Convergence of spectral sequences

by

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Introduction

In this paper we shall study the convergence of spectral sequences. The problem is roughly the following. If in some nice abelian category there is given an exact couple

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

and if the corresponding spectral sequence is denoted by $\{E^r\}$, when will there exist an integer r_o such that $E^{r_o} \simeq E^{\infty}$?

Our main result is the

<u>Theorem (3.2)</u> Suppose that $D = \{D_p\}$ is graded and suppose i is of degree 1 . Let $\eta_p^{p-k} : D_{p-k} \to D_p$ be the restriction of $i^{(k)}$ to D_{p-k} , then the following conditions are equivalent

(i) there exists an integer r_o such that $E^{r_o} \simeq E^{\infty}$

(ii) for each p and each k we have isomorphisms

$$\operatorname{coker}\{\ker \eta_p^{p-k-r_o} \to \ker \eta_p^{p-k}\} \simeq \operatorname{coker}\{\ker \eta_p^{p-k-k'} \to \ker \eta_p^{p-k}\}$$

$$\ker\{\operatorname{coker} \eta^p_{p+k} \to \operatorname{coker} \eta^p_{p+k+r_o}\} \simeq \ker\{\operatorname{coker} \eta^p_{p+k} \to \operatorname{coker} \eta^p_{p+k+k'}\}$$

for all $k' \geq r_o$.

If one of these conditions holds we shall say that $\{E_r\}$ converges uniformly. Using (3.2) we prove

<u>Corollary(3.7)</u> If $\{E^r\}$ converges uniformly then there exist exact sequences

$$0 \to \ker(H_{p-1} \to H_p) \to E_p^{\infty} \to \operatorname{coker}(H^{p-1} \to H^p) \to 0$$

$$0 \to \varinjlim_p{}_{(1)} D_p \to \varinjlim_p H^p \to \varprojlim_p D_p \to \varinjlim_p D_p \to \varprojlim_p H_p \to \varprojlim_p{}^{(1)} D_p \to 0$$

where $H^p = \ker(\varprojlim_{p'} D_{p'} \to D_p)$

$$H_p = \operatorname{coker}(D_p \to \varinjlim_{p'} D_{p'}) .$$

The above results generalize theorems of Serre [8] and Shih [9]. See also Grothendieck [4], Chap. 0, §13.

If $(F_pC^{\cdot})_{p \leq 0}$ is a complete filtration of a complex $C^{\cdot}$ in the sense of Eilenberg, Moore [3] and if the spectral sequence $\{E^r\}$ associated, converges uniformly then using (3.7) we prove,

$$E_p^{\infty} \simeq \ker(H^{\cdot}_{p-1} \to H^{\cdot}_p)$$

$$H^n(C^{\cdot}) = \varprojlim_p H^n_p$$

where $H^{\cdot}_p$ is graded as a quotient-object of $\varinjlim H^{\cdot}(F_pC^{\cdot}) = H^{\cdot}(C^{\cdot})$.

This generalizes Corollary 6.3 of [3].

The first section contains some results on the functors $\varprojlim$ and $\varinjlim$. In particular we prove a theorem characterizing the projective systems D for which $\varprojlim^{(1)} D = 0$.

The second section is concerned with the relationship between E^{∞} and the filtration $\{H^p\}$ of $\varprojlim D$ and the cofiltration $\{H_p\}$ of $\varinjlim D$.

In the third section we prove the theorem stated above and we deduce some corollaries.

The last section contains some results on morphisms of exact couples.

A first version of this paper was written in the spring of 1966 and some of the results were presented to the International Congress of Mathematicians in Moscow the same year. Since then Eckmann and Hilton have published two papers [1], [2] on spectral sequences ~~associated to projective systems~~ in a more general setting, proving some of our results, such as lemma (2.1). However their goals seem to be somewhat different from ours, and their methods do not involve the study of the higher derived functors of $\varprojlim$ and $\varinjlim$ which is essential for the results of this paper.

§1. Some results on projective and inductive limits.

Let $\underline{c}$ be an abelian category with exact denumerable products and sums. Denote by $\underline{c}_Z$ the category of projective systems in $\underline{c}$ indexed by the integers Z. An object D of $\underline{c}_Z$ is then a sequence of morphisms in $\underline{c}$

$$D: \quad \cdots \to D_{p-1} \xrightarrow{\eta_p^{p-1}} D_p \xrightarrow{\eta_{p+1}^{p}} D_{p+1} \to \cdots .$$

We know, see Roos [7], that under these assumptions, the functors,

$$\varprojlim \text{ and } \varinjlim : \underline{c}_Z \to \underline{c}$$

exist together with their satelites $\varprojlim^{(i)}$ and $\varinjlim_{(i)}$, and it is easy to prove the following properties

(1) $\varprojlim^{(i)} = \varinjlim_{(i)} = 0$ for $i \geq 2$.

(2) if all η_p^{p-1} are epimorphic resp. monomorphic then

$$\varprojlim^{(1)} D = 0 \quad \text{resp.} \quad \varinjlim_{(1)} D = 0 .$$

<u>Definition (1.1).</u> If D is an object of $\underline{c}_Z$ we define the <u>completion</u> $\bar{D}$

resp. the cocompletion $\underline{D}$ of D by

$$\bar{D}_p = \varprojlim_k \operatorname{coker} \eta_p^{\,p-k} \quad \text{resp.} \quad \underline{D}_p = \varinjlim_k \ker \eta^p_{\,p+k}$$

we have natural morphisms

$$D \to \bar{D} \qquad \text{resp.} \quad \underline{D} \to D$$

Let ${}^1D(1)$ resp ${}_1D(1)$ be the kernel resp. the cokernel of this morphism, and define inductively ${}^iD(1)$ resp ${}_iD(1)$ as the kernel resp. cokernel of

$$ {}^{i-1}D(1) \to {}^{i-1}\bar{D}(1) \quad \text{resp.} \quad {}_{i-1}\underline{D}(1) \to {}_{i-1}D(1)$$

In this way we obtain a filtration resp. a cofiltration of D

$$D = {}^0D(1) \hookleftarrow {}^1D(1) \hookleftarrow \cdots \hookleftarrow {}^iD(1) \hookleftarrow {}^{i+1}D(1) \hookleftarrow \cdots$$

$$\text{resp. } D = {}_0D(1) \twoheadrightarrow {}_1D(1) \twoheadrightarrow \cdots \twoheadrightarrow {}_iD(1) \twoheadrightarrow {}_{i+1}D(1) \twoheadrightarrow \cdots$$

Since all definitions and all results in this section, except for (1.8) and (1.10) have obvious duals we shall omit these duals.

The filtration (1) will be called the <u>1-fold canonical filtration of</u> D, and the subobject ${}^1D(2) = \varprojlim_i {}^iD(1)$ will be called the ∞-<u>term</u> of the 1-fold canonical filtration.

Inductively we define the n-fold canonical filtration of D

$$D = {}^0D(n) \hookleftarrow {}^1D(n) \hookleftarrow \cdots \hookleftarrow {}^iD(n) \hookleftarrow {}^{i+1}D(n) \hookleftarrow \cdots$$

as follows: ${}^{i+1}D(n)$ is the ∞-term of the (n-1)-fold canonical filtration of ${}^iD(n)$, and the subobject ${}^1D(n+1) = \varprojlim_i {}^iD(n)$ is called the ∞-term of the n-fold canonical filtration.

<u>Definition (1.2)</u> We shall say that the n-fold canonical filtration is

complete if

$$\text{(i)} \quad 0 \to {}^{k+1}D(1) \to {}^{k}D(1) \to {}^{k}\overline{D}(1) \to 0 \quad \text{is exact for all} \quad k \geq 0$$

$$\text{(ii)} \quad D/{}^{1}D(m+1) = \varprojlim_{k} D/{}^{k}D(m) \quad \text{for all} \quad 1 \leq m \leq n\ .$$

<u>Definition (1.3)</u> We shall say that D is stable (satisfy the Mittag-Leffler condition) if for every p there exists a $r_p \in Z^+$ such that

$$\operatorname{coker} \eta_p^{p-r_p} = \operatorname{coker} \eta_p^{p-k} \quad \text{for all} \quad k \geq r_p\ .$$

We shall call the number r_p the <u>height</u> of D at p, and we shall say that D is stable of uniform height r if we can choose all r_p in the above definition equal to r.

The following lemma is trivial.

<u>Lemma (1.4)</u> If for some n and some k, ${}^{k}D(n)$ is stable then ${}^{k+1}D(n)$ is epimorphic and ${}^{\ell}D(n) = {}^{k+1}D(n)$ for all $\ell \geq k+1$. If, on the other hand for some n and k, ${}^{\ell}D(n) = {}^{k}D(n)$ for all $\ell \geq k$ then ${}^{k}D(n)$ is epimorphic.

<u>Lemma (1.5)</u> Suppose D is stable then

$$\varprojlim{}^{(1)} D = 0\ .$$

<u>Proof.</u> Consider the projective system H on $Z \times Z$ defined by $H_{m,n} = \operatorname{im} \eta_{\max(m,n)}^{\min(m,n)}$.

H restricted to the diagonal Δ in $Z \times Z$ is isomorphic to D and H restricted to $\Delta_r = \{(p, p-r_p) \mid p \in Z\}$ is epimorphic. As both Δ and Δ_r are cofinal in $Z \times Z$ the result follows from (ii) above.

QED.

