UiO : Department of Mathematics
University of Oslo

Orientation Reversing Gauge Transformations

Jørgen Olsen Lye

Master's Thesis, Spring 2015

Abstract

I investigate whether or not a vector bundle admits an orientation reversing gauge transformation (aka. bundle automorphism). I introduce principal bundles and an equivariant formulation to aid the investigation and show that the answer is always yes for odd rank bundles. I give definite answers for all vector bundles over $\mathbb{S}^n$ for $1 \leq n \leq 4$ more or less directly and give an explicit computation for tangent bundles over $\mathbb{S}^n$ for all n. I Give definite answers for oriented rank n bundles over n dimensional CW complexes X with $H^n(X; \mathbf{Z})$ torsion free and find that in that situation the answer is that the Euler class of the bundle has to vanish. I show that the answer for tangent bundles over $\mathbb{RP}^n$ for all even n is negative. I provide an inconclusive discussion of the problem's relation to the question "when does an even rank bundle split as the sum of odd rank bundles?". I finish by a series of unresolved questions.

Acknowledgements

I would first and foremost like to thank John Rognes for giving me a riveting project. His patience, expert guidance, and enthusiasm have all been invaluable. Many of the better ideas of this thesis are due to him. I regret any omitted attributions should they still abound. It is also regrettable that more of the modern machinery of algebraic topology which I often had the pleasure seeing deftly wielded didn't make it into the thesis. In particular the reader should know that they will *not* be treated to a proof of theorem 3.3 which boils down to comparing spectral sequences. The fault is entirely my own.

Thanks are also in order to my fellow students for creating a friendly environment in which to study and for providing lively debates, mathematical or otherwise.

Introduction

The question I set out to investigate in this thesis is the following. Given a rank k real vector bundle E over a topological space X and a map $\psi : X \to \mathbb{R}^*$, does there exist a gauge transformation (aka. bundle automorphism) $\phi : E \to E$ such that $\det(\phi_x) = \psi(x)$ for all points $x \in X$?

The lay of the land

The structure of the thesis is as follows. Chapter 1 sets the scene and gives some background information. I also say a few words about the complex version of the problem as a contrast to the real version. I then outline how the problem can be recast as an equivariant lifting problem. With the equivariant formulation the problem will be readily shown to have a solution when the rank is odd. I end the first chapter by sketching some related variants of the problem (beyond the complex variant).

Chapter 2 is a differential topological digression where I specialize to bundles over $\mathbb{S}^n$ before specialising to tangent bundles. I perform explicit computations for arbitrary bundles over $\mathbb{S}^n$ for $n = 1, 2, 3, 4$ before carrying out a somewhat messy computation for the tangent bundle of $\mathbb{S}^n$ for all n.

Chapter 3 returns to the general setting. I define the Euler class and use it along with some results from non-equivariant obstruction theory to give definite answer to the problem in some cases (the tangent bundle of orientable manifolds for instance) and a partial answer in other cases (non-orientable bundles in particular).

Chapter 4 is somewhat akin to an appendix where I go through some obstruction theory in order to prove some of the key results in chapter 3. At the very least I try to motivate and explain the setting of results I do not prove in obstruction theory. I end chapter 4 with a nod in the direction of equivariant obstruction theory.

I end the thesis with a series of questions that arose along the way but which I have not dealt with in a satisfactory manner. Here I also include questions tangentially related to my main investigation.

Prerequisites

I will throughout assume that the reader is familiar with characteristic classes (as introduced in Milnor and Stasheff's book [12]) and rudimentary algebraic topology (on the level of Allan Hatcher's textbook [3]). It could be argued that my demands on the reader are somewhat uneven. I assume familiarity with Stiefel-Whitney classes and use these quite liberally in examples, but I go to some length introducing the Euler class from scratch.

This is not intended as a textbook in geometric topology, and as such there will be numerous statements without proof. I have tried to formulate whatever I am using precisely and to provide a reference. I appologise in advance to any reader who either feels I am referencing too much, not enough, or simply waving away all the interesting details whilst painstakingly plowing through tedious details.

Notation

I will refer to a vector bundle $E \to X$ whose fibers have dimension n as a rank n vector bundle. This conforms with the usage amongst differential geometers but goes against topologists, who tend to use the term n-dimensional vector bundle. I will mostly assume that the vector bundle in question is real and that the rank is the dimension of the fibers as real vector spaces unless otherwise specified or directly implied by the setting.

Whenever I use the word manifold I will have have a C^∞ manifold in mind, but what I need is for its tangent bundle to exits, so the reader can mentally substitute C^1-manifold if they are so inclined.

Contents

Chapter 1

Background information

In this chapter I outline and in some cases elaborate on what was known about the problem when I was given the formulation. I of course start by stating the problem before moving on to the homotopy invariance of the problem and some easy consequences. Principal bundles are introduces and the equivariant formulation is developed. The problem is shown to have solutions for odd rank bundles and I discuss a procedure for locally defining a gauge transformation. The chapter ends with the formulation of a few related problems.

The Problem

Let X be a paracompact, connected and path connected Hausdorff space and $E \xrightarrow{\pi} X$ a vector bundle with fibers vectors spaces over $\mathbb{K}$ where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ (the original problem had $\mathbb{K} = \mathbb{C}$). A bundle automorphism (or gauge transformation) means a map $\phi : E \to E$ which is a linear automorphism in each fiber and which makes the following diagram commute.

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & X & \end{array}$$

In particular, ϕ restricts to an automorphism ϕ_x on each fiber $p^{-1}(\{x\}) = E_x \subset E$ with a well-defined determinant $\det(\phi_x) : X \to \mathbb{K}^*$. The problem is: Assume a map $\psi : X \to \mathbb{K}^*$ is given. Does there exist a bundle automorphism $\phi : E \to E$ such that $\det(\phi) = \psi$?

Just to be clear, the map ϕ_x I am talking about is given via a trivialisation $t : E_{|U} \to U \times \mathbb{K}^n$ and $t\phi_{|U}t^{-1}(x, v) = (x, \phi_x v)$ in the diagram.

$$\begin{array}{ccc} E_{|U} & \xrightarrow{t} & U \times \mathbb{K}^n \\ {\scriptstyle\phi_{|U}}\downarrow & & \downarrow{\scriptstyle (x,v) \mapsto (x, \phi_x v)} \\ E_{|U} & \xrightarrow[t]{} & U \times \mathbb{K}^n \end{array}$$

On another overlapping trivialising neighbourhood V the function ϕ_x will differ by a conjugation, $\tilde{\phi}_x = g(x)\phi_x g(x)^{-1}$, $g : U \cap V \to GL(n, \mathbb{K})$ so $\det(\phi_x)$ is independent of local trivialisation. This is the map I refer to as $\det(\phi) : X \to \mathbb{K}^*$.

Reduction to unit norm

The first reduction is to study norm 1 maps, i.e. $|\tilde{\psi}(x)| = 1$ for all $x \in X$. Assume the problem has a solution for such maps. Let $\psi : X \to \mathbb{K}^*$ be any continuous map and define $\tilde{\psi} = \frac{\psi}{|\psi|}$. Find a corresponding $\tilde{\phi}$ such that $\det(\tilde{\phi}) = \tilde{\psi}$. Set $\phi = |\psi|^{1/n}\tilde{\phi}$ where $n = \text{rk } E$. This of course means to multiply ϕ_x by $|\psi(x)|^{1/n}$ in each fiber. Then $\det(\phi) = |\psi|\tilde{\psi} = \psi$.

In the complex case this means that the problem is equivalent to asking: given a map $\psi : X \to U(1)$ and a complex bundle $E \to X$, is there a bundle automorphism ϕ such that $\det(\phi) = \psi$? The real version is this. Given a map $\psi : X \to O(1) = \{\pm 1\}$, is there a gauge transformation ϕ such that $\det(\phi) = \psi$?

Here the big difference between $\mathbb{R}^*$ and $\mathbb{C}^*$ becomes apparent: $\mathbb{R}^*$ is non-connected whereas $\mathbb{C}^*$ is.

Reduction to a constant map

In the real case there is a further reduction. Since $\psi : X \to O(1)$ is continuous and X is assumed to be connected ψ has to be constant. $\psi = 1$ can be solved by the identity $\phi = \text{id} : E \to E$. So the interesting case is $\psi(x) = -1$. I.e. gauge transformations with $\det(\phi_x) = -1$. These will be referred to as orientation reversing gauge transformations or simply orientation reversals for short. The problem I will be looking at is thus whether or nor a given vector bundle admits an orientation reversing gauge transformation.

Homotopy Invariance

Complex case

Given continuous family of maps $\psi_t : X \to \mathbb{C}^*$ for $0 \le t \le 1$ and a bundle automorphism $\phi_0 : E \to E$ such that $\psi_0 = \det(\phi_0)$, then there is a family of bundle automorphism $\phi_t : E \to E$ such that $\psi_t = \det(\phi_t)$. A proof of this statement can be found in [1], and I omit it since I will not be looking into the complex case beyond this chapter. The homotopy invariance implies that existence of a gauge transformation ϕ with $\psi = \det(\phi)$ will only depend on the homotopy class of the map ψ, $[\psi] \in [X, \mathbb{C}^*] \cong [X, \mathbb{S}^1] = [X, K(\mathbf{Z}, 1)] \cong H^1(X; \mathbf{Z})$. The easy consequences of this is that if $H^1(X, \mathbf{Z}) = 0$ (for instance if $\pi_1(X)$ is 0 or a torsion group), then $[\psi] = 0$, hence $\psi \cong 1$ and $\phi = \text{id} : E \to E$ will represent the homotopy class of the desired automorphism.

