THE FORMAL RIEMANN-ROCH THEOREM FOR (NON-COMPACT) TOPOLOGICAL MANIFOLDS

by

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INTRODUCTION.

In this paper we will prove the so called differentiable Riemann-Roch theorem without using the differentiable structure of the manifold, the proof is actually formulated for topological manifolds.

The proof of the original differentiable Riemann-Roch theorem /2/ consists of two parts. In the first there is given a map

$$f: X \to Y$$

of manifolds and a <u>transfer</u> homomorphism

$$f_!: KU(X) \to KU(Y)$$

that depends on several choices, is constructed.

In the second and most important part of the proof properties of KU-theory and the Chern character are considered. When combined with the transfer homomorphism, the Chern character gives the R.-R. formula

$$ch(f_!(x)Td(Y) = f_!^s(ch(x)Td(X))$$

where $f_!^s$ is the transfer homomorphism in singular cohomology. X and Y are now taken to be (weakly almost) complex manifolds and Td is the Todd class.

Today the second part of the proof belongs to general cohomology theory. Some of the properties of the Chern character are used as a definition of a multiplicative cohomology transformation

$$m: k^{**} \to h^{**}$$

where k^* and h^* are multiplicative cohomology theories, and a so called <u>formal</u> Riemann-Roch theorem

$$m(f_!^k(x))\underline{m}(Y) = f_!^h(m(x)\underline{m}(X))$$

arises. Here $\underline{m}$ is the "Todd" class defined using m, and the manifolds are supposed to be k^*-orientable.

A similar formula is valid when f but not neccesarily Y is k^*-orientable. In this form the theorem is proved in /10/.

Concerning the first part of the paper /2/ it is natural to ask if the transfer homomorphism is functorial in some sense. (That this is not a useless question is illustrated by the proof of our theorem (5.1).) This question was answered in the affirmative in /11/ and /5/ in the case of differentiable manifolds.

The main part of this paper is concerned with the category on which the transfer homomorphism is a functor, and the proof that it is a functor on this category.

Although the proof of /11/ is formulated for differentiable manifolds only, the stable isotopy uniqueness of normal bundles /12/ implies that /11/ is equally valid in the topological case. We follow another line of proof and do not use /12/.

The paper is divided into five paragraphs:

§ 1.

Here we consider the Pontryagin-Thom map determined by an open imbedding of bundle spaces. We assume that the imbedding is given and let the bases of the bundles be general spaces. The main result is the well known homotopy commutativity of a certain diagram (1.6). This expresses a stability property of the Pontryagin-Thom map and formulas like

$$f_!(f^*(y)x) = yf_!(x)$$

are corollaries.

In this § we clearly see why f has to be proper, (1.1).

§ 2.

In this § we prove some lemmas on fibre isotopies of $(R^q,0)$-bundles that will be needed later on. The main result is that each isotopy class of fibrepreserving open bundlespace imbeddings contains exactly one isotopy class of bundle isomorphisms, (2.6). From this we deduce that when two bundles are contained in the same microbundle, there is a canonical isotopy class of isomorphisms of the two bundles. This is a strengthening of the uniqueness part of the Kister-Mazur theorem that states that the two bundles are isomorphic. The proofs of this § are based on the relative version of the Kister-Mazur theorem proved in /15/.

§ 3.

Here we consider k^*-oriented proper maps and define the category whose objects are manifolds and whose morphisms are proper oriented maps under a homotopy relation.

We also prove a theorem on k^*-thomclasses that generalizes theorem (1.1) of /1/ and has a much simpler proof. This theorem states that when $U \in k^*(T(b),pt)$ where $T(b)$ is the thomspace of the bundle b , and U has the property that $1/n\ U$ is a thomclass for b in the cohomology theory $n^{-1}k^*$ obtained from k^* by localizing in the multiplicative system $(1,n,n^2,\dots)$, some Whitney multiple

$$n^t b$$

has a k^*-thomclass. (3.3).

§ 4.

In this § we prove that the transfer homomorphism is well defined and a functor on the category of § 3. (4.10).

§ 5.

Here we first apply theorem (4.10) to prove a well known theorem of Atiyah /4/ in the case of not neccesarily compact manifolds and proper homotopy equivalences. (5.1.5.2). This is a real strengthening of Atiyah's theorem because his proof depends strongly on notions like reducibility and S-duality that do not seem to work well in the non-compact case.

Next we consider general properties of multiplicative cohomology transformations, following /3/. The result is, of course, the formal Riemann-Roch and Wu theorems stated in (5.11,14,15,16).

We introduce the homology theory k_* corresponding to k^* and prove without using S-duality that the compact manifold X has a k_*-fundamental class if and only if its tangent bundle has a k^*-thomclass. (5.18). We go on to prove that the transfer homomorphism determined by

$$X \to \text{Point}$$

is evaluation on a fundamental class. We write down the Wu formula in this case and obtain the classical Wu formula for the Stiefel-Whitney class as a corollary, 5.21).

§ 1. The Pontryagin-Thom construction.

Constructing the transfer maps we shall take a map

$$f: X \to Y$$

of manifold and lift it to an imbedding

$$i_o : X \to E$$

where E is a bundle space over Y. X will then have a tubular neighbourhood in some $E \times R^n$. [20].

In this § we start in the latter situation. Let X and Y be spaces with bundles ξ and η and let

$$i : E\xi \to E\eta$$

be an open topological imbedding. Then we have the Pontryagin-Thom map

$$D^i: Y^\eta \to X^\xi$$

defined by $$D^i = i^{-1} \quad \text{on} \quad i(E\xi)$$

and $$D^i(Y^\eta - i(E\xi)) = pt \; .$$

We say that the imbedding i is <u>bounded</u> iff D^i is continuous.

(1.1) <u>Lemma.</u>

When i is bounded, the map $f = pr_\eta \, i \, s_\xi : X \to Y$ is proper. When f is proper, there is a nbd. U of X in $E\xi$ such that whenever $\alpha : E\xi \to E\xi$ is a bundle-imbedding with image in U, the imbedding

$$i\alpha : E\xi \to E\eta$$

is bounded.

<u>Proof.</u> Choose a locally finite covering of Y consisting of relatively compact sets and argue by point-set topology. Note that if X is compact, every i is bounded.

(1.2) Lemma.

Let X, Y and Z be base spaces of the bundles ξ, η and μ . Let

$$i : E\xi \to E\eta \quad \text{and} \quad j : E\eta \to E\mu$$

be open imbeddings. Then

$$D^{ji} = D^i D^j .$$

Proof. Trivial.

(1.3) Lemma.

Let $i,j : E\xi \to E\eta$ be open imbeddings which are boundedly pseudo-isotopic in the sense that there is an open bounded imbedding

$$J : E(\xi \times I) \to E(\eta \times I)$$

which is i over the 0-slice and j over the 1-slice.

Then D^i and D^j are homotopic rel.pt.

Proof.

The following map is a homotopy.

$$Y^{\eta} \times I \to (Y \times I)^{\eta\times I} \xrightarrow[D^J]{} (X \times I)^{\xi\times I} \to X^{\xi} .$$

Next we shall consider stability properties of the maps D^i . We shall, however, not stabilize our bundles by adding new bundles, we prefer to use composition of bundles [20] which is a more flexible tool.

Recall that when ξ and η are microbundles with diagrams

$$\xi : \qquad X \to E \to X$$

$$\eta : \qquad E \to F \to E ,$$

the microbundle $\xi\circ\eta$ has the diagram

$$X \to F \to X$$

where the maps are

$$X \to E \to F \to E \to X .$$

$\xi o \eta$ is the composite micro-bundle and

$$pr_{\xi o \eta} = pr_{\xi} pr_{\eta} .$$

When both ξ and η are bundles, an easy application of the homotopy covering theorem [20] shows that $\xi o \eta$ is a bundle. The correspondence

$$\eta \longrightarrow \xi o \eta$$

takes bundles on $E\xi$ to bundles on X. It is a functor preserving bundle isomorphisms and isotopies of such.

Note that

$$E(\xi o \eta) = E\eta .$$

It is easily seen that the map

$$(pr_{\eta}, id) : E(\xi o \eta) \to E\xi \times E\eta$$

extends to a diagonal map

$$d : X^{\xi o \eta} \to X^{\xi} \wedge (E\xi)^{\eta} .$$

The inclusion $X^{\eta} \subset (E\xi)^{\eta}$ is a homotopy equivalence because $X \subset E\xi$ is. Any map

$$\Delta : X^{\xi o \eta} \to X^{\xi} \wedge X^{\eta}$$

such that the diagram (+) homotopy - commutes rel.pt is called a diagonal map.

$$\begin{tikzcd}[row sep=small] & X^{\xi} \wedge (E\xi)^{\eta} \\ X^{\xi o \eta} \arrow[ur, "d"] \arrow[dr, "\Delta"'] & \cup \\ & X^{\xi} \wedge X^{\eta} \end{tikzcd} \qquad (+)$$

(1.4) Definition.

The diagonal map Δ defined above is unique in homotopy rel.pt.

Now let $i : E\xi \to E\eta$ be an open imbedding of bundlespaces, and let μ be a bundle on $E\eta$. The map

$$i_b : Ei^*\mu \to E\mu$$

is then an open imbedding. Considered as an imbedding of bundlespaces, it will be called an <u>induced</u> imbedding and denoted

$$\bar{i} : E\xi o i^*\mu \to E\eta o\mu .$$

(1.5) <u>Lemma.</u> When i is bounded, $\bar{i}$ is also bounded.

We omit the proof.

(1.6) <u>Proposition.</u>

Let $i : E\xi \to E\eta$ be a bounded imbedding of bundles over the spaces X and Y. Let μ be a bundle on $E\eta$ and let $\bar{i}$ be the induced imbedding and suppose that $\mu = pr_\eta^*\mu$. Then the diagram homotopy-commutes rel.pt.

$$\begin{tikzcd}
Y^{\eta o\mu} \arrow[r, "\Delta"] \arrow[d, "D^{\bar{i}}"] & Y^\eta \wedge Y^\mu \arrow[dr, "D^i \wedge id"] & \\
& & X^\xi \wedge Y^\mu \\
X^{\xi o i^*\mu} \arrow[r, "\Delta"] & X^\xi \wedge X^{f^*\mu} \arrow[ur, "id \wedge f_b"'] &
\end{tikzcd}$$

<u>Proof.</u> Note that $i^*\mu \mid X = s_\xi^* i^* pr_\eta^*\mu = f^*\mu$ where $f = pr_\eta\, i\, s_\xi$.

Consider the diagram (*).

$$\begin{tikzcd}
Y^{\eta o\mu} \arrow[r, "d"] \arrow[d, "D^{\bar{i}}"] & Y^\eta \wedge (E\eta)^\mu \arrow[dr, "D^i \wedge id"] & \\
& & X^\xi \wedge (E\eta)^\mu \\
X^{\xi o i^*\mu} \arrow[r, "d"'] & X^\xi \wedge (E\xi)^{i^*\mu} \arrow[ur, "id \wedge i_b"'] &
\end{tikzcd} \qquad (*)$$

This diagram commutes by definition of d and D^i.

Next consider the diagram (**) containing (*).

$$\begin{tikzcd}
Y^{\eta\circ\mu} \arrow[r, "d"] \arrow[dd, "D^{\bar{i}}"'] \arrow[ddr, phantom, "(*)"] & Y^{\eta} \wedge (E\eta)^{\mu} \arrow[r, phantom, "\supset"] \arrow[d] & Y^{\eta} \wedge Y^{\mu} \arrow[d, "D^{i} \wedge id"] \\
& X^{\xi} \wedge (E\eta)^{\mu} \arrow[r, phantom, "\supset"] & X^{\xi} \wedge Y^{\mu} \\
X^{\xi\circ i^{*}\mu} \arrow[r, "d"] & X^{\xi} \wedge (E\xi)^{i^{*}\mu} \arrow[r, phantom, "\supset"] \arrow[u] & X^{\xi} \wedge X^{f^{*}\mu} \arrow[u, "id \wedge f_b"']
\end{tikzcd} \qquad (**)$$

Clearly this diagram commutes except the lower right square.

In the following diagram,

$$\begin{tikzcd}
(E\eta)^{\mu} \arrow[r, "\supset", "r"'] & Y^{\mu} \\
(E\xi)^{i^{*}\mu} \arrow[u, "i_b"] \arrow[r, phantom, "\supset"] & X^{f^{*}\mu} \arrow[u, "f_b"']
\end{tikzcd} \qquad (***)$$

which appears in (**), the retraction r is given by the fact that $\mu = pr_{\eta}^{*}\mu$. After deleting the inclution in the upper line from (***), the diagram commutes. Because r is a homotopy-equivalence, (***) homotopy-commutes after deleting r. Hence (**) homotopy-commutes. In view of the definition of the diagonal maps Δ , the proposition is proved.

Q.E.D.

In the rest of this § we draw some consequences of the proposition.

(1.7) <u>Corollary.</u>

With the notation of the proposition, let $\mu = \varepsilon^{q}$. Then under the identifications

$$X^{\xi\circ\varepsilon^{q}} = X^{\xi} \wedge S^{q} \quad \text{and} \quad Y^{\eta\circ\varepsilon^{q}} = Y^{\eta} \wedge S^{q} ,$$

we get the identification

$$D^{\bar{i}} = D^{i} \wedge S^{q} .$$

Proof.

