A NORMALFORM IN

FIRST ORDER ARITHMETIC

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We work with extensions of the systems of natural deductions NM, NJ, NK introduced by Gentzen [1] and Johansson [3] . The systems are extended by quantifierfree axioms and inductionrule. We call the extended systems NMA, NJA, NKA. Prawitz [5] has shown how to define a normalform for prooftrees in NM, NJ, NK. For the case where the logical symbols are &, ⊃, ∀ he does it by giving certain local reductions in a natural way. A prooftree is in normalform if it has no local reductions. He shows that any prooftree can be brought to a normalform by a finite number of local reductions. In the usual correspondence between NM, NJ, NK and LM, LJ, LK the prooftrees in normalform correspond to cutfree proofs.

In this paper we extend Prawitz's result to NMA, NJA, NKA. It is well known that there can be no full cut-elimination result for elementary number theory (first order arithmetic.) We can see this in minimal and intuitionistic elementary number theory as follows :
From a cutfree proof of a Π_2^0 formula $\forall x\ \exists y\ A\ (x.y)$ we can read off a primitive recursive function $f(x)$ such that $\forall x\ A\ (x,\ f(x)\ )$ is true. We know that we can prove that the Ackermann - function is total ; but since it is not primitive recursive, we cannot give a cutfree proof that it is total.

In spite of this we give an extension of the normalform results to NMA, NJA, NKA . We give in a natural way local reductions of prooftrees. We show that all reduction-sequences starting with a given prooftree terminate after a finite number of reductions in the same normalform. This normalform is strong enough to conclude the consi - stency of NMA, NJA, NKA . In NMA and NJA we show how to derive the results of Harrop [2] .

The proof is influenced by Sanchis [6] . I have been given much help by Dag Prawitz. With his normalform for iterated inductive defi - nitions Per Martin - Løf has since given a theorem stronger than at least some of the the results in this paper.

2   As mentioned above NMA, NJA, NKA are the systems NM, NJ, NK

extended by quantifierfree axioms and inductionrule.

We have an unlimited list of free variables a,b,c, •••• ; and

an unlimited list of bound variables x,y,z, •••••

Our connectives and quantifiers are &, v, ⊃, $\forall$, ∃ .

We have symbols for zero 0   , successor ′ , and relations

including = , S (sum) , P (product) .

We write λ for 0 = 0′ .

The numerals are 0, 0 ′ , 0 ″ , •••••• •

Terms , atomic formulae , formulae , and closed formulae are

defined in the usual way.

Our proofs are written in treeform. We call them prooftrees. The

downmost formula (or rather the formula at the downmost node of the

tree) is the conclusion of the prooftree. The topmost formulae are

either axioms or assumptions. The assumptions are either open or closed.

(Prawitz [5] calls a closed assumption discharged.) We read the

prooftree as : From the open assumptions A_1 , •••• A_n we get the

conclusion B . A closed assumption is written with a squarebracket

around.

The axioms are the quantifierfree formulae with true universal

closure.

To each connective and quantifier we have rules for introduction

and elimination. These are the same as in NM, NJ, NK , In the usual

shorthand (see Prawitz [5]) the rules are given by :

&I

$$\frac{A \qquad B}{A \& B}$$

&E)

$$\frac{A \& B}{A} \qquad\qquad \frac{A \& B}{B}$$

∨I

$$\frac{A}{A \vee B} \qquad \frac{B}{A \vee B}$$

∨E)

$$\frac{A \vee B \qquad \overset{[A]}{C} \qquad \overset{[B]}{C}}{C}$$

⊃I

$$\frac{\overset{[A]}{B}}{A \supset B}$$

⊃E)

$$\frac{A \qquad A \supset B}{B}$$

∀I

$$\frac{Aa}{\forall x Ax}$$

∀E)

$$\frac{\forall x Ax}{At}$$

∃I

$$\frac{At}{\exists x \; Ax}$$

∃E)

$$\frac{\exists x Aa \qquad \overset{[Aa]}{B}}{B}$$

in addition we have an inductionrule

IND)

$$\frac{Ao \qquad \overset{[Aa]}{Aa'}}{At}$$

We have the usual restriction on the eigenvariable a in
∀I, ∃E, IND.
These rules are common for all three systems and are all the rules
of NMA. To get NJA we add the rule

∧I)

$$\frac{\wedge}{A}$$

To get NKA we add the more general rule

∧K)

$$\frac{\overset{[A \supset \wedge]}{\wedge}}{A}$$

In both $\wedge_I$ and $\wedge_K$ we assume that A is different from $\wedge$. In case we have only &, $\supset$, $\vee$ as logical signs, we can assume that A is atomic.