Real case

As in the complex case, $\psi_t : X \to \mathbb{R}^*$ for $0 \le t \le 1$ along with a bundle automorphism $\phi_0 : E \to E$ will give rise to a family of bundle automorphisms ϕ_t with $\psi_t = \det(\phi_t)$, but here the proof is trivial since ψ_t has to be a constant for all t hence $\phi_t = \phi_0$ will do the job. The vanishing of the group $H^1(X; \mathbf{Z})$ will turn out not to suffice in the real case, since there are bundles over $\mathbb{S}^{2n}$ for all $n \geq 1$ which do not allow orientation reversals. I will prove this in chapters 2 and 3.

Reduction to isomorphism class

For isomorphic bundles I can move gauge transformations from one to the other. As a proposition this reads

Proposition 1.1 (Proof delayed)**.** *Let $E, E' \to X$ be two bundles over X. If $E \cong E'$ and one of them admits an orientation reversal then so does the other.*

In chapter 3 - lemma 3.3 - I will discuss pulling back gauge transformations and then this proposition will be a corollary.

This is the result which allows me to focus on isomorphism classes of bundles rather than bundles proper, and I will be using it without further comment in the remainder of the thesis.

Principal bundles

Since I will from time to time find it convenient to analyse the problem using principal bundles I will spend some time introducing these and their relation to vector bundles. I follow closely the exposition of [7], which in turn is close to how [11] handles it.

Let G be a topological group (G will be a Lie group in all my examples). A principal G-bundle P is a topological space (it will be a smooth manifold in my examples) along with the following.

- A continuous action of G on P from the right.
- A surjective map $\pi : P \to X$ with $\pi(p \cdot g^{-1}) = \pi(p)$ for all $p \in P$ and $g \in G$.
- For any $x \in X$ there is a neighbourhood U of x and a G-equivariant map $\phi : P_{|U} \to U \times G$ which is such that if $\phi(p) = (\pi(p), h(p))$ for some $h(p) \in G$, then $\phi(pg^{-1}) = (\pi(p), h(p)g^{-1})$.

As usual $P_{|U} = \pi^{-1}(U)$.

Relation to vector bundles

The relation between $O(n)$-bundles (or $U(n)$-bundles) over X and real (respectively complex) vector bundles over X is very close. The two (principal and vector) bundles are both determined by the same data. This can be made precise as follows. Assume a real rank n vector bundle $E \to X$ is given and equip E with an inner product (which I can do as X is assumed to be paramcompact and Hausdorff). Define $P_{O(n)} \subset \bigoplus_{k=1}^{n} E$ to be the set $P_{O(n)} = \{(e_1, \cdots, e_n) | \{e_1, \cdots e_n\}$ restricts to an orthonormal basis over each $x \in X\}$. An $O(n)$-action is given by $(e_1, \cdots e_n) \mapsto \left(\sum_{k=1}^n g_{k1}^{-1} e_k, \cdots \sum_{k=1}^n g_{kn}^{-1} e_k\right)$. Let $\phi_U : E_{|U} \to U \times \mathbb{R}^n$ be a local trivialisation for E. This gives rise to a trivialisation $\bigoplus_{k=1}^{n} \phi_U : \left(\bigoplus_{k=1}^{n} E_{|U}\right) \to U \times R^{n^2}$. When restricted to $P_{O(n)}$ this gives a trivialisation

$P_{O(n)} \xrightarrow{\cong} U \times O(n)$ where $O(n)$ is identified as n-tuples of orthonormal vectors.

Finally, and perhaps most importantly, we have the fact that the transition functions are the same. By this I mean the following. Let U_α and U_β be trivialising neighbourhoods, $E_{|U_\alpha} \xrightarrow{t_\alpha} U_\alpha \times R^n$ and $E_{|U_\beta} \xrightarrow{t_\beta} U_\beta \times \mathbb{R}^n$ and assume $U_\alpha \cap U_\beta = U_{\alpha\beta} \neq \varnothing$. The situation is encoded in the diagram. I have omitted writing restrictions on t_α and t_β.

$$\begin{array}{ccc} E_{|U_{\alpha\beta}} & \xrightarrow{t_\alpha} & U_{\alpha\beta} \times \mathbb{R}^n \\ \Big\| & & \Big\downarrow t_\beta \circ t_\alpha^{-1} \\ E_{|U_{\alpha\beta}} & \xrightarrow{t_\beta} & U_{\alpha\beta} \times \mathbb{R}^n \end{array}$$

The right map is $t_\beta \circ t_\alpha^{-1}(x,v) = (x, g_{\alpha\beta}(x)v)$ for some transition function $g_{\alpha\beta} : U_{\alpha\beta} \to O(n)$. For the associated principal bundle the situation is given in the diagram below - where I allow myself to write t_α for the map induced from t_α and I write P for $P_{O(n)}$.

$$\begin{array}{ccc} P_{|U_{\alpha\beta}} & \xrightarrow{t_\alpha} & U_{\alpha\beta} \times O(n) \\ \Big\| & & \Big\downarrow t_\beta \circ t_\alpha^{-1} \\ P_{|U_{\alpha\beta}} & \xrightarrow{t_\beta} & U_{\alpha\beta} \times O(n) \end{array}$$

Here the map $t_\beta \circ t_\alpha^{-1}$ is given by $(x, v_1, v_2, \cdots v_n) \mapsto (x, g_{\alpha\beta}(x)v_1, \cdots g_{\alpha\beta}(x)v_n)$. Writing $h = (v_1, v_2, \cdots v_n) \in O(n)$ this becomes $(x,h) \mapsto (x, g_{\alpha\beta}(x)h)$.

The fact that the vector bundle E and the associated principal bundle share an open cover of X of trivialising neighbourhoods along with transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to O(n)$ is what is meant by the statement that both kinds of bundles are defined using the same data.

The same discussion goes through in the complex case with $U(n)$ instead of $O(n)$ where the complex vector bundle is assumed to be equipped with an Hermitian inner product.

Vector bundles from principal bundles

I should perhaps mention how one goes from a principal $O(n)$-bundle to a real vector bundle or a principal $U(n)$-bundle to a complex vector bundle[1].

Let $P \to X$ be a principal $O(n)$-bundle and let $\rho : O(n) \hookrightarrow GL(n, \mathbb{R})$ be the defining or fundamental (or whatever other name you have) representation. Define $E = P \times_\rho \mathbb{R}^n = (P \times \mathbb{R}^n)/\sim$ with the equivalence relation $(p,v) \sim (pg^{-1}, \rho(g)v)$. E is then a fiber bundle with fiber $\mathbb{R}^n$, which I claim is a vector bundle.

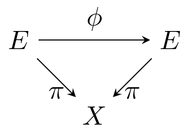
[1]This is a special case of a more general construction for an arbitrary topological group G. I do not intend to use the general setup here hence I do not introduce it. The interested reader may consult [7] or [11].

To see this, one can go through the following. The projection map $\pi : P \to X$ extends to $P \times \mathbb{R}^n \to X$ with $\pi(p, v) = \pi(p)$ and this becomes a map on the quotient since $\pi(pg^{-1}) = \pi(p)$. Let $\phi : P_{|U} \to U \times O(n)$ be a local trivialisation, and define $\psi : P_{|U} \to O(n)$ as being ϕ followed by a projection to the second factor. Define $\Phi : (P \times_\rho \mathbb{R}.)_{|U} \to U \times \mathbb{R}^n$ as the map sending the class of $(p, v) \mapsto (\pi(p), \rho(\psi(p))v)$. This is a bijection since the inverse is given by $(x, v) \mapsto (\phi^{-1}(x, 1), v)$ where I mean the equivalence class of the latter. To see that this is the inverse is a small computation: $(x, v) \mapsto (\phi^{-1}(x, 1), v) \mapsto (\pi(\phi^{-1}(x, 1)), \rho(\psi(\phi^{-1}(x, 1)))v)$. Per definition we have that $\psi(\phi^{-1}(x, 1)) = 1$ and $\rho(1) = 1$. Also by definition is the fact that $\pi(\phi^{-1}(x, 1)) = x$. I.e. $(x, v) \mapsto (x, v)$. The other composition is similar. $(p, v) \mapsto (\pi(p), \rho(\psi(p))v) \mapsto (\phi^{-1}(\pi(p), 1), \rho(\psi(p))v) = (p, v)$ up to equivalence $(pg^{-1}, \rho(g)v) \sim (p, v)$.