It suffices to consider the following diagram, and use the proposition.

$$\begin{tikzcd}
 & & Y^{\eta} \wedge S^{q} \arrow[rrdd, "D^{i} \wedge id"] & & \\
Y^{\eta o \epsilon^{q}} \arrow[rru, leftrightarrow] \arrow[rr, "\Delta"] \arrow[dd, "D^{\bar{i}}"] & & Y^{\eta} \wedge Y^{\epsilon^{q}} \arrow[u] \arrow[rd] & & \\
 & & & X^{\xi} \wedge Y^{\epsilon^{q}} \arrow[r] & X^{\xi} \wedge S^{q} \\
X^{\xi o \epsilon^{q}} \arrow[rr, "\Delta"'] \arrow[rrd, leftrightarrow] & & X^{\xi} \wedge X^{\epsilon^{q}} \arrow[ru, "id \wedge f_{b}"'] \arrow[d] & & \\
 & & X^{\xi} \wedge S^{q} \arrow[rru, leftrightarrow] & &
\end{tikzcd}$$

Note that the diagonal map defined in (1.4) defines a pairing

$$h^{*}(X^{\xi},pt) \otimes h^{*}(X^{\eta},pt) \to h^{*}(X^{\xi o \eta},pt)$$

given by

$$\alpha\beta = \Delta^{*}(\alpha \wedge \beta).$$

(1.8) Corollary.

With the notation of the proposition, let

$$\alpha \in h^{*}(X^{\xi},pt) \quad \text{and} \quad \beta \in h^{*}(Y^{\mu},pt).$$

Then

$$(D^{\bar{i}})^{*}(\alpha \cdot f_{b}^{*}(\beta)) = (D^{i})^{*}(\alpha)\cdot\beta$$

and

$$(D^{\bar{i}})^{*}(f_{b}^{*}(\beta)\alpha) = \beta(D^{i})^{*}(\alpha) .$$

Proof.

$$(D^{\bar{i}})^{*}(\alpha f_{b}^{*}(\beta)) = (D^{\bar{i}})^{*}\Delta^{*}(id \wedge f_{b})^{*}(\alpha \wedge \beta) =$$

$$\Delta^{*}(D^{i} \wedge id)^{*}(\alpha \wedge \beta) = (D^{i})^{*}(\alpha)\beta \quad \text{according to}$$

proposition (1.6).

The second equation is obtained using the diagonal map

$$Y^{\eta o \mu} \longrightarrow Y^{\mu} \wedge Y^{\eta}$$

and the version of (1.6) valid for this map.

Q.E.D.

(1.9) Corollary.

With the notation of proposition (1.6) we have

$$(D^i)^*(f^*(y)\alpha) = y(D^i)^*(\alpha)$$

where $y \in h^*(Y)$ and $\alpha \in h^*(X^\xi, pt)$, and the pairing is the usual action of $h^*(\text{base})$ on $h^*(\text{Thom space, pt})$.

Proof.

Identifying $h^*(X)$ with $h^*(X^0, pt)$ it is easily verified that the action of $h^*(X)$ is determined by a diagonal map

$$\Delta : X^\xi \longrightarrow X^0 \wedge X^\xi .$$

Taking $\mu = 0$ in proposition (1.6) we find that the asserted equation is a special case of the second equation in (1.8) because $i = \bar{i}$.

We now give the first definition of a transfer map.

(1.10) Definition.

Let ξ , η be bundles over the spaces X , Y and let U , V be Thom classes for these bundles. (see § 3). Also let $i : E\xi \to E\eta$ be a bounded imbedding. The transfer map

$$t = t(i,U,V) : h^*(X) \to h^*(Y)$$

is defined by

$$(D^i)^*(xU) = t(x)V .$$

(1.11) Lemma.

When $i : E\xi \to E\eta$ and $j : E\eta \to E\mu$ are open bounded imbeddings, and U, V and W are Thom classes for ξ, η and μ , we have

$$t(ji,U,W) = t(j,V,W)t(i,U,V) .$$

Proof.

Write this equation as $t'' = t't$. Then

$$t''(x)W = (D^{ji})^*(xU) = (D^j)^*(D^i)^*(xU) =$$
$$(D^j)^*(t(x)V) = t'(t(x))W.$$

Cancelling the Thom class W , we get $t'' = t't$.

(1.12) Proposition.

When $t : h^*(X) \to h^*(Y)$ is the map defined in (1.10), and $f : X \to Y$ is as defined in (1.1), we have

$$t(f^*(y)x) = yt(x)$$

for all $y \in h^*(Y)$, $x \in h^*(X)$.

That is, t is a $h^*(Y)$-module homomorphism.

Proof.

Using corollary (1.9) we get

$$t(f^*(y)x)V = (D^i)^*(f^*(y)xU) = y(D^i)^*(xU) = yt(x)V.$$

Now cancel V.

(1.13) Corollary.

With the notation of proposition (1.6), let A be a Thom class for μ . Then f_b^*A is a Thom class for $f^*\mu$ and

$$t(i,U,V) = t(\bar{i},Uf_b^*(A),VA) .$$

Proof.

Let us write this equation $t = t'$.

$$t'(x)VA = (D^{\bar{i}})^*(xUf_b^*(A)) = (D^i)^*(xU)A = t(x)VA$$

according to corollary (1.8).

Q.E.D.

Note in particular that

$$t(i,U,V) = t(i,f^*(a)U,aV)$$

whenever $a \in h^0(Y)$ is such that aV is a Thom class. This follows either from (1.13) or from (1.12).

Finally we mention a corollary of (1.6) of a somewhat different type. It will not be used later.

(1.14) Proposition.

Let X be a space dominated by a finite-dimensional CW complex. Let ξ and η be bundles on X and suppose that there is an open bounded imbedding

$$i : E\xi \to E\eta$$

which is the identity on the zero-section. Then $J(\xi) = J(\eta)$.

The proof is practically the same as in (5.1). When X is a manifold, ξ and η are stably isomorphic.

§ 2. Fibre Isotopies of Topological Bundles.

In this §, we prove a theorem (2.10) on fibre-isotopies of bundle-imbeddings. We use the relative version of the Kister-Mazur theorem proved in [15] :

Theorem. (P. Holm)

When a numerable microbundle over a space X contains a bundle over a halo of a closed set $F \subset X$, it contains a bundle over all of X that equals the given bundle over F.

In this § all fibrebundles and microbundles shall be numerable [9] with no restriction on the base space.

(2.1) Lemma.

Let $\alpha_0 : E\xi \to E\eta$ be a bundle-imbedding over the base space X and let $U \subset E\eta$ be a microbundle nbd. of X. Then there is an isotopy

$$\alpha_t : E\xi \to E\eta$$

of bundle-imbeddings such that

$$\alpha_t(E\xi) \subset \alpha_0(E\xi)$$

for all t, and $\alpha_1(E\xi) \subset U$.

Proof.

In the bundle $\eta \times I$ there is the microbundle nbd.

$$M = \alpha_0(E\xi) \times [0,\tfrac{1}{2}] \cup (\alpha_c(E\xi) \cap U) \times [\tfrac{1}{2}\ 1] \quad \text{of} \quad X \times I .$$

The theorem quoted from [15] now gives a bundle μ contained in M equal to $\alpha_0(E\xi)$ over $X \times \{0\}$. We then use the fact that

M is a microbundle over $X \times I$. The homotopy covering theorem (for bundles) now gives an isomorphism

$$\xi \times I \to \mu$$

such that the composite

$$\alpha : E\xi \times I \to E\mu \subset M \subset E\eta \times I$$

equals α_0 over the 0-slice. Clearly α is the required isotopy.

(2.2) Lemma.

Let $\alpha_0, \alpha_1 : E\xi \to E\eta$ be isotopic bundle-imbeddings.

Then there is an isotopy from α_0 to α_1 whose image at each stage is contained in

$$\alpha_0(E\xi) \cup \alpha_1(E\xi) .$$

Proof.

Let $\quad \alpha : E\xi \times I \to E\eta \times I$

be some isotopy from α_0 to α_1. In lemma (2.1) take $X \times I$ as base space and define $U \subset E\eta \times I$ by

$$U = [\alpha_0(E\xi) \cup \alpha_1(E\xi)] \times I .$$

Lemma (2.1) now gives an isotopy

$$\beta_t : E\xi \times I \to E\eta \times I$$

with

$$\beta_t(E\xi \times I) \subset \alpha(E\xi \times I)$$

for each t, and

$$\beta_1(E\xi \times I) \subset U .$$

The isotopy β_t corresponds to a bundle-imbedding

$$\beta : E(\xi \times I \times I) \to E(\eta \times I \times I)$$

over $X \times I \times I$.

Write $R = \{0,1\} \times I \cup I \times \{1\}$. Restricting β to $X \times R$ we obtain a bundle-imbedding

$$\beta : E(\xi \times R) \to E(\eta \times R)$$

which may be regarded as an isotopy of bundle-imbeddings $E\xi \to E\eta$ parametrized by R. Note that R is an interval. We assert that this is the required isotopy. First we have $(0,0), (1,0) \in R$ and

$$\beta_{(0,0)} = \alpha_0 , \quad \beta_{(1,0)} = \alpha_1 .$$

Then we must show that

$$\beta(E\xi \times R) \subset [\alpha_0(E\xi) \cup \alpha_1(E\xi)] \times R .$$

This is a consequence of the following three inclusion relations.

I. $\beta(E\xi \times \{0\} \times I) \subset \alpha(E\xi \times I) \times I \cap E\eta \times \{0\} \times I = \alpha_0(E\xi) \times \{0\} \times I$.

II. Similarly $\beta(E\xi \times \{1\} \times I) \subset \alpha_1(E\xi) \times \{1\} \times I$.

III. $\beta(E\xi \times I \times \{1\}) = \beta_1(E\xi \times I) \times \{1\} \subset U \times \{1\} =$

$$= [\alpha_0(E\xi) \cup \alpha_1(E\xi)] \times I \times \{1\} .$$

(2.3) <u>Corollary.</u>

When two bundle imbeddings

$$r, z : E\xi \to E\eta$$

are isotopic, there are bundle imbeddings

$$\alpha, \beta : E\xi \to E\xi$$

both isotopic to the identity such that

$$r\beta = z\alpha .$$

<u>Proof.</u>

$N = r(E\xi) \cap z(E\xi)$ is a microbundle nbd. in η . Hence, by lemma (2.1), there is a bundle-imbedding

$$q : E\xi \to E\eta$$

isotopic to r with image in N. By the same lemma we can choose q so that there is an isotopy q_t from $q_0 = q$ to $q_1 = r$ with image in $N \cup r(E\xi) = r(E\xi)$.

With $\beta = r^{-1}q$, we get that β is isotopic by $r^{-1}q_t$ to $r^{-1}q_1 = \mathrm{id}$.

But q is also isotopic to z and lemma (2.2) gives an isotopy q^t from $q^0 = q$ to $q^1 = z$ with image in $N \cup z(E\xi) = z(E\xi)$ Hence $\alpha = z^{-1}q$ is isotopic to id, and

$$r\beta = r(r^{-1}q) = q = z(z^{-1}q) = z\alpha$$

In the above lemmas we have constructed small bundle imbeddings and small isotopies. We now go in the opposite direction to obtain bundle isomorphisms from bundle-imbeddings.

(2.4) Lemma.

Any bundle-imbedding is isotopic to a bundle isomorphism.

Proof.

Let $\alpha_0 : E\xi \to E\eta$ be a bundle-imbedding. The bundle $\eta \times I$ contains the bundle

$$\alpha_0(E\xi) \times [0, \tfrac{1}{2}\rangle \cup E\eta \times \langle \tfrac{1}{2}, 1]$$

over the halo

$$X \times [I - \{\tfrac{1}{2}\}] \quad \text{of} \quad X \times \{0, 1\}.$$

By the theorem quoted from [15] there is a bundle μ contained in $\eta \times I$ which equals $\alpha_0(E\xi)$ over the 0-slice and $E\eta$ over the 1-slice. Now take an isomorphism

$$\xi \times I \to \mu$$

such that the composite

$$E\xi \times I \to E\mu \subset E\eta \times I$$

is an isotopy as required.

(2.5) Lemma.

When two bundle isomorphisms $\beta_o, \beta_1 : \xi \to \eta$ are isotopic as bundle imbeddings, they are isotopic as bundle isomorphisms.

Proof. We may suppose that there is an isotopy

$$\beta : E\xi \times I \to E\eta \times I$$

from β_o to β_1 which is stationary over $[0,\frac{1}{3}]$ and $[\frac{2}{3},1]$. In the bundle $\eta \times I \times I$ there is contained the bundle

$$\lambda = \beta(E\xi \times I) \times [0,\tfrac{1}{2}\rangle \cup E(\eta \times I) \times \langle\tfrac{1}{2},1]$$
$$\cup (E\eta \times ([0,\tfrac{1}{3}\rangle \cup \langle\tfrac{2}{3},1])) \times I$$

whose base is a halo of $X \times \partial(I \times I)$. Hence, by [15], $\eta \times I \times I$ contains a bundle μ equal to λ over $X \times \partial(I \times I)$. Let

$$\xi \times I \times I \to \mu$$

be a bundle isomorphism equal to β over $X \times I \times \{0\}$. Then the composite

$$B : E\xi \times I \times I \to E\mu \subset E\eta \times I \times I$$

restricted to $X \times R$ where

$$R = \{0,1\} \times I \cup I \times \{1\}$$

gives an isotopy of bundle isomorphisms from β_o to β_1 parametrized by R.