We distinguish between major and minor premisses. In &I, &E, $\vee$I, $\supset$I, $\forall$I, $\forall$E, $\exists$I, $\wedge_I$, $\wedge_K$ all the premisses are major. In $\vee$E the one to the left is major, the two to the right minor. In $\exists$E the one to the left is major, the one to the right is minor. In $\supset$E the one to the right is major, the one to the left is minor. In IND both premisses are minor.

A branch in a prooftree which can be traced up from the conclusion to an axiom or an assumption, through major permisses is called a main branch. (or consequence of IND)

Being reasonably careful with the free variables in a prooftree, we can substitute a term for a free variable not used as an eigenvariable.

A main subtree of a prooftree is a subtree with conclusion a minor premiss of a rule with consequence in a main branch of the original prooftree.

3. We can consider an introductionrule as giving a sufficient reason for introducing a connective, and an eliminationrule as an inverse to the introductionrule. If we first apply an introdutionrule to introduce a connective and then the eliminationrule, we essentially restore the original situation. It was not necssary to use the two rules. We have another redundancy when the inductionterm in IND is either zero or successor. To each such redundancy we have a reduction.

DEFINITION

A maximal formula in a prooftree is a formula which occurs as either both consequence of an introductionrule and major premiss of an eliminationrule, or consequence of IND with inductionterm either zero or successor.

DEFINITION

A prooftree is in normalform if it does not contain any maximal formula.

Our problem now is to give a systematic transformation of any prooftree to a prooftree in normalform with the same conclusion and not more open assumptions. This will be done by the reductions defined below. We will see that for each maximal formula there is a natural way of getting rid of it, but at the possible expense of creating new maximal formulae.

Observe that in the reductions defined below the condusions remain the same and we do not get new open assumptions. The reductions are as follows :

&-reduction:

$$
\frac{\dfrac{\begin{array}{cc}\Sigma_1 & \Sigma_2\\ A & B\end{array}}{A \,\&\, B}}{A}\qquad\text{is reduced to}\qquad\begin{array}{c}\Sigma_1\\ A\\ \Sigma_3\end{array}
$$
$$\Sigma_3$$

$$
\frac{\dfrac{\begin{array}{cc}\Sigma_1 & \Sigma_2\\ A & B\end{array}}{A \,\&\, B}}{B}\qquad\text{is reduced to}\qquad\begin{array}{c}\Sigma_2\\ B\\ \Sigma_4\end{array}
$$
$$\Sigma_4$$

∨-reduction:

$$
\frac{\begin{array}{ccc}\begin{array}{c}\Sigma_1\\ A\end{array} & \begin{array}{c}[A]\\ \Sigma_2\\ C\end{array} & \begin{array}{c}[B]\\ \Sigma_3\\ C\end{array}\\ \hline A \vee B & C & C\end{array}}{C}\qquad\text{is reduced to}\qquad\begin{array}{c}\Sigma_1\\ A\\ \Sigma_2\\ C\\ \Sigma_4\end{array}
$$
$$\Sigma_4$$

$$\frac{\dfrac{\begin{array}{c}\Sigma_5\\ B\end{array}}{A \lor B} \quad \dfrac{[A]}{\begin{array}{c}\Sigma_2\\ C\end{array}} \quad \dfrac{[B]}{\begin{array}{c}\Sigma_2\\ C\end{array}}}{\begin{array}{c}C\\ \Sigma_4\end{array}} \qquad \text{is reduced to} \qquad \begin{array}{c}\Sigma_5\\ B\\ \Sigma_3\\ C\\ \Sigma_4\end{array}$$

⊃ - reduction:

$$\dfrac{\begin{array}{c}\Sigma_1\\ A\end{array} \qquad \dfrac{\begin{array}{c}[A]\\ \Sigma_2\\ B\end{array}}{A \supset B}}{\begin{array}{c}B\\ \Sigma_3\end{array}} \qquad \text{is reduced to} \qquad \begin{array}{c}\Sigma_1\\ A\\ \Sigma_2\\ B\\ \Sigma_3\end{array}$$

∀-reduction:

$$\dfrac{\dfrac{\begin{array}{c}\Sigma_1\\ Aa\end{array}}{\forall x\, Ax}}{\begin{array}{c}At\\ \Sigma_2\end{array}} \qquad \text{is reduced to} \qquad \begin{array}{c}\Sigma_1'\\ At\\ \Sigma_2\end{array}$$

where Σ_1' is obtained from Σ_1 by substituting t for a.