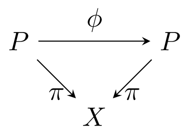
For the complex case, let $\rho : U(n) \hookrightarrow GL(n, \mathbb{C})$ be the fundamental representation and go through the previous steps. If instead one uses $\rho^c : U(n) \to GL(n, \mathbb{C})$, $\rho^c(a) = a^*$ one ends up with the complex conjugated complex vector bundle instead.

Bundle automorphisms

A vector bundle automorphism was a map making the following diagram commute



The principal bundle variant is a G-equivariant ϕ which makes this diagram commute



If it's necessary to keep these apart, then I suppose that the first kind should be called vector bundle automorphisms and the second kind a G-bundle automorphisms or gauge transformation. I will use the terms interchangeably, with a bias towards "gauge transformation" due to its predominance in physics (see [19] for instance).

A fun fact is that the set of bundle automorphisms $\mathrm{Aut}_X(P)$ of a principal bundle is a topological group called the Gauge group. I will not need this fact, however. What I will need is that gauge transformations are in bijection with sections of $P \times_{Ad} G = P \times G/\sim$ where the equivalence is $(p, g) \sim (ph^{-1}, hgh^{-1})$ for any $h \in G$. These sections again are in bijection with G-maps $\tilde{\phi} : P \to G^{ad}$ where I mean maps such that $\tilde{\phi}(pg^{-1}) = g\tilde{\phi}(p)g^{-1}$. In words, these are G maps with right G-action on P and adjoint action on G. As a lemma this reads

Lemma 1.1. *There is a bijection $Map_G(P, G^{ad}) \to Aut_X(P)$*

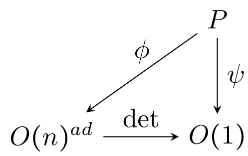
Proof. The correspondence is quite neat to write out. Given a G-map $\tilde{\phi} : P \to G^{ad}$, let $\phi(p) = p\tilde{\phi}(p)$. This is a gauge transformation $\phi : P \to P$ since $\phi(ph^{-1}) = ph^{-1}\tilde{\phi}(ph^{-1}) = ph^{-1}h\tilde{\phi}(p)h^{-1} = \phi(p)h^{-1}$, and clearly $\pi(\phi(p)) = \pi(p)$.

The map $\mathrm{Map}_G(P, G^{ad}) \to \mathrm{Aut}_X(P)$ defined above is surjective as the requirement $\pi(\phi(p)) = p$ forces $\phi(p)$ to be on the form $\phi(p) = p\tilde{\phi}(p)$ for some map $\tilde{\phi} : P \to G$. Since $\phi(ph^{-1}) = ph^{-1}\tilde{\phi}(ph^{-1}) = p\tilde{\phi}(p)h^{-1}$ we must have that $\tilde{\phi}(ph^{-1}) = h\tilde{\phi}(p)h^{-1}$, i.e. $\tilde{\phi} : P \to G^{ad}$ as a G-map.

The map is injective since if $\phi(p) = p\tilde{\phi}_1(p) = p\tilde{\phi}_2(p)$ then $\phi(p)\tilde{\phi}_1(p)^{-1} = p = p\tilde{\phi}_2(p)\tilde{\phi}_1(p)^{-1}$. Hence $\tilde{\phi}_2(p)\tilde{\phi}_1(p)^{-1} = e$ by the freeness of the G-action on P. □

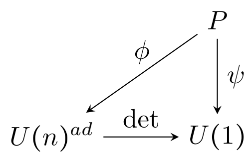
The above lemma yields the correspondence with the promised equivariant lifting problem.

Proposition 1.2. *Let $P \to X$ be a principal $O(n)$-bundle associated to the real vector bundle $E \to X$. Assume $\phi : X \to O(1)$ is given. Then a bundle automorphism $\phi : E \to E$ with $\det(\phi) = \psi$ corresponds bijectively to an $O(n)$-equivariant map $\phi : P \to O(n)^{ad}$ which makes the following diagram commute*



The complex version looks very similar:

Proposition 1.3. *Let $P \to X$ be a principal $U(n)$-bundle associated to the complex vector bundle $E \to X$. Assume $\psi : X \to U(1)$ is given. Then a bundle automorphism $\phi : E \to E$ with $\det(\phi) = \psi$ corresponds bijectively to a $U(n)$-equivariant map $\phi : P \to U(n)^{ad}$ which makes the following diagram commute*



Both problems above will be referred to the equivariant lifting formulation of the problem.

$O(2k+1)$-bundles

For $O(2k+1)$-bundles, the answer to the main question is "yes".

Proposition 1.4. *Assume $P \to X$ is a principal $O(2k+1)$-bundle. Then P admits an orientation reversing gauge transformation.*

Proof. The proof is elementary given the equivariant formulation: Let $\phi(p) = \psi(p)1$ where $1 \in O(2k+1)$ is the matrix. Then $\det(\phi(p)) = \psi(p)^{2k+1} = \psi(p)$ since $\psi(p) \in O(1) \cong \{\pm 1\}$. □

Remark 1.1. Note how this is a real phenomenon, as it's generally not true that $z = z^n$ for $z \in U(1)$ unless $n = 1$.

Locally defining gauge transformations

It's possible to see that odd bundles admit orientation reversals without the equivariant formulation as well. Let $E \to X$ be a real vector bundle of odd rank. Let $\mathcal{U} = \{U_\alpha\}_\alpha$ be a collection of trivialising neighbourhoods for E and write $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Start with some U_α and define $\phi_\alpha : E_{|U_\alpha} \to E_{|U_\alpha}$ via the trivialisations; $\phi_\alpha(e) = t_\alpha^{-1}(x, \psi(x)v)$ where $t_\alpha(e) = (x, v)$. This is the content of this diagram.

$$\begin{array}{ccc} E_{|U_\alpha} & \xrightarrow{t_\alpha} & U_\alpha \times \mathbb{R}^n \\ {\scriptstyle \phi_\alpha}\downarrow & & \downarrow{\scriptstyle f} \\ E_{U_\alpha} & \xrightarrow[t_\alpha]{} & U_\alpha \times \mathbb{R}^n \end{array}$$

f is defined as $f(x,v) = (x, \psi(x)v)$. Locally defining a map like this requires checking that there are no conflicts on overlapping trivialisations, and that's where the centrality of $\psi(x) \cdot 1$ comes in. By this I mean that on $U_\alpha \cap U_\beta$ we can write $\phi_\alpha = t_\alpha^{-1} f t_\alpha$ and $\phi_\beta = t_\beta^{-1} f t_\beta$. Demanding that these are equal gives $f = t_\beta t_\alpha^{-1} f t_\alpha t_\beta^{-1}$. Recall that $t_\beta t_\alpha^{-1}(x,v) = (x, g_{\alpha\beta} v)$, and as such the composition reads $f = t_\alpha^{-1} t_\beta f t_\beta^{-1} t_\alpha(x,v) = (x, g_{\alpha\beta}^{-1}(\psi \cdot 1) g_{\alpha\beta} v) = (x, \psi(x)v)$. I.e. the locally defined function patches together to a well-defined global function.

The insight that I could locally define an orientation reversal if it could be chosen to commute with the transition functions $g_{\alpha\beta}$ on each intersection of some trivialising cover might be formulated as a lemma

Lemma 1.2. *Let $E \to X$ be a real rank n vector bundle. E admits an orientation reversing bundle automorphism if there is an open cover $\mathcal{U} = \{U_\alpha\}_\alpha$ of X of trivialising neighbourhoods for E and a choice of constant element $r \in O(n)_-$ which commutes with all the transition functions of E. In particular, when n is odd, this can always be done by choosing $r = -1$.*

There is a sort of converse to the above as well

Lemma 1.3. *Assume $\phi : E \to E$ is a given bundle automorphism with $\phi_x \in O(n)_-$ for each $x \in X$. On intersecting trivialising neighbourhoods this has to restrict to compatible maps which requires that $g_{\alpha\beta}(x)\phi_x g_{\alpha\beta}(x)^{-1} = \phi_x$ at each point $x \in X$.*

Isomorphic bundles do not have to have the same transition functions. The transition functions only have to be related by similarity transformations at each point. The precise relationship is encoded in theorem 2.7 on page 63 i [11].

Proposition 1.5 (No proof)**.** *Let E, E' be two rank n vector bundles over X. Pick a common trivialising cover $\{U_\alpha\}_\alpha$. Let the transition functions for E and E' be $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{K})$ and $g'_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{K})$ respectively. Then $E \cong E'$ if and only if there are maps $S_\alpha : U_\alpha \to GL(n, \mathbb{K})$ such that $g_{\alpha\beta}(x) = S_\alpha(x)^{-1} g_{\alpha\beta}(x) S_\alpha(x)$ for each $x \in U_\alpha \cap U_\beta$.*

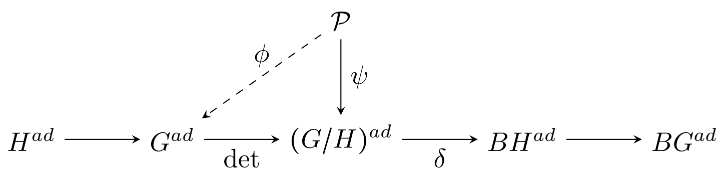
Proposition 1.6 (Locally defining automorphisms)**.** *Let $E \to X$ be a rank n real bundle and let $\mathcal{U} = \{U_\alpha\}_\alpha$ be a cover of X by trivialising neighbourhoods of E*

with transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to O(n)$. Then E admits an orientation reversing bundle automorphism $\phi : E \to E$ if and only if there are maps $S_\alpha : U_\alpha \to GL(n, \mathbb{R})$ such that $(S(x)^{-1}_\alpha g_{\alpha\beta}(x) S_\alpha(x))\phi_x(S_\alpha(x)^{-1} g_{\alpha\beta}(x)^{-1} S_\alpha(x)) = \phi_x$. Using commutator notation this can be written as $[g_{\alpha\beta}, \phi_x] = 1$ with the understanding that g can be changed by a similarity transformation.