(2.6) Proposition.

Each isotopy class of bundle imbeddings $\xi \to \eta$ contains exactly one isotopy class of bundle isomorphisms $\xi \to \eta$.

Proof.

This is the content of (2.4) and (2.5).

We now draw a conclusion of (1.1) and (2.3) concerning boundedness of imbeddings.

(2.7) Lemma.

Let ξ and η be bundles over the spaces X and Y and let $i : E\xi \to E\eta$ be an open imbedding such that $f = pr_\eta \, i \, s_\xi$ is proper. Then there exist bundle imbeddings $r : E\xi \to E\xi$ isotopic to id such that ir is bounded. Any two such imbeddings are boundedly isotopic.

Proof.

Let U be the nbd. of X in $E\xi$ described in (1.1). According to (2.1) there is a bundle imbedding

$$r : E\xi \to E\xi$$

with image in U and which is isotopic to id. By (1.1) ir is bounded. Let iz be an other bounded imbedding with z isotopic to id. Then r is isotopic to z so that there are bundle imbeddings α and β both isotopic to id and, according to (2.3), such that $z\alpha = r\beta$. Now

$$ir \simeq ir\beta = iz\alpha \simeq iz$$

where the isotopy is bounded because ir and iz are bounded.

(2.8) Definition.

Let i, f and r be the maps defined in (2.7). Then we define

$$D^i : Y^\eta \to X^\xi$$

to be the homotopy class of the map D^{ir}. This is a well defined homotopy class according to (1.3) and (2.7), provided that f is proper.

That is, when f is proper, we can define D^i as a homotopy class, even when i is not bounded.

(2.9) Definition.

Let $X \to M \to X$ be a microbundle over an arbitrary space X; denote it by M. When ξ is a bundle over X, a bundle imbedding $\xi \to M$ or $E\xi \to M$ is a topological imbedding $E\xi \to M$ that preserves fibres and zerosection. It is then clear how to define isotopies.

We define a category Bundle $(X)/M$ where an object is a bundle ξ on X together with an isotopy class of bundle imbeddings $\xi \to M$. We denote an object by

$$\xi \to M$$

A morphism $(\xi \to M) \to (\eta \to M)$ is an isotopy-class of bundle imbeddings $\xi \to \eta$ such that

$$\begin{tikzcd}[column sep=small] \xi \arrow[rr] \arrow[dr] & & \eta \arrow[dl] \\ & M & \end{tikzcd} \qquad (*)$$

commutes. To be accurate, the isotopy-classes in (*) shall have representatives that make the diagram commute as a diagram of spaces

$$\begin{tikzcd}[column sep=small] E\xi \arrow[rr] \arrow[dr] & & E\eta \arrow[dl] \\ & M & \end{tikzcd} \qquad (**)$$

Given two morphisms

$$\begin{tikzcd}[column sep=small] \xi \arrow[rr] \arrow[dr] & & \eta \arrow[dl] \\ & M & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] \eta \arrow[rr] \arrow[dr] & & \lambda \arrow[dl] \\ & M & \end{tikzcd} \qquad (+)$$

the composite morphism

$$\begin{tikzcd}[column sep=small] \xi \arrow[rr] \arrow[dr] & & \lambda \arrow[dl] \\ & M & \end{tikzcd}$$

is defined by composing maps representing the morphisms (+). We must show that this defines a morphism which is unique.

Let

$$\begin{tikzcd} E\xi \arrow[rr, "a"] \arrow[dr, "b"'] & & E\eta \arrow[dl, "c"] \\ & M & \end{tikzcd} \qquad \begin{tikzcd} E\eta \arrow[rr, "a'"] \arrow[dr, "b'"'] & & E\lambda \arrow[dl, "c'"] \\ & M & \end{tikzcd}$$

be diagrams representing (+). Then

$$\begin{tikzcd} E\xi \arrow[rr, "a"] \arrow[dr, "b'a"'] & & E\eta \arrow[dl, "b'"] \\ & M & \end{tikzcd}$$

is also a representing diagram because $b'a \simeq ca = b$. Hence the diagram

$$\begin{tikzcd} E\xi \arrow[rr] \arrow[dr, "b'a"'] & & E\lambda \arrow[dl, "c'"] \\ & M & \end{tikzcd} \tag{++}$$

represents the composite morphism. Moreover the imbeddings a' and a belong to welldefined isotopy classes so that the composite morphism is unique. Hence to compose the morphisms (+), we may simply delete a copy of M and write

$$\begin{tikzcd} \xi \arrow[r] \arrow[dr] & \eta \arrow[r] \arrow[d] & \lambda \arrow[dl] \\ & M & \end{tikzcd}$$

It is then clear that identity maps exist.

(2.10) <u>Theorem.</u>

When x and y are objects in

the category Bundle $(X)/M$,
the set Hom (x,y)

has exactly one element.

Proof.

Let $\xi \to M$ and $\eta \to M$ be two objects. When we represent these isotopy-classes by imbeddings $\alpha : E\xi \to M$ and $\beta : E\eta \to M$, we may according to (2.1) choose α such that

$$\mathrm{Im}(\alpha) \subset \mathrm{Im}(\beta).$$

Hence there is a map $\varphi : E\xi \to E\eta$ such that

$$\alpha = \beta\varphi.$$

φ is then a bundle imbedding and represents a morphism

$$\begin{tikzcd} \xi \arrow[rr] \arrow[dr] & & \eta \arrow[dl] \\ & M & \end{tikzcd}$$

When we know that the sets Hom(x,y) are nonempty, it suffices to show that Hom(x,x) has only one element to conclude the proof, as the reader easily verifies.

That is, when given a commutative diagram

$$\begin{tikzcd} E\xi \arrow[rr, "a"] \arrow[dr, "b"'] & & E\xi \arrow[dl, "c"] \\ & M & \end{tikzcd}$$

where $b \simeq c$, we must show that $a \simeq \mathrm{id}$. To this end we note that (2.1), (2.2) and (2.3) which deal with bundle imbeddings $E\xi \to E\eta$, are equally valid for bundle imbeddings

$$E\xi \to M .$$

In fact the proofs are valid when we replace $E\eta$ by M everywhere.

Because b and c are isotopic, there are, by (2,3), bundle imbeddings $r, z : E\xi \to E\xi$ both isotopic to id such that $bz = cr$. But $b = ca$ so that

$$caz = cr \text{ , and}$$

$$az = r$$

because c is injective. Because $z, r \simeq id$, we get

$$a \simeq id .$$

Q.E.D.

Note that (2.10) and (2.6) implies that when ξ and η are bundles contained in the microbundle M, there is a <u>canonical</u> isotopy class of isomorphisms

$$\xi \to \eta .$$

(2.11)

When X is a manifold, we define

$$\tau X = \mathrm{Bundle}(X)/TX$$

where TX is the tangent microbundle space. By abuse of notation, we shall denote objects in this category by τX. In view of (2.10) we may do so.

For later use we need some standard facts about fibre bundles. The fibre bundles shall be numerable with some fixed fibre and structural group.

(2.12) <u>Lemma.</u>

Let ξ and η be fibre bundles over X and let

$$a, b : \xi \to \eta$$

be two isomorphisms. Suppose that a and b are isotopic when restricted to a set $A \subset X$ such that X can be deformed into A. Then a and b are isotopic over all of X.

<u>Proof:</u> Standard.

(2.13) <u>Corollary.</u>

Suppose that $A \subset X$ is a deformation retract and let

$r : X \to X$ be a retraction onto A. When ξ is a fibre bundle on X there are isomorphisms

$$\xi \to r^*\xi$$

which are the identity over A, and these are isotopic.

Proof: Standard.

When $f : X \to Y$ is a map of spaces, it is clear how to define a bundle map (or isomorphism) covering f (or over f). When

$$H : X \times I \to Y$$

is a homotopy, an isotopy of bundle maps covering the homotopy H is defined in the same way.

(2.14) Lemma.

Let $A \subset X$ be a strong deformation retract and let $r : X \to X$ be a retraction onto A. Let

$$H : X \times I \to X$$

be a homotopy from id_X to r rel. A. Let

$$a : \xi \to \xi$$

be a fibre bundle map covering r and isotopic to id_ξ by an isotopy covering H. Then the restricted map

$$a : \xi|A \to \xi|A$$

is isotopic to the identity and this fact uniquely determines the isotopy class (over r) of a.

Proof. Standard.

Now let X be a manifold and ξ a bundle on X. Let $pr_2 : X \times X \to X$ denote the second projection.

(2.15) <u>Lemma.</u>

The microbundles

$$T(E\xi)\,|\,X \quad \text{and} \quad (TX) \circ pr_2{}^*\xi$$

are isomorphic by an isomorphism given by a canonical homeomorphism of the total spaces.

<u>Proof.</u>

The diagram of $pr_2{}^*\xi$ is

$$X \times X \xrightarrow{(id,\, s_\xi pr_2)} (X \times X) \times_X E\xi \xrightarrow{pr} X \times X$$

where the fibered product is formed by means of the maps

$$X \times X \xrightarrow[pr_2]{} X \xleftarrow[pr_\xi]{} E\xi \ .$$

Hence the diagram of $(TX) \circ pr_2{}^*\xi$ is the upper line in the following diagram.

$$\begin{tikzcd}
X \arrow[r, "\Delta"] & X \times X \arrow[r, "{(id,\, s_\xi pr_2)}"] \arrow[dr, "{(id,\, s_\xi)}"'] & (X \times X) \times_X E\xi \arrow[r, "pr"] \arrow[d, "pr_1 \times id"] & X \times X \arrow[r, "pr_1"] & X \\
& & X \times E\xi \arrow[urr, "pr_1"'] & &
\end{tikzcd}$$

Because the diagram commutes, the vertical map is a homeomorphism which is canonical, and the broken line is the diagram of $T(E\xi)\,|\,X$, the proof is complete.

(2.16)

In practice, we choose bundles in the microbundles and apply (2.15) to these bundles. We can do so in view of (2.10).

§ 3. Orientations.

When h^* is a multiplicative cohomology theory and ξ is a bundle over X, a h^*-Thom class for ξ is an element

$$U \in h^*(X^\xi, pt)$$

such that when $F \subset X$ is a closed set over which ξ has constant rank, the restricted class

$$U_F \in h^*(F^\xi, pt)$$

is homogeneous.

When $F = P$, a point in X, we also claim that U_P shall be one of the canonical generators of the $h^*(pt)$-module $h^*(P^\xi, pt)$. There are at most two such generators corresponding to the two homotopy equivalences

$$P^\xi, pt \to S^q, pt \qquad \text{for some} \quad q \geq 0.$$

(When $q = 0$ there is only one such equivalence.) Note that if a class $U \in h^*(X^\xi, pt)$ restricts to some other generator in $h^*(P^\xi, pt)$ when $P \subset X$ is a point, there is a unit $\alpha \in h^0(X)$ such that αU restricts to a canonical generator for all $P \subset X$.

The bundle ξ is said to be h^*-orientable iff it has a h^*-Thom class. When A is a ring (commutative with 1), a $H^*(-;A)$-Thom class is called an A-Thom class.

(3.1) <u>Lemma.</u>

I. When U is a h^*-Thom class for ξ, all other h^*-Thom classes can be written

$$aU$$

where $a \in h^0(X)$ is unique and restricts to $\pm 1 \in h^0(P)$ when $P \subset X$ is a point.

II. When ξ and η are bundles on X and U and W are Thom classes for ξ and $\xi \oplus \eta$, there is a unique Thom class V for η such that

$$W = UV .$$

Proof.

I. By the Thom isomorphism theorem, any Thom class can be written uniquely as $a \cdot U$ where $a \in h^0(X)$. Let subscript $_P$ denote restriction to P and the Thom space over P. Then

$$a_P U_P \quad \text{and} \quad U_P$$

are canonical generators, hence $a_P = \pm 1$.

Conversely, when $a \in h^0(X)$ and $a_P = \pm 1$ for all $P \subset X$, aU is clearly a Thom class.

II. The product UV is determined by the diagonal map

$$\Delta : X^{\xi \oplus \eta} \to X^{\xi} \wedge X^{\eta} .$$

Choose some Thom class V' for η. Then

$$W = a(UV') \quad \text{and}$$
$$V = aV' \qquad \text{uniquely,}$$

according to I.

(3.2)

When $k \geq 1$ is an integer, we define a k-adic Thom class to be a class

$$U \in h^*(X^{\xi}, pt)$$

that fullfils all the requirements to a Thom class except that it restricts to k times a canonical generator in the fibres of the Thom space. A k-adic Thom class becomes a Thom class when we localize our theory in the multiplicative system $\{1, k, k^2, \cdots\} \subset k^0(pt)$.

We now make a digression to prove (3.3) and (3.4).

(3.3) The k-adic Dold theorem.

Let X be a finite-dimensional connected cell complex. When ξ is a bundle on X that has a k-adic Thom class, some Whitney-multiple $k^{\alpha}\cdot\xi$ is orientable provided either (i) or (ii) is true.

(i) X is simply connected.

(ii) $2k$ does not map to zero under $\mathbb{Z} \to h^{0}(pt)$.

Proof.

When $n \geq 1$ our induction hypotesis will be that some Whitney-multiple $k^{\alpha}\cdot\xi$ restricted to the n-sceleton X_n is orientable. We first prove this for $n = 1$.