∃ - reduction:

$$\dfrac{\dfrac{\begin{array}{c}\Sigma_1\\ At\end{array}}{\exists x\, Ax} \qquad \dfrac{[Aa]}{\begin{array}{c}\Sigma_2\\ B\end{array}}}{\begin{array}{c}B\\ \Sigma_3\end{array}} \qquad \text{is reduced to} \qquad \begin{array}{c}\Sigma_1\\ At\\ \Sigma_2'\\ B\\ \Sigma_3\end{array}$$

where Σ_2' is obtained from Σ_2 by substituting t for a.

IND-reduction (zero-case):

$$\dfrac{\begin{array}{c}\Sigma_1\\ Ao\end{array} \qquad \dfrac{[Aa]}{\begin{array}{c}\Sigma_2\\ Aa'\end{array}}}{\begin{array}{c}Ao\\ \Sigma_3\end{array}} \qquad \text{is reduced to} \qquad \begin{array}{c}\Sigma_1\\ Ao\\ \Sigma_3\end{array}$$

$$\frac{\dfrac{\Sigma_5}{B} \quad [A] \quad [B]}{\dfrac{A \lor B \quad \dfrac{\Sigma_2}{C} \quad \dfrac{\Sigma_2}{C}}{\dfrac{C}{\Sigma_4}}}$$

is reduced to

$$\begin{array}{c} \Sigma_5 \\ B \\ \Sigma_3 \\ C \\ \Sigma_4 \end{array}$$

⊃ - reduction:

$$\frac{\Sigma_1 \quad \dfrac{[A]}{\dfrac{\Sigma_2}{B}}}{\dfrac{A \quad A \supset B}{\dfrac{B}{\Sigma_3}}}$$

is reduced to

$$\begin{array}{c} \Sigma_1 \\ A \\ \Sigma_2 \\ B \\ \Sigma_3 \end{array}$$

∀-reduction:

$$\frac{\dfrac{\Sigma_1}{Aa}}{\dfrac{\forall x\, Ax}{\dfrac{At}{\Sigma_2}}}$$

is reduced to

$$\begin{array}{c} \Sigma_1' \\ At \\ \Sigma_2 \end{array}$$

where Σ_1' is obtained from Σ_1 by substituting t for a .

∃ - reduction:

$$\frac{\dfrac{\Sigma_1}{At} \quad \dfrac{[Aa]}{\dfrac{\Sigma_2}{B}}}{\dfrac{\exists x\, Ax \qquad B}{\dfrac{B}{\Sigma_3}}}$$

is reduced to

$$\begin{array}{c} \Sigma_1 \\ At \\ \Sigma_2' \\ B \\ \Sigma_3 \end{array}$$

where Σ_2' is obtained from Σ_2 by substituting t for a .

IND-reduction (zero-case):

$$\frac{\dfrac{\Sigma_1}{Ao} \quad \dfrac{[Aa]}{\dfrac{\Sigma_2}{Aa'}}}{\dfrac{Ao}{\Sigma_3}}$$

is reduced to

$$\begin{array}{c} \Sigma_1 \\ Ao \\ \Sigma_3 \end{array}$$

IND-reduction (successor-case):

$$\begin{array}{ccc} & [Aa] & \\ \Sigma_1 & \Sigma_2 & \\ \underline{Ao \quad Aa'} & & \\ At' & & \\ \Sigma_3 & & \end{array} \qquad \text{is reduced to} \qquad \begin{array}{cc} & [Aa] \\ \Sigma_1 & \Sigma_2 \\ \underline{Ao \quad Aa'} \\ At \\ \Sigma_2' \\ At' \\ \Sigma_3 \end{array}$$

where Σ_2' is obtained from Σ_2 by substituting t for a .

These are all the reductions.

DEFINITION

A reductionsequence is a sequence of prooftrees such that each prooftree reduces to the next in the sequence.

4   We want to prove that all reductionsequences terminate. Say we would first prove that to each prooftreee there is a reduction-sequence which terminates in a prooftree in normalform. It is not hard to show that if this is true, we can always do with the parti-cular reductionsequences we get by always reducing one of the down-most maximal formulae in the main branch, and then after the main branches are cleared up go to the main branches in the main subtrees etc. So we concentrate on those reductionsequences. It becomes soon apparent that the major obstacles to a proof are the induction-rules with inductionterm a free variable. To take care of those we add the obvious ω-reduction : Substitute any numeral for a free variable in the inductionterm. We are forced to add two other rather trivial reductions so that the prooftree remains a prooftree after the substitution. There are additional problems when we have v or ∃ . To take care of those we introduce the collapsed proof-

trees. We then prove over several lemmata, all obvious except the main lemma, that the relation given by all those extra reductions is well-founded. We can then prove the weaker version of the theorem mentioned above by induction over the well-founded relation. Being a little more clever we can prove that all reductionsequences terminate. The main problem here is that in some reduction we may cut off a whole subtree and will therefore not be able to keep track of what could be going on in that subtree. To take care of this we introduce the associated subtrees of a maximal formula.