I will be using (and deriving) a special case of this statement for bundles over $\mathbb{S}^n$ in chapter 2.

Related problems

From the equivariant formulation there is another class of closely related problems, namely given a pair of Lie groups $H \subset G$, does the G-equivariant lifting problem



admit natural interpretations and/or a definite answer?

I am assuming that the map $G \xrightarrow{\det} G/H$ makes sense, so G and H can of course not be chosen arbitrarily. Note that the Quaternions do not furnish such an example directly, as there is no SSp(n), since $a \in Sp(n)$ has $\det(a) = 1$ already.

Inspired by $G = U(n)$ and $G = O(n)$, it's tempting to suggest $G = U(p, q)$ and $G = O(p, q)$. These are the matrix groups preserving the bilinear forms (or pseudo inner products)

$$\langle u, v\rangle = -\sum_{i=1}^{p} \overline{u_i} v_i + \sum_{j=p+1}^{p+q} \overline{u_j} v_j$$

and

$$\langle u, v\rangle = -\sum_{i=1}^{p} u_i v_i + \sum_{j=p+1}^{p+q} u_j v_j$$

on $\mathbb{C}^{p+q}$ and $\mathbb{R}^{p+q}$ respectively.

Some homogeneous spaces

For some of these groups there are obvious choices of principal bundles. Analogously to how a sphere can be defined as a homogeneous space via its principal frame bundle $SO(n) \to SO(n+1) \to \mathbb{S}^n$ it's possible to define spaces using $O(p, q)$ or $SO(p, q)$. For instance $SO(1, n)_e / SO(n) \cong H^n$, n-dimensional hyperbolic space. Since H^n is contractible this is an uninteresting example for the lifting problem but the next two spaces I will define are not contractible. $AdS_n = O(2, n)/O(1, n)$ Anti de Sitter space and $dS_n = O(1, n)/O(1, n-1)$ de Sitter space. These are part of larger families with

$$\mathbb{S}^n_s = \{x \in \mathbb{R}^{n+1,s} \big| \langle x, x\rangle = +1\}$$

and
$$\mathbb{H}^n_s = \{x \in \mathbb{R}^{n+1,s+1} \big| \langle x, x\rangle = -1\}$$
where the notation comes from [18]. Topologically these aren't too exotic.

Lemma 1.4. *There are diffeomorphisms*
$$a : \mathbb{S}^n_s \to \mathbb{R}^s \times \mathbb{S}^{n-s}$$
$$a(x) = \left(x_1, \cdots x_s, \frac{x_{s+1}}{p(x)}, \cdots \frac{x_{n+1}}{p(x)} \right)$$
with $p(x) = \sqrt{1 + \sum_{i=1}^{s} x_i^2}$ *and*
$$b : \mathbb{H}^n_s \to \mathbb{S}^s \times \mathbb{R}^{n-s}$$
$$b(x) = \left(\frac{x_1}{q(x)}, \cdots, \frac{x_{s+1}}{q(x)}, x_{s+2}, \cdots x_{n+1} \right)$$
with $q(x) = \sqrt{1 + \sum_{i=s+2}^{n+1} x_i^2}$.

Proof. Both statements are from rewriting the definitions:
$$x \in \mathbb{S}^n_s \Leftrightarrow \sum_{i=s+1}^{n+1} x_i^2 = 1 + \sum_{i=1}^{s} x_i^2$$
and
$$x \in \mathbb{H}^n_s \Leftrightarrow \sum_{i=1}^{s+1} x_i^2 = 1 + \sum_{i=s+2}^{n+1} x_i^2$$
The maps a and b are obviously smooth. The inverse maps are
$$a^{-1}(x) = (x_1, \cdots x_s, p(x)x_{s+1}, \cdots p(x)x_{n+1})$$
and
$$b^{-1}(x) = (q(x)x_1, \cdots, q(x)x_{s+1}, x_{s+2}, \cdots, x_{n+1})$$
These are also seen to be smooth. □

The above is lemma 2.4.6 in [18].

To realise these as quotients of groups I suggest looking at the fiber sequences
$$O(p,q) \xrightarrow{i} O(p, q+1) \xrightarrow{\pi} \mathbb{S}^{q-1}_p$$
$i(A) = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ and $\pi(A) = A(e_{p+q+1})$. This reproduces the frame bundle of the Riemannian n-sphere $\mathbb{S}^n$ when $(p,q) = (0,n)$ and de Sitter space dS_n for $(p,q) = (1, n-1)$.

For the hyperbolic spaces, I can offer the fiber sequence
$$O(p,q) \xrightarrow{i} O(p+1, q) \xrightarrow{\pi} \mathbb{H}^{q-1}_p$$

$i(A) = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$ and $\pi(A) = A(e_1)$. The famous cases are $(p,q) = (0, n+1)$ with $\mathbb{H}^n_0 \cong \mathbb{H}^n \times \mathbf{Z}/2$ (two copies of Riemannian hyperbolic space) and $(p,q) = (1,n)$ giving $\mathbb{H}^{n-1}_1 \cong AdS_n$. [18] doesn't formulate the above as fiber sequences, but he gets sufficiently close to it for me to suspect that the above is known to practitioners of pseudo-Riemannian geometry.

The point of the above spaces is that they furnish examples of principal bundles with structure group $O(p,q)$ and topologically well-known base spaces occurring naturally. To simplify matters, it would be possible to ask for $O(p,q)$ bundles which admit a reduction of the structure group to either $SO(p,q)$ or $SO(p,q)_e$ (the connected component).

I have not had time to investigate these cases in this thesis, but I think it's interesting to know that there are questions related to my main question which become apparent with the equivariant formulation.

The problem is unstable

I would like to elaborate on an observation I owe to John Rognes, namely that the problem is not stable under adding a trivial bundle. By this I mean that if E is a (real or complex) vector bundle over X which may or may not allow a solution to either the real or complex problem, the bundle $E \oplus \epsilon^n$ will allow a solution for any $n \geq 1$. Here, as in Milnor-Stasheff [12], $\epsilon^n = X \times \mathbb{K}^n$.

In particular, suppose c is some stable total characteristic class satisfying the Whitney-sum axiom $c(E \oplus E') = c(E)c(E')$ and $c(\epsilon^n) = 1$. Then $c(E \oplus \epsilon^n) = c(E)$, even though $E \oplus \epsilon^n$ allows a desired bundle automorphism whereas E might not. This means that there will in general not be a quick answer to whether or not a bundle allows solution to the real or complex variant of the question in terms of a stable characteristic class.

This is somewhat modified in chapter 3, where I for have a definite answer in special situations using a Chern class (but not the total Chern class!), but I claim that an analogy with the situation of orientability and spin structure will be impossible. By this I mean that we know that $E \to X$ is orientable if and only if $w_1(E) = 0$ and $E \to X$ admits a spin structure if and only if $w_2(E) = 0$, regardless of the rank of E and dimension of X (if X is a CW complex). What I claim is that there can be no equally simple statement about orientation reversal for an arbitrary rank bundle over a paracompact Hausdorff space X.

The interesting range

When looking for interesting examples there are a couple of things to keep in mind. For the real case, only even rank bundles are interesting. For orientable real bundles I can narrow the search a bit. If X is an n-dimensional CW-complex then the interesting area is when $\mathrm{rk}(E) \leq n$, due to a statement I take from [8] (where it appears as corollary 14.2 on page 514).

Proposition 1.7. *Assume $E \to X$ is a rank k oriented vector bundle over a CW complex X of dimension $n < k$. Then there exists a rank n vector bundle $E' \to X$ such that $E \cong E' \oplus \epsilon^{n-k}$.*

I will give a proof of this in chapter 4 after I have established some obstruction theory. I have an independent argument for this when the X is a sphere in the next chapter.

Chapter 2

Spheres as examples

In this chapter I will have a look at what the situation is for the spheres $\mathbb{S}^n$. I will start by saying something about clutching functions on spheres. I follow Husemoller's computation to derive the clutching function of the tangent bundle of spheres. I then perform specialized computations for arbitrary k-bundles over $\mathbb{S}^n$ for $1 \le n \le 4$. I end the chapter by a completely brute force computation for tangent bundles of $\mathbb{S}^n$ for all n. There will be a more elegant approach in the next chapter, but which relies on the machinery of characteristic classes, whereas the main arguments of this chapter only need homotopy theory.