Lemma (1.6) Let D be an object of $\underline{c}_Z$, then

$$\varprojlim{}^{(i)}\, \overline{D} = 0 \text{ for } i \geq 0 .$$

Proof. Put $F_{m,n} = D_{\max(m,n)}/\mathrm{im}\,\eta^{\min(m,n)}_{\max(m,n)}$. Then F is a projective system defined on the ordered set $Z \times Z$. Since F restricted to the diagonal Δ is zero it follows that

$$\varprojlim_{Z\times Z}{}^{(i)} F = 0 \quad \text{for all } i \geq 0 .$$

Now $\varprojlim\limits_{Z\times Z}{}^{(i)} F$ is the abutment of the spectral sequence given by:

$$E^2_{p,q} = \varprojlim_m{}^{(p)} \varprojlim_{n \leq m}{}^{(q)} \; D_m/\mathrm{im}\,\eta^n_m$$

see [6] or [7], thus giving us isomorphisms

$$\varprojlim_m \varprojlim_{n \leq m} D_m/\mathrm{im}\,\eta^n_m \simeq \varprojlim_{Z\times Z} F$$

$$\varprojlim_m{}^{(1)} \varprojlim_{n \leq m} D_m/\mathrm{im}\,\eta^n_m \simeq \varprojlim_{Z\times Z}{}^{(1)} F$$

QED.

Theorem (1.7) Let D be an object of $\underline{c}_Z$ then the following statements are equivalent

(i) $\varprojlim{}^{(1)} D = 0$

(ii) For all $n \geq 1$ the n-fold canonical filtration of D is complete and

$$\varprojlim{}^{(1)}\, 1_D(n+1) = 0 .$$

Proof. We shall prove that (i) is equivalent to (ii) with $n = 1$, leaving the more general statement as an easy exercise.

Suppose $\varprojlim^{(1)} D = 0$, then applying the functor $\varprojlim$ to the two exact sequences

$$0 \to \ker \eta^n_m \to D_n \to \operatorname{im} \eta^n_m \to 0$$
$$0 \to \operatorname{im} \eta^m_m \to D_m \to \operatorname{coker} \eta^n_m \to 0$$

we easily deduce an exact sequence

$$0 \to {}^1D(1) \to D \to \overline{D} \to 0 .$$

Using Lemma (1.6) we find $\varprojlim^{(k)} D \simeq \varprojlim^{(k)} {}^1D(1)$ for $k \geq 0$ so we may continue, proving that for all i the sequence

$$0 \to {}^{i+1}D(1) \to {}^iD(1) \to {}^i\overline{D(1)} \to 0$$

is exact and

(3) $$\varprojlim^{(k)} {}^iD(1) \simeq \varprojlim^{(k)} D \quad \text{for all } k \geq 0 .$$

Now, let H be the projective system on $Z \times Z^{+\circ}$ defined by

$$H_{n,i} = {}^iD_n(1)$$

we know that $\varprojlim_{Z \times Z^\circ}{}^{(i)} H$ is the abutment of two spectral sequences given by

$$'E^2_{p,q} = \varprojlim_i{}^{(p)} \varprojlim_n{}^{(q)} {}^iD_n(1)$$
$$''E^2_{p,q} = \varprojlim_n{}^{(p)} \varprojlim_i{}^{(q)} {}^iD_n(1) .$$

Using (3) we find $\varprojlim {}^1D(2) \simeq \varprojlim D$ and

$$'E^2_{p,q} = 0 \quad \text{for} \quad p \neq 0\ ,\ q \neq 0$$

thus

$$\varprojlim_{Z^+ \times Z^+}{}^{(1)} H = 0$$

From this it follows that

$$''E^2_{1,0} = ''E^2_{0,1} = 0$$

i.e.

$$(4) \qquad \begin{gathered} \varprojlim{}^{(1)}\ {}^{1}D(2) = \varprojlim_{n}{}^{(1)}\ \varprojlim_{i}{}^{i}D_n(1) = 0 \\ \varprojlim_{n}\ \varprojlim_{i}{}^{(1)}\ {}^{i}D_n = 0 \end{gathered}$$

We are going to prove that for all $n \leq m$ the morphism

$$(5) \qquad \varprojlim_{i}{}^{(1)}\ {}^{i}D_n(1) \to \varprojlim_{i}{}^{(1)}\ {}^{i}D_m(1)$$

is an epimorphism. Together with (4), (5) implies

$$\varprojlim_{i}{}^{(1)}\ {}^{i}D_n = 0 \quad \text{for every} \quad n \in Z$$

and this gives us the isomorphism $D_n/{}^{1}D_n(2) \simeq \varprojlim_{i} D_n/{}^{1}D_n(1)$.

Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & {}^{\infty}D_n \arrow[r] & {}^{j}D_n \arrow[r] \arrow[d] & \varprojlim_{k \geq j+2} {}^{j}D_n/{}^{k}D_n \arrow[r] \arrow[d, "\psi"] & \varprojlim_{k}{}^{(1)}\ {}^{k}D_n \arrow[r] \arrow[d, "\phi"] & 0 \\
0 \arrow[r] & {}^{\infty}D_m \arrow[r] & {}^{j}D_m \arrow[r] & \varprojlim_{k \geq j+2} {}^{j}D_m/{}^{k}D_m \arrow[r, "\rho"] & \varprojlim_{k}{}^{(1)}\ {}^{k}D_m \arrow[r] & 0 \\
0 \arrow[r] & {}^{\infty}D_m \arrow[r] \arrow[u, equal] & {}^{j+1}D_m \arrow[r] \arrow[u] & \varprojlim_{k \geq j+2} {}^{j+1}D_m/{}^{k}D_m \arrow[r] \arrow[u, "\wr"'] & \varprojlim_{k}{}^{(1)}\ {}^{k}D_m \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

in which the horizontal sequences are exact. We have to prove that ϕ is epimorphic. But by the commutativity of the lower right square, we know that

$$\operatorname{im} \nu \to \varprojlim{}^{(1)}\, {}^{k}D_m$$

is epimorphic, and since $\operatorname{im} \nu \subseteq \operatorname{im} \psi$ we find that $\rho \circ \psi$ is epimorphic. By the commutativity of the upper right square this proves that ϕ is epimorphic.

We have therefore proved (i) → (ii).

To prove the converse part of the theorem, we first note that (ii) together with Lemma (1.6) proves that

$$\text{(6)} \qquad \varprojlim{}^{(\cdot)}\, D \simeq \varprojlim{}^{(\cdot)}\, {}^{i}D(1) \quad \text{for every} \quad i \in Z^{+}$$

$$\varprojlim_{i}{}^{(1)}\, {}^{i}D_n(1) = 0 \quad \text{for all} \quad n \in Z .$$

Considering the projective system H above, using the spectral sequence $'E$ and the isomorphism (6) we find

$$\varprojlim_{Z^{+}\times Z^{+}}{}^{(1)}\, H \simeq \varprojlim_{Z^{+}}{}^{(1)}\, D$$

But the spectral sequence $''E$ degenerates, therefore $\varprojlim_{Z^{+}\times Z^{+}}{}^{(1)}\, H = 0$.

QED.

<u>Lemma (1.8)</u> Let A be a noetherian ring of finite dimension and M a finitely generated A-module. Suppose M is filtered by submodules $\{M_i\}$, $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_i \supseteq M_{i+1} \supseteq \cdots$ then there exists an integer i such that

$$\dim V(M_i/M_\ell) < \dim V(M_i) \quad \text{for all} \quad \ell \geq i \, .$$

Proof. Recall that $V(M) = \{\mathfrak{p} \in \operatorname{spec}(A) \mid M_{\mathfrak{p}} \neq 0\}$. The filtration of M induces a filtration of ordered sets

$$V(M) \supseteq V(M_1) \supseteq \cdots \supseteq V(M_i) \supseteq V(M_{i+1}) \supseteq \cdots .$$

Since each of the ordered sets $V(M_i)$ has a finite number of minimal elements and since A is noetherian there must exist an i_1 such that

$$V(M_{i_1}) = V(M_k) \quad \text{for all} \quad k \geq i_1 \, .$$

Let $\{\mathfrak{p}_s\}_{s=1}^{m}$ be the minimal elements of $V(M_{i_1})$ then $(M_{i_1})_{\mathfrak{p}_s}$ has finite length. Therefore there exists an $i \geq i_1$ such that

$$(M_i)_{\mathfrak{p}_s} \simeq (M_\ell)_{\mathfrak{p}_s} \quad \text{for all} \quad \ell \geq i \text{ and all } s = 1, \cdots, m \, .$$

This means that $\mathfrak{p}_s \notin V(M_i/M_\ell)$ for all $s = 1, \cdots, m$ and all $\ell \geq i$. Thus

$$\dim V(M_i/M_\ell) < \dim V(M_i) \quad \text{for all} \quad \ell \geq i \, .$$

QED.

Definition (1.9) We shall say that the n-fold canonical filtration is trivial if ${}^{i}D(n) = {}^{1}D(n)$ for all $i \geq 1$.

Theorem (1.10) Let A be a noetherian ring of finite Krull dimension n. Let D be a projective system of finitely generated A-modules, then the following statements are equivalent

(i) $\varprojlim^{(1)} D = 0$

(ii) the $(n+1)$-fold canonical filtration is trivial and complete.

Proof. If the $(n+1)$-fold canonical filtration is trivial, then by Lemma (1.4)

${}^{1}D(n+1)$ will be epimorphic and therefore $\varprojlim^{(1)} {}^{1}D(n+1) = 0$. Suppose the $(n+1)$-fold canonical filtration is nontrivial, then using Lemma (1.8) we find $\dim V(D_m) > n$ for some m, which contradicts the assumption that $\dim A = n$. The rest follows from Theorem (1.7).

QED.

