NOTE ON THE PROJECTIVE LIMIT ON SMALL CATEGORIES

by

O. A. Laudal

In the Bull. of the Amer. Math. Soc. 74 (1968) p.1129-1132) Oberst formulated a conjecture on the exactness of the projective limit functor on the category of functors on a small category with values in the category of abelian groups.

In this note we give a proof of his theorem.

Some of the lemmas seem to have been proved by Oberst and Isbel by other methods.

<u>Theorem.</u> Let X be a small connected category, <u>Ab</u> the category of abelian groups, then the two following conditions are equivalent

(i) For all $F \in \mathrm{ob}\ \underline{\mathrm{Ab}}^X$ $\quad \varprojlim^{(i)} F = 0 \quad \forall i \geq 1.$

(ii) $\exists\, y \in \mathrm{ob}\, X$ such that

1. $\forall x\, \exists\, \xi \in X(y,x)$

2. If $\begin{tikzcd} x \arrow[dr] & & x' \arrow[dl] \\ & u & \end{tikzcd}$ is in X then it can be completed

   to a commutative diagram $\begin{tikzcd} & y \arrow[dl] \arrow[dr] & \\ x \arrow[dr] & & x' \arrow[dl] \\ & u & \end{tikzcd}$

3. $\exists\, \varepsilon \in X(y,y)$ such that $\forall\, \xi \in X(y,y)$

   $\xi\varepsilon = \varepsilon.$

Proof. Since (ii) implies that

$$\lim_{\substack{\leftarrow \\ X}} F = H^0(X(y,y),F(y)) = \{\alpha\in F(y) \mid F(\varepsilon)(\alpha) = \alpha\}$$

$= \{F(\varepsilon)(\beta) \mid \beta \in F(y)\}$, it is trivial to see that (ii)=> (i).

To prove that (i) implies (ii), let F be the object of $\underline{Ab}^X$ defined by $F(x) = \coprod_{\xi\in ob(X/x)} \mathbb{Z}\xi$ with $\mathbb{Z}\xi = \mathbb{Z}\ \forall\ \xi$

Consider the obvious epimorphism

$$\rho\colon F \to \mathbb{Z} \quad \text{with } \mathbb{Z} \text{ the constant obj. of } \underline{Ab}^X.$$

Since $\lim\limits_{\substack{\leftarrow \\ X}}$ is exact we have that $\rho^*\colon \lim\limits_{\substack{\leftarrow \\ X}} \mathbb{Z} \to \lim\limits_{\substack{\leftarrow \\ X}} \mathbb{Z} = \mathbb{Z}$ is epi.

Therefore $\exists \alpha \in \lim\limits_{\substack{\leftarrow \\ X}} F$ with $\phi^*(\alpha) = 1$.

If $\pi_x\colon \lim\limits_{\substack{\leftarrow \\ X}} F \to F(x)$ is the canonical homomorphism, then

$\forall x \in X$, $\alpha_x = \pi_x(\alpha) \in F(x)$ is non-zero.

Now

$$\alpha_x = \sum_{i,j=1}^{n,m} \alpha_x^j(y_i)\xi_{ij}^x \quad \text{with } \xi_{ij}^x \in X(y_i,x)$$

and

$$\alpha_x^j(y_i) \in \mathbb{Z}, \qquad \sum_{i,j} \alpha_x^j(y_i) = 1.$$

For at least one i we must have $\sum_{j=1}^{m} \alpha_x^j(y_i) \neq 0$ and we may assume that $\alpha_x^j(y_i) \neq 0$ for $1 \leq j \leq m' \leq m$.

If the diagram
$$\begin{tikzcd} x \arrow[dr, "\phi"'] & & x' \arrow[dl, "\psi"] \\ & u & \end{tikzcd}$$
is in X, then $\phi^*\alpha_x = \alpha_u = \psi^*\alpha_{x'}$

and therefore

$$\Sigma\alpha^j_x = \Sigma\alpha^j_u = \Sigma\alpha^j_{x'} \qquad \text{with} \qquad \alpha^j_{\cdot} = \alpha^j_{\cdot}(y_i).$$

Since X is connected it follows that $\alpha^1_x(y_i) \neq 0$ for all $x \in X$ and furthermore there exist $\xi^u_{ij} \in X(y_i,u)$ with the corresponding $\alpha^j_u(y_i) \neq 0$.