DEFINITION

The associated subtree of a prooftree with respect to a maximal formula is the subtree which can be cut off in the reduction. So for example

$$\text{to} \quad \dfrac{\overset{\Sigma_1}{A} \qquad \overset{\Sigma_2}{B}}{\dfrac{A \ \& \ B}{\underset{\Sigma_3}{A}}} \qquad \text{we associate} \qquad \overset{\Sigma_2}{B}$$

$$\text{and to} \quad \overset{\Sigma_1}{A} \quad \dfrac{\overset{[A]}{\underset{\Sigma_2}{B}}}{\dfrac{A \supset B}{\underset{\Sigma_3}{B}}} \qquad \text{we associate} \qquad \overset{\Sigma_1}{A}$$

DEFINITION

To each $\vee E$ and $\exists E$ we define the collapsed prooftrees

$$\dfrac{\overset{\Sigma_1}{A \vee B} \quad \overset{[A]}{\underset{\Sigma_2}{C}} \quad \overset{[B]}{\underset{\Sigma_3}{C}}}{\underset{\Sigma_4}{C}} \qquad \text{is collapsed to} \qquad \overset{A}{\underset{\Sigma_4}{\underset{C}{\Sigma_2}}} \quad \text{and} \quad \overset{B}{\underset{\Sigma_4}{\underset{C}{\Sigma_3}}}$$

$$\text{and} \qquad \dfrac{\overset{\Sigma_1}{x\,Ax} \quad \overset{[Aa]}{\underset{B}{\Sigma_2}}}{B} \qquad \text{is collapsed to} \qquad \begin{array}{c} Aa \\ \Sigma_2 \\ B \\ \Sigma_3 \end{array}$$
$$\underset{\Sigma_3}{B}$$

DEFINITION

We define a binary relation $\mathcal{R}$ between prooftrees. $\Sigma_1 \,\mathcal{R}\, \Sigma_2$ is defined by cases depending on the free variables in the main branches of Σ_1 and on the downmost formula in the main branches of Σ_1 which is not the consequence of an eliminationrule, of $\bigwedge_I$, nor of $\bigwedge_K$. (Not that it matters, but there is at most one such formula.)

a) A formula in the main branches contains a free variable a not used as an eigenvariable.

$\Sigma \,\mathcal{R}\, \Sigma'$ where Σ' is obtained from Σ by substituting any numeral for a.

b) Case a does not apply and the last rule used is an introductionrule. Depending on whether we have one or two premisses we get

$$\underset{A}{\Sigma} \,\mathcal{R}\, \Sigma$$

or $\qquad \dfrac{\Sigma_1 \quad \Sigma_2}{A} \,\mathcal{R}\, \Sigma_1 \,,\, \Sigma_2 \qquad \left(\text{i.e.} \; \dfrac{\Sigma_1 \quad \Sigma_2}{A} \,\mathcal{R}\, \Sigma_i \qquad (i=1,2)\right)$

c) Case a and case b do not apply and there is a downmost maximal formula in the main branches. Then

$$\Sigma \,\mathcal{R}\, \Sigma_1 \,,\, \Sigma_2$$

where Σ_1 is the reduction of Σ and Σ_2 is the associated subtree.

d) Case a and case b do not apply and there is a $\vee E$ or

$\exists E$ in the main branches. Then

$$\Sigma \; \mathcal{R} \; \Sigma_1$$

where Σ_1 is the collapsed prooftree.

e) Case a, b, c do not apply. (i.e. all the formulae in the main branch are consequences of eliminationrules or $\wedge_I$ or $\wedge_K$.) Then

$$\Sigma \; \mathcal{R} \; \Sigma_1, \ldots, \Sigma_n \quad \text{where} \quad \Sigma_1, \ldots, \Sigma_n \text{ are the main subtrees}$$

(i.e. subtrees above a minor premiss with consequence in the main branch.)

This concludes the definition of $\mathcal{R}$.

DEFINITION

$\succ$ is the transitive closure of $\mathcal{R}$.