If $\{F_pC^\cdot\}$ is a complete filtration of a complex $C^\cdot$ and if $D_p = H_n(F_pC^\cdot)$ we shall see in §3 and §4 that ${}^{1}D(1) \neq 0$ if and only if dF_pC^{n-1} is not closed in the topology of C^n generated by the filtration $\{F_pC^n\}$. Moreover we will have $\varprojlim^{(i)} D = 0$ for $i \geq 0$. Thus if $\mathfrak{a}$ is an ideal of a complete $\mathfrak{q}$-adic ring A and if the completion $\hat{\mathfrak{a}}$ of $\mathfrak{a}$ in the $\mathfrak{q}$-adic topology of $\mathfrak{a}$ has a nonclosed image in A then the projective system

$$D_k = \mathfrak{q}^k/\mathrm{im}(\widehat{\mathfrak{q}^k\mathfrak{a}} \to \mathfrak{q}^k)$$

will have the properties:

$$\varprojlim^{(i)} D = 0 \quad \text{for} \quad i \geq 0 \quad \text{and} \quad {}^{1}D(1) \neq 0 .$$

An example of this sort is the ideal $\mathfrak{a}$ generated by the elements $x_1 - x_i^i$ of the formal power series ring $k[[x_1, \cdots, x_i, \cdots]]$ in a countable number of variables over a field.

2. Spectral sequences.

Let D be an object of $\underset{\sim}{c}_Z$

$$D : \cdots \to D_{p-1} \xrightarrow{i_{p-1}} D_p \xrightarrow{i_p} D_{p+1} \to \cdots .$$

For each $p \in Z$ we can find one, but in general lots of, objects E_p and morphisms j_p and k_p in $\underline{c}$ such that the diagram

(1)
$$\begin{array}{ccc} D_{p-1} & \xrightarrow{i_{p-1}} & D_p \\ & {}_{k_p}\nwarrow \quad \swarrow_{j_p} & \\ & E_p & \end{array}$$

is an exact couple. It suffices, in fact, to find an object E_p and morphisms j'_p and k'_p such that the following sequence becomes exact

$$0 \to \text{coker } i_{p-1} \xrightarrow{j'_p} E_p \xrightarrow{k'_p} \ker i_{p-1} \to 0$$

This is obviously the same as picking an element E_p from

$$\text{Ext}^1 (\ker i_{p-1} , \text{coker } i_{p-1}) .$$

Thus the set

$$S(D) = \prod_{p \in Z} \text{Ext}^1 (\ker i_{p-1} , \text{coker } i_{p-1})$$

is in one-to-one correspondence with the set of all, up to isomorphisms, graded exact couples

$$\begin{array}{ccc} D & \xrightarrow{i} & D \\ & {}_{k}\nwarrow \quad \swarrow_{j} & \\ & E & \end{array}$$

with

$$D = \coprod_{p \in Z} D_p , \ E = \coprod_{p \in Z} E_p$$

where i, j and k have degrees $+1$, 0 and -1 respectively.

Given an object D in $\underline{c}_Z$ and an exact couple $E \in S(D)$, we would like to ~~calculate~~ compute

$$\varprojlim^{(i)} D \quad \text{and} \quad \varinjlim_{(i)} D$$

using only the spectral sequence

$$(E^r)_{r \in Z^+} .$$

Let us first introduce some notations. If

$$D : \cdots \to D_{p-1} \xrightarrow{i_{p-1}} D_p \xrightarrow{i_p} D_{p+1} \to \cdots$$

is an object of $\underline{c}_Z$ and if $p' \leq p$ let

$$\eta^{p'}_p : D_{p'} \to D_p$$

be the obvious composition of the i_p 's .

We put

$$\left.\begin{array}{l} {}^iH(D) = \varprojlim^{(i)} D \\ {}_iH(D) = \varinjlim_{(i)} D \end{array}\right. \quad \text{for } i = 0, 1 .$$

We define a canonical <u>filtration</u> $(H^p(D))_{p \in Z}$ of ${}^oH(D)$ and a canonical <u>cofiltration</u> $(H_p(D))_{p \in Z}$ of ${}_oH(D)$, by

$$H^p(D) = \ker \pi^p$$

$$H_p()D = \operatorname{coker} \mu_p$$

where $\pi^p : \varprojlim D \to D_p$ and $\mu_p : D_p \to \varinjlim D$ are the canonical morphisms.

Now for $p' < p$, consider the diagram of exact sequences

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ 0 & \to & \ker \eta^{p'}_{p} & \to & D_{p'} & \to & \operatorname{im} \eta^{p'}_{p} & \to & 0 \\ & & & & & & \downarrow & & \\ & & & & & & D_p & & \\ & & & & & & \downarrow & & \\ & & & & & & \operatorname{coker} \eta^{p'}_{p} & & \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

Applying the functors $\varprojlim\limits_{p'}$ resp. $\varinjlim\limits_{p}$ we easily deduce:

$$H^p(D) \approx \varprojlim_{p'} \ker \eta^{p'}_{p} \qquad\qquad H_p(D) \approx \varinjlim_{p'} \operatorname{coker} \eta^{p}_{p'}$$

and we put:

$$^1H^p(D) = \varprojlim_{p'}{}^{(1)} \ker \eta^{p'}_{p} \qquad\qquad {}_1H_p(D) = \varinjlim_{p'}{}_{(1)} \operatorname{coker} \eta^{p}_{p'}$$

In the $r^{th.}$ derived of the exact couple (1)

$$\begin{array}{ccc} D^r & \xrightarrow{i^r} & D^r \\ & {}_{k^r}\nwarrow \quad \swarrow_{j^r} & \\ & E^r & \end{array}$$

we shall consider D^r as a subobject of D and E^r as a subquotient of the graded object E. Thus:

$$D^r{}_p = \operatorname{im} \eta_p^{p-r-1}$$

Using the same methods as in the proof of (1.5) we easily prove that:

$$^iH(D^r) = {}^iH(D) \qquad\qquad {}_iH(D^r) = {}_iH(D)$$

$$^iH^p(D^r) = {}^iH^p(D) \qquad\qquad {}_iH_p(D^r) = {}_iH_p(D)$$

for all $p \in Z$ and $i = 0, 1$.

Now, look at the exact sequence deduced from the r^{th} derived exact couple,

$$(2) \qquad \to D^r_{p+r-2} \xrightarrow{i^r} D^r_{p+r-1} \xrightarrow{j^r} E^r_p \xrightarrow{k^r} D^r_{p-1} \xrightarrow{i^r} D^r_p \to \quad .$$

<u>Lemma (2.1)</u> For every $k \geq 0$ the sequence (2) induces an exact sequence

$$0 \to \operatorname{coker} i^r_{p+r-2} \to Z^r_{p,k} \to \ker i^{r+k}_{p-1} \to 0$$

where $Z^r_{p,k}$ is the sup. of the subobjects of E^r_p for which E^{r+k}_p is a quotient (see [5]).

<u>Proof.</u> As coker $i^r_{p+r-2} = \ker k^r_p$ the inclusion

$$\operatorname{coker} i^r_{p+r-2} \subseteq Z^r_{p,k} \quad \text{for all} \quad k \geq 0$$

is evident.

Now look at the commutative diagram:

$$\begin{tikzcd}
Z^r_{p,k-1} \arrow[r, phantom, "\subseteq"] \arrow[d, "\phi"', "\text{surj.}"] & E^r_p \arrow[r, "k^r_p"] & D^r_{p-1} \arrow[d, phantom, sloped, "\supseteq"] & \\
Z^{r+k-1}_{p,1} \arrow[r, phantom, "\subseteq"] \arrow[d, "\text{surj.}"] & E^{r+k-1}_p \arrow[r, "k^{r+k-1}_p"] & D^{r+k-1}_{p-1} & \\
E^{r+k}_p \arrow[rr, "k^{r+k}_p"] & & D^{r+k}_{p-1} \arrow[u] \arrow[r, "i^{r+k}_{p-1}"] & D^{r+k}_p
\end{tikzcd}$$

Taking into account the definition of $Z^r_{p,k} = \phi^{-1}(Z^{r+k-1}_{p,1})$ it becomes fairly evident that k^r_p maps $Z^r_{p,k}$ onto $\ker i^{r+k}_{p-1}$.

QED.

Now apply the functor $\varprojlim_{k \in Z}$ to the exact sequence of (2.1).

Since the projective system coker i^r_{p+r-2} is constant with respect to

$k \in Z$ we obtain the exact sequence:

(3) $$0 \to \operatorname{coker} i^r_{p+r-2} \to \varprojlim_{k \in Z} Z^r_{p,k} \to \varprojlim_{k \in Z} \ker i^{r+k}_{p-1} \to 0$$

and the isomorphism

(4) $$\varprojlim_{k \in Z}{}^{(1)} Z^r_{p,k} \simeq \varprojlim_{k \in Z}{}^{(1)} \ker i^{r+k}_{p-1}$$
$$\simeq \varprojlim_{k \in Z}{}^{(1)} \ker i^{k}_{p-1}$$

In particular we find that the projective systems indexed by $r \in Z$, $\varprojlim_{k \in Z} \ker i^{r+k}_{p-1} = \varprojlim_{k \in Z} \ker i^{k}_{p-1}$ and $\varprojlim_{k \in Z}{}^{(1)} Z^r_{p,k}$ are constant. Remembering that in the notations of [5]:

$$\varprojlim_{k \in Z} Z^r_{p,k} = \bigcap_{k \in Z} Z^r_{p,k} = \overline{E}^r_p$$

and, by definition,

$$E^{\infty}_p = \varinjlim_{r \in Z} \overline{E}^r_p = \varinjlim_{r \in Z} \varprojlim_{k \in Z} Z^r_{p,k} .$$

Since $\varprojlim_{k \in Z}{}^{(1)} Z^r_{p,k}$ is constant with respect to r we may define:

(5) $${}^1E_p = \varprojlim_{k \in Z}{}^{(1)} Z^r_{p,k}$$

Then using the functor $\varinjlim_{r \in Z}$ on the sequence (3) we get an exact sequence:

(6) $$0 \to \varinjlim_{r \in Z} \operatorname{coker} i^r_{p+r-2} \to E^{\infty}_p \to \varprojlim_{k \in Z} \ker i^{k}_{p-1} \to 0$$

and isomorphisms:

$$(7) \qquad {}^{1}E_p^{\infty} \simeq \varprojlim_{k \in Z}{}^{(1)} \ker i_{p-1}^{k} \quad \text{and} \quad {}_{1}E_p^{\infty} \simeq \varinjlim_{r \in Z^{o}}{}^{(1)} \operatorname{coker} i_{p+r-2}^{r}$$

where in analogy with the definition above we have put:

$$(8) \qquad {}_{1}E_p^{\infty} = \varinjlim_{r \in Z^{o}}{}^{(1)} \bar{E}_p^{r} .$$

Now, look at the commutative diagrams of exact sequences

$$\begin{array}{ccccccccc}
0 & \to & \ker \eta_{p-1}^{p-k} & \to & D_{p-k} & \to & D_{p-1}^{k} & \to & 0 \\
 & & \text{inj.} \downarrow & & \| & & \downarrow i_{p-1}^{k} & & \\
0 & \to & \ker \eta_{p}^{p-k} & \to & D_{p-k} & \to & D_{p}^{k} & &
\end{array}$$

$$\begin{array}{ccccccccc}
 & & D_{p+r-2}^{r} & \to & D_{p+r-1} & \to & \operatorname{coker} \eta_{p+r-1}^{p-1} & \to & 0 \\
 & & i_{p+r-2}^{r} \downarrow & & \| & & \downarrow \text{surj.} & & \\
0 & \to & D_{p+r-1}^{r} & \to & D_{p+r-1} & \to & \operatorname{coker} \eta_{p+r-1}^{p} & \to & 0 .
\end{array}$$

Using the snake lemma we get exact sequences:

$$(10) \qquad \begin{array}{l}
0 \to \ker \eta_{p-1}^{p-k} \to \ker \eta_{p}^{p-k} \to \ker i_{p-1}^{k} \to 0 \\
0 \to \operatorname{coker} i_{p+r-2}^{r} \to \operatorname{coker} \eta_{p+r-1}^{p-1} \to \operatorname{coker} \eta_{p+r-1}^{p} \to 0
\end{array}$$

Applying the functors $\varprojlim_{k \in Z}$ resp. $\varinjlim_{r \in Z}$ on these sequences we are left with the exact sequences:

$$(10) \qquad \begin{array}{l}
0 \to H^{p-1} \to H^{p} \to \varprojlim_{k} \ker i_{p-1}^{k} \to {}^{1}H^{p-1} \to {}^{1}H^{p} \to {}^{1}E_p \to 0 \\
0 \to {}_{1}E_p \to {}_{1}H_{p-1} \to {}_{1}H_p \to \varinjlim_{r} \operatorname{coker} i_{p+r-2}^{r} \to H_{p-1} \to H_p \to 0
\end{array}$$

Together (6) and (10) give us,

Theorem (2.2) For any $E \in S(D)$ we have the following diagram of exact sequences:

$$\begin{tikzcd}[column sep=small]
 & & & & 0 \arrow[d] & & & & \\
0 \arrow[r] & {}_1E_p^{\infty} \arrow[r] & {}_1H_{p-1} \arrow[r] & {}_1H_p \arrow[r] & \times \arrow[r] \arrow[d] & H_{p-1} \arrow[r] & H_p \arrow[r] & 0 & \\
 & & & & E_p^{\infty} \arrow[d] & & & & \\
 & 0 \arrow[r] & H^{p-1} \arrow[r] & H^p \arrow[r] & \times \arrow[r] \arrow[d] & {}^1H^{p-1} \arrow[r] & {}^1H^p \arrow[r] & {}^1E_p^{\infty} \arrow[r] & 0 \\
 & & & & 0 & & & &
\end{tikzcd}$$

3. Convergence of spectral sequences.

The following theorems are the main results in this paper.

Theorem (3.1) Suppose $E \in S(D)$, then the following conditions are equivalent

(i) For every $p \in Z$ there exists a $r_p \geq 1$ such that

$$E_p^{r_p} \simeq E_p^{\infty}$$

(ii) For every $p \in Z$ the projective system

$$\{\ker \eta_p^{p-k}\}_{k \in Z^+} \tag{1}$$

is stable, and the projective system

$$\{\operatorname{coker} \eta_{p+k}^{p}\}_{k \in Z^+} \tag{2}$$

is costable.

If one of these conditions is satisfied we shall say that the spectral sequence $\{E^r\}$ converge.

Proof. Consider the exact sequences (see §.2).

(3) $$0 \to \operatorname{coker} i^r_{p+r-2} \to Z^r_{p,k} \to \ker i^{r+k}_{p-1} \to 0$$

(4) $$0 \to \ker \eta^{p-k-r}_{p-1} \to \ker \eta^{p-k-r}_{p} \to \ker i^{r+k}_{p-1} \to 0$$

(5) $$0 \to \operatorname{coker} i^r_{p+r-2} \to \operatorname{coker} \eta^{p-1}_{p+r-1} \to \operatorname{coker} \eta^{p}_{p+r-1} \to 0$$

If (1) is stable we must have that

(6) $$\{\ker i^{r+k}_{p-1}\}_{k \in Z^+}$$

is stable, but being monomorphic it has to be constant for big k 's .

A dual argument shows that if (2) is costable, then

(7) $$\{\operatorname{coker} i^r_{p+r-2}\}_{r \in Z^+}$$

is constant for big r 's .

As (3) is exact we have proved that (ii) imply that the projective system

(8) $$\{Z^r_{p,k}\}_{r,k \in Z^+}$$

is constant for big r and k 's .

This means that there exists $r_o, k_o \in Z^+$ such that

$$E^{\infty}_p \simeq Z^{r_o}_{p,k_o} \simeq E^{r_o+k_o}_p$$

thus (ii) ==> (i).

To prove (i) $\Rightarrow$ (ii) we start by observing that (i) is, in fact, equivalent to (8) being constant for $r \geq r_p$ and $k \geq 0$. So suppose (8) is constant, for $r \geq r_p$ and $k \geq 0$, then using the exactness of (3) we find that (6) and (7) are constant for $k \geq r_p$ resp. $r \geq r_p$. Now suppose we are given a $k \geq 0$ and let us choose a $k' \geq k$ such that

(9) $\ker i^s_{p-j} = \ker i^{k'}_{p-j}$ for all $s \geq k'$ and $1 \leq j \leq k$.

(We may put $k' = \max\{r_p, r_{p-1}, \cdots, r_{p-k+1}\}$.)

For each $1 \leq j \leq k$ consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker \eta^{p-s}_{p-j} \arrow[r] \arrow[d] & \ker \eta^{p-s}_{p-j+1} \arrow[r] \arrow[d] & \ker i^s_{p-j} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker \eta^{p-k'}_{p-j} \arrow[r] \arrow[d] & \ker \eta^{p-k'}_{p-j+1} \arrow[r] \arrow[d] & \ker i^{k'}_{p-j} \arrow[r] \arrow[d, "\text{mono}"] & 0 \\
0 \arrow[r] & \ker \eta^{p-k}_{p-j} \arrow[r] & \ker \eta^{p-k}_{p-j+1} \arrow[r] & \ker i^k_{p-j} \arrow[r] & 0
\end{tikzcd}$$

in which each horizontal sequence is exact.

Using the snake lemma we get a diagram

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \operatorname{coker}\{\ker\eta^{p-s}_{p-j} \to \ker\eta^{p-k}_{p-j}\} \arrow[r] \arrow[d, "\phi^s_j"'] & \operatorname{coker}\{\ker\eta^{p-s}_{p-j+1} \to \ker\eta^{p-k}_{p-j+1}\} \arrow[r] \arrow[d, "\phi^s_{j-1}"'] & \operatorname{coker}\{\ker i^s_{p-j} \to \ker i^k_{p-j}\} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{coker}\{\ker\eta^{p-k'}_{p-j} \to \ker\eta^{p-k}_{p-j}\} \arrow[r] & \operatorname{coker}\{\ker\eta^{p-k'}_{p-j+1} \to \ker\eta^{p-k}_{p-j+1}\} \arrow[r] & \operatorname{coker}\{\ker i^{k'}_{p-j} \to \ker i^k_{p-j}\} \arrow[r] & 0
\end{tikzcd}$$

in which the sequences are exact.

Now for all $s \geq k'$ ϕ^s_k is an isomorphism, both sides being zero, thus ϕ^s_{k-1} is an isomorphism for all $s \geq k'$. Continuing we readily find that ϕ^s_0 is an isomorphism for all $s \geq k'$, thus proving that (1) is stable.

A dual argument may be used to prove that (2) is costable, thus finishing the proof.

QED.

<u>Theorem (3.2)</u> Suppose $E \in S(D)$ then the following conditions are

equivalent

(i) There exists a $r \geq 1$ such that

$$E^r \simeq E^\infty$$

(ii) There exists a $r \geq 1$ such that for every $p \in Z$ the projective system

$$\{\ker \eta_p^{p-k}\}_{k \in Z^+}$$

is stable of uniform height r, and the projective system

$$\{\operatorname{coker} \eta_{p+k}^{p}\}_{k \in Z^+}$$

is costable of uniform depth r.

If one of these conditions is satisfied we shall say that the spectral sequence $\{E^r\}$ converges uniformly.

Proof. In the proof of (3.1) we may put $r_p = r$ and k' can be chosen equal to r. This proves the theorem.

QED.

Proposition (3.3) Suppose that $\{E^r\}$ converges uniformly, then

$\underline{D}$ is stable

and

$\overline{D}$ is costable.