Clutching functions

The arguably easiest way to specify a vector bundle over an n-sphere is to write $\mathbb{S}^n = D^n_- \cup D^n_+$ with $D^n_- \cap D^n_+ \cong \mathbb{S}^{n-1}$ and note that any vector bundle $E \to \mathbb{S}^n$ restricted to $D^n_\pm$ will be trivial since the disk is contractible. So the data needed to determine an oriented bundle is the transition function $g : \mathbb{S}^{n-1} \to SO(n)$, or rather the homotopy class of this map.

For principal G-bundles there is the analogous statement, namely that a principal G-bundle over $\mathbb{S}^n$ is determined by the homotopy class of a map $g : \mathbb{S}^{n-1} \to G$. This is for instance Corollary 8.4 on page 98 in [11] which states the following:

Theorem 2.1 (No proof)**.** *Let G be a path connected group. The isomorphism classes of locally trivial principal G-bundles ξ over $\mathbb{S}^n$ are classified by elements $[c_\xi] \in \pi_{n-1}(G)$.*

I will mainly need this for the case where $G = SO(n)$, corresponding to oriented real vector bundles. The complex case would be $G = U(n)$, but as promised I will not delve into that setting. It's worth noting that I will a priori need a stronger result to say something about non-orientable bundles since then $G = O(n)$ which is not connected. However, as I point out the next chapter, there are no non-orientable bundles over $\mathbb{S}^n$ for $n \ge 2$.

Since I'm mostly working with orientable vector bundles $\leftrightarrow$ principal $SO(n)$-bundles I could have gotten the same result from the classification result of $Vect^k_+(\mathbb{S}^n) \cong [\mathbb{S}^n, \tilde{Gr}_k] \cong \pi_n(\tilde{Gr}_k)$ along with the realisation of the oriented Grassmannian as a homogeneous space $\tilde{Gr}_k(N) \cong SO(N)/(S(O(k)\times O(N-k)))$

and the associated long-exact sequence in homotopy. Here N can be taken to be at least $n+k$. One also needs to know that $\pi_l(SO(n)) \xrightarrow{i_*} \pi_l(SO(n+k))$ is an isomorphism when $l \leq k+2$ which follows from the fiber sequence $SO(n) \to SO(n+1) \to \mathbb{S}^n$ along with the fact that $\pi_l(\mathbb{S}^n) = 0$ for $l < n$.

Maximal interesting rank of bundles over spheres

If the reader dislikes me referencing obstruction theory in the form of proposition 1.7 from the previous chapter I can offer an argument based on theorem 2.1 instead.

Proposition 2.1. *Let $E \to \mathbb{S}^n$ be a rank $(n+k)$-bundle with $k \geq 1$. Then there is an $(n+k-1)$-bundle E' such that $E \cong E' \oplus \epsilon^1$.*

Proof. The isomorphism class of an $(n+k)$-bundle over $\mathbb{S}^n$ is determined by an element $[g] \in \pi_{n-1}(SO(n+k))$. By the long exact sequence in homotopy associated to the fiber sequence $SO(k) \xrightarrow{i} SO(k+1) \to \mathbb{S}^k$, it's apparent that $\pi_{n-1}(SO(n+k)) \xrightarrow{i_*} \pi_n(SO(n+k+1)$ is an isomorphism for $k \geq 1$, surjective for $k=0$ and $[g] \mapsto i_*[g]$ is represented by $\begin{pmatrix} g(x) & 0 \\ 0 & 1 \end{pmatrix}$. Given an $(n+k)$-bundle for $k \geq 1$ with clutching function $g : \mathbb{S}^{n-1} \to SO(n+k)$ there is a function $h : \mathbb{S}^{n-1} \to SO(n+k-1)$ with $i_*([h]) = [g]$, i.e. $[g]$ can be represented by a map $\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$, corresponding to $E \cong E' \oplus \epsilon^1$ for some $(n+k-1)$-bundle E'. □

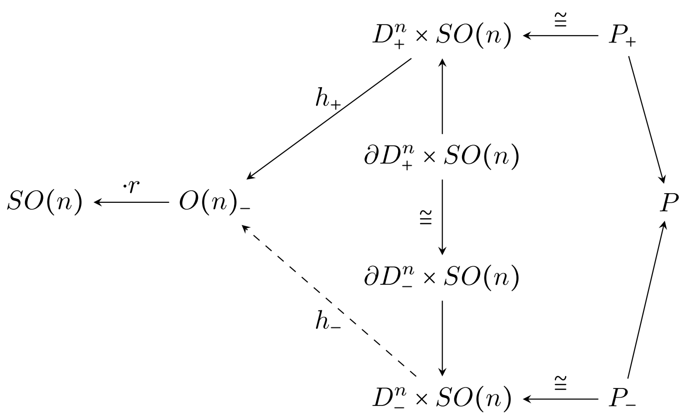
There is a "lower bound" on the rank of interesting examples of bundles over spheres. For $n > 2$ there aren't any non-trivial rank 2 bundles over $\mathbb{S}^n$, simply because $\pi_{n-1}(SO(2)) \cong \pi_{n-1}(\mathbb{S}^1) = 0$. This doesn't repeat for the next even rank as $\pi_{n-1}(SO(4)) \cong \pi_{n-1}(\mathbb{S}^3) \times \pi_{n-1}(\mathbb{S}^3) \neq 0$ for all $n-1 > 3$.

The setup

The idea of this chapter[1] is that one can define an equivariant map on $P_{|D^n_+} \cong D^n_+ \times SO(n) \to O(n)_-$ (say), which restricts to a to an equivariant map $P_{|\mathbb{S}^{n-1}} \cong \mathbb{S}^{n-1} \times SO(n) \to O(n)_-$. This corresponds to a non-equivariant map $\mathbb{S}^{n-1} \to O(n)_-$ which one can clearly construct as a constant, $x \mapsto r$ where $\det r = -1$. For general $\mathbb{S}^n$ I will be using $r = \mathrm{diag}(+1, +1, \cdot, -1)$ but for $n = 4$ I find it convenient to use $r = (-1, 1, \cdots 1)$. This does not correspond to a constant map when the other trivialization is used, but rather picks up an adjoint action $h(x) = g(x) r g(x)^{-1}$ where $g : \mathbb{S}^{n-1} \to SO(n)$ is the clutching function of the bundle. h extends from $\mathbb{S}^{n-1}$ to D^n_- if and only if it is homotopic to a constant. Multiplying h with r to the right gives a map $f = h \cdot r : \mathbb{S}^{n-1} \to SO(n)$ and I'm interested in whether or not this is homotopic to a constant function, meaning if $[f] \in [\mathbb{S}^{n-1}, SO(n)] \cong \pi_{n-1}(SO(n))$ represents 0. Note that I'm here implicitly using the fact that $SO(n)$ is a Lie group (or more generally an H-space) to say that π_1 acts trivially on π_{n-1}, thus creating no problem in identifying unbased homotopy classes with a homotopy group.

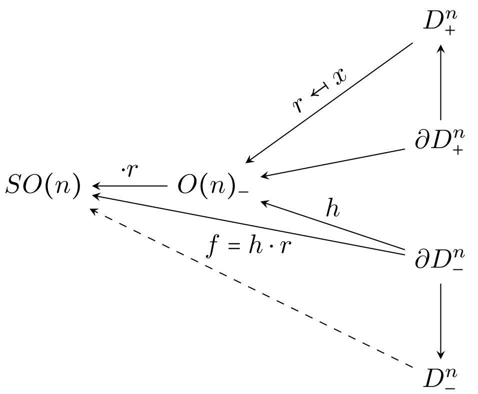
The diagram should clarify the situation. I use the following abbreviations: $P_\pm = P_{|D^n_\pm}$, $P_{+-} = P_{|D^n_+ \cap D^n_-} \cong P_{|\mathbb{S}^{n-1}}$.

[1]Which I owe to John Rognes.



In the diagram $h_+((x,\rho)) = \rho r \rho^{-1}$.

The non-equivariant diagram of interest corresponding to the above is as follows



As in the text above, $h : \partial D^n_- \to O(n)_-$, $h(x) = g(x) r g(x)^{-1}$.

What has been shown here is the following.

Lemma 2.1. *Assume $E \to \mathbb{S}^n$ is an oriented vector bundle with clutching function $g : \mathbb{S}^{n-1} \to SO(n)$. Then E admits an orientation reversing automorphism if and only if the function $f : \mathbb{S}^{n-1} \to SO(n)$ given by $f(x) = g(x) r g(x)^{-1} r$ is nullhomotopic where r is an element of $O(n)_-$.*

Remark 2.1. The above lemma can be compared to Proposition 1.6 in the previous chapter, where the sphere allows a cover of only 2 trivialising neighbourhoods with 1 corresponding transition function.

There is an algebraic fact which is somewhat relevant here, namely if r can be central.

Lemma 2.2. *The center of $O(n)$ is $O(1) = \{\pm 1\}$. Furthermore* $\det(-1) = (-1)^n$ *so $O(n)_-$ has a central element if and only if n is odd.*

This reinforces what I said about odd bundles, where r can be chosen to be -1 and $f(x) = 1$ for all x.