Collapsing a maximal tree in X_1, we may assume that X_1 is a boquet of 1-spheres. If (i) is true, the map $X_1 \subset X$ is null-homotopic, and ξ is trivial over X_1. In any case it suffices to assume $X_1 = S^1$. If the orientation system of ξ restricted to $X_1 = S^1$ is trivial, there is still nothing to prove. In case it is nontrivial, we deduce that $2kg = 0$ where

$$g \in h^{q}(S^{q},pt)$$

is a generator. Hence $2k = 0$ in $h^{0}(pt)$, but this is not the case according to (ii).

Hence the induction hypotesis is true for $n = 1$ with $\alpha = 0$. Assume that it is true for some $n \geq 1$ and some α.

When A is the given k-adic Thom class for ξ, it is clear that $R = A^{\beta}$ is a k^{β}-adic Thom class for $\beta\cdot\xi$, we choose $\beta = k^{\alpha}$. By hypothesis there is a Thom class V for $\beta\cdot\xi$ over X_n. We may choose the sign of V such that

$$R_x = k^{\beta}V_x \quad \text{when} \quad x \in X^{n}.$$

We also assume that X_{n+1} is obtained from X_n by imbedding a boquet S of n-spheres and attaching the corresponding boquet D of $(n+1)$ - cells.

$$X_{n+1} = X_n \cup D, \quad X_n \cap D = S.$$

D being contractible, there is a Thom class U for $\beta \cdot \xi$ over D. We choose the sign of U such that

$$U_x = V_x \text{ when } x \in S .$$

(As before, restriction to F and the Thom space over F is denoted by subscript $_F$).

U_S and V_S are both Thom classes over S, hence

$$U_S = (1 + \epsilon)V_S \tag{I}$$

where $\epsilon \in h^0(S)$ and $\epsilon_x = 0$ for all $x \in S$.

By a similar argument, we get

$$R_{X_n} = (k^\beta + \delta)V \tag{II}$$

where $\delta \in h^0(X_n)$ is such that $\delta_x = 0$ for all $x \in X_n$ because we have $R_x = k^\beta V_x$. Taking $x \in S$ and using equations (I) and (II), we get

$$R_x = k^\beta V_x = k^\beta U_x.$$

Hence over the contractible space D we get

$$R_D = k^\beta U . \tag{III}$$

Now using equations (II), (III) and (I) successively, all restricted to S, we get

$$(k^\beta + \delta_S)V_S = R_S = k^\beta U_S = k^\beta(1 + \epsilon)V_S.$$

Hence

$$\delta_S = k^\beta \varepsilon .$$

Recall that $\varepsilon \in h^0(S,pt)$, so that $\varepsilon^2 = 0$ because $\Delta : S \to S \wedge S$ is nullhomotopic. Putting $m = k^\beta$ we get in $h^*(S^{m\beta\xi},pt)$ the following identity

$$\begin{aligned}(U^m)_S &= (U_S)^m = (1+\varepsilon)^m (V_S)^m = \\ (1+m\varepsilon)(V_S)^m &= (1+\delta_S)(V_S)^m = \\ &=((1+\delta)V^m)_S .\end{aligned}$$

Thus the two Thom classes U^m for $m\beta\xi$ over D and $(1+\delta)V^m$ for $m\beta\xi$ over X_n coincide over $S = D \cap X_n$. This implies that there is a Thom class for $m\beta\xi$ over X_{n+1}. Because $m\beta = k^{\beta+\alpha}$, we are done.

(3.4) Corollary. (Adams)

Let $E \to X$ be a q-sphere bundle over a connected finite dimensional cell complex X. Assume that there is a map

$$E \to S^q$$

of degree $\pm k$ in each fibre where $k \geq 1$. Then some k^α-fold Whitney join of E with itself is fibrehomotopically trivial.

Proof.

We use stable cohomotopy theory and observe that the map $E \to S^q$ determines a k-adic Thom class for the discbundle spanned by E. We also note that condition (ii) of (3.3) is fullfilled by cohomotopy theory. We denote the discbundle spanned by a sphere-bundle E by $\overline{E}$. Then

$$\overline{E * F} = \overline{E} \oplus \overline{F}$$

where $*$ is the Whitney join and $\oplus$ is the Whitney sum (or product). By (3.3) some multiple Whitney sum

$$\beta \cdot \overline{E}$$

is orientable where $\beta = k^{\alpha}$. Increasing α so that $\dim X + 2 \leq k^{\alpha}(q + 1)$ we are in the stable range and the cohomotopy Thom class of

$$\beta \cdot \overline{E} = \overline{*^{\beta} E}$$

is represented by a map

$$*^{\beta}E \to S^{\beta(q+1)-1}$$

of degree ± 1 in each fibre. By a theorem of Dold [8], $*^{\beta}E$ is fibrehomotopically trivial.

It is easily seen that corollary (3.4) is equivalent to theorem (1.1) of [1].

(3.5) Definition.

Let ξ, η be bundles on X, Y and let

$$\alpha : \xi \to \eta$$

be a bundle imbedding covering $f : X \to Y$.

We define

$$\alpha^{*} : h^{*}(Y^{\eta}, pt) \to h^{*}(X^{\xi}, pt)$$

as follows. Take a bundle-isomorphism α_1 isotopic to α over f. Then the map of Thom spaces

$$T(\alpha_1) : X^{\xi} \to Y^{\eta}$$

is defined. We put

$$\alpha^{*} = T(\alpha_1)^{*} .$$

Then α^{*} is well-defined because the isotopy class (over f) of α_1 as an isomorphism is uniquely determined according to (2.6).

When η is a bundle on Y and μ is a bundle on $E\eta$, we have the natural homotopy equivalence

$$Y^{\mu} \subset (E\eta)^{\mu} .$$

By means of this equivalence we identify the corresponding cohomology groups. In particular a Thom class for μ and a Thom class for $\mu|Y$ is the same thing.

(3.6) <u>Definition.</u>

Let λ and η be bundles on Y and let μ be a bundle on $E\eta$. Then there is a diagonal map

$$\Delta : Y^{\lambda\oplus(\eta o \mu)} \to Y^{\lambda\oplus\eta} \wedge Y^{\mu}$$

defined in the same way as the diagonal map of (1.4). The diagonal map is unique in homotopy. When S is a Thom class for $\lambda\oplus\eta$ and V is a Thom class for μ, we get a Thom class

$$SV = \Delta^*(S \wedge V)$$

for $\lambda \oplus (\eta o \mu)$.

We now give a definition of an orientation of a map $f : X \to Y$ of manifolds that does <u>not</u> use imbeddings of X in $Y \times \mathbb{R}^q$

(3.7) <u>Definition.</u>

A <u>Thom class</u> for a map $f : X \to Y$ of manifolds is a triple (α, U, V) Where α is a bundle imbedding covering f

$$\alpha : \tau X \, o \, \xi \to \tau Y \, o \, \eta$$

and U, V are Thom classes for the bundles ξ, η on $E\tau X, E\tau Y$.

We say that two Thom classes (α, U, V) and (α', U', V') for f are equivalent if there is a bundle λ on Y and Thom classes R for $f^*\lambda \oplus \tau X$ and S for $\lambda \oplus \tau Y$ such that

$$RU = (f_b \oplus \alpha)^*(SV)$$

and $$RU' = (f_b \oplus \alpha')^*(SV') .$$

Here we use definition (3.5) and $f_b \oplus \alpha$ is the bundle imbedding

$$f^*\lambda \oplus \tau X \circ \xi \to \lambda \oplus \tau Y \circ \eta .$$

Using (2.6) it is not difficult to see that this relation between Thom classes is an equivalence relation.

(3.8) Definition.

An orientation of a map $f : X \to Y$ is an equivalence class of Thom classes for f. An oriented map $X \to Y$ is a pair (f,ω) where f is a map $X \to Y$ and ω is an orientation of f.

This definition is justified by the following lemma.

(3.9) Lemma.

Let $(f,\omega) : X \to Y$ be an oriented map of manifold and let

$$\alpha : \tau X \circ \xi \to \tau Y \circ \eta$$

be a bundle imbedding covering f. When V is a Thom class for η, there is exactly one Thom class U for ξ such that (α,U,V) represents ω.

We omit the proof that U exists. Suppose that (α,U,V) and (α,U',V) are equivalent. With the notation of (3.7) we have

$$RU = (f_b \oplus \alpha)^*(SV) = RU'$$

and hence $U = U'$.

(3.10) Definition.

Given maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ of manifolds with Thom classes (α,U,V) and (β,V,W) where

$$\tau X \circ \xi \xrightarrow{\alpha} \tau Y \circ \eta \xrightarrow{\beta} \tau Z \circ u$$

are bundle imbeddings covering f and g. The composite Thom class is defined to be

$$(\beta\alpha,U,W) .$$

(3.11) Lemma.

Definition (3.10) gives a definition of composition of oriented maps by composing representing Thom classes. This composition operation is associative so that we have a category of manifolds and oriented maps.

Proof.

Let $(f,\omega) : X \to Y$ and $(g,\zeta) : Y \to Z$ be oriented maps. It is a consequence of (3.9) that ω and ζ may be represented by Thom classes as in (3.10). Suppose that

$$\tau X \circ \xi' \xrightarrow{\alpha'} \tau Y \circ \eta' \xrightarrow{\beta'} \tau Z \circ \mu'$$

and (α', U', V') , (β', V', W') is another such representation. We then have two Thom classes representing ω, and two representing ζ . It is then easily seen that there is a bundle λ on Z and Thom classes R,S,T for

$$f^*g^*\lambda \oplus \tau X, \quad g^*\lambda \oplus \tau Y, \quad \lambda \oplus \tau Z$$

such that

$$RU = (f_b \oplus \alpha)^*(SV) \ , \ RU' = (f_b \oplus \alpha')^*(SV')$$

and

$$SV = (g_b \oplus \beta)^*(TW) \ , \ SV' = (g_b \oplus \beta') \ (TW')$$

when the notation is as in (3.7). Note that

$$f_b \oplus \alpha \ : \ f^*g^*\lambda \oplus \tau X \circ \xi \to g^*\lambda \oplus \tau Y \circ \eta \ .$$

The four equations above imply

$$RU = (g_b f_b \oplus \beta\alpha)^*(TW)$$

and

$$RU' = (g_b f_b \oplus \beta'\alpha')^*(TW') \ .$$

That is, the two composed Thom classes are equivalent, and hence the composed orientation is well defined.

Associativity is clear from the definition and the neutral oriented map

$$(id,1) : X \to X$$

clearly exist.

(3.12) Definition.

When $f : X \to Y$ and $g : A \to B$ are maps with Thom classes (α, U, V) and (β, P, Q), we define the product Thom class for $f \times g$ to be

$$(\alpha \times \beta,\ U \wedge P,\ V \wedge Q).$$

(3.13) Lemma.

The above definition gives a product of oriented maps

$$(f, \omega) \times (g, \zeta) = (f \times g,\ \omega \times \zeta).$$

Proof. Trivial.

(3.14) Example.

When X is an oriented manifold, and P is a point, there is a canonical orientation for any map $f : P \to X$. The map

$$f_b : f^* \tau X \to \tau X$$

may be considered as a map

$$\alpha : \tau P \circ f^* \tau X \to \tau X \circ \varepsilon^0 .$$

When Ω is a Thom class for τX,

$$(\alpha, f_b{}^* \Omega, 1)$$

is a Thom class for $f : P \to X$.

Because f is canonically oriented when X is oriented, we shall consider f as an oriented map. Note that S^1 is canonically oriented so that any map $f : P \to S^1$ will be considered as oriented.

We now define a relation of homotopy in the category of manifolds and oriented maps.

(3.15) Definition.

Two oriented maps (f,ω) and (g,λ) : $X \to Y$ are homotopic if there is an oriented map

$$(H,v) : Y \times S^1 \to Y$$

such that $(f,\omega) = (H,v)(1_X \times j_1)$

and $(g,\lambda) = (H,v)(1_X \times j_2)$.

Here $(1_X \times j_1)$ denotes, by abuse of notation the composite oriented map

$$X = X \times P \xrightarrow[1_X \times j_1]{} X \times S^1 .$$

(3.16) Lemma.

The relation of homotopy defined in (3.15) is an equivalence relation on the set of oriented maps $X \to Y$. It is compatible with composition so that we get a homotopy category of oriented maps.

Proof.

The proof goes just as for the ordinary homotopy category.

(3.17) Remark.

We have defined homotopy classes of oriented maps $X \to Y$, but not oriented homotopy classes of maps $X \to Y$. In general homotopy classes of continuous maps can not be oriented. Let, for example, P be a point and X a connected manifold, nonorientable over $\mathbb{Z}$. Then every map $P \to X$ is orientable over $\mathbb{Z}$, and all maps $P \to X$ are homotopic. There is, however, no reasonable definition of an orientation for this homotopy class, as the reader will see by letting P trace through a loop in X that has no lifting to the orientation cover of X.

(3.18) Lemma.

When two continuous maps

$$f, g : X \to Y$$

are homotopic, and ω is an orientation for f, there exists an orientation λ for g such that (f,ω) is homotopic to (g,λ) .

Proof.

Let $H : X \times S^1 \to Y$ be a map such that $f = HJ_1$ and $g = HJ_2$ where

$$J_i : X \to X \times S^1$$

is the imbedding $J_i(x) = (x, z_i)$. The maps J_i are canonically oriented as in (3.15). We may also choose H so that it is homotopic to $f\,pr$ where $pr : X \times S^1 \to X$. Clearly pr and hence $f\,pr$ are orientable, hence H is orientable.

