(y) \cap \Sigma(f) = \{x_1,\ldots,x_k\}$. Let $\Sigma^{I_i}(f)$ be the Thom-Boardman stratum through x_i. At each x_i choose a neighbourhood U_i with $f|U_i \cap \Sigma^{I_i}(f)$ a one to one immersion. Let $y' \sim y$. Since f is proper, $f^{-1}(y') \cap \Sigma(f) \subset \bigcup_{i=1}^{m} U_i$ for y' sufficiently close to y. Now since $y' \sim y$, $f|\Sigma^{I_i}(f) \cap U_i$ is one to one, $\# f^{-1}(y) \cap \Sigma^{I_i}(f) = 1$, we have $\# f^{-1}(y') \cap \Sigma^{I_i}(f) = 1$. This yields $y' \in \bigcap_{i=1}^{m} f(\Sigma^{I_i}(f) \cap U_i)$. Since S is a refinement of T, the S stratum through y is locally contained in $\bigcap_{i=1}^{m} f(\Sigma^{I_i}(f)) \cap U_i)$.

§ 4. Proof of the main theorem.

4.1. Stable unfoldings and Thom-Boardman stratifications.

Since the set of mappings of finite singularity type as well as the set of Boardman mappings contains an open dense subset of $C^\infty(N,P)$, we can find an open dense set of mappings with both properties. Let $f : N \to P$ be such a mapping and let

$$\begin{array}{ccc} N & \xrightarrow{i} & N' \\ \downarrow f & & \downarrow F \\ P & \xrightarrow[j]{} & P' \end{array}$$

be a stable unfolding. Since F is stable, F is also a Boardman mapping. Since $F \pitchfork j(P)$, F will be of the form $F(x,t) = (y \circ F(x,t),t)$ locally at points in $i(N)$. Here (x,t) is local chart in N' where $(x,0)$ is chart in $i(N)$ and (y,t) is local chart in P' where $(y,0)$ is chart in $j(P)$.

Since F locally is a mapping preserving the t coordinate and the Thom-Boardman stratification is given by repeted rank stratifications, it is clear that $\Sigma^I(F) \cap N = \Sigma^I(f)$ for each index $I = (i_1,\ldots,i_n)$. (Here we think of N as a submanifold of N' through i).

<u>Proposition 4.2. Let N be compact $f : N \to P$ a Boardman mapping of finite singularity type. Let</u>

$$\begin{array}{ccc} N & \xrightarrow{i} & N' \\ \downarrow f & & \downarrow F \\ P & \xrightarrow[j]{} & P' \end{array}$$

<u>be a stable unfolding of f. Let S be the Whitney stratification of P' associated to F as defined in §3. Then $j \pitchfork S$ implies that f is a multi-Boardman mapping.</u>

Proof. Let $\dim N' = n + k$, $\dim P' = p + k$ where $n = \dim N$, $p = \dim P$. Let $y \in P$ and $\{x_1,\ldots,x_m\} \subset f^{-1}(y)$. Let $\Sigma^{I_i}(f)$ be the Thom-Boardman stratum through x_i. We have to show that $\{f(\Sigma^{I_i}(f))\}_{i=1}^m$ intersect in general position. Since f is a submersion outside $\Sigma(f)$ we can assume $x_i \in \Sigma(f)$ $i=1,\ldots,m$. Through i,j we have $N \subset N'$, $P \subset P'$ as submanifolds. Let U be the S stratum through y, $H_i = df(x_i)T_{x_i}\Sigma^{I_i}(f)$, $H_i' = dF(x_i)T_{x_i}\Sigma^{I_i}(F)$. Then H_i, H'_i, T_yU, T_yP are all subspaces of T_yP'. Since $j \pitchfork S$, T_yP intersects T_yU transversally, but locally at y $U \subset \bigcap_{i=1}^m f(\Sigma^{I_i}(f))$, hence T_yP intersects $\bigcap_{i=1}^m H'_i$ transversally. This fact combined with the fact that f is stable and hence a multi Boardman mapping gives us