DEFINITION

An $\mathcal{R}$ -sequence is a sequence of prooftrees each in $\mathcal{R}$ -relation to the next.

5. We will show that $\succ$ is well-founded, i.e. all $\mathcal{R}$ -sequences are finite.

DEFINITION

A prooftree is regular if all $\mathcal{R}$ -sequences starting with it are finite.

We have the following obvious lemma:

LEMMA 1

i) If Σ is regular and $\Sigma \; \mathcal{R} \; \Sigma'$, then Σ' is regular.

ii) If all prooftrees in $\mathcal{R}$ -relation to Σ are regular, then Σ is regular.

Note that we can do induction over the ordering $\succ$ restricted to regular prooftrees. We call this $\mathcal{R}$-induction. The following is also obvious.

LEMMA 2

i) A prooftree consisting of only an axiom or an assumption is regular. (We call such a prooftree trivial.)

ii) Regularity is closed under $\wedge_I$, $\wedge_K$ and introductionrules.

iii) Regularity is closed under &E and $\forall E$.

iv) Regularity is closed under $\supset E$, $\vee E$, $\exists E$ provided there are only trivial prooftrees above the minor premisses.

Proof:

We indicate how to prove first part of iii.

We use $\mathcal{R}$-induction.

Assume $\begin{array}{c} \Sigma \\ A \& B \end{array}$ regular. Want $\begin{array}{c} \Sigma \\ A \& B \\ \hline A \end{array}$ regular. Two cases to consider — either the last rule in $\begin{array}{c} \Sigma \\ A \& B \end{array}$ is an introductionrule or it is not. Both cases equally obvious.

MAIN LEMMA

If $\begin{array}{c} \Pi \\ A \end{array}$ and $\begin{array}{c} A \\ \Sigma \end{array}$ are regular, then also $\begin{array}{c} \Pi \\ A \\ \Sigma \end{array}$. (We may here put Π over more than one open assumption A.)

Proof:

The proof is by a double induction. A primary induction over the length of A and a secondary $\mathcal{R}$-induction over either A with less than or equal $\begin{array}{c} \Pi \\ A \end{array}$ or over $\begin{array}{c} \Pi \\ A \end{array}$ with less than or equal $\begin{array}{c} \Sigma \\ \end{array}$ A . To the initial step obseve that the least prooftrees in the $\mathcal{R}$-relation consist of a single branch with only &E , $\vee E$, $\wedge_I$, $\wedge_K$ used as rules. So we use lemma 2 to get $\begin{array}{c} \Pi \\ A \\ \Sigma \end{array}$ regular.

Now to the induction step. Suppose there is an infinite $\mathcal{R}$-sequence starting with $\overset{\Pi}{\underset{\Sigma}{A}}$. We divide up into cases depending on the first $\mathcal{R}$-relation in the sequence.

i)   We use case a in the definition of $\mathcal{R}$.

ii) We use case b .

iii) We use case c with either associated subtree or reduction with maximal formula different from A .

iv) We use case d .

v)   We use case e .

vi) We use case c and reduction with maximal formula A .

Cases i - v are obvious. So we go to case vi . First observe that we can put $\overset{\Pi}{A}$ over all the open assumptions A in $\overset{A}{\Sigma}$ not in the main branch by an argument as in cases i - v to get the regular prooftree $\overset{A}{\Sigma'}$. We want to put $\overset{\Pi}{A}$ over the open assumption A in the main branch of $\overset{A}{\Sigma'}$. We now have subcases depending on the principal logical symbol in A . We give the argument for $A = B \supset C$, $A = B \vee C$. The remaining subcases are similar.

$$
\underset{\Sigma'}{\overset{\Pi}{B \supset C}} \quad = \quad \dfrac{\underset{B}{\overset{\Sigma_1}{B}} \qquad \dfrac{\overset{[B]}{\overset{\Pi'}{C}} \quad B \supset C}{}}{\underset{\Sigma_2}{C}}
$$

Observe that $\overset{\Sigma_1}{B}$, $\overset{B}{\underset{C}{\Pi'}}$, $\overset{}{\underset{\Sigma_2}{}}$ are regular. By induction the reduction of

$$
\underset{\Sigma'}{\overset{\Pi}{B \supset C}} \qquad \qquad \overset{\Sigma_1}{\underset{\Sigma_2}{\overset{B}{\underset{C}{\Pi'}}}} \qquad \qquad \text{is regular.}
$$

$$
\underset{\Sigma'}{\overset{\Pi}{B \vee C}} \quad = \quad \dfrac{\dfrac{\overset{\Pi'}{B}}{B \vee C} \qquad \overset{[B]}{\underset{D}{\Sigma_1}} \qquad \overset{[C]}{\underset{D}{\Sigma_2}}}{\underset{\Sigma_3}{D}}
$$