Another quick observation is that this correctly predicts that if g is homotopic to a constant (meaning that the bundle is trivial) then f is homotopic to a constant and the bundle admits an orientation reversal.

I will show for the 4-sphere that neither of the above situations are necessary, as the 4-sphere has non-trivial 4-bundles which admit orientation reversals. Before I get started on the 4 lowest dimensional spheres let me find the clutching function for the tangent bundle of the n-sphere.

The tangent bundle of $\mathbb{S}^n$ as a principal bundle

The spaces $\mathbb{S}^n$ are homogeneous spaces can fit into the following fiber sequence

$$SO(n) \xrightarrow{i} SO(n+1) \xrightarrow{p} \mathbb{S}^n$$

Here i is the inclusion $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ and p is the map $p(A) = A(e_{n+1})$ where $e_{n+1} \in \mathbb{R}^{n+1}$ is the last element of the standard basis. This fiber sequence determines the frame bundle of $\mathbb{S}^n$ by $P = SO(n+1)$ with "the obvious" action of $SO(n)$ from the right.

The transition function will be of utmost importance for my purposes both in this chapter and the next, so let me take some time deriving it. This computation will be the one found in [11] or [10] with little added. Be warned: since [11] has $\{e_0, \cdots, e_n\}$ as standard basis for $\mathbb{R}^{n+1}$ whereas I have $\{e_1, \cdots, e_{n+1}\}$ there will be a some shifts in my indices compared to his.

For $a, b \in \mathbb{S}^{n-1}$ define $R(a,b) \in SO(n)$ by demanding that if $\langle a, y\rangle = \langle b, y\rangle = 0$ then $R(b,a)y = y$ and that $R(b,a)a = b$ where the rotation is along the shortest great circle from a to b. The formula for R is

$$R(a,b)y = y - \frac{\langle a+b, y\rangle}{1 + \langle a, b\rangle}(a+b) + 2\langle a, y\rangle b$$

Define $\phi : D^n_+ = \mathbb{S}^n \smallsetminus \{-e_{n+1}\} \to SO(n+1)$ by $\phi(x) = R(x, e_{n+1})$. Let $r(x) = \phi(e_n)^2 x$. It turns out that $r(x_1, \cdots, x_{n+1}) = (x_1, \cdots, x_{n-1}, -x_n, -x_{n+1})$ which shows that r is an involution, $r^2 = 1$. With these auxiliary functions out of the way I can start defining local trivialisations.

Define open sets $D^n_+ = \mathbb{S}^n \smallsetminus \{-e_{n+1}\}$ and $D^n_- = \mathbb{S}^n \smallsetminus \{e_{n+1}\}$. Define maps $h_+ : D^n_+ \times SO(n) \to SO(n+1)$ and $h_- : D^n_- \times SO(n) \to SO(n+1)$ by $h_+(x,u) = \phi(x)u$ and $h_-(x,u) = r\phi(r(x))u$. These fit into commutative diagrams

$$\begin{array}{ccc} D^n_+ \times SO(n) & & \\ \downarrow h_+ & \searrow pr_1 & \\ SO(n+1) & \xrightarrow[p]{} & \mathbb{S}^n \end{array}$$

$$\begin{array}{ccc} D^n_- \times SO(n) & & \\ \downarrow h_- & \searrow pr_1 & \\ SO(n+1) & \xrightarrow[p]{} & \mathbb{S}^n \end{array}$$

That these really do commute is checked by small computations. $ph_+(x,u) = p(\phi(x)u) = (\phi(x)u)e_{n+1} = R(e_{n+1},x)e_{n+1} = x$ and $ph_-(x,u) = (r\phi(r(x))u)(e_{n+1}) = r\phi(r(x))e_{n+1} = rR(r(x),e_{n+1})e_{n+1} = r^2(x) = x$. Note that $u = \begin{pmatrix} \tilde{u} & 0 \\ 0 & 1 \end{pmatrix}$ per definition of the inclusion $SO(n) \to SO(n+1)$ so $u(e_{n+1}) = e_{n+1}$. It should be clear that both maps commute with $SO(n)$-action from the right.

On $D^n_+ \cap D^n_-$ the two trivialisations are related by a transition function $g : \mathbb{S}^n \setminus \{e_{n+1}, -e_{n+1}\}$ which has to satisfy $h_+(x,g(x)u) = h_-(x,u)$, or $\phi(x)g(x)u = r\phi(r(x))u \implies g(x) = \phi(x)^{-1}r\phi(r(x))$. Let $g(x)$ also denote the restriction to $\mathbb{S}^{n-1} \subset \mathbb{S}^n$. For this map to be useful I want the identity $g(x) = \alpha(x)\alpha(e_n)$ where $\alpha(x)$ is the reflection in the hyperplane whose normal vector is x. [11] does this via $g(x) = R(x,e_n)^2$ and $R(x_1,x_2)^2 = \alpha(x_1)\alpha(x_2)$. Let me just take this identity for granted and summarise:

Proposition 2.2. *The clutching function for the tangent bundle $T\mathbb{S}^n \to \mathbb{S}^n$ or its associated frame bundle $SO(n+1) \to \mathbb{S}^n$ can be written as $g : \mathbb{S}^{n-1} \to SO(n)$, $g(x) = \alpha(x)\alpha(e_n)$ where $\alpha(x)$ is the reflection in the hyperplane orthogonal to x.*

The case $X = \mathbb{S}^1$

$\mathbb{S}^1$ is a bit special as it admits non-orientable bundles. It's not terribly interesting on the other hand, seeing how $\pi_1(BO(k)) = \mathbf{Z}/2$ for all $k \geq 1$, meaning that there are precisely 2 isomorphism classes of rank k bundles over $\mathbb{S}^1$. For any $k \geq 1$ there are two non-isomorphic bundles over $\mathbb{S}^1$, namely $\gamma_1 \times \epsilon^{k-1}$ and ϵ^k where $\gamma_1 \to \mathbb{S}^1$ is the Mobius bundle. These are clearly not isomorphic, the one being orientable and the other not. By the classification theorem these have to be the only ones up to isomorphism. This establishes my result for $\mathbb{S}^1$

Proposition 2.3. *All vector bundles over $\mathbb{S}^1$ admit orientation reversals.*

The case $X = \mathbb{S}^2$

A clutching function for a rank 2 bundle over $\mathbb{S}^2$ is a map $g : \mathbb{S}^1 \to SO(2) \cong \mathbb{S}^1$, and it's well known that such maps are all of the form $g(z) = z^k$ with $\deg(g) = k$ for $k \in \mathbf{Z}$. The map $f = grg^{-1}r$ can easily be computed:

$$rg^{-1}r \cong \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos(k\theta) & \sin(k\theta) \\ -\sin(k\theta) & \cos(k\theta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos(k\theta) & -\sin(k\theta) \\ \sin(k\theta) & \cos(k\theta) \end{pmatrix} = g$$

Hence $f(z) = g(z)^2 = z^{2k}$. This is nullhomotopic if and only if $2k = 0$ (this statement is due Hopf, and it's generalisation can for instance be found as corollary 4.25 on page 361 in [3]). The conclusion is:

Proposition 2.4. *A rank 2 vector bundle over $\mathbb{S}^2$ admits an orientation reversing gauge transformation if an only if the bundle is trivial.*

More can be said, actually. Line bundles are trivial over $\mathbb{S}^2$, and if I assume proposition 1.7 or use Proposition 2.1, I know that bundles of rank 3 and above will split off trivial bundles and as such admit orientation reversals.

Proposition 2.5. *Any rank k bundle over $\mathbb{S}^2$ will admit an orientation reversal for $k \neq 2$. For $k = 2$ only trivial bundles admit orientation reversals.*

The exceptional $X = \mathbb{S}^3$

$\mathbb{S}^3$ is very special amongst the spheres when it comes to bundles. For the question at hand we have the following

Proposition 2.6. *Any real bundle $E \to \mathbb{S}^3$ admits an orientation reversal.*

Proof. The only even rank bundle of rank < 3 is a rank 2 bundle which is trivial as $n = 3 > 2 = k$. □

The above proposition could be seen as a special case of a much more general statement.

Theorem 2.2. *Let G be a Lie group (not necessarily compact). Then $\pi_2(G) = 0$.*

I could equally well have taken theorem 2.2 as given, but I couldn't find a nice source with a proof. If I assume some other facts about Lie groups which I can find references for and a (deep) Morse theory result, theorem 2.2 can actually be proven. The idea of consulting [21] and looking at ΩG I got from [24].

Proof. I require a some statements which I take for granted. The first reduction is that I can assume G is connected since topologically $G \cong G_0 \times \pi_0(G)$ where $\pi_0(G)$ is discrete. Discrete spaces do not contribute to π_2, so I can restrict my attention to G_0. Next up is a statement saying I can restrict to compact groups.

Proposition 2.7 (No proof)**.** *Any non-compact Lie group deformation retracts onto a compact Lie group. In fact, a connected Lie group is homeomorphic (as a space) to $K \times \mathbb{R}^N$ for some $N \geq 0$ and K a compact subgroup.*

This is theorem 6 in [20].