Consequently there exist orientable bundles ξ on X and η on Y and bundle imbeddings

$$\tau X \oplus \epsilon^1 \oplus \xi \xrightarrow{\alpha} \tau(X \times S^1) \oplus (\xi \times S^1) \xrightarrow{\beta} \tau Y \oplus \eta$$

covering J_1 and H. Take a Thom class representing ω of the form $(\beta\alpha, \Sigma\, U, V)$ where Σ is the canonical Thom class of ϵ^1 . Let v be the orientation of H represented by

$$(\beta,\ pr_b^*\, U, V).$$

Then

$$(f,\omega) = (H,v)J_1$$

where J_1 has the canonical orientation. We define λ by

$$(g,\lambda) = (H,v)J_2\ .$$

Then (g,λ) and (f,ω) are homotopic by definition.

§ 4. The Transfer Homomorphism.

In § 3 we defined in (3.16) a category consisting of manifolds and oriented maps. We also defined the corresponding homotopy category. To define the transfer homomorphism, we need the corresponding categories based on proper maps and proper homotopies.

(4.1) Definition.

The category of manifolds and oriented proper maps and the corresponding proper homotopy category are defined just as in (3.16), with the restriction that all maps be proper. It should be noted that the map H defined in (3.18) in this setting may be chosen as a proper map so that (3.18) is still true for the proper category.

Next comes the main construction of this §.

Given a proper map $f : X \to Y$ of manifolds and a bounded imbedding $i : E\xi \to E\eta$ of bundle spaces lifting f, that is so that $f = pr_\eta\, i\, s_\xi$. We shall construct a bundle imbedding

$$\alpha : \tau X \circ pr_2{}^*\xi \to \tau Y \circ pr_2{}^*\eta$$

covering f and unique in isotopy (over f). Here pr_2 denotes one of the composite maps

$$E\tau X \subset X \times X \xrightarrow{pr_2} X$$

$$E\tau Y \subset Y \times Y \xrightarrow[pr_2]{} Y$$

The imbedding

$$i\, s_\xi \times i : X \times E\xi \to E\eta \times E\eta$$

determines a bundle imbedding

$$(\tau E\xi)|X \to \tau E\eta$$

covering $i\, s_\xi$, when restricted to bundle spaces in the microbundles. Also there is a bundle-isomorphism covering pr_η

$$\tau(E\eta) \to (\tau E\eta) \mid Y$$

which is the identity on $Y \subset E\eta$. (2.13). The bundle imbedding α is the composite

$$\tau X \circ pr_2{}^*\xi = (\tau E\xi)\,|\,X \to \tau(E\eta) \to (\tau E\eta)\,|\,Y = \tau Y \circ pr_2{}^*\eta$$

where the first and the last isomorphism are defined in (2.17). Clearly α covers $pr_\eta i s_\xi = f$ in the base.

(4.2) <u>Definition.</u>

The bundle imbedding α defined above will be called the bundle imbedding determined by i .

(4.3) <u>Definition.</u>

When ω is an orientation of f , we define

$$t(i,\omega) : h^*(X) \to h^*(Y)$$

as follows:

When V is a Thom class for η , let U be the unique (3.9) Thom class for ξ such that (α,U,V) represents ω. Then define

$$t(i,\omega) = t(i,U,V) \text{ .} \qquad (1.10)$$

This definition is independent of the choice of V. When we take another Thom class aV for η with $a \in h^0(Y)$, ω is represented by $(\alpha,\ f^*(a)U,\ aV)$. This is an easy consequence of the definition (3.7) of equivalent Thom classes. From (1.13) we conclude that

$$t(i,U,V) = t(i,f^*(a)U,\ aV).$$

(4.4) <u>Proposition.</u>

Let $(f,\omega) : X \to Y$ be an oriented proper map and $i : E\xi \to E\eta$ an open bounded imbedding lifting f.

When μ is a bundle on $E\eta$ (such that $\mu = pr_\eta{}^*\mu$) , we have

$$t(i,\omega) = t(\tau,\omega)$$

where τ is the induced imbedding determined by μ . (1.4).

Proof:

We recall that τ is an imbedding

$$\tau : E(\xi \circ i^*\mu) \to E(\eta \circ \mu) .$$

Let (α, U, V) be a Thom class representing ω. Here α is the bundle imbedding determined by i. (4.2) When A is a Thom class for μ, there is a Thom class $(\bar{\alpha}, UB, VA)$ representing ω where B is a Thom class for $i^*\mu$ and $\bar{\alpha}$ is the bundle imbedding determined by τ. Suppose that $B = i^*A$. Then we get from (1.13)

$$t(\tau,\omega) = t(\tau, U_i{}^*A, VA) = t(i,U,V) = t(i,\omega)$$

because $$Ui^*A = Uf_b{}^*A .$$

Hence it suffices to prove that $B = i^*A$ or in other words that (α, U, V) and $(\bar{\alpha}, Ui^*A, VA)$ are equivalent Thom classes. The proof of this statement is tedious and requires a lot of diagrams, hence we skip it. The proof goes as follows: Choose R and S so that

$$RU = (f_b \oplus \alpha)^*(SV) \quad \text{in the notation of (3.7)}$$

and prove that

$$RUi^*A = (f_b \oplus \bar{\alpha})^*(SVA) .$$

(4.5) Proposition.

(i) Let $(f,\omega) : X \to Y$ and $(g,u) : Y \to Z$ be proper oriented maps and let $i : E\xi \to E\eta$, $j : E\eta \to E\lambda$ be bounded open imbeddings of bundle spaces lifting f and g respectively. Then

$$t(ji, u\omega) = t(j,u)t(i,\omega)$$

provided there is a splitting $\eta = \eta_1 \oplus \eta_2$ such that

$$is_\xi(X) \subset E\eta_1$$

and

$$g\, pr_\eta = pr_\lambda j \quad \text{restricted to } E\eta_1 .$$

(ii) The conclusion $t(ji,u\omega) = t(j,u)t(i,\omega)$ also holds in case $is_\xi(X) \subset s_\eta(Y)$.

(iii) When the maps f and g are given, we can always find imbeddings i and j with the properties described in (i).

Proof.

(i) $u\omega$ is an orientation of gf . In order that the proposition make sense, we must have $gf = pr_\lambda j i s_\xi$. But $g\, pr_\eta i s_\xi(X) = pr_\lambda j i s_\xi(x)$ because $is_\xi(X) \subset E\eta_1$. That is $gf(x) = pr_\lambda jis_\xi(x)$.

Let α, β and γ be the bundle imbeddings determined by i, j and ji . Then there are Thom classes U, V and W for ξ, η and λ such that $t(i,\omega) = t(i,U,V)$, $t(j,u) = t(j,V,W)$ and hence $t(j,u)t(i,\omega) = t(ji,U,W)$ according to (1.11). Now $(\beta\alpha, U,W)$ is a Thom class representing $u\omega$. If γ is isotopic (over gf) to $\beta\alpha$, (γ,U,W) is another representing Thom class for $u\omega$, $t(ji,U,W) = t(ji,u\omega)$, and (i) is proved.

To prove that γ and $\beta\alpha$ are isotopic, we consider a diagram defining α and β .

$$\begin{tikzcd}
X \times E\xi \arrow[r, "a"] \arrow[dr, "\alpha"'] & E\eta \times E\eta \arrow[d, shift right, "p"'] \arrow[dr, "j \times j"] & \\
 & Y \times E\eta \arrow[u, shift right, "c"'] \arrow[r, "b"] \arrow[dr, "\beta"'] & E\lambda \times E\lambda \arrow[d, "q"] \\
 & & Z \times E\lambda
\end{tikzcd}$$

In this diagram (that does not commute) $a = is_\xi \times i$, $b = js_\eta \times j$, $c = s_\eta \times id$, p is a bundle isomorphism covering pr_η equal to the identity over Y and q is a similar isomorphism covering pr_λ . By definition $\alpha = pa$ and $\beta = qb$.

Although some of the maps in the diagram are defined on the whole total space, they should be considered as bundle maps defined on some nbd. of the zero section. That is, the diagram looks like

$$\begin{tikzcd}
(\tau E\xi) \mid X \arrow[r] \arrow[dr] & \tau E\eta \arrow[d, shift right] \arrow[dr] & \\
& (\tau E\eta) \mid Y \arrow[u, shift right] \arrow[r] & \tau E\lambda \arrow[d] \\
& & (\tau E\lambda) \mid Z .
\end{tikzcd}$$

Now let $D_t : E\eta \to E\eta$ be a deformation of $E\eta$ rel Y such that

$$D_1 = \mathrm{id} \ , \quad D_o = s_\eta \, pr_\eta \ .$$

Then there is an isotopy

$$A_t : \tau E\eta \to \tau E\eta$$

covering D_t and equal to the identity over $s_\eta(Y)$. According to (2.16) we may choose p such that $cp = A_o$. This gives an isotopy

$$q(j \times j)A_t a$$

from $q(j \times j)a = \gamma$ to $q(j \times j)cpa = qb\alpha = \theta\alpha$ covering $pr_\lambda j \, D_t \, is_\xi : X \to Z$. If this homotopy is stationary, the proof is finished. We choose the deformation D_t such that the fibres of $E\eta_1 \subset E\eta$ are preserved. Then

$$pr_\eta \, D_t = pr_\eta \quad \text{on} \quad E\eta_1$$

and $\qquad D_t \, is_\xi(X) \subset E\eta_1$. Consequently

$$pr_\lambda j \, D_t \, is_\xi = g \, pr_\eta \, D_t \, is_\xi = g \, pr_\eta \, is_\xi = gf \ .$$

(ii) Trivial. Take $\eta_1 = 0$, $\eta_2 = \eta$, and use (i) .

(iii) Let λ_1 be a bundle on Z such that there is an imbedding

$$i_1 : X \to Eg^* \lambda_1$$

lifting f and admitting a normal bundle. (λ_1 may be chosen as a trivial bundle.) Let

$$j_2 : E\eta_2 \to E\lambda_2$$

be an open bounded imbedding lifting g. We have the induced imbedding

$$j = \overline{j_2} : E(\eta_2 \circ j_2^* \, pr_{\lambda_2}^* \, \lambda_1) \to E(\lambda_2 \oplus \lambda_1) .$$

Using the isomorphism

$$\eta_2 \circ j_2^* \, pr_{\lambda_2}^* \, \lambda_1 \cong \eta_2 \oplus \eta_1$$

with $\eta_1 = g^*\lambda_1$, the composite imbedding

$$X \to E\eta_1 \subset E(\eta_2 \oplus \eta_1)$$

visibly maps X to $E\eta_1$. We must also show that

$$g \, pr_\eta = pr_\lambda \, j \quad \text{over} \quad E\eta_1$$

when

$$\eta = \eta_2 \oplus \eta_1 \quad \text{and} \quad \lambda = \lambda_2 \oplus \lambda_1 .$$

Because $g = pr_\lambda \, j \, s_\eta$, we must show that

$$pr_\lambda \, j \, s_\eta \, pr_\eta = pr_\lambda \, j \quad \text{over} \quad E\eta_1 .$$

This equation simply states that each fibre of $E\eta_1$ is mapped by j into some fibre of $E\lambda$. Because j is an induced imbedding, this is clearly true.

(4.6) Lemma.

Let ξ be an orientable bundle on the manifold X and let

$$i : E\xi \to E\xi$$

be an open bounded imbedding that restricts to the identity on $X \subset E\xi$. Then

$$t(i,1) = id$$

when 1 is the neutral orientation of the identity map $X \to X$.

Proof.

Let

$$\alpha : \tau X \circ pr_2{}^*\xi \to \tau X \circ pr_2{}^*\xi$$

be the bundle imbedding determined by i and let (α, U, V) be a Thom class representing 1. Because

$$t(i,1)(x) = xt(i,1)(1)$$

according to (1.12), it suffices to prove that $t(i,1) = 1$, or that $V = (D^i)^*(U)$. (It is true more or less trivially that $t(i,1)$ is a unit in $h^0(X)$).

Because (α, U, V) represents 1, there is a bundle ν on X such that when

$$id \oplus \alpha : \nu \oplus (\tau X \circ pr_2{}^*\xi) \circlearrowleft ,$$

we have

$$RU = (id \oplus \alpha)^* RV$$

when R is a Thom class for $\nu \oplus \tau X$. The microbundle diagram for $\nu \oplus (\tau X \circ pr_2{}^*\xi)$ is

$$X \xrightarrow{(s_\nu, s_\xi)} E\nu \times E\xi \xrightarrow{pr_\nu pr_1} X ,$$

and the map $id \oplus \alpha$ is simply

$$id \times i : E\nu \times E\xi \to E\nu \times E\xi$$

on the total space. The map

$$\wedge : X^{\nu\oplus(\tau X o pr_2{}^*\xi)} \to X^{\nu\oplus\tau X} \wedge X^\xi$$

that determines the product Thom classes is the Thom space map derived from a closed imbedding

$$\beta : E(\nu \oplus (\tau X \circ pr_2{}^*\xi)) \to E(\nu \oplus \tau X) \times E\xi$$

restricting to the diagonal map $X \to X \times X$ in the base. On the total spaces β is given by

$$\beta : E\nu \times E\xi \to (E\nu \times X) \times E\xi$$

where

$$\beta(v,w) = (v, pr_\xi(w), w) .$$

This fact follows from the definition (3.6) after some calculation. Note that the diagram of $\nu \oplus \tau X$ is

$$X \xrightarrow{(s_\nu, id)} E\nu \times X \xrightarrow{pr_\nu} X .$$

To obtain a Thom space map from β , we collapse closed sets whose complements are sufficiently small bundle nbd's.