We have $\begin{matrix}\Pi'\\B\end{matrix}$ and $\begin{matrix}B\\\Sigma_1\\D\\\Sigma_3\end{matrix}$ regular so by induction the reduction of

$$\begin{matrix}\Pi\\B \lor C\\\Sigma'\end{matrix} \qquad \begin{matrix}\Pi'\\B\\\Sigma_1\\D\\\Sigma_3\end{matrix} \qquad \text{is also regular.}$$

This concludes the proof of the main lemma.

In case our prooftrees contain only formulae built up from & , $\supset$, $\lor$ we can prove that last case in a simpler way. We assume here that in $\bigwedge_I$, $\bigwedge_K$ A is atomic.

Let $\begin{matrix}A\\\Sigma_1\\A_1\end{matrix},\ldots,\begin{matrix}A\\\Sigma_n\\A_n\end{matrix}$ be the main subtrees of $\begin{matrix}A\\\Sigma\end{matrix}$.

By induction $\begin{matrix}\Pi\\A\\\Sigma_1\\A_1\end{matrix},\ldots,\begin{matrix}\Pi\\A\\\Sigma_n\\A_n\end{matrix}$ are regular.

Using the assumptions we see that the lengths of $A_1,\ldots,A_n$ are less than the length of A .

Cut of $\begin{matrix}A\\\Sigma\end{matrix}$ at $A_1,\ldots,A_n$ to get the

prooftree $A \quad \begin{matrix}A_1 \ldots A_n\\\overline{}_*\\\Sigma\end{matrix}$

Using lemma 2 we get $\begin{matrix}\Pi\\A \quad \begin{matrix}A_1 \ldots A_n\\\overline{}_*\\\Sigma\end{matrix}\end{matrix}$ regular.

By induction

$$\begin{matrix}\Pi\\A\\\Sigma\end{matrix} = \begin{matrix}\Pi\\A\end{matrix} \quad \begin{matrix}\Pi_1\\A_1\end{matrix} \ldots \begin{matrix}\Pi_n\\A_n\end{matrix}$$

is regular and we are done.

Now using the main lemma and lemma 2 we immediately have

LEMMA 3

i) Regularity is closed under $\supset$E , $\lor$E , E .

ii) Regularity is closed under IND with inductionterm a numeral.

iii) Regularity is closed under IND .

THEOREM 1

    All prooftrees are regular.

6. We will now prove the normalform theorem. By $\mathcal{R}$-induction it is easy to prove that all prooftrees can be reduced to a prooftree in normalform. We want to show that all reductionsequences terminate.

DEFINITION

The order of a formula in a prooftree is the number of minorpremisses we must go through to get down from the formula to the conclusion of the prooftree.

DEFINITION

Given two formulae A, B in a prooftree. A dominates B if either of the following holds

i)   the order of A < the order of B ;

ii)   they have the same order and the branch through A is to the left of the branch through B ; or

iii) they have the same order and are on the same branch and A is below B .

LEMMA 4

    A dominates B is a linear well-ordering.

To each reduction ρ of a prooftree we assign the maximal formula,

LEMMA 5

    Let ρ be a reduction of Π to Π' and A the assigned

formula in Π . Then the ordering of Π up to but not including A is an initial segment of the ordering in Π' .

To each reduction we assign the initial segment given by lemma 5.

LEMMA 6

Let ρ_0 be a reduction of Π_0 to Π_1 with segment I_0 and ρ_1 a reduction of Π_1 to Π_2 with segment I_1 . Assume I_1 is more than one formula less than I_0 . We can then apply ρ_1 to Π_0 to get Π_2' with segment I_1 . If Π_2' is different from Π_2 , then there are reductions $\sigma_0, \sigma_1, \ldots, \sigma_n$ such that σ_0 takes Π_2' to Σ_1 , σ_1 takes Σ_1 to $\Sigma_2, \ldots \ldots, \sigma_n$ takes Σ_n to Π_2 . The reductions $\sigma_0, \ldots, \sigma_n$ have segments larger than I_1 and they are all of the same kind as the reduction ρ_0 .

DEFINITION

A reductionsequence is standard, if the segment of any reduction ρ is not more than one formula shorter than the one of the reduction which precedes ρ in the sequence.

