

NOTE ON THE PROJECTIVE LIMIT ON SMALL CATEGORIES

by

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In the Bull. of the Amer. Math. Soc.⁷⁴ (1968) p.1129-1132) Oberst formulated a conjecture on the exactness of the projective limit functor on the category of functors on a small category with values in the category of abelian groups.

In this note we give a proof of his theorem.

Some of the lemmas seem to have been proved by Oberst and Isbel by other methods.

Theorem. Let X be a small connected category, \underline{Ab} the category of abelian groups, then the two following conditions are equivalent

(i) For all $F \in \text{ab } \underline{Ab}^X$ $\varprojlim^{(1)} F = 0 \quad \forall i \geq 1.$

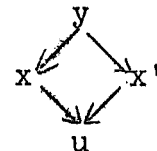
(ii) $\exists y \in \text{ob } X$ such that

① $\forall x \exists \xi \in X(y, x)$

② If $x \searrow x'$ is in X then it can be completed

u

to a commutative diagram



③ $\exists \epsilon \in X(y, y)$ such that $\forall \xi \in X(y, y)$

$\xi \epsilon = \epsilon.$

Proof. Since (ii) implies that

$$\lim_{\leftarrow X} F = H^0(X(y,y), F(y)) = \{\alpha \in F(y) \mid F(\epsilon)(\alpha) = \alpha\}$$

$= \{F(\epsilon)(\beta) \mid \beta \in F(y)\}$, it is trivial to see that (ii) \Rightarrow (i).

To prove that (i) implies (ii), let F be the object of $\underline{\text{Ab}}^X$ defined by $F(x) = \coprod_{\xi \in \text{ob}(X/x)} \mathbb{Z} \xi$ with $\mathbb{Z} \xi = \mathbb{Z} \forall \xi$

Consider the obvious epimorphism

$$\rho: F \rightarrow \mathbb{Z} \quad \text{with } \mathbb{Z} \text{ the constant obj. of } \underline{\text{Ab}}^X.$$

Since $\lim_{\leftarrow X}$ is exact we have that $\rho^*: \lim_{\leftarrow X} \mathbb{Z} \rightarrow \lim_{\leftarrow X} \mathbb{Z} = \mathbb{Z}$ is epi.

Therefore $\exists \alpha \in \lim_{\leftarrow X} F$ with $\phi^*(\alpha) = 1$.

If $\pi_x: \lim_{\leftarrow X} F \rightarrow F(x)$ is the canonical homomorphism, then

$\forall x \in X, \alpha_x = \pi_x(\alpha) \in F(x)$ is non-zero.

Now

$$\alpha_x = \sum_{i,j=1}^{n,m} \alpha_x^j(y_i) \xi_{ij}^x \quad \text{with } \xi_{ij}^x \in X(y_i, x)$$

and

$$\alpha_x^j(y_i) \in \mathbb{Z}, \quad \sum_{i,j} \alpha_x^j(y_i) = 1.$$

For at least one i we must have $\sum_{j=1}^m \alpha_x^j(y_i) \neq 0$ and we may

assume that $\alpha_x^j(y_1) \neq 0$ for $1 \leq j \leq m' \leq m$.

If the diagram

$$\begin{array}{ccc} x & & x' \\ \phi \searrow & & \swarrow \psi \\ & u & \end{array}$$

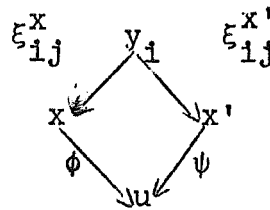
is in X , then $\phi^* \alpha_x = \alpha_u = \psi^* \alpha_{x'}$

and therefore

$$\Sigma \alpha_x^j = \Sigma \alpha_u^j = \Sigma \alpha_{x'}^j, \quad \text{with} \quad \alpha_x^j = \alpha_x^j(y_1).$$

Since X is connected it follows that $\alpha_x^1(y_1) \neq 0$ for all $x \in X$ and furthermore there exist $\xi_{1j}^u \in X(y_1, u)$ with the corresponding $\alpha_u^j(y_1) \neq 0$.

Consequently $\exists \xi_{1j}^x \in X(y_1, x), \xi_{1j}^{x'} \in X(y_1, x')$ with $\phi \circ \xi_{1j}^x = \xi_{1j}^u = \psi \circ \xi_{1j}^{x'}$ i.e. the above diagram can be completed to



We have proved (ii) ②, and at the same time (ii) ①. We are therefore reduced to prove (ii) ③.