$$\operatorname{codim}\left(\bigcap_{i=1}^m H'_i \cap T_yP, T_yP'\right) = \sum_{i=1}^m \operatorname{codim}(H'_i, T_yP') + \operatorname{codim}(T_yP, T_yP')$$

$$= h + \sum_{i=1}^m \operatorname{codim}(H_i', T_yP') \quad *)$$

Now $\bigcap_{i=1}^m H_i' \cap T_yP = \bigcap_{i=1}^m H_i$. Since $\operatorname{codim}(\Sigma^{I_i}(f),N) = \operatorname{codim}(\Sigma^{I_i}(F),N')$

and $f|\Sigma^{I_i}(f), F|\Sigma^{I_i}(F)$ both are immersions, we get $\operatorname{codim}(H_i', T_yP') = \operatorname{codim}(H_i, T_yP)$. Furthermore we have $\operatorname{codim}(\bigcap_{i=1}^{m} H_i, T_yP') = k + \operatorname{codim}(\bigcap_{i=1}^{m} H_i, T_yP)$. Substituting this in *) we get $\operatorname{codim}(\bigcap_{i=1}^{m} H_i, T_yP') = k + \operatorname{codim}(\bigcap_{i=1}^{m} H_i, T_yP) = k + \sum_{i=1}^{m} \operatorname{codim}(H_i, T_yP)$ which yields $\operatorname{codim}(\bigcap_{i=1}^{m} H_i, T_yP) = \sum_{i=1}^{m} \operatorname{codim}(H_i, T_yP)$, and hence $\{H_i\}_{i=1}^{m}$ intersect in general position. This shows that f is a multi Boardman mapping.

4.3. Theorem. Let N be compact and $C^\infty(N,P)$ the space of smooth mappings $f: N \to P$. The set of multi Boardman mappings contains an open dense set of $C^\infty(N,P)$.

Proof: As we have pointed out earlier, the set of Boardman mappings of finite singularity type contains an open dense subset of $C^\infty(N,P)$. Let $f: N \to P$ be in this set. It suffices to show that f can be approximated by a mapping which is an interior point in the set of multi Boardman mappings. Let

$$\begin{array}{ccc} N & \xrightarrow{i} & N \times U = N' \\ f \downarrow & & \downarrow F \\ P & \xrightarrow[j]{} & P \times U = P' \end{array}$$

be a stable unfolding of f as described in §2. Let S be the Whitney stratification of F described in §3. By §1 j can be approximated by j' transverse to S. If j' is sufficiently close to j, j' will be a closed embedding. We will show that $F \pitchfork j'(P)$. Letting $N_1 = F^{-1}(j'(P))$ we will construct a diffeomorphism $i': N \to N_1$ close to i as a mapping $N \to N'$. Let $f' = (j')_o^{-1} F o i'$. We will prove that f' will be close to f, if j' is sufficiently close to j. Hence it follows that f' is an interior point in the set of Boardman mappings of finite singularity type.

Furthermore

$$\begin{array}{ccc} N & \xrightarrow{i'} & N' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle F} \\ P & \xrightarrow[j']{} & P' \end{array}$$

is a stable unfolding of f' with $j' \pitchfork S$, hence proposition 4.2 gives that f' is a multi Boardman mapping. If f'' is sufficiently close to f' there is by proposition 2.1 an unfolding

$$\begin{array}{ccc} N & \xrightarrow{i''} & N' \\ {\scriptstyle f''}\downarrow & & \downarrow{\scriptstyle F} \\ P & \xrightarrow[j'']{} & P' \end{array}$$

where j'' is close to j'. Since the set of mappings transverse to S is open, we can assume $j'' \pitchfork S$ hence, f'' will be a multi Boardman mapping. This shows that f' is an interior point in the set of multi Boardman mappings.