Proposition 2.8 (No proof)**.** *Let G be a connected Lie group. Then the canonical manifold structure on the universal covering group $\tilde{G}$ (the covering space given a group structure which turns out to be unique up to isomorphism) makes $\tilde{G}$ into a connected Lie group such that the covering map is a smooth homomorphism.*

A reference is [23] proposition 1.99 on page 89. He refers to connected Lie groups as analytic groups.

The above 2 facts allow me to specialise to compact, connected, and simply connected Lie groups, as $\pi_2(G) \cong \pi_2(\tilde{G})$ for the universal cover $\tilde{G}$ and $\pi_2(G) \cong \pi_2(K \times \mathbb{R}^N) \cong \pi_2(K)$. I plan to use the fiber sequence $\Omega G \to \mathcal{P}G \to G$ where ΩG is the loop space of G and $\mathcal{P}G$ is the path space. Since $\mathcal{P}G$ is contractible the long exact sequence in homotopy gives the famous result $\pi_n(G) = \pi_{n-1}(\Omega G)$. For $n = 2$ this says $\pi_2(G) = \pi_1(\Omega G)$. For $\pi_1(\Omega G)$ I propose using a result due to Bott, stated as Theorem 21.7 on page 116 in [21].

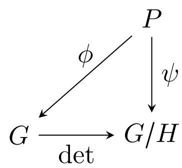
Theorem 2.3 (No proof)**.** *Let G be a compact simply connected Lie group. Then ΩG has the homotopy type of a CW-complex with no odd dimensional cells, and only finitely many λ-cells for each even value of λ. Thus $H_\lambda(\Omega G)$ is 0 for λ odd and free Abelian for even λ.*

The Hurewicz isomorphism applies since $\pi_0(\Omega G) = \pi_1(G) = 0$ per assumption so $\pi_1(\Omega G) \to H_1(\Omega G) = 0$ is an isomorphism. □

Corollary 2.3.1. *Let $P \to \mathbb{S}^3$ be a principal G-bundle with G a Lie group. Then P is trivial.*

Proof. Follows from theorem 2.1 and theorem 2.2. □

One could go on to say that the equivariant lifting problem sketched at the end of chapter 1 always admits a solution for G a Lie group and principal G-bundle $P \to X = \mathbb{S}^3$.



This is simply because $P \cong \mathbb{S}^3 \times G$.

The case $X = \mathbb{S}^4$

A clutching function for $\mathbb{S}^4$ is described by a map $g : \mathbb{S}^3 \to SO(4)$, and I will use the spin cover $\pi_3(SO(4)) \cong \pi_3(\mathrm{spin}(4)) \cong \pi_3(SU(2) \times SU(2)) \cong \pi_3(SU(2)) \times \pi_3(SU(2)) = \pi_3(\mathbb{S}^3) \times \pi_3(\mathbb{S}^3)$ where the double covering can be realized through quaternions.[2] In fact, according to [13], (which in turn relies on Hirtzebruch) any element of $\pi_3(\mathrm{spin}(4)) \cong \pi_3(\mathbb{S}^3) \times \pi_3(\mathbb{S}^3) \cong \mathbf{Z} \times \mathbf{Z}$ can be thought to act on a vector $v \in \mathbb{R}^4 \cong \mathbb{H}$ via $g_{hj}(p)(v) = p^h v p^j$ where $h, j \in \mathbf{Z}$ and $p \in \mathbb{S}^3 \subset \mathbb{H}$ is a unit quaternion.

The goal is to write the function $f : \mathbb{S}^3 \to SO(4)$, $f(x) = g(x) r g(x)^{-1} r$ using quaternions. I will for convenience let r be the reflection in the first unit vector in $\mathbb{H}$, meaning a reflection in the central generator $1 \in \mathbb{H}$. Writing a quaternions as $q = (q_0, q_1, q_2, q_3)$ and its conjugate by $q^* = (q_0, -q_1, -q_2, -q_3)$, the action of r can be written $r(q) = -q^*$. Computing the action of f on $v \in \mathbb{H}$ can then look like this.

$$f(p)(v) = g(p) r g(p)^{-1}(-v^*) = -g(p) r(p^{-h} v^* p^{-j}) = g(p)(p^j v p^h) = p^{j+h} v p^{j+h}$$

In particular, for $v = 1 \cong e_1$ we get $f(p)(1) = p^{2(j+h)}$. This is nullhomotopic if and only if $2(j+h) = 0$, by the same result as for $\mathbb{S}^1$. But $(j+h)\iota = e(E)$ according to [13] where ι generates $H^4(\mathbb{S}^4; \mathbf{Z}) \cong \mathbf{Z}$. Hence this statement.

Proposition 2.9. *A rank 4 vector bundle over $\mathbb{S}^4$ (hence an orientable vector bundle) admits an orientation reversing gauge transformation if and only if the Euler class of E, $e(E) \in H^4(\mathbb{S}^4; \mathbf{Z})$ vanishes $e(E) = 0$.*

I will have quite a bit more to say about the Euler class in chapter 3. As an aside, [13] claims that the first Pontryagin class of E, $p_1(E)$, equals $\pm 2(h-j)\iota$. So the existence of an orientation reversing gauge transformation forced the

[2] My thanks to John Rognes for his insistence that I do the $\mathbb{S}^4$ calculation using quaternions and for pointing out that [13] is a good reference here.

Euler class to vanish but only restricted the Pontryagin class to be a multiple of 4 times the generator of $H^4(\mathbb{S};\mathbf{Z})$. The fact that the Pontryagin class is of little use should be seen in light of my brief discussion of stable characteristic classes in chapter 1. What it does show however is that unless $h = j = 0$ the bundle will not be trivial. So there are non-trivial rank 4 bundles over $\mathbb{S}^4$ which do admit orientation reversing gauge transformations.

By either assuming prop. 1.7 or using prop. 2.1 again we get a similar statement as for the 2-sphere.

Proposition 2.10. *Let $E \to \mathbb{S}^4$ be a rank k vector bundle. Then E admits an orientation reversal for any rank $k \neq 4$.*

Tangent bundle of $\mathbb{S}^4$

It's known that the tangent bundle of $\mathbb{S}^4$ has Euler class 2ι. I can demonstrate this for the 4-sphere using the fact that the cluthing function for the tangent bundle can given by[3] $g(x)v = \alpha(x)\alpha(e_1)v = \alpha(x)\alpha(1)v$ which can be written using quaternions as simply

$$g(x)v = -\alpha(x)v^* = xvx$$

I.e. $p = x$ and $h = j = 1$. If one didn't know that $(h + j)\iota = e(E)$ but knew that $e(T\mathbb{S}^4) = 2\iota$, then this computation would have told you about the relation between $e(E)$ and the integers j and h.

Direct computation in all dimensions

The plan of this section is to write down an explicit expression for the clutching function for the tangent bundle of spheres and use these to calculate an expression for $f : \mathbb{S}^{n-1} \to SO(n)$. I will then determine whether this can be nullhomotopic for n even.

Following Husemoller [11], let $\alpha : \mathbb{S}^{n-1} \to O(n)$ be defined by $\alpha(x)y = y - 2\langle y, x\rangle x$, i.e. $\alpha(x)$ is the reflection through the hyperplane orthogonal to x. I assume we know that $g(x) = \alpha(x)\alpha(e_n)$, $g : \mathbb{S}^{n-1} \to SO(n)$ is the characteristic map or clutching function. Husemoller writes c_n for this map. I want to calculate

$$f(x) = g(x)rg(x)^{-1}r$$

with $r = \mathrm{diag}(1, 1, \cdots, -1)$. That is to say I think of f as a map $\mathbb{S}^{n-1} \to SO(n)$. I will work in coordinates and employ the summation convention.

First I need a coordinate expression for $g(x)$:

$$g(x)y = \alpha(x)\alpha(e_n)y = y - 2\langle y, e_n\rangle e_n - 2\langle y, x\rangle x + 4\langle y, e_n\rangle \langle e_n, x\rangle x$$

$$g(x)y = y^j e_j - 2y_n e_n - 2y^k x_k x^j e_j + 4y_n x_n x^j e_j$$

In terms of $y^j \mapsto g^j{}_k y^k$, this is

$$g^j{}_k = \delta^j{}_k - 2\delta^j{}_n \delta_{nk} - 2x_k x^j + 4x_n x^j \delta_{nk}$$

[3]This corresponds to the fiber bundle $SO(4) \to SO(5) \xrightarrow{p} \mathbb{S}^4$ with $p(u) = u(e_1)$.