Let F_1 be the object of \underline{Ab}^X defined by

$$F_1(x) = \coprod_{\xi \in X(y, x)} \mathbb{Z}\xi$$

with $y = y_1$ (i.e. the y_1 picked above).

By (ii) ① there \exists an epimorphism in \underline{Ab}^X

$$F_1 \rightarrow \mathbb{Z}$$

Since by assumption \mathbb{Z} is projective as an object of \underline{Ab}^X \mathbb{Z} is a direct summand of F_1 , therefore \mathbb{Z} is a direct summand of $F_1(y)$, as an $X(y, y)$ -module. But $F_1(y) = \mathbb{Z}[X(x, y)]$ and it therefore follows that the cohomology of the monoid $M = X(y, y)$ is trivial.

Lemma A. If a monoid M is cohomological trivial then $\exists \epsilon \in M$ such that $\forall \xi \in M \quad \xi\epsilon = \epsilon$.

Proof. Look at the epimorphism $\mathbb{Z}[M] \rightarrow \mathbb{Z}$. Since cohomology is trivial, the corresponding homomorphism

$$H^0(M, \mathbb{Z}(M)) \rightarrow H^0(M, \mathbb{Z}) = \mathbb{Z}$$

is epimorphic. Now

$$H^0(M, \mathbb{Z}(M)) = \left\{ \sum_{i=1}^m \alpha_i \xi_i \mid \alpha_i \in \mathbb{Z}, \xi_i \in M \text{ such that } \forall \xi \in M \right.$$

$$\left. \sum_{i=1}^n \xi_i \xi_i = \xi_j \text{ for some } j_\xi \text{ and } \alpha_i = \alpha_{j_\xi} \right\}$$

It follows that there $\exists \sum_{i=1}^n \alpha_i \xi_i \in \mathbb{Z}(M)$ with $\sum_{i=1}^n \alpha_i = 1$ such

that $\forall \xi \in M, \xi \xi_i = \xi_{\sigma_\xi(i)}$ where σ_ξ is a permutation of $\{1, 2, \dots, n\}$.

The correspondence $\xi \mapsto \sigma_\xi \in S(n)$ gives us a homomorphism

$$\sigma: M \rightarrow S(n)$$

$$\text{since } \xi' \xi \xi_i = \xi' \xi_{\sigma_\xi(i)} = \xi_{\sigma_\xi(\sigma_\xi(i))}.$$

Let $H = \text{im } \sigma$ and let G be the subgroup of $S(n)$ generated by H .

Sublemma B. If $M \xrightarrow{\sigma} H$ is an epimorphic homomorphism of monoids and if M is cohomologically trivial then H is cohomologically trivial.

Proof. This follows from $H^0(M, -) = H^0(H, -)$ in the category of H -modules.

QED.

Sublemma C. If H is cohomologically trivial then G is cohomologically trivial.

Proof. It is well known that $H^1(G, -) \simeq H^1(H, -)$.

QED.

Sublemma D. If a group G is cohomologically trivial, then $G = \{1\}$.

Proof. As above $\exists \sum_{i=1}^n \alpha_i \xi_i \in \mathbb{Z}(G)$ such that

$$\forall \xi \in G, \quad \xi \xi_1 = \xi_{\sigma_\xi(1)} \text{ and } \alpha_1 = \alpha_{\sigma_\xi(1)}, \quad \sum_{i=1}^n \alpha_i = 1.$$

Since $\forall i, j \exists \xi$ with $\sigma_\xi(i) = j$ we have $\alpha_i = \alpha_j$ for all i, j and $G = \{\xi_1, \dots, \xi_n\}$.

$$1 = n \cdot \alpha_1 = \#G \cdot \alpha_1$$

It obviously follows that $\#G = 1$ and $\alpha_1 = 1$, so $G = \{1\}$.

QED.

Combining B, C, D we find that $\sigma_\xi = 1$ for all $\xi \in M$, this of course means that $\forall \xi \in M \quad \xi \xi_1 = \xi_1$.

Put $\epsilon = \xi_1$ for some i , and we have proved A.

QED.

And this ends the proof of Theorem since $A \Rightarrow (ii) \text{ (3)}$.

QED.