Proof. As for each $s \in Z^+$ the projective system

$$\{\ker \eta_{p+s}^{p-k}\}_{k \in Z^+}$$

is stable of uniform height r we have for every s and k and every $t \geq r$ an isomorphism

$$\operatorname{coker}\{\ker \eta^{p-k-r}_{p+s} \to \ker \eta^{p-k}_{p+s}\} \simeq \operatorname{coker}\{\ker \eta^{p-k-t}_{p+s} \to \ker \eta^{p-k}_{p+s}\} \quad .$$

Now $\varinjlim$ and coker commute, thus

$$\operatorname{coker}\{\varinjlim_s \ker \eta^{p-k-r}_{p+s} \to \varinjlim_s \ker \eta^{p-k}_{p+s}\} \simeq \operatorname{coker}\{\varinjlim_s \ker \eta^{p-k-t}_{p+s} \to \varinjlim_s \ker \eta^{p-k}_{p+s}\}$$

and by definition of $\underline{D}$ this is the same as

$$\operatorname{coker}\{\underline{D}_{p-k-r} \to \underline{D}_{p-k}\} \simeq \operatorname{coker}\{\underline{D}_{p-k-t} \to \underline{D}_{p-k}\}$$

but this means that $\underline{D}$ is stable.

A dual argument shows that $\bar{D}$ is costable.

QED.

<u>Corollary (3.4)</u> Suppose that $\{E^r\}$ converges, then

(i) For all $p \in Z$

$$^1H^p = {}_1H_p = 0$$

(ii) For every $p \in Z$ we have an exact sequence

$$0 \to \ker\{H_{p-1} \to H_p\} \to E_p^\infty \to \operatorname{coker}\{H^{p-1} \to H^p\} \to 0$$

(iii) For every $p \in Z$ there are exact sequences

$$0 \to \varinjlim{}_{(1)}\ {}^1D(1) \to \varinjlim_p H^p \to {}^oH \to \varinjlim\ {}^1D(1) \to 0$$

$$0 \to \varprojlim\ {}_1D(1) \to {}_oH \to \varprojlim H_p \to \varprojlim{}^{(1)}\ {}_1D(1) \to 0$$

(iv) For every $p \in Z$ there are exact sequences

$$0 \to {}^1D_p(1) \to D_p \to \bar{D}_p \to {}^1H \to 0$$

$$0 \to {}_1H \to \underline{D}_p \to D_p \to {}_1D_p(1) \to 0$$

(v) $^1D(1)$ is epimorphic and $_1D(1)$ is monomorphic.

Proof. By (3.1) we know that $\{\ker \eta_p^{p-k}\}_{k \in Z^+}$ is stable, thus by (1.5):

$$^1H^p \simeq \varprojlim_k{}^{(1)} \ker \eta_p^{p-k} = 0 .$$

Dually, we find $_1H_p = 0$. Together this gives us (i), and (ii) follows immediately from (2.2) and (i).

Now using (i) and the exact sequences

$$(10) \qquad \begin{array}{l} 0 \to \ker \eta_p^s \to D_s \to \operatorname{im} \eta_p^s \to 0 \\ 0 \to \operatorname{im} \eta_s^p \to D_s \to \operatorname{coker} \eta_s^p \to 0 \end{array}$$

we get the exact sequences

$$(11) \qquad \begin{array}{l} 0 \to H^p \to {}^oH \to {}^1D_p \to 0 \\ 0 \to {}_1D_p \to {}_oH \to H_p \to 0 \end{array}$$

and the isomorphisms

$$(12) \qquad {}^1H \simeq \varprojlim_s{}^{(1)} \operatorname{im} \eta_p^s \,, \quad {}_1H \simeq \varinjlim_s{}_{(1)} \operatorname{im} \eta_s^p \ .$$

From the exactness of the sequences (11) we deduce that ${}^1D$ is epimorphic and ${}_1D$ is monomorphic. Applying respectively $\varinjlim_p$ and $\varprojlim_p$ to the same sequences (11) we get (iii), and applying $\varinjlim_p$ and $\varprojlim_p$ to the exact sequences of (10), using (12), we finally deduce (iv).

QED.

Corollary (3.5) Suppose $\{E_r\}$ converges and suppose further that

$$_oH = {}_1H = 0$$

then for each $p \in Z$ there is an exact sequence

(i) $$0 \to H^{p-1} \to H^{p} \to E_p^{\infty} \to 0 .$$

Moreover we have an exact sequence

(ii) $$0 \to \varinjlim H^{p} \to {}^{0}H \to \varinjlim_{(1)} \bar{D} \to 0$$

and the isomorphism

(iii) $${}^{1}H \simeq \varinjlim \bar{D} .$$

Proof. Consider the exact sequence of (3.4)

$$0 \to {}^{1}D(1) \to D \to \bar{D} \to {}^{1}H \to 0 .$$

This may be split into two exact sequences

$$0 \to {}^{1}D(1) \to D \to K \to 0$$
$$0 \to K \to \bar{D} \to {}^{1}H \to 0 .$$

Now $\varinjlim_{(\cdot)} D = 0$ and $\varinjlim_{(1)} {}^{1}H = 0$ so we deduce

$$\varinjlim_{(1)} {}^{1}D(1) = 0 \quad \varinjlim K = 0$$
$$\varinjlim \bar{D} \simeq {}^{1}H \qquad \varinjlim {}^{1}D(1) \simeq \varinjlim_{(1)} K \simeq \varinjlim_{(1)} \bar{D}$$

and this together with (3.4) proves the corollary.

QED.

Corollary (3.6) Suppose $\{E_r\}$ converges and suppose

$$H^{\cdot} = {}^{1}H^{\cdot} = 0$$

then for each $p \in Z$ there is an exact sequence

(i) $$0 \to E_p^{\infty} \to H_{p-1} \to H_p \to 0 .$$

Moreover we have an exact sequence

(ii) $$0 \to \varprojlim^{(1)} \underline{D} \to {}_0H \to \varprojlim_p H_p \to 0$$

and the isomorphism

(iii) $${}_1H \cong \varprojlim \underline{D} \ .$$

Proof. Dual to that of (3.5).

Corollary (3.7) Suppose that $\{E^r\}$ converges uniformly then

(i) For all $p \in Z$

$$^1H^p = {}_1H_p = 0$$

(ii) For every $p \in Z$ we have an exact sequence

$$0 \to \ker\{H_{p-1} \to H_p\} \to E_p^{\infty} \to \operatorname{coker}\{H^{p-1} \to H^p\} \to 0$$

(iii) For every $p \in Z$ we have an exact sequence

$$0 \to {}_1H \to \varinjlim_p H^p \to {}^0H \to {}_0H \to \varprojlim_p H_p \to {}^1H \to 0 \ .$$

Moreover we have isomorphisms

(iv) $$\varinjlim H^p \simeq \varprojlim \underline{D} \quad \text{and} \quad \varprojlim H_p \simeq \varinjlim \overline{D} \ .$$

Proof. Using (3.3) and (1.5) we know that

$$\varinjlim_{(1)} \overline{D} = 0 \ , \quad \varprojlim^{(1)} \underline{D} = 0 \ .$$

Now from (iv) of (3.4) we deduce two exact sequences:

$$0 \to {}^1D(1) \to D \to K \to 0$$
$$0 \to K \to \overline{D} \to {}^1H \to 0 \ .$$

From the last one we conclude:

$$\varinjlim_{(1)} K = 0$$

and the sequence

$$0 \to \varinjlim K \to \varinjlim \bar{D} \to {}^1H \to 0$$

is exact.

From the first we then get the exact sequence

$$0 \to \varinjlim {}^1D(1) \to {}_0H \to \varinjlim K \to 0$$

and the isomorphism

$$\varinjlim_{(1)} {}^1D \simeq {}_1H \quad .$$

Putting things together, using (iii) of (3.4) we get the following exact sequence

$$0 \to {}_1H \to \varinjlim_{p} H^p \to {}^0H \to {}_0H \to \varinjlim \bar{D} \to {}^1H \to 0 \ .$$

Dually we find the exact sequence

$$0 \to {}_1H \to \varprojlim \underline{D} \to {}^0H \to {}_0H \to \varprojlim_{p} H_p \to {}^1H \to 0 \ .$$

The 5-lemma then concludes the proof.

QED.

Remark (3.8). Let D be a monomorphic projective system of abelian groups, and suppose $D_p \simeq D_0$ for all $p \geq 0$. Then we know that

$${}^1H = \varprojlim^{(1)} D \neq 0$$

if and only if D_0 is nondiscrete, but not complete, in the topology induced by the subgroups D_p for $p \leq 0$. As the projective system $\{\ker \eta_p^{p-k}\}_{k \in Z^+}$ is zero and the projective system $\{\operatorname{coker} \eta_{p+k}^{p}\}_{k \in Z^+}$ is monomorphic, the condition (ii) of (3.2) is satisfied. Thus the condition (i) of (3.2) does not exclude the situation ${}^1H \neq 0$. If we change the projective system D by imposing $D_p = 0$ for $p > 0$, then we find that the condition (ii) of (3.1) is satisfied if and only if for some p_o we have $D_p \simeq D_{p_o}$ for all $p \leq p_o$.

If this last condition is not satisfied we will not be able to obtain E_1^∞ after a finite number of steps, i.e. as an E_1^r.

Remark (3.9) By (3.1) we find that if for some $E \in S(D)$ (E_r) converges then the same is true for any $E \in S(D)$.