Finally we will prove that $F \pitchfork j'(P)$, construct a diffeomorphism $i' : N \to N_1$, close to i and show that f' is arbitrary close to f if j' is chosen sufficiently close to j. First, since $j : P \to P'$ is a closed embedding, j is stable hence if j' is close to j, j' is a closed embedding of the form $j' = h \circ j \circ k^{-1}$ where $h \in \mathrm{Diff}\, P'$, $k \in \mathrm{Diff}\, P$ are close to the identity mappings $id_{P'}$, id_P. Let $G = h^{-1} \circ F$. We have $F \pitchfork j'(P) \Leftrightarrow G \pitchfork j(P)$. If j' is close to j, G will be close to F, and since the set of mappings transverse to $j(P)$ is open, we can assume that $G \pitchfork j(P)$ and consequently that $F \pitchfork j'(P)$.

Now we want to construct $i' : N \to N_1$. Note that F is of the form $F(x,t) = (F_1(x,t), t)$ as a mapping $N \times U \to P \times U$ where U is open in $\mathbb{R}^k$ and G is of the form $G(x,t) = (G_1(x,t), G_2(x,t))$. For each $x \in N$ define $G_x : U \to U$

by $G_x(t) = G_2(x,t)$. If G is sufficiently close to F, G_x will be arbitrarily close to the identity map $id_u : U \to U$ for all $x \in N$. This yields that G_x can be chosen as a diffeomorphism for all x. In particular we will have G_x one to one, surjective, and $[\frac{dG}{dt}2\ (x,t)]$ is non singular for all x and t. Note that $N_1 = G_2^{-1}(o)$, and let $\pi : N \times U \to N$ be the projection. Consider the restriction $\pi : N_1 \to N$. Since G_x is one to one surjective and $[\frac{dG}{dt}2\ (x,t)]$ is non singular, it will follow that $\pi \mid N_1$ is one to one surjective and a local diffeomorphism. Hence $\pi|N_1$ is a diffeomorphism. Now we define i' to be $(\pi|N_1)^{-1}oi$. We have to show that i' is close to i as a mapping $N \to N' = N \times U$. $i' : N \to N \times U$ is by construction of the form $x \to (x,\phi(x))$. We have to show that ϕ is close to the zero mapping. Note that $G_2oi' = 0$. Since N is compact, the mapping $C^\infty(N',U) \to C^\infty(N,U)$ given by $H \to Hoi'$ is continuous ([2] prop. 3.9, chapter II). Let $F_2 : N' \to U$ be defined by $F_2(x,t) = t$. Since F is close to G, F_2 is close to G_2 and $F_2oi' = \phi$ is close to the zero mapping.

Finally $f' = (j')^{-1}oFoi'$ will be close to $f = j^{-1}Foi$ because i', j' is close to i, j and N is compact. Hence all the claims are proved.

References

[1] Gibson, C.G., Wirthmüller, K., du Plessis, A.A., Loijenga, E.J.N.: Topological stability of smooth mappings. Springer lecture notes 552, Berlin 1976.

[2] Golubitsky, M., Guillemin, V.: Stable mappings and their singularities. Graduate texts in math. 14, Springer Verlag, New York 1973.

[3] Mather, J.N.: Infinitesimal stability implies stability. Annals of Math. Vol. 89, p.254-291. 1969.

[4] Mather, J.N.: Finitely determined map germs. Publ. I.H.E.S. 35, p. 127-156. 1969.

[5] Mather, J.N.: Transversality. Advances in Math. 4, p.301-336. 1970.

[6] Mather, J.N.: How to stratify mappings and jetspaces. Singularités d'applications differentiable. Plans sur Bex 1975, Springer lecture notes 535, p.128-176. Berlin 1976.

[7] Mather, J.N.: On Thom-Boardman singularities. Dynamical systems, Academic Press, New York and London 1973, p. 223-248.