We shall prove that the diagram of bundlespaces

(+)
$$\begin{tikzcd} E\nu \times E\xi \arrow[r, "\beta"] \arrow[d, "id \times i"] & (E\nu \times X) \times E\xi \arrow[d, "id \times id \times i"] \\ E\nu \times E\xi \arrow[r, "\beta"'] & (E\nu \times X) \times E\xi \end{tikzcd}$$

gives rise to a homotopy commutative diagram

(*)
$$\begin{tikzcd} X^{\nu\oplus(\tau X \circ pr_2{}^*\xi)} \arrow[r, "\Delta"] \arrow[d, "T(id\oplus\alpha)"] & X^{\nu\oplus\tau X} \wedge X^\xi \\ X^{\nu\oplus(\tau X \circ pr_2{}^*\xi)} \arrow[r, "\Delta"] & X^{\nu\oplus\tau X} \wedge X^\xi \arrow[u, "id \wedge D^i"'] \end{tikzcd} .$$

If this is so, we get

$$RU = \Delta^*(R \wedge U) = T(id\oplus\alpha)^*\Delta^*(id \wedge D^i)^*(R \wedge U) = (id\oplus\alpha)^*(R(D^i)^*(U)).$$

But

$$RU = (id \oplus\alpha)^*(RV), \text{ hence } RV = R(D^i)^*(U) \text{ and } V = (D^i)^*(U).$$

To prove the proposed homotopy commutativety of the diagram (*), we first deform β . Let $D_t : E\xi \to E\xi$ be a deformation rel. X with $D_1 = id$, $D_o = s_\xi\, pr_\xi$. Then

$$\beta_t(v,w) = (v, pr_\xi D_t i(w), w)$$

is a closed imbedding

$$E\nu \times E\xi \to (E\nu \times X) \times E\xi$$

preserving the zero section. We get a homotopy of Thom space maps $T(\beta_t)$ with $T(\beta_o) = \Lambda$ because $\beta_o = \beta$. Thus it suffices to prove that Λ is homotopic to

$$T(\mathrm{ide} \oplus \alpha)T(\beta_1)(\mathrm{id} \wedge D^i) .$$

But this is a consequence of the fact that

$$(+\ 1) \qquad \begin{array}{ccc} E\nu \times E\xi & \xrightarrow{\beta_1} & (E\nu \times X) \times E\xi \\ \big\downarrow \mathrm{id} \times i & & \big\downarrow \mathrm{id} \times \mathrm{id} \times i \\ E\nu \times E\xi & \xrightarrow[\beta]{} & (E\nu \times X) \times E\xi \end{array}$$

commutes.

<u>Note:</u> Lemma (4.6) is a direct consequence of (4.4) and (4.9) when we use a theorem of M.W. Hirsch [12] on the stable isotopy-uniqueness of normal bundles.

(4.7) <u>Corollary.</u>

Let $(f,\omega) : X \to Y$ be an oriented proper map. Let ξ and λ be bundles on X and let η be a bundle on Y. When

$$i : E\xi \to E\eta \quad \text{and} \quad j : E\lambda \to E\eta$$

are open bounded imbeddings lifting f such that $is_\xi = js_\lambda$, we have $t(i,\omega) = t(j,\omega)$.

<u>Proof.</u>

Obviously the bundles ξ and λ have the same stable class. Because an induced imbedding gives the same transfer map (4.4), we may suppose that ξ and λ are isomorphic, by adding a trivial bundle to η. Now let $k : E\xi \to E\lambda$ be a <u>bundle</u> imbedding so small that $ir = jk$ for some open imbedding $r : E\xi \to E\xi$. (That is so small that $jk(E\xi) \subset i(E\xi)$.) Then r is the identity on X. Because $(f,\omega) = (f,\omega)(\mathrm{id}, 1)$, we get from (4.5 ii)

$$t(i,\omega) = t(i,\omega)t(r,1) = t(ir,\omega) = t(jk,\omega) = t(j,\omega)t(k,1) = t(j,\omega).$$

Here we used (4.6) to get $t(r,1) = id$ and the trivial fact that $t(k,1) = id$. (Note that k preserves fibres.)

(4.8) Proposition.

The transfer homomorphism is independent of the bounded imbedding used to define it. Precisely :

Let $(f,\omega) : X \to Y$ be a proper oriented map, let $i : E\xi \to E\eta$ and $j : E\lambda \to E\mu$ be open bounded imbeddings lifting f. When η and μ are orientable, we have

$$t(i,\omega) = t(j,\omega).$$

Proof.

Because induced imbeddings give the same transfer maps, we may assume that η and μ are vectorbundles, trivial for instance. In case $is_\xi(X)$ intersects the zero section in η, we may add a trivial bundle to η and isotope X and its normal bundle away from the zero section without changing the transfer map according to (4.4), (4.7) and (4.9). Hence we may assume that $is_\xi(X)$ does not intersect the zero section in η. Now define

$$J : X \times \mathbb{R} \to E(\eta \oplus \mu \oplus \varepsilon^2)$$

by

$$J(x,t) = (t\cdot is_\xi(x),\ (1-t)\cdot js_\lambda(x),\ t(1-t), t^2(1-t)).$$

Here ε^2 denotes the trivial bundle of dimension 2. Clearly J is a closed imbedding such that

$$J_0 = (0,\ js_\lambda,\ 0)$$

$$J_1 = (is_\xi,\ 0,\ 0)$$

correspond to imbeddings induced from i and j. After adding trivial bundles, we may assume that J admits a normal bundle. This implies that the isotopy J_t of X may be extended to an isotopy of normal bundles. Because the actual choice of normal bundle does not matter according to (4.7), we may write

$$t(i,\omega) = t(is_\xi,\ \omega) = t(J_1,\omega) = t(J_0,\omega) = t(js_\lambda,\omega) = t(j,\omega),$$

using (4.9).

(4.9) Lemma.

Let $i_t : E\xi \to E\eta$ be an isotopy of bounded imbeddings lifting $f : X \to Y$. Then

$$t(i_0,\omega) = t(i_1,\omega) .$$

Proof.

This is an immediate consequence of the definition (4.2) of the bundle imbedding α_t determined by i_t and the definition (4.3) of $t(i,\omega)$.

(4.10) Theorem.

On the proper homotopy category of manifolds and oriented maps defined in (4.1) there is a covariant functor taking the map

$$(f,\omega) : X \to Y$$

to the homomorphism $f_\omega : h^*(X) \to h^*(Y)$.

When $i : E\xi \to E\eta$ is an open bounded imbedding of bundle spaces lifting f (that is $f = pr_\eta\, i\, s_\xi$) , we have

$$f_\omega = t(i,\omega)$$

as defined in (4.3), provided η , and hence ξ , is orientable.

The following equation holds

$$f_\omega(f^*(y)x) = yf_\omega(x)$$

showing that f_ω is a $h^*(Y)$-module homomorphism.

Proof.

Proposition (4.8) shows that f_ω is well defined by $f_\omega = t(i,\omega)$. Proposition (4.5 iii) implies that f_ω is a functor on the category of manifolds and proper oriented maps. The homotopy invariance of f_ω is seen as follows, we use the notation of (3.15).

Two properly homotopic oriented maps may be put in the form

$$(H,v)(1_X \times j_1) \quad \text{and} \quad (H,v)(1_X \times j_2) .$$

Because the transfer homomorphism is a functor, it suffices to show that the "universal" homotopic maps $1_X \times j_1$ and $1_X \times j_2$ give the same transfer. But this is clear from their definition (3.15).

§ 5. Applications.

The Riemann-Roch Theorem.

Our main application of the transfer homomorphism will be a proof of Atiyah's theorem [4] that when $f : X \to Y$ is a homotopy-equivalence of compact manifolds,

$$J(\tau X) = f^* J(\tau Y) \ .$$

We shall prove this theorem without the compactness condition, provided f is a proper homotopy equivalence.

(5.1) Theorem.

Let $f : X \to Y$ be a proper map of connected manifolds of the same dimension. If f admits a one-sided proper homotopy inverse,

$$J(\tau X) = f^* J(\tau Y) \ .$$

Moreover f induces an isomorphism in any multiplicative cohomology theory.

Proof.

We may assume that f has a left inverse g so that gf is properly homotopic to the identity on X. We must prove that the conclution of the theorem is true for both f and g.

We first apply singular cohomology mod 2. Then every map has a unique orientation (use lemma (3.9)) and we denote the transfer homomorphism by $f_!$. Also gf is homotopic to id as an oriented map, according to (3.18). Consequently

$$g_!(f_!(1)) = (gf)_!(1) = 1 \ , \ f_!(1) \neq 0 \quad \text{and} \quad f_!(1) = 1$$

because $H^0(Y, \mathbb{Z}_2) = \mathbb{Z}_2$.

Now let η be a bundle on Y of stable class $g^*(\tau X) - \tau Y$, and let $i : E\xi \to E\eta$ be an open bounded imbedding lifting f.

Then the stable class of ξ is

$$\xi = f^*(\tau Y + \eta) - \tau X = f^*(\tau Y + g^*\tau X - \tau Y) - \tau X = f^*g^*\tau X - \tau X = 0 .$$

By adding a trivial bundle to η and taking an induced imbedding, we may assume that ξ is trivial. Note that

$$J(\eta) = g^*J(\tau X) - J(\tau Y)$$

and

$$f^*J(\eta) = J(\tau X) - f^*J(\tau Y) .$$

Hence the main assertion of the theorem is equivalent to $J(\eta) = 0$.

Let

$$D^i : Y^\eta \to X^\xi$$

be the Thom map, and let u,v be the mod 2 Thom classes for ξ,η. Then

$$(D^i)^*(xu) = f_!(x)v \quad \text{and}$$

$$(D^i)^*(u) = v .$$

Because ξ is trivial, u is the reduction of an integral Thom class U. We define $V = (D^i)^*(U)$. In the "fibres" of the Thom space Y^η the class V does not restrict to 0 because its mod 2 reduction v does not. Hence η is orientable over Q, consequently over $\mathbb{Z}$. Let V' be a Thom class for η such that $V = mV'$, $m > 0$. Let z be the orientation of f determined by U and V' and let ω be some $\mathbb{Z}$-orientation of g. Note that g is orientable over $\mathbb{Z}$ because its normal bundle is η. Then $(g,\omega)(1,z)$ is homotopic to (id,χ) for some orientation χ of id, according to (3.18). We have $g_\omega f_z = (gf)_{\omega z} = id_\chi$ and

$$g_\omega(f_z(1)) = id_\chi(1) = \pm 1 .$$

But

$$(D^i)^*(U) = f_z(1)\cdot V' = V = mV'$$

and

$$g_\omega(f_z(1)) = g_\omega(m) = mg_\omega(1) = \pm 1 .$$

Because $H^0(X,\mathbb{Z}) = \mathbb{Z}$, we get $m = \pm 1$ and $m = 1$. Thus $V = V'$ is a Thom class for η over $\mathbb{Z}$. Let q be the fibre dimension of ξ and let

$$p : X^\xi \to S^q$$

be the canonical map. We may suppose that $U = p^*(U_0)$ where

$$U_0 \in \tilde{H}^q(S^q)$$

is a generator. The map

$$pD^i : Y^\eta \to S^q$$

has the property that $(pD^i)^*(U_0) = V$ is a $\mathbb{Z}$-Thom class for η. Consequently pD^i has degree 1 when restricted to a "fibre" in the Thom space. A theorem of Dold [8] implies that $J(\eta) = 0$.

When h^* is a multiplicative cohomology theory, let $U_1 \in \tilde{h}^q(S^q)$ be the suspension of the unit. Also let $V_1 = (D^i)^* \, p^*(U_1)$. Then $p^*(U_1)$ and V_1 together with i determine an h^*-orientation ρ of f. We get

$$f_\rho f^*(y) = y f_\rho(1) = y$$

because

$$(D^i)^*(p^* U_1) = f_\rho(1) V_1 = V_1$$

by definition of ρ.

Hence $f_\rho f^* = \mathrm{id}$. We also have $f^* g^* = \mathrm{id}$. That is

$$g^* = f_\rho = (f^*)^{-1} .$$

(5.2) <u>Theorem.</u>

When X and Y are manifolds and $f : X \to Y$ is a proper homotopy-equivalence,

$$f^* J(\tau Y) = J(\tau X) .$$

<u>Proof.</u>

We may assume that X and Y are connected. The conclusion follows from (5.1) if $\dim X = \dim Y$. Using cohomology mod 2, we note that $\deg f_! = \dim Y - \dim X$. Let g be an inverse of f in the proper homotopy category. We get

$$f_!(1) \neq 0 \quad \text{and} \quad g_!(1) \neq 0 \quad \text{because} \quad g_! f_!(1) = f_! g_!(1) = 1 .$$

Consequently $\deg f_!(1) \geq 0$ and $\deg g_! \geq 0$ which implies $\dim X = \dim Y$.

Q.E.D.