Remark (3.10) It is easy to show that if for $p_1 < p_2$ the projective system $\{\ker \eta^k_{p_2}\}_{k < p_2}$ resp. $\{\operatorname{coker} \eta^{p_1}_k\}_{k > p_1}$ is stable resp. costable then so is also the projective system $\{\ker \eta^k_{p_1}\}_{k < p_1}$ resp. $\{\operatorname{coker} \eta^{p_2}_k\}_{k > p_2}$. Moreover if $\{\ker \eta^k_p\}_{k < p}$ is stable then using the exact sequence

$$0 \to \ker \eta_p^{p-k} \to D_{p-k} \to \operatorname{im} \eta_p^{p-k} \to 0$$

we find an exact sequence

$$0 \to H^p \to {}^oH \to {}^1D(1) \to 0 .$$

Suppose that D is a projective system of graded objects from $\underline{c}$, and let $\underline{c}^*_Z$ be the subcategory of $\underline{c}_Z$ consisting of such objects and morphisms of degree 0. Then of course, ${}^iH$, and ${}_iH$ are graded objects. Moreover the filtration

$$\{H^p\}_{p \in Z}$$

of ${}^oH$, and the cofiltration

$$\{H_p\}_{p \in Z}$$

of ${}_oH$ are graded, i.e. for every $p \in Z$ the morphisms

$$\begin{array}{ccccc} H^{p-1} & \to & H^p & \to & {}^0H \\ {}_0H & \to & H_{p-1} & \to & H_p \\ {}^1H^{p-1} & \to & {}^1H^p & \to & {}^1H \\ {}_1H & \to & {}_1H_{p-1} & \to & {}_1H_p \end{array}$$

are of degree zero.

Let E be an element of $S(D)$ and assume for each $p \in Z$ that E_p is graded, so that the resulting exact couple

$$\begin{array}{ccc} D & \xrightarrow{i} & D \\ & k\nwarrow \quad \swarrow j & \\ & E & \end{array}$$

is bigraded. The set of such E will be denoted by $S^*(D)$. We shall call the $\underline{p}$ in E_p, respectively D_p, the <u>primary degree</u>, and the $\underline{n}$ in the graduations $\{E_{p,n}\}_{n \in Z}$ and $\{D_{p,n}\}_{n \in Z}$ of E_p respectively D_p the <u>total degree</u>.

Thus i will always have total degree 0. Suppose j have total degree u and k have total degree v, then the total degree of $j^{(r)}$ is u and the total degree of $k^{(r)}$ is v.

In particular the morphisms in the exact sequence of (2.1) have total degree u and v respectively, and the same must be true for the morphisms in the exact sequences (3) and (6) of §2.

In the same way we find that the isomorphisms (7) of §2 have total degree v and $+u$ respectively.

As the total degree of i is 0 we find, moreover, that the morphisms in the sequences (1) of §2 have total degree 0. Together this gives us the following bigraded version of (2.2).

Theorem (3.11) For any object D in $\underline{c}_Z^*$ and for any $E \in S^*(D)$ such that the total degrees of j and k are u and v respectively, we have the following diagram of exact sequences

$$\begin{array}{ccccccccccccc}
 & & & & & & 0 & & & & & & \\
 & & & & & & \downarrow & & & & & & \\
0 \to {}_1E^{\infty}_{p,n-v-u} & \to & {}_1H_{p-1,n-u} & \to & {}_1H_{p,n-u} & \to & * & \to & H_{p-1,n-u} & \to & H_{p,n-u} & \to & 0 \\
 & & & & & & \downarrow & & & & & & \\
 & & & & & & E^{\infty}_{p,n} & & & & & & \\
 & & & & & & \downarrow & & & & & & \\
 & & 0 \to H^{p-1,n+v} & \to & H^{p,n+v} & \to & * & \to & {}^1H^{p-1,n+v} & \to & {}^1H^{p,n+v} & \to & {}^1E^{\infty}_{p,n+v+u} \to 0 \\
 & & & & & & \downarrow & & & & & & \\
 & & & & & & 0 & & & & & &
\end{array}$$

Now looking carefully at the proof of (3.1) we find the following theorems:

Theorem (3.12) Suppose D is an object of $\underline{c}_Z^*$ and let $E \in S^*(D)$, then the following conditions are equivalent

(i) For every $p \in Z$ there exists an $r_{p,n} \in Z$ such that

$$E^{r_{p,n}}_{p,n} \simeq E^{\infty}_{p,n}$$

(ii) For every $p \in Z$ the projective system

$$\{\ker \eta_p^{p-k}(n+v)\}_{k \in Z^+}$$

is stable, and the projective system

$$\{\operatorname{coker} \eta^{p}_{p+k}(n-u)\}_{k \in Z^+}$$

is costable.

Here we have denoted by $\eta_p^{p'}(n)$ the n' th homogeneous component of $\eta_p^{p'}$.

Theorem (3.13) Suppose D is an object in $\underline{c}_Z^*$ and let $E \in S_Z^*$, then the following conditions are equivalent.

(i) There exists an $r_n \in Z$ such that

$$E^{r_n}_{p,n} \simeq E^{\infty}_{p,n} \quad \text{for every} \quad p \in Z \ .$$

(ii) There exists an $r_n \in Z^+$ such that for every $p \in Z$ the projective system

$$\{\ker \eta^{p-k}_{p}(n+v)\}_{k \in Z^+}$$

is stable of uniform height r_n, and the projective system

$$\{\operatorname{coker} \eta^{p}_{p+k}(n-u)\}_{k \in Z^+}$$

is costable of uniform depth r_n.

Remark (3.14) Using the remark (3.10) and the theorem (3.12) one may easily deduce the theorem of Shih [9], see also Proposition (13.7.4) of [4].

Let $c.$ be a complex in $\underline{c}$ with differential d of degree -1. Then a system of complexes $(F_p c.)_{p \in Z}$ is called a filtration of $c.$ if there are given for every $p \in Z$ monomorphisms

$$F_{p-1}\, c. \to F_p\, c. \to c. \quad .$$

Dually we say that a system of complexes $\{K_p c.\}_{p \in Z}$ is a cofiltration of $c.$ if there are given for every $p \in Z$ epimorphisms

$$c. \to K_{p-1}\, c. \to K_p\, c.\, .$$

We shall assume that for every filtration

$$\varinjlim_{p} F_p c. \simeq c.$$

and for every cofiltration

$$\varprojlim K_p\, c_\bullet \simeq c_\bullet \quad .$$

From this we deduce the relations

$$\varinjlim_p{}^{(\cdot)} \operatorname{coker}(F_p\, c_\bullet \to c_\bullet) = 0 \ , \ \varinjlim_p{}^{(1)} F_p\, c_\bullet = 0$$

$$\varprojlim_p{}^{(\cdot)} \ker(c_\bullet \to K_p\, c_\bullet) = 0 \ , \ \varprojlim_p{}^{(1)} K_p\, c_\bullet = 0 \quad .$$

Using the general theory of the functors $\varinjlim$ and $\varprojlim$ (see [6] and [7]) and spectral sequences we get in the case of a filtration the following diagrams of exact sequences.

$$\begin{tikzcd}
 & & 0 & & \\
 & & H_n(c_\bullet) \arrow[u] & & \\
0 \arrow[r] & \varinjlim\limits_{p \in Z} H_n(F_p\, c_\bullet) \arrow[r] & * \arrow[r] \arrow[u, "\cong"'] & \varinjlim\limits_{p \in Z}{}^{(1)} H_{n-1}(F_p\, c_\bullet) \arrow[r] & 0 \\
 & & 0 = H_{n-1}(\varinjlim{}^{(1)} F_p\, c_\bullet) \arrow[u] & & \\
 & & 0 \arrow[u] & &
\end{tikzcd}$$

(13)

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & H_n(\varprojlim F_p\, c_\bullet) \arrow[d] & & \\
0 & \varprojlim H_n(F_p\, c_\bullet) \arrow[l] & * \arrow[l] \arrow[d] & \varprojlim{}^{(1)} H_{n+1}\, F_p\, c_\bullet) \arrow[l] & 0 \arrow[l] \\
 & & H_n(\varprojlim{}^{(1)} F_p\, c_\bullet) \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

and in the case of a cofiltration dual diagrams.

Now, look at the projective systems of graded objects

$$F\colon \cdots \to H_\bullet\,(F_{p-1}\, c_\bullet) \to H_\bullet\,(F_p\, c_\bullet) \to H_\bullet\,(F_{p+1}\, c_\bullet) \to \cdots$$

$$K\colon \cdots \to H_\bullet\,(K_{p-1}\, c_\bullet) \to H_\bullet\,(K_p\, c_\bullet) \to H_\bullet\,(K_{p+1}\, c_\bullet) \to \cdots \quad .$$

The problem is to calculate $H.(c.)$ by using spectral sequences associated to the projective systems F and K.

There exist natural exact couples in $S^*(F)$ and $S^*(K)$, given by

$$E = \{E_p\} \quad \text{with} \quad E_p = H.(\operatorname{coker}\{F_{p-1}\, c. \to F_p\, c.\})$$

$$J = \{J_p\} \quad \text{with} \quad J_p = H.\ (\ker\ \{K_{p-1}\, c. \to K_p\, c.\}) \ .$$

The total degree of j_E is 0, the total degree of k_E is -1, the total degree of j_J is -1 and that of k_J is 0, the notations being evident.