LEMMA 7

If we have a finite reductionsequence which takes Π to Π' , then there is another finite standard reductionsequence which takes Π to Π' . All the reductions in the new sequence are of types used in the old sequence.

LEMMA 8

Given an infinite reductionsequence starting with Π , then there is another infinite standard reductionsequence starting with Π .

THEOREM 2

All reductionsequences terminate.

Proof:

We prove by $\mathcal{R}$ -induction over Π that all reductionsequences starting with Π terminate. Obvious if Π is not in $\mathcal{R}$-relation to any prooftree. Assume we have a standard infinite reductionsequence starting with Π . The proof goes now by cases:

i)   The main branch of Π has a free variable not used as an eigenvariable.

ii) The last rule in Π is an introductionrule.

iii) The main branch contains a maximal formula and cases i, ii do not apply.

iv) Cases i, ii, and iii do not apply.

Here i, ii, and iv are immediate. For iii either the downmost maximal formula in the main branches is reduced in the first step in the reductionsequence or it is contained in the segment of any reduction in the sequence. (Here we used that the sequence was standard.) The first alternative is obvious. We give an example of what to do with the second. Say our maximal formula is $A \supset B$, and Π is

$$
\frac{\Pi_1 \qquad \dfrac{\begin{array}{c}[A]\\ \Pi_2\\ B\end{array}}{A \supset B}}{\begin{array}{c} B \\ \Pi_3 \end{array}}
$$

Since $A \supset B$ is included in all the segments, it will separate each prooftree in the reductionsequence in two parts. There must be either an infinite reductionsequence starting with $\begin{array}{c}\Pi_1\\ A\end{array}$ or one starting with $\begin{array}{c}A\\ \Pi_2\\ B\end{array}$. In the last case we also get an infinite

reductionsequence starting with $\begin{matrix}\Pi_1\\ A\\ \Pi_2\\ B\\ \Pi_3\end{matrix}$. Now $\Pi\,\mathcal{R}_A^{\Pi_1}$, $\begin{matrix}\Pi_1\\ A\\ \Pi_2\\ B\\ \Pi_3\end{matrix}$ and we

are done. Observe that we had to have $\Pi\,\mathcal{R}_A^{\Pi_1}$ since it could

happen that there were no assumptions A above B . We do a

similar analysis when the maximal formula is A & B, A ∨ B,

∀x Ax, or ∃x Ax . This concludes the proof.

A standard reductionsequence ending in a prooftree in normalform

is uniquely determined. One always chooses the reduction with

the least segment. Hence:

THEOREM 3

Given a prooftree Π . Then all reductionsequences starting

with Π and ending in a normalform, end in the same normalform.

Prawitz [5] gives also a stronger normalform.

DEFINITION

A segment in a prooftree Π is a sequence $A_1,\ldots,A_n$ of conse-

cutive formula occurrences in a branch such that

i) A_1 is not the consequence of ∨E or ∃E ;

ii) A_i for each i < n is a minor premiss of ∨E or ∃E ; and

iii) A_n is not the minor premiss of ∨E or ∃E .

DEFINITION

A maximal segment is a segment that begins with a consequence of

an I-rule or $\wedge_I$ and ends with a major premiss of an E-rule.

DEFINITION

A redundant application of ∨E or ∃E is an application which

has a minor premiss where no assumption is closed.

DEFINITION

A prooftree is strongly normal if it contains neither maximal
segments nor redundant applications of ∨E or ∃E . (Observe
that a maximal formula is a maximal segment so that a strongly
normal prooftree is normal.)

DEFINITION

A redundant variable in a prooftree is a free variable not used
as an eigenvariable and which does neither occur in the conclu-
sion nor in any open assumption.

Using redundant variables there are trivial transformations of
prooftrees to prooftrees in (strong) normalform. For instance
take the prooftree $\begin{array}{c} \Pi_1 \\ A \\ \Pi_2 \end{array}$ over into

$$
\cfrac{
\cfrac{\begin{array}{c}\Pi_1 \\ A\end{array} \quad 0 = 0}{A \,\&\, 0 = 0}
\qquad
\cfrac{\begin{array}{c}\Pi_1 \\ A\end{array} \quad a = a}{A \,\&\, a = a}
}{
\cfrac{A \,\&\, b = b}{\begin{array}{c} A \\ \Pi_2 \end{array}}
}
$$

Observe that the reduction of a prooftree without redundant vari-
ables is without redundant variables.

THEOREM 4

To all prooftrees Π we can find a normal prooftree with
no redundant variables and with the same conclusion and not more
open assumptions as in Π .