Consequently $\exists\, \xi^x_{ij} \in X(y_i x)$, $\xi^{x'}_{ij} \in X(y_i,x')$ with $\phi \circ \xi^x_{ij} = \xi^u_{ij} = \psi \circ \xi^{x'}_{ij}$ i.e. the above diagram can be completed to

$$\begin{tikzcd} & y_i \arrow[dl, "\xi^x_{ij}"'] \arrow[dr, "\xi^{x'}_{ij}"] & \\ x \arrow[dr, "\phi"'] & & x' \arrow[dl, "\psi"] \\ & u & \end{tikzcd}$$

We have proved (ii) 2, and at the same time (ii) 1. We are therefore reduced to prove (ii) 3.

Let F_1 be the object of $\underline{Ab}^X$ defined by

$$F_1(x) = \coprod_{\xi \in X(y,x)} \mathbb{Z}\xi$$

with $y = y_i$ (i.e. the y_i picked above).

By (ii) 1 there $\exists$ an epimorphism in $\underline{Ab}^X$

$$F_1 \to \mathbb{Z}$$

Since by assumption $\mathbb{Z}$ is projective as an aobject of $\underline{Ab}^X$ $\mathbb{Z}$ is a direct summand of F_1, therefore $\mathbb{Z}$ is a direct summand of $F_1(y)$, as an $X(y,y)$-module. But $F_1(y) = \mathbb{Z}[X(x,y)]$ and it therefore follows that the cohomology of the monoid $M = X(y,y)$ is trivial.

<u>Lemma A.</u> If a monoid M is cohomological trivial then $\exists\, \varepsilon \in M$ such that $\forall\, \xi \in M$ $\xi\varepsilon = \varepsilon$.

Proof. Look at the epimorphism $\mathbb{Z}[M] \to \mathbb{Z}$. Since cohomology is trivial, the corresponding homomorphism

$$H^o(M,\mathbb{Z}(M)) \to H^o(M,\mathbb{Z}) = \mathbb{Z}$$

is epimorphic. Now

$$H^o(M,\mathbb{Z}(M)) =' \{ \sum_{i=1}^{m} \alpha_i \xi_i \mid \alpha_i \in \mathbb{Z},\ \xi_i \in M \text{ such that } \forall\, \xi \in M$$

$$\forall_i \quad \xi\xi_i = \xi_j \qquad \text{for some } j_\xi \text{ and } \alpha_i = \alpha_{j\xi}\}$$

It follows that there $\exists$ $\sum_{i=1}^{n} \alpha_i \xi_i \in \mathbb{Z}(M)$ with $\sum_{i=1}^{n} \alpha_i = 1$ such that $\forall\, \xi \in M$, $\xi\xi_i = \xi_{\sigma_\xi(i)}$ where σ_ξ is a permutation of $\{1,2,\ldots,n\}$.

The correspondence $\xi \to \sigma_\xi \in S(n)$ gives us a homomorphism

$$\sigma : M \to S(n)$$

since $\xi'\xi\xi_i = \xi'\xi_{\sigma_\xi(i)} = \xi_{\sigma_{\xi'}(\sigma_\xi(i))}$.

Let $H = \operatorname{im} \sigma$ and let G be the subgroup of $S(n)$ generated by H.

Sublemma B. If $M \xrightarrow{\sigma} H$ is an epimorphic homomorphism of monoids and if M is cohomologically trivial then H is cohomologically trivial.

Proof. This follows from $H^o(M,-) = H^o(H,-)$ in the category of H-modules.

QED.

<u>Sublemma C.</u> If H is cohomologically trivial then G is cohomologically trivial.

<u>Proof.</u> It is well known that $H^i(G,-) \simeq H^i(H,-)$.

QED.

<u>Sublemma D.</u> If a group G is cohomologically trivial, then $G =' \{1\}$.

<u>Proof.</u> As above $\exists \sum_{i=1}^{n} \alpha_i \xi_i \in \mathbb{Z}(G)$ such that $\forall \xi \in G, \quad \xi\xi_i = \xi_{\sigma_\xi(i)}$ and $\alpha_i = \alpha_{\sigma_\xi(i)}, \quad \sum_{i=1}^{n} \alpha_i = 1.$

Since $\forall i,j \ \exists \xi$ with $\sigma_\xi(i) = j$ we have $\alpha_i = \alpha_j$ for all i,j and $G =' \{\xi_1, \dots, \xi_n\}$.

$$1 = n \cdot \alpha_1 = \# G \cdot \alpha_1$$

It obvisouly follows that $\# G = 1$ and $\alpha_1 = 1$, so $G = \{1\}$.

QED.

Combining B,C,D we find that $\sigma_\xi = 1$ for all $\xi \in M$, this of course means that $\forall \xi \in M \quad \xi\xi_i = \xi_i$.

Put $\varepsilon = \xi_i$ for some i, and we have proved A.

QED.

And this ends the proof of <u>Theorem</u> since A => (ii) 3.

QED.