By (3.2) we then have the following diagrams of exact sequences

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & & & & \\
0 \arrow[r] & {}_1E^\infty_{p,n+1} \arrow[r] & {}_1H_{p-1,n} \arrow[r] & {}_1H_{p,n} \arrow[r] & * \arrow[r] \arrow[d] & H_{p-1,n} \arrow[r] & H_{p,n} \arrow[r] & 0 & \\
 & & & & E^\infty_{p,n} \arrow[d] & & & & \\
 & 0 \arrow[r] & H^{p-1,n-1} \arrow[r] & H^{p,n-1} \arrow[r] & * \arrow[r] \arrow[d] & {}^1H^{p-1,n-1} \arrow[r] & {}^1H^{p,n-1} \arrow[r] & {}^1E^\infty_{p,n-1} \arrow[r] & 0 \\
 & & & & 0 & & & &
\end{tikzcd}$$

and:

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & & & & \\
0 \arrow[r] & {}_1J^\infty_{p,n+1} \arrow[r] & {}_1H_{p-1,n+1} \arrow[r] & {}_1H_{p,n+1} \arrow[r] & * \arrow[r] \arrow[d] & H_{p-1,n+1} \arrow[r] & H_{p,n+1} \arrow[r] & 0 & \\
 & & & & J^\infty_{p,n} \arrow[d] & & & & \\
 & 0 \arrow[r] & H^{p-1,n} \arrow[r] & H^{p,n} \arrow[r] & * \arrow[r] \arrow[d] & {}^1H^{p-1,n} \arrow[r] & {}^1H^{p,n} \arrow[r] & {}^1J^\infty_{p,n-1} \arrow[r] & 0 \\
 & & & & 0 & & & &
\end{tikzcd}$$

The following results generalize the Corollary (6.3) of [3].

Theorem (3.15) Suppose $\varprojlim^{(.)} F_p c. = 0$ and suppose for some $p_o \in Z^+$ that $F_p c. = F_{p_o} c.$ for all $p \geq p_o$. If $\{E^r\}$ converges uniformly, then

$$E_p^{\infty} = \ker\{H_{p-1} \to H_p\} \ , \ H.\ (c.) = \varprojlim H_p \ .$$

Theorem (3.16) Suppose $\varinjlim_{(.)} K_p c. = 0$ and suppose for some $p_o \in Z^+$ $K_p c. = K_{p_o} c.$ for all $p \leq p_o$. If $\{\mathcal{J}^r\}$ converges uniformly, then;

$$\mathcal{J}_p^{\infty} = \operatorname{coker}\{H^{p-1} \to H^p\} \ , \ H.\ (c.) = \varinjlim H^p \ .$$

Proof. Using (3.7) and the diagrams (13) these results become obvious.

QED.

We are now going to explicit the conditions (ii) of (3.12) and (3.13) to the case of a filtered complex.

Proposition (3.17) The condition (ii) of (3.12) for the projective system F above, is equivalent to the following

(i) For every $p \in Z$ and $k \in Z^+$ there exists a $k' \in Z^+$ such that for every $k'' \geq k'$

(14) $$Z^n_{p-k''} \cap B^n_p + B^n_{p-k} = Z^n_{p-k'} \cap B^n_p + B^n_{p-k}$$

(15) $$B^n_{p+k''} \cap Z^n_{p+k} + Z^n_p = B_{p+k'} \cap Z^n_{p+k} + Z^n_p$$

where $Z^n_p = \ker\{F_p c_n \xrightarrow{d} F_p c_{n-1}\}$, $B^n_p = \operatorname{im}\{F_p C_{n+1} \xrightarrow{d} F_p c_n\}$.

Proof. We have

$$\ker \eta^{p-k}_p = Z_{p-k} \cap B_p / B_{p-k}$$

$$\operatorname{coker} \eta^p_{p+k} = Z_{p+k} / Z_p + B_{p+k} \ .$$

Using this we readily show that (1) is equivalent to $\{\ker \eta_p^{p-k}(n)\}_{k \in Z^+}$ being stable, and (2) is equivalent to $\{\operatorname{coker} \eta_{p+k}^{p}(n)\}_{k \in Z^+}$ being costable.

QED.

Now as intersection and kernel commute we have:

$$Z^n_{p-k''} \cap B^n_p = F_{p-k''}\, c_n \cap B^n_p$$

$$B^n_{p+k''} \cap Z^n_{p+k} = B^n_{p+k''} \cap F_{p+k}\, c_n \quad .$$

Moreover (15) is equivalent to

$$(16) \qquad B^n_{p+k''} \cap Z^n_{p+k} + F_p\, c_n = B_{p+k'} \cap Z^n_{p+k} + F_p\, c_n \quad .$$

Proposition (3.18) The conditions (ii) of (3.13) for the projective system F above, is equivalent to the following (ii)' . There exists an $r_n \in Z^+$ such that for every $p \in Z$, $k,s \in Z^+$

$$F_{p-r_n}\, c_n \cap d(F_{p+k}\, c_{n+1}) \subseteq F_{p-r_n-s}\, c_n + d(F_p\, c_{n+1}) \; .$$

Proof. Let r_n be the number in (3.13), then the condition (ii) of (3.13) is equivalent to the following condition

$$F_{p-r_n-k-s}\, c_n \cap B^n_p + B^n_{p-k} = F_{p-k-r_n}\, c_n \cap B^n_p + B^n_{p-k}$$

(17)

$$B^n_{p+k+r_n+s} \cap F_{p+k}\, c_n + F_p\, c_n = B^n_{p+k+r_n} \cap F_{p+k}\, c_n + F_p\, c_n$$

for every $p \in Z$ and $k,s \in Z^+$.

But (17) is equivalent to

$$(18)\quad \begin{aligned} F_{p-k-r_n} c_n \cap B^n_p &\subseteq F_{p-k-r_n-s}\, c_n + B^n_{p-k} \\ B_{p+k+r_n+s} \cap F_{p+k}\, c_n &\subseteq B^n_{p+k+r_n} + F_p\, c_n \end{aligned}$$

If we in the first formula put p for $p-k$ and in the second put p for $p+k+r_n$ we find that (18) is equivalent to

$$F_{p-r_n} c_n \cap B^n_{n+k} \subseteq F_{p-r_n-s} + B^n_p \quad .$$

QED.

Corollary (3.19) Suppose that $c.$ is a complex of R-modules, R being a commutative ring, and suppose the filtration $\{F_p\, c.\}_{p \in Z}$ is bounded to the right, i.e. $F_p\, c. = F_{p_o}\, c.$ for some p_o and all $p \geq p_o$. Suppose further that the submodules B^n_p of c_n is closed in the topology of c_n induced by the submodules $\{F_p\, c_n\}_{p \in Z}$. Then the condition (ii) of (3.13) is equivalent to

(ii)' There exists an $r_n \in Z^+$ such that for every $p \in Z$

$$F_{p-r_n} c_n \cap d(c_{n+1}) \subseteq d(F_p\, c_{n+1}) \quad .$$

Proof. Use (3.18) and remember that $\bigcap_{s \in Z^+} (F_{p-s}\, c_n + B^n_p) = B^n_p$, since B^n_p is closed.

QED.

Remark (3.20) The above corollary generalizes a result of Serre, [8], II-15.

If we suppose $\varprojlim F_p\, c. = 0$ we may avoid the condition that B^n_p be closed in c_n. In fact using (2.10) and (4.6) we prove that if $\{E^r\}$ converges then B^n_p is closed in c_n, thus (ii) $\Longrightarrow$ (ii)'. On the other hand (ii)' obviously imply (ii).

4. Induced morphisms.

Let

$$\phi : C \to D$$

be a morphism in $\underline{c}_Z$, and suppose given exact couples $E \in S(C)$, $I \in S(D)$ and a morphism of graded objects

$$\psi : E \to I$$

compatible with ϕ .

Then evidently we get morphisms

$$\phi : {}^iH(C) = \varprojlim{}^{(i)} C \to \varprojlim{}^{(i)} D = {}^iH(D)$$

$$\phi : {}_iH(C) = \varinjlim{}_{(i)} C \to \varinjlim{}_{(i)} D = {}_iH(D)$$

$$\psi^r : E^r \longrightarrow I^r .$$

Let

$$C : \cdots \to C_{p-1} \xrightarrow{k_{p-1}} C_p \to \cdots \qquad C = \{C_p, \xi_p^{p'}\}$$

$$D : \cdots \to D_{p-1} \xrightarrow{i_{p-1}} D_p \to \cdots \qquad D = \{D_p, \eta_p^{p'}\}$$

be the two objects of $\underline{c}_Z$.

Lemma (4.1) If the morphism ϕ induces isomorphisms

$$\phi : \ker k_p^r \to \ker i_p^r$$

for every $p \in Z$ and $r \in Z^+$ then ϕ induces an isomprhism

$$\underline{\phi} : \underline{C} \longrightarrow \underline{D} .$$

Proof. By §2 (9) we have a diagram of exact sequences

$$(1)\qquad \begin{array}{ccccccccc} 0 & \to & \ker \rho^{p-k}_{p-1} & \to & \ker \rho^{p-k}_{p} & \to & \ker k^{k}_{p-1} & \to & 0 \\ & & \alpha^{k-1}_{p-1}\downarrow & & \alpha^{k}_{p}\downarrow & & \gamma^{k}_{p}\downarrow & & \\ 0 & \to & \ker \eta^{p-k}_{p-1} & \to & \ker \eta^{p-k}_{p} & \to & \ker i^{k}_{p-1} & \to & 0 \ . \end{array}$$

As γ^k_p is an isomorphism for all $p \in Z$ and $k \in Z^+$ and as $\alpha^1_p = \gamma^1_p$ an inductive argument shows that α^k_p is an isomorphism for all $p \in Z$ and $k \in Z^+$. But then

$$\underline{C}_p = \lim_{\vec{k}} \ker \rho^{p}_{p+k}$$

and

$$\underline{D}_p = \lim_{\vec{k}} \ker \eta^{p}_{p+k}$$

must be isomorphic.

QED.

Lemma (4.2) If the morphism ϕ induces isomorphisms

$$\phi : \operatorname{coker} k^r_p \to \operatorname{coker} i^r_p$$

for every $p \in Z$ and $r \in Z^+$ then ϕ induces an isomorphism

$$\bar{\phi} : \bar{C} \to \bar{D} \ .$$

Proof. Dual to that of (4.1).

QED.