THEOREM 5

To all prooftrees Π we can find a strongly normal proof-
tree with the same conclusion and not more open assumtions and
no redundant variables.

Proof:

By $\mathcal{R}$-induction over Π .

Obvious if Π is not in $\mathcal{R}$-relation to any other prooftree.

We divide up into cases as usual:

i)   The main branches of Π contain a free variable not used as an eigenvariable.

ii)   The last rule used is an introductionrule.

iii)   In the main branches we have a $\bigwedge_I$ followed by an elimination rule.

iv)   In the main branches we have a maximal formula.

v)   In the main branches we have a redundant $\vee E$ or E .

vi)   In the main branch we have only $\bigwedge_I, \bigwedge_K$, &E, $\supset$E, $\vee$E .

vii)   In the main branch we have only $\bigwedge_I, \bigwedge_K$ and E-rules but at least one $\vee$E or E , and case i does not apply.

Here i - vi are straightforward. Now to vii . Take the topmost of the $\vee$E or E in the main branch. Say it was $\vee$E . Our prooftree Π is

$$\frac{\overset{\Pi_1}{A \vee B} \quad \overset{\Pi_2}{C} \quad \overset{\Pi_3}{C}}{\underset{\Pi_4}{C}}$$

By $\mathcal{R}$-induction and the assumption we can get strongly normal prooftrees without redundant variables

$$\overset{\Pi_1'}{A \vee B} \quad , \quad \begin{matrix} \Pi_2' \\ \Pi_4' \\ D \end{matrix} \quad , \quad \begin{matrix} \Pi_3' \\ \Pi_4' \\ D \end{matrix} \quad .$$

If $\begin{matrix} \Pi_2' \\ \Pi_4' \\ D \end{matrix}$ has not A as open assumption, take it as the result.

Similary if $\begin{matrix} \Pi_3' \\ \Pi_4' \\ D \end{matrix}$ has not B as open assumption. Else take

$$\frac{A \lor B \quad \begin{matrix}\Pi_1' & \Pi_2' & \Pi_3' \\ & \Pi_4' & \Pi_4' \\ & D & D\end{matrix}}{D}$$

where we close the open assumptions A above D in the first

minor premiss and the open assumptions B above D in the second.

The proof when the topmost of the ∨E or ∃E in the main branch

is ∃E , is similar. This concludes the proof.

There are difficulties in getting equivalents to theorem 2

and 3 for the strong normalform. Firstly it is not clear how

we should reduce a maximal segment. Secondly there are no rea-

sons why we should get uniqueness. For example

$$\frac{\begin{matrix}\Pi_1 & [A] & \\ A & \Pi_2 & \Pi_3 \\ A \lor B & C & C\end{matrix}}{C}$$

has both $\begin{matrix}\Pi_3 \\ C\end{matrix}$ and $\begin{matrix}\Pi_1 \\ A \\ \Pi_2 \\ C\end{matrix}$ as natural reductions.

7.   We will now use the normalform and the strong normalform

for prooftrees en NMA, NJA, NKA . The consistency of the systems

is obvious since a normal proof without redundant variables of

0 = 0' cannot contain free variables, quantifiers, induction-

rules. Using the strong normalform we can prove the results of

Harrop [2] for NMA and NJA . In fact we can follow step for

step the proof of Prawitz [5] of Harrops results for NM and NJ.

Of course we do not have the subformulaproperty for normal and

strongly normal proofs.   The result of Kreisel [4, page 331]

gives hope for improved results in extended systems.

We prove our normalform theorem by use of bar induction.

We could also have used ordinalassignments and transfinite induction. One such assignment can be given as follows:

The assumptions and the axioms are assigned 1 .

If in a proof we have assigned to the premisses of a rule α, β, γ , assign to the consequence $\varphi_m(\alpha \; \beta \; \gamma)$ where $\#$ is Hessenbergs natural sum, $\varphi_0 = \lambda x \, (x+1)$, φ_m for $m > 0$ is the m-th Bachmann function, and m is

i) 0 in &I, &E, $\vee$I, $\supset$I, $\forall$I, $\vee$E, I, $\bigwedge_I$, $\bigwedge_K$;

ii) length of minor premiss in $\supset$E ;

iii) length of major premiss in $\vee$E, $\exists$E ; and

iv) $1 +$ length of premiss in IND .

This assignment does not seem to be too economical. On the other hand it is unreasonable to expect an assignment which gives the normalform theorem by ε_0-induction without further analysis. This since we can add the ω-rule for $\forall$-introduction without affecting the proof of the normalform theorem.

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