So the components of the matrix $(g(x))^i{}_j = g^i{}_j$. Clearly $(r)^j{}_k = \delta^j{}_k - 2\delta^j{}_n\delta_{kn}$. I then just need an expression for $g(x)^{-1}$. Since $\alpha(x)^2 = 1$, it's easy to see that $g(x)^{-1}y = \alpha(e_n)\alpha(x)y$. The components of this matrix is thus (writing $\overline{g} = g^{-1}$)

$$\overline{g}^j{}_k = \delta^j{}_k - 2x_k x^j - 2\delta_{nk}\delta^j{}_n + 4x_k x_n \delta^j{}_n$$

For an arbitrary element $g \in GL(n, \mathbb{K})$ we have the following:

$$f^i{}_j = g^i{}_k r^k{}_l \overline{g}^l{}_s r^s{}_j = g^i{}_k(\delta^k{}_l - 2\delta^k{}_n\delta_{nl})\overline{g}^l{}_s(\delta^s{}_j - 2\delta^s{}_n\delta_{nj})$$

$$f^i{}_j = (g^i{}_l - 2g^i{}_n\delta_{ln})(\overline{g}^l{}_j - 2\overline{g}^l{}_n\delta_{jn}) = \delta^i{}_j - 2\delta^i{}_n\delta_{jn} - 2g^i{}_n\overline{g}_{nj} + 4g^i{}_n\overline{g}_{nn}\delta_{nj}$$

For $g : \mathbb{S}^{n-1} \to SO(n)$ the clutching function as above we have these relations:

$$g^i{}_n = \delta^i{}_n - 2\delta^i{}_n - 2x^i x_n + 4x_n x^i = -\delta^i{}_n + 2x^i x_n$$
$$\overline{g}_{jn} = \delta_{nj} - 2x_n x_j - 2\delta_{nj} + 4x_n x_j = -\delta_{nj} + 2x_n x_j$$

Using these transition functions gives the expression

$$f(x)_{ij} = \delta_{ij} + 4x_n(x_i\delta_{nj} + x_j\delta_{ni}) - 8x_n^2(x_i x_j + \delta_{nj}\delta_{in}) + 8x_n x_i(2x_n^2 - 1)\delta_{nj}$$

To help a reader not familiar with index notation navigate, the element $f(x) \in SO(n)$ can be written out as a (quite unenlightening) matrix.

$$\begin{pmatrix} 1 - 8x_n^2x_1^2 & -8x_n x_1 x_2 & \cdots & 4x_n x_1(2x_n^2 - 1) \\ -8x_n^2 x_1 x_2 & 1 - 8x_n^2 x_2^2 & \cdots & 4x_n x_2(2x_n^2 - 1) \\ \vdots & & & \vdots \\ -4x_n x_1(2x_n^2 - 1) & -4x_n x_2(2x_n^2 - 1) & \cdots & 1 + 8x_n^2(x_n^2 - 1) \end{pmatrix}$$

To say something useful about f I propose to compose this with the projection p in the fiber sequence

$$SO(n-1) \to SO(n) \xrightarrow{p} \mathbb{S}^{n-1}$$

$p(u) = u(e_n)$ to get a map $p(f) : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$. I will compute the degree of this. In index notation I can write

$$p(u)^i = u^i{}_j\delta^j{}_n$$

$$p(f)^i = f^i{}_k\delta^k{}_n = +f^i{}_n$$
$$pf(x)^i = \delta_n{}^i(1 - 4x_n^2) + 4x_n x^i(2x_n^2 - 1)$$

In terms of the matrix above, p picks out the right-hand column, so

$$pf(x) = \begin{pmatrix} 4x_n x_1(2x_n^1 - 1) \\ \vdots \\ 4x_n x_{n-1}(2x_n^2 - 1) \\ 1 + 8x_n^2(x_n^2 - 1) \end{pmatrix}$$

The fiber of pf over e_1 will be 4 points, as is seen from inspection. I compute the degree of pf by computing the local degrees at these 4.

The 4 points are found by solving

$$\delta^i{}_1 = pf(x)^i = \delta_n{}^i(1 - 4x_n^2) + 4x_n x^i(2x_n^2 - 1)$$

giving $x_n = \epsilon_1\sqrt{\frac{1}{2} + \epsilon_2\frac{\sqrt{2}}{4}}$, $x_1 = \frac{1}{4x_n(2x_n^2-1)}$ and $x_i = 0$, $1 < i < n$ with $\epsilon_1, \epsilon_2 = \pm 1$.

To compute the degree I will compute the local degree of pf by checking the determinant of its Jacobian.

The Jacobian of pf looks like this

$$J(pf) = \begin{pmatrix} 4x_n(2x_n^2-1) & 0 & \cdots & 8x_n x_1(2x_n^2+1) - 4x_1 \\ 0 & 4x_n(2x_n^2-1) & 0 \quad \cdots & 8x_n x_2(2x_n^2-1) - 4x_2 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & 16x_n(2x_n^2-1) \end{pmatrix}$$

with determinant (notice that the Jacobian is upper-triangular)

$$\det(J(pf)) = 4^{n+1}x_n^n(2x_n^2-1)^n$$

Inserting the critical points $x_n = \frac{\epsilon_2}{\sqrt{2}}\sqrt{1+\frac{\epsilon_1}{\sqrt{2}}}$ yields the expression

$$\det(J(pf)) = 2^{n+2}\left(1+\frac{\epsilon_1}{\sqrt{2}}\right)^n(\epsilon_1\epsilon_2)^n$$

The sign of this is $(\epsilon_1\epsilon_2)^n$. When n is even, this is always positive, so the local degrees are all +1. When n is odd, there are 2 points with local degree −1, 2 with +1, hence

$$\deg(pf) = \begin{cases} 4 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

What I conclude from the above is that since $\deg(pf) \neq 0$ for even n, f can not be homotopic to a constant. The converse did not have to hold, since p might be homotopic to a constant without any implications for f. I will actually argue below that p is homotopic to a constant for all odd n, although one of my arguments references quite esoteric knowledge of homotopy groups of $SO(n)$. However, the discussion I had about odd bundles in chapter 1 deals with the odd n cases in terms of bundles and the discussion of p is more of a curiosity. Combined this is what I have shown.

Proposition 2.11. *The tangent bundle $T\mathbb{S}^n \to \mathbb{S}^n$ or equivalently the frame bundle $SO(n) \to SO(n+1) \to \mathbb{S}^n$ admits an orientation reversing gauge transformation if and only if n is odd.*

Some smarter approaches

Note that in the notation of [11] $r = -\alpha(e_n)$, so the function $f : \mathbb{S}^{n-1} \to SO(n)$ can, when the bundle is the frame bundle of the n-sphere with clutching function $g(x) = \alpha(x)\alpha(e_n)$, be written as

$$f(x) = g(x)rg(x)^{-1}r = \alpha(x)\alpha(e_n)\alpha(e_n)\alpha(e_n)\alpha(x)\alpha(e_n) = (\alpha(x)\alpha(e_n))^2$$

So f is nullhomotpic if g is. This is not "if and only if", since g is nullhomotpic if and only if $\mathbb{S}^n$ is parallelizable which is known to happen if and only if $n = 1, 3, 7$, whereas f is nullhomotpic for all odd n but not for any even n.

This can be analysed in some more detail, since $[f] = [g^2] = 2[g]$ thanks to $SO(n)$ being a Lie group. It's also known what the homotopy groups

n	2	3	4	5	6	7
$\pi_{n-1}(SO(n))$	$\mathbf{Z}$	0	$\mathbf{Z}\times\mathbf{Z}$	$\mathbf{Z}/2$	$\mathbf{Z}$	0

Table 2.1: A list of $\pi_{n-1}(SO(n))$ for $2 \leq n \leq 7$, taken from [16].

n	8s	8s+1	8s+2	8s+3	8s+4	8s+5	8s+6	8s+7
$\pi_{n-1}(SO(n))$	$\mathbf{Z}\times\mathbf{Z}$	$\mathbf{Z}/2\times\mathbf{Z}/2$	$\mathbf{Z}\times\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}\times\mathbf{Z}$	$\mathbf{Z}/2$	$\mathbf{Z}$	$\mathbf{Z}/2$

Table 2.2: A list of $\pi_{n-1}(SO(n))$ for $s \geq 1$, taken from [16].

$\pi_{n-1}(SO(n))$ are for all n. [16] has a list which I will permit myself to use. The relevant data is given in tables 2.1 and 2.2. Note that for all the odd-n groups $[f] = 2[g] = 0$ (which we knew), but as I mentioned above $[g] \neq 0$. Actually, the tables along with this knowledge tells us that $[g]$ generates $\pi_{n-1}(SO(n)$ for odd $n \neq 8s+1$. For n even, the reader will note that the only groups with 2-torsion are $\pi_{8s+1}(SO(8s+2))$ so for $n \neq 8s+2$ we can conclude that if $[f] = 0$ then $[g] = 0$ which is false.

Without reference to the homotopy groups of $SO(n)$ it would also have been possible to proceed somewhat like I did in my brute force approach, but quicker. Applying $p_* : \pi_{n-1}(SO(n)) \to \pi_{n-1}(\mathbb{S}^{n-1})$ to $[f]$ we get an element of $\pi_{n-1}(\mathbb{S}^{n-1})$, but $p_*([f]) = p_*([2g]) = 2p_*([g])$ and then proceed to compute the degree of $p_*([g])$. Husemoller [11] and Steenrod [10] both have geometric arguments, but it's also highly tractable to simply perform a simplified version of the computation above; component i of $pg(x)$ is $(pg(x))_i = g(x)_{in} = \delta_{in} - 2\delta_{in} - 2x_nx_i + 4x_nx_i = -\delta_{in} + 2x_nx_i$. Written out this says that

$$pg(x) = \begin{pmatrix} 2