Theorem (4.3) Suppose ϕ induces either isomorphisms

$$\ker\{{}^1H^{p-1}(C) \to {}^1H^p(C)\} \to \ker\{{}^1H^{p-1}(D) \to {}^1H^p(D)\}$$

$$\operatorname{coker}\{H^{p-1}(C) \to H^p(C)\} \to \operatorname{coker}\{H^{p-1}(D) \to H^{p-1}(D)\}$$

or isomorphisms

$$\operatorname{coker}\{{}_1H_{p-1}(C) \to {}_1H_p(C)\} \to \operatorname{coker}\{{}_1H_{p-1}(D) \to {}_1H_p(D)\}$$

$$\ker\{H_{p-1}(C) \to H_p(C)\} \to \ker\{H_{p-1}(D) \to H_p(D)\}$$

then if ψ^{r_o} is an isomorphism, ϕ will induce isomorphism

$$\overline{C}^{r_o} \to \overline{D}^{r_o}$$

$$\underline{C}^{r_o} \to \underline{D}^{r_o} .$$

Proof. Look at the commutative diagram of exact sequences (see (2.1)).

$$\begin{array}{ccccccccc}
0 & \to & \operatorname{coker} k^r_{p+r-2} & \to & Z^r_{p,k}(C) & \to & \ker k^{r+k}_{p-1} & \to & 0 \\
 & & \downarrow \alpha^r_p & & \downarrow \psi^r_{p,k} & & \downarrow \gamma^{r+k}_p & & \\
0 & \to & \operatorname{coker} i^r_{p+r-2} & \to & Z^r_{p,k}(D) & \to & \ker i^{r+k}_{p-1} & \to & 0
\end{array}$$

where α^r_p and γ^{r+k}_p are induced by ϕ and $\psi^r_{p,k}$ is induced by ψ^r.

For $r \geq r_o$ and $k \in Z^+$ we know that $\psi^r_{p,k}$ is an isomorphism. Applying $\varprojlim_k$ and $\varinjlim_r$ on this diagram we get the following commutative diagram (see §2 (6)).

$$\begin{array}{ccccccccc}
0 & \to & \varinjlim_r \operatorname{coker} {}'k^r_{p+r-2} & \to & E^\infty_p & \to & \varprojlim_k \ker {}'k^k_{p-1} & \to & 0 \\
 & & \downarrow \alpha_p & & \downarrow \psi & & \downarrow \gamma_p & & \\
0 & \to & \varinjlim_r \operatorname{coker} {}'i^r_{p+r-2} & \to & I^\infty & \to & \varprojlim_k \ker {}'i^k_{p-1} & \to & 0 .
\end{array}$$

Now, from the diagram (1) above we deduce the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccccc}
0 & \to & H^{p-1}(C) & \to & H^p(C) & \to & \varprojlim_k \ker {}'k^k_{p-1} & \to & {}^1H^{p-1}(C) & \to & {}^1H^p(C) & \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \gamma_p & & \downarrow & & \downarrow & \\
0 & \to & H^{p-1}(D) & \to & H^p(D) & \to & \varprojlim_k \ker {}'i^k_{p-1} & \to & {}^1H^{p-1}(D) & \to & {}^1H^p(D) & \to \cdots
\end{array}$$

and dually we get the commutative diagram of exact sequences

$$\begin{array}{ccccccccccc}
\cdots \to & {}_1H_{p-1}(C) & \to & {}_1H_p(C) & \to & \varinjlim_r \operatorname{coker} {}'k^r_{p+r-2} & \to & H_{p-1}(C) & \to & H_p(C) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow \alpha_p & & \downarrow & & \downarrow & \\
\cdots \to & {}_1H_{p-1}(D) & \to & {}_1H_p(D) & \to & \varinjlim_r \operatorname{coker} {}'i^r_{p+r-2} & \to & H_{p-1}(D) & \to & H_p(D) & \to 0 \; .
\end{array}$$

The conditions of the theorem guarantee that for every $p \in Z$, either γ_p or α_p is an isomorphism.

From this we easily deduce that for every $p \in Z$, $r,k \in Z^+$, $r \geq r_o$, the morphisms

$$\alpha^r_p \text{ and } \gamma^{r+k}_p$$

are isomorphisms. The conclusion then follows from (4.1) and (4.2).

QED.

<u>Corollary (4.4)</u> Suppose either $C = \bar{C}$ and $D = \bar{D}$ or $C = \underline{C}$ and $D = \underline{D}$, and suppose further that for some $r_o \in Z^+$, ψ^{r_o} is an isomorphism. Then

$$\phi : C^{r_o} \to D^{r_o}$$

is an isomorphism, and in particular

$${}_iH(C) \simeq {}_iH(D)$$

$${}^iH(C) \simeq {}^iH(D) \; .$$

<u>Proof.</u> If $C = \bar{C}$ and $D = \bar{D}$ we know that

$$\varprojlim{}^{(i)} D = \varprojlim{}^{(i)} D = 0 \quad \text{for} \quad i \geq 0$$

and

$${}^1C(1) = {}^1D(1) = 0 \; .$$

Thus we deduce from the exact sequences

$$0 \to \ker \rho_p^{p-k} \to C_{p-k} \to \operatorname{im} \rho_k^{p-k} \to 0$$

$$0 \to \ker \eta_p^{p-k} \to D_{p-k} \to \operatorname{im} \eta_p^{p-k} \to 0$$

$${}^{i}H^{p}(C) = {}^{i}H^{p}(D) = 0 \quad \text{for all} \quad p \in Z,\ i \geq 0 \ .$$

The conclusion now follows trivially from (4.3) and the fact that if $C = \bar{C}$ and $D = \bar{D}$ then $C^{r_o} = \bar{C}^{r_o}$ and $D^{r_o} = \bar{D}^{r_o}$. If $C = \underset{\sim}{C}$ and $D = \underset{\sim}{D}$ we may use a dual argument.

QED.

<u>Corollary (4.5)</u> Suppose that E and I converge and suppose for some $r_o \in Z^+$ ψ^{r_o} is an isomorphism. Then if either

$${}^{o}H(C) = {}^{o}H(D) = 0 \quad \text{or} \quad {}_{o}H(C) = {}_{o}H(D) = 0$$

ϕ will induce isomorphisms

$$\bar{C}^{r_o} \simeq \bar{D}^{r_o}$$
$$\underset{\sim}{C}^{r_o} \simeq \underset{\sim}{D}^{r_o} \ .$$

<u>Proof.</u> By (3.1) we know that ${}^{1}H^{p} = {}_{1}H_{p} = 0$ for all $p \in Z$, so the conclusion follows trivially from (4.3).

QED.

Let C. and D. be filtered complexes with filtrations $\{F_p\ C.\}_{p \in Z}$ and $\{G_p\ D.\}_{p \in Z}$. Suppose given a morphism

$$\phi : C. \to D.$$

respecting the filtrations. We shall say that ϕ has filtration degree w if ϕ induces morphisms

$$F_p\, C_\circ \to G_{p+w}\, D_\circ \quad .$$

In this case ϕ induces a morphism of projective systems

$$\phi : F \to G$$

and morphisms

$$\psi^r : E^r F \to E^r G$$

where $E^r F$ and $E^r G$ are the natural spectral sequences associated to the filtrations of $C_\circ$ and $D_\circ$.

Lemma (4.6) Suppose $\varprojlim_p F_p\, C_\circ = 0$ then the following conditions are equivalent.

(i) $$^1F(1) = 0$$

(ii) For every $p \in Z$ we have

$$d(F_p\, C_\circ) = \varprojlim_k \left(d(F_p\, C_\circ) + F_{p-k}\, C_\circ\right) \quad .$$

Proof. As for every $p \in Z$

$$F_p = Z_p/B_p$$

we have

$$^1F_p = \varprojlim_k \left(Z_{p-k} + B_p/B_p\right)$$

so that $^1F_p = 0$ if and only if

$$\varprojlim_k \left(Z_{p-k} + B_p\right) \simeq B_p \quad ,$$

and this is seen to be equivalent with

$$\varprojlim_k \left(F_{p-k}\, C_\circ + B_p\right) \simeq B_p$$

by using the exact sequence

$$0 \to Z_{p-k} + B_p \to F_{p-k}\, C_\cdot + B_p \to B_{p-k} \to 0$$

and the fact that $\varprojlim_k B_{p-k} = 0$.

QED.

Remark (4.7) The condition (ii) above says that $d(F_p\, C_\cdot)$ is closed in the topology of $C_\cdot$ defined by the subobjects $\{F_p\, C_\cdot\}_{p \in Z}$.

Theorem (4.8) Suppose $\varprojlim^{(i)} F_p\, C_\cdot = \varprojlim^{(i)} G_p\, D_\cdot = 0$ for $i = 0, 1$ and suppose for each $p \in Z$ that $d(F_p\, C_\cdot)$ and $d(G_p\, D_\cdot)$ are closed in the topology of $C_\cdot$ respectively $D_\cdot$ defined by the filtrations. Then, if for some $r_o \in Z^+$, ψ^{r_o} is an isomorphism ϕ will induce isomorphisms

$$F^{r_o} \simeq G^{r_o}$$

and in particular we will have

$$H_\cdot\, (C_\cdot) \simeq H_\cdot\, (D_\cdot) \; .$$

Proof. By (4.6) we have

$${}^1F(1) = {}^1G(1) = 0$$

so by (1.7)

$$F = \overline{F} \quad , \quad G = \overline{G} \; .$$

The conclusion then follows from (4.4).

QED.

Remark (4.9) If $E^r(F)$ converge and $\varprojlim F = 0$ then by (2.10) we know that ${}^1F(1) = 0$. Thus by (4.6) $d(F_p\, C_\cdot)$ is closed in the topology of $C_\cdot$ defined by the filtration.

In [3] Eilenberg and Moore prove that the last conclusion of (4.8) holds without the condition that $d(F_p\, C_\cdot)$ and $d(G_p\, D_\cdot)$ be closed.

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