We note that Atiyah's proof does not work in the case of non-compact manifolds. His proof uses S-duality, and the theory of S-duality does not work as it should for non-compact manifolds. As an example, let M be a manifold with boundary $\partial M \neq \emptyset$. The usual S-dual of $\operatorname{int} M$ and of M is $M^\nu/(\partial M)^\nu$ where ν is a normal bundle. But the space we have used to define the transfer homomorphism is $(\operatorname{int} M)^\nu$ or M^ν. This fact suggests a definition of S-duality for locally compact non-compact spaces.

(5.3) <u>Theorem.</u>

The tanget bundle of the manifold X is J-equivalent to a vector bundle if and only if some product

$$X \times \mathbb{R}^q$$

has the proper homotopy type of a differentiable manifold.

<u>Proof.</u>

If $X \times \mathbb{R}^q$ has the proper homotopy type of a differentiable manifold, theorem (5.2) implies that $\tau(X \times \mathbb{R}^q)$ and hence τX is J-equivalent to a vector bundle.

Now suppose that $J(\tau X) = J(\beta)$ where β is a vector bundle on X. Also let ν be a normal bundle of X in some euclidean space. Then $E\nu$ has a differentiable structure, and so has $E(\nu \oplus \beta)$ which is the total space of a vectorbundle on $E\nu$. We have $J(\nu \oplus \beta) = J(\nu \oplus \tau X) = 0$. Adding a suitable trivial bundle to β, we conclude that $E(\nu \oplus \beta)$ has the proper homotopy type of $X \times \mathbb{R}^q$. In fact they have the same proper fiber-homotopy type.

We are now going to prove the usual Riemann-Roch and Wu theorems. [5] The proofs are entirely formal. We assume that a multiplicative transformation of cohomology theories is given, define the corresponding "Todd class" of an oriented map, and the R.-R. theorem drops out. The arguments are well established in the litterature [2], [3], [11], and our proofs will be short.

Up to this point we have assumed that a fixed multiplicative cohomology theory h^* is given. Now let k^* be another such theory, and let $\lambda : k^{**} \to k^{**}$ be a multiplicative cohomology transformation. That is :

(i) $\lambda(X,A) : k^{**}(X,A) \to h^{**}(X,A)$ is a natural ring homomorphism,

(ii) λ commutes with the coboundary homomorphisms in the long cohomology sequences.

(iii) $\lambda(pt) : k^{**}(pt) \to h^{**}(pt)$ satisfies $\lambda(pt)(1) = 1$. Here

$$k^{**}(X,A) \subseteq \prod_{q=-\infty}^{+\infty} k^q(X,A)$$

denotes either the subring of elements vanishing below some non-fixed degree or the subring of elements vanishing above some non-fixed degree.

(5.4) <u>Lemma.</u>

2λ preserves the $\mathbb{Z}_2$-gradings of $k^{**}(X)$ and $h^{**}(X)$.

Proof.

Let $z \in k^1(S^1)$ and $w \in h^1(S^1)$ be the suspension of the units. Then $\lambda(z) = w$. When $b \in h^{**}(X)$ is a homogeneous element, we know that

$$T^*(b \times w) = (-1)^{\deg(b)} w \times b$$

when $T : S^1 \times X \to X \times S^1$ is the twisting map. Consequently $2b$ is even (odd) if and only if

$$T^*(b \times w) = \epsilon w \times b$$

where $\epsilon = 1$ $(\epsilon = -1)$ and $b \in h^{**}(X)$ is arbitrary. Now let $a \in k^{**}(X)$ be an even (odd) element. Then

$$T^*(a \times z) = \epsilon z \times a .$$

Using λ we get

$$T^*(\lambda(a) \times w) = \epsilon w \times \lambda(a) ,$$

hence $2\lambda(a)$ is an even (odd) element.

(5.5) Definition.

$\lambda' : k^* \to h^*$ is the cohomology transformation defined by $\lambda'(x) =$ the homogeneous component of $\lambda(x)$ in degree n, when $x \in k^n(X)$.

λ' has all the properties of λ, except multiplicativity. In many well known cases λ' is the identity transformation. Because λ preserves suspensions of units, λ' does so too. When U is a k^*-Thom class for a bundle, $\lambda'(U)$ is a h^*-Thom class.

(5.6) Definition.

When ξ is a bundle on a space X and U is a k^*-Thom class for ξ, let

$$\underline{\lambda}(\xi,U) \in h^{**}(X)$$

be defined by

$$\underline{\lambda}(\xi,U)\lambda'(U) = \lambda(U) .$$

(5.7) Lemma.

(i) $2\underline{\lambda}(\xi,U)$ is an even element.

(ii) $\underline{\lambda}$ is natural, $f^*\underline{\lambda}(\xi,U) = \underline{\lambda}(f^*\xi, f_b^*(U))$ when $f : B \to X$ is a map to the base of ξ .

(iii) $\underline{\lambda}(\xi,U)$ is invertible in $h^{**}(X)$.

Proof.

(i) We have $2\underline{\lambda}(\xi,U)\lambda'(U) = 2\lambda(U)$ where $\lambda'(U)$ is a Thom class and the elements U, $\lambda'(U)$ and $2\lambda(U)$ have the same $\mathbb{Z}_2$-degree according to (5.4).

(ii) Obvious.

(iii) Let τ be a bundle on X such that $\xi \oplus \tau$ is trivial and let V be a k^*-Thom class for τ such that the Thom class UV for $\xi \oplus \tau$ is a suspension of the unit pulled up to the Thom space. Then $\lambda(UV) = \lambda'(UV)$, We have

$$\underline{\lambda}(\xi,U)\lambda'(U) = \lambda(U)$$

and

$$\underline{\lambda}(\tau,V)\lambda'(V) = \lambda(V) ,$$

consequently

$$\underline{\lambda}(\xi,U)\underline{\lambda}(\tau,V)\lambda'(U)\lambda'(V) = \lambda(UV) = \lambda'(UV) ,$$

where we use (i) to conclude that $\underline{\lambda}(\tau,V)$ commutes with every other element. In this equation, bot $\lambda'(U)\lambda'(V)$ and $\lambda'(UV)$ are h^*-Thom classes. This implies that $\underline{\lambda}(\xi,U)\underline{\lambda}(\tau,V)$ and hence $\underline{\lambda}(\xi,U)$ is a unit.

(5.8) <u>Lemma.</u>

Let $f : X \to Y$ be a proper k^*-orientable map of manifolds. Let

$$i : E\xi \to E\eta$$

be an open bounded imbedding of bundlespaces lifting f, and let U and V be k^*-Thom classes for ξ and η. When we put

$$t^k = t(i,U,V) \quad \text{and} \quad t^h = t(i,\lambda'(U),\lambda'(V)) ,$$

we get

$$\lambda(t^k(x))\cdot\underline{\lambda}(\eta,V) = t^h(\lambda(x)\cdot\underline{\lambda}(\xi,U)$$

and

$$\lambda(t^k(x)) = t^h(\lambda(x)\cdot a)$$

where

$$a = \underline{\lambda}(\xi,U)f^*\underline{\lambda}(\eta,V)^{-1} .$$

<u>Proof.</u>

$$\lambda(t^k(x))\underline{\lambda}(\eta,V)\lambda'(V) = \lambda(t^k(x))\lambda(V) =$$

$$\lambda(t^k(x)V) = \lambda(D^i)^*(xU) = (D^i)^*\lambda(xU) =$$

$$(D^i)^*(\lambda(x)\underline{\lambda}(\xi,U)\lambda'(U)) = t^h(\lambda(x)\cdot\underline{\lambda}(\xi,U)\lambda'(V) .$$

The conclusion follows because $\lambda'(V)$ is a Thom class. As a consequence

$$t^h(\lambda(x)\cdot a)\underline{\lambda}(\eta,V) = t^h(\lambda(x)a\, f^*\underline{\lambda}(\eta,V)) =$$

$$t^h(\lambda(x)\underline{\lambda}(\xi,U)) = \lambda(t^k(x))\underline{\lambda}(\eta,V) ,$$

where we use (1.12) and (5.7, i). Because $\underline{\lambda}(\eta,V)$ is a unit in $h^{**}(Y)$, it may be cancelled. We note that $\underline{\lambda}(\eta,V)^{-1}$ is well-defined because $\underline{\lambda}(\eta,V)$ is in the centre of $h^{**}(Y)$ according to (5.7, i)

We would like the following assertion to be true.

(5.9) <u>Assertion.</u>

When ξ, ξ', $\xi \oplus \xi'$ are bundles with Thom classe U,U' and

UU', we have

$$\underline{\lambda}(\xi,U)\underline{\lambda}(\xi',U') = \underline{\lambda}(\xi \oplus \xi', UU') .$$

From definition (5.6) we see that the assertion holds if λ' is multiplicative. We note that λ' is multiplicative if λ is monotonic in the sense that when $x \in k^n(X)$, $\lambda(x)$ is zero in degrees $< n$.

The Steenrod operations Sq and P in cohomology mod 2 and mod p are monotonic. The Chern character

$$ch : KU^*(X) \to H^{**}(X,Q)$$

is not monotonic. This is a consequence of the formula $ch(\beta x) = ch(x)$ where $\beta \in KU^{-2}(pt)$ is a Bott element. Nevertheless (5.9) is true for ch .

(5.10) Lemma.

If assertion (5.9) is true, a k^*-oriented map gives rise to a h^*-oriented map in a canonical way.

When (f,ω) is k^*-oriented, we obtain an element $\underline{\lambda}(\omega) \in h^{**}(X)$ to be defined below.

Proof.

When $f : X \to Y$ is a map of manifolds, and (α,U,V) is a Thom class representing its k^*-orientation, we let $(\alpha,\lambda'(U),\lambda'(V))$ represent the h^*-orientation. This gives a well defined h^*-orientation according to (5.9), (5.7 ii) and (3.7).

We define

$$\underline{\lambda}(\omega) = \underline{\lambda}(\xi,U)f^*\underline{\lambda}(\eta,V)^{-1}$$

where $\alpha : \tau X \circ \xi \to \tau Y \circ \eta$ is the bundle-isomorphism in (α,U,V). That this is independent on the choice of Thom class-representing ω follows from (5.9), (5.7,ii) and (3.7).

(5.11) Theorem. (Riemann-Roch).

$$\lambda(f_\omega(x)) = f_{\lambda(\omega)}(\lambda(x)\cdot\underline{\lambda}(\omega))$$

where $(f,\lambda(\omega))$ is the h^*-oriented map obtained from (f,ω) by means of λ, assuming (5.9) to be true.

Proof.

This is an immediate consequence of (5.8) using the definitions of (5.10). Note that when (5.9) is not true, we may regard (5.8) as the R.-R. theorem.

We now consider k^*-oriented manifolds and continuous maps $f : X \to Y$ of such manifolds. A k^*-oriented manifold is a manifold X together with a Thom class U in the tangent bundle. When Y is k^*-oriented by the tangent bundle Thom class V and

$$\alpha : \tau X \oplus \xi \to \tau Y \oplus \eta$$

is a bundle isomorphism covering f, we say that a Thom class (α, U_1, V_1) for f is determined by X and Y in case

$$UU_1 = \alpha^*(VV_1)$$

where α also denotes the map of Thom spaces.

(5.12) Proposition.

The above definition of a Thom class for a continuous map $f : X \to Y$ of k^*-oriented manifolds determines a functor from the category whose objects are k^*-oriented manifolds and whose morphisms are usual proper homotopy classes of continuous maps, to the proper homotopy category of manifolds and oriented maps defined in (4.1).

Proof.

This is a straightforward verification based on the definitions (3.7) and (3.15).

(5.13) Definition.

When ξ is a bundle on X with a k^*-Thom class U, we define $\underline{\lambda}(-\xi, U^{-1})$ as follows. Let η be a bundle on X and

$$\alpha : \xi \oplus \eta \to \varepsilon^q$$

be a trivialization. Let V be a Thom Class for η such that UV corresponds under α to the suspension of the unit. We put

$$\underline{\lambda}(-\xi, U^{-1}) = \underline{\lambda}(\eta, V) .$$

This definition is independent of the choices made. By the calculation in the proof of (5.7, iii) we know that

$$\underline{\lambda}(-\xi, U^{-1})\underline{\lambda}(\xi, U)$$

is a unit in $h^0(X)$. If (5.9) is true, this unit is 1.

When (X,U) is an oriented manifold, we define

$$\underline{\lambda}(X, U^{-1}) = \underline{\lambda}(-\tau X, U^{-1}) .$$

(5.14) Theorem. (Riemann-Roch).

On the category of k^*-oriented manifolds and proper homotopy classes of continuous maps there is a functor taking

$$f : X \to Y$$

to

$$f_!^k : k^*(X) \to k^*(Y)$$

and another functor taking f to

$$f_!^h : h^*(X) \to h^*(Y) .$$

The R.-R. relation

$$\lambda(f_!^k(x))\underline{\lambda}(Y, V^{-1}) = f_!^h(\lambda(x)\underline{\lambda}(X, U^{-1}))$$

is valid when U and V are the Thom classes orienting X and Y.

Proof.

The first of the two functors is the composition of the functor of (5.12) composed with the transfer functor f (4.10). The second functor is obtained by first using the functor taking the k^*-oriented manifold (X,U) to the h^*-oriented manifold $(X,\lambda'(U))$ and then proceeding as above. To obtain the R.-R. relation, we use (5.8): Let

$$i : E\xi \to E\eta$$

be a bounded open imbedding of bundle-spaces lifting f. Suppose that there is an isomorphism

$$\beta : \tau Y \oplus \eta \to \varepsilon^q .$$

Let

$$\alpha : \tau X \circ pr_2{}^*\xi \to \tau Y \circ pr_2{}^*\eta$$

be the isomorphism covering f determined by i. Let B be a Thom class for η such that VB corresponds to the suspension of the unit under β. When A is a Thom class for ξ such that $UA = \alpha^*(VB)$, it is clear that VA corresponds to the suspension of the unit under $\beta\alpha$. Consequently

$$\underline{\lambda}(X, U^{-1}) = \underline{\lambda}(\xi,A) \quad \text{and} \quad \underline{\lambda}(Y, V^{-1}) = \underline{\lambda}(\eta,B) .$$

Also

$$f_!^k = t(i,A,B) \quad \text{and} \quad f_!^h = t(i,\lambda'(A),\lambda'(B)) ,$$

according to proposition (5.12). Now (5.8) reads

$$\lambda(f_!^k(x))\cdot\underline{\lambda}(\eta,B) = f_!^h(\lambda(x)\cdot\underline{\lambda}(\xi,A))$$

or

$$\lambda(f_!^k(x))\cdot\underline{\lambda}(Y,V^{-1}) = f_!^h(\lambda(x)\cdot\underline{\lambda}(X, U^{-1}))$$

which is the R.-R. relation.

The reader should note that this theorem is independent of the truth of (5.9), in contrast to theorem (5.11).

Defining

$$\underline{\lambda}(f) = \underline{\lambda}(X,\ U^{-1})f^*\underline{\lambda}(Y,\ V^{-1})^{-1}$$

that obviously depends on U and V, we obtain just as in (5.8):

(5.15) Corollary.

$$\lambda(f_!^k(x)) = f_!^h(\lambda(x)\cdot\lambda(f)) .$$

Assuming that λ is an epimorphism, we define

$$\mathrm{Wu}\ (X,U) \in k^{**}(X)$$
$$\mathrm{Wu}\ (Y,V) \in k^{**}(Y)$$

and

$$\mathrm{Wu}\ (f) \in k^{**}(X)$$

to be elements satisfying

$$\lambda\ \mathrm{Wu}\ (X,U) = \underline{\lambda}(X,\ U^{-1})$$
$$\lambda\ \mathrm{Wu}\ (Y,V) = \underline{\lambda}(Y,\ V^{-1})$$

and

$$\lambda\ \mathrm{Wu}\ (f) = \underline{\lambda}(f) .$$

(5.16) Theorem. (Wu.)

Assume that λ is an epimorphism and let the data of (5.14) be given. Then the following Wu formulae hold.

$$\lambda(f_!^k(x)\cdot\mathrm{Wu}(Y,V)) = f_!^h\lambda(x\cdot\mathrm{Wu}(X,V)),$$
$$\lambda(f_!^k(x)) = f_!^h\lambda(x\cdot\mathrm{Wu}(1)) .$$

Proof.

Immediate from (5.14,15).

We assume that the cohomology theories k^* and h^* are represented by spectra and let k_* and h_* be the non-reduced homology theories defined by the same spectra, so that we have slant and Kronecker products defined as in [23].

When X is a compact connected manifold without boundary, a fundamental class for X is an element

$$\Omega \in h_n(X)$$

such that there is a map $q : X \to S^n$ such that

$$q_*(\Omega)$$

is the suspension of the unit.

(5.17) <u>Definition.</u>

We have defined a h_*-fundamental class Ω for the n-manifold X.

It is known that the fundamental classes for X correspond in a 1-1 way with the h^*-Thom classes for τX, and the correspondence is canonical. A proof using S-duality is given in [5]. We shall not prove that much, but the following lemma is proved without using S-duality.

(5.18) <u>Lemma.</u>

Let X be a connected compact manifold without boundary. Then τX has a h^*-Thom class if and only if X has a h_* fundamental class.

<u>Proof.</u>

Let a fundamental class

$$\Omega \in h_n(X)$$

be given. Let $u : X^o \wedge X^\nu \to S^{n+k}$

be defined as follows. (This definition is given in [16].) ν is a bundle on X such that $\tau X \oplus \nu$ is a trivial bundle of rank $n + k$. Let

$$d : X \to X \times E\nu$$

be the diagonal imbedding. A normal microbundle of d is

$$X \xrightarrow[d]{} X \times E\nu \xrightarrow[pr_\nu pr_2]{} X.$$

This microbundle equals $\tau_2(X) \oplus \nu$ where $\tau_2(X)$ is the second tangent bundle

$$X \to X \times X \xrightarrow[pr_2]{} X .$$

Consequently the normal bundle of d is trivial. This gives a map

$$X \times E\nu \to S^{n+k}$$

that collapses the complement of a tubular nbd. of $d(X)$ and has degree 1 on the "fibres" in the normalbundle Thom space.

This map extends to a map

$$u : X^o \wedge X^\nu \to S^{n+k} .$$

Let

$$U \in h^k(X^\nu, pt)$$

be defined by

$$U = \Omega \setminus u^*(\sigma)$$

where σ is the suspension of the unit in $h^{n+k}(S^{n+k}, pt)$. Let $\varphi : S^k \to X^\nu$ be the inclusion of a "fibre". Then the composite

$$X^o \wedge S^k \xrightarrow{id \wedge \varphi} X^o \wedge X^\nu \xrightarrow{u} S^{n+k}$$

is visibly a map of degree 1 from the relative manifold $X^O \wedge S^k, pt$ to S^{n+k}, according to a well-known characterization of such maps [22]. This gives, when $\tau \in h_k(S^k, pt)$ is the suspension of the unit,

$$< \varphi^* U, \tau > = < \Omega \setminus u^*(\sigma), \varphi_* \tau > =$$

$$\pm < u^*(\sigma), \Omega \wedge \varphi_*(\tau) > =$$

$$\pm < \sigma, u_*(id \wedge \varphi)_*(\Omega \wedge \tau) > = \pm 1$$

because $u(id \wedge \varphi)$ has degree 1 and Ω is a fundamental class. (We use the theorem of Hopf stating that the maps of degree 1 generate the cohomotopy group $[Y/B, S^n]$ when Y,B is a compact relative n-manifold.)

We have proved that ν has a Thom class and consequently that τX has a Thom class.

To prove the converse, let

$$E\nu \subset S^{n+k}$$

be an open imbedding and let

$$p : S^{n+k} \to X^{\nu}$$

be the resulting map. Also let $D \subset X$ be an imbedded compact n-disc and let

$$q : X \to D/\partial D$$

be the resulting map. Then there is a homotopy commutative diagram

$$\begin{tikzcd} & X^{\nu} \arrow[r, "\Delta"] \arrow[d, "\beta"] & X^{o} \wedge X^{\nu} \arrow[d, "q\wedge id"] \\ S^{n+k} \arrow[ur, "p"] \arrow[r, "\alpha"'] & D/\partial D \wedge S^{k} \arrow[r, "id\wedge\varphi"'] & D/\partial D \wedge X^{\nu} \end{tikzcd}$$

where α is a homotopy-equivalence. To see this, note that $(q \wedge id)\Delta$ sends the complement of $D^{\nu} \subset X^{\nu}$ to the base-point. Because ν is trivial over D, we obtain the map β. β may be chosen such that there is an open set

$$N \subset D/\partial D \wedge S^{k}$$

such that the restricted map

$$\beta p : p^{-1}\beta^{-1} N \to N$$

is a homeomorphism. Because $\alpha = \beta p$ is a map of spheres, it is a homotopy-equivalence. Let

$$U \in h^{k}(X^{\nu}, pt)$$

be a Thom class and define

$$\Omega \in h_n(X)$$

by

$$\Omega = \Delta_* p_*(\gamma)/U$$

where we identify

$$h_n(X^o, pt) = h_n(X) \quad \text{and} \quad \gamma \in h^{n+k}(S^{n+k}, pt)$$

is the suspension of the unit. Let

$$\delta \in h^n(D/\partial D, pt)$$

be the suspension of the unit. Then

$$\begin{aligned} < \delta, q_*\Omega > &= < q^*\delta, \Delta_* p_*(\gamma)/U > = \\ &\pm <p^*\Delta^*(q \wedge id)^*(\delta \wedge U), \gamma > = \\ &\pm < \alpha^*(id \wedge \varphi)^*(\delta \wedge U), \gamma > = \\ &\pm < \delta \wedge \varphi^*(U), \alpha_*(\gamma) > = \pm 1 \end{aligned}$$

because U is a Thom class and α is a homotopy equivalence.

Consequently $q_*(\Omega)$ is $\pm$ the suspension of the unit, and Ω is a fundamental class.

Q.E.D.

The maps $S^{n+k} \to X^o \wedge X^\nu$ and $X^o \wedge X^\nu \to S^{n+k}$ used above are both S-duality maps. This is proved in [23] and [16].

(5.19) Theorem.

When (X,U) is a k^*-oriented compact connected manifold without boundary, there is a k_*-fundamental class Ω for X such that when

$$c : X \to P$$

is the map from X to a point that has its canonical k^*-orientation, we have

$$c_!^k(x) = < x, \Omega > .$$

Proof.

Consider the diagram

$$S^{n+k} \xrightarrow{p} X^{\nu} \xrightarrow{\Delta} X^{o} \wedge X^{\nu}$$

used in the proof of (5.18). Let $\gamma \in k_{n+k}(S^{n+k},pt)$ be the suspension of the unit and let U_1 be the Thom class for ν used to define $c_!^k$. Then

$$\Omega = \Delta_* p_*(\gamma)/U_1$$

is a fundamental class according to the proof of (5.18). We have, when $x \in k^*(X)$,

$$< x, \Omega > = < x, \Delta_* p_*(\gamma)/U_1 > =$$

$$< p^*\Delta^*(x \wedge U_1), \gamma > = < p^*(xU_1), \gamma > .$$

Now let $\gamma_1 \in k^{n+k}(S^{n+k},pt)$ be the suspension of the unit. Then

$$< \gamma_1, \gamma > = 1 \quad \text{and} \quad c_!^k(x)\gamma_1 = p^*(xU_1) .$$

$$< x, \Omega > = < c_!^k(x)\gamma_1, \gamma > =$$

$$c_!^k(x) < \gamma_1, \gamma > = c_!^k(x) .$$

We let $k = \dim U_1$ be even to avoid sign trouble.

(5.20) Theorem. (The Wu formula).

Let (X,U) be a compact connected k^*-oriented manifold without boundary. Let $\lambda : k^{**} \to h^{**}$ be a multiplicative cohomology transformation. Then there is a k_*-fundamental class Ω and a h_*-fundamental class T for X such that

$$\lambda < x, \Omega > = < \lambda(x)\underline{\lambda}(X, U^{-1}), T >$$

for all $x \in k^*(X)$.

Proof.

Let $$c : X \to P$$

be the point map and let V be the canonical Thom class for τP. According to theorem (5.19) we can choose Ω and T so that

$$c_!^k(x) = \langle x, \Omega \rangle \quad \text{and} \quad c_!^h(y) = \langle y, T \rangle .$$

The R.-R. formula (5.14) now reads

$$\lambda \langle x, \Omega \rangle \cdot \underline{\lambda}(P, V^{-1}) = \langle \lambda(x)\underline{\lambda}(X, U^{-1}), T \rangle .$$

But obviously $\underline{\lambda}(P, V^{-1}) = 1$.

(5.21) Corollary.

When $W(X)$ is the total Stiefel-Whitney class of τX, we have the relation

$$\langle x, [X] \rangle = \langle Sq(x) \cdot W(X)^{-1}, [X] \rangle$$

for all $x \in H^*(X, \mathbb{Z}_2)$.

Proof.

In (5.20) we let $\lambda = Sq = 1 + Sq^1 + Sq^2 + \ldots$ and $k^* = h^* = H^*(-, \mathbb{Z}_2)$. Then X has only one fundamental class and the formula of (5.20) reads

$$Sq \langle x, [X] \rangle = \langle Sq(x)\underline{Sq}(X, U^{-1}), [X] \rangle .$$

Hence it suffices to note that Sq is the identity on a point and that

$$\underline{Sq}(X, U^{-1}) = \underline{Sq}(X, U)^{-1} = W(X)^{-1} .$$

Notations and definitions.

- Bounded imbedding; p. 5.
- Bundle; a fibre bundle with fibre $\mathbb{R}^q$ and the group of homeomorphisms of $(\mathbb{R}^q, 0)$ as structural group. q need not be constant.
- Bundle imbedding; an open topological imbedding of bundle-spaces commuting with the bundle projections and restricting to the identity on the zero section.
- Bundle imbedding determined by i ; def. (4.2).
- Composite bundle; p. 6.
- Composite Thom class; def. (3.10).
- D^i ; p. 5.
- Diagonal map; def. (1.4) and (3.6).
- $E\xi$; total space of the bundle ξ .
- f_b ; when $f : X \to Y$ is a map and ξ is a bundle on Y , we get maps $Ef^*\xi \to E\xi$ and $X^{f^*\xi} \to Y^\xi$ both denoted by f_b .
- $f_\omega(\omega \nparallel b)$; see theorem (4.10).
- Fundamental class; def. (5.17).
- Homotopy of oriented maps; def. (3.15).
- Induced imbedding; p. 8.
- $J(\xi)$; the stable spherical fibration determined by the bundle ξ .
- λ' ; def. (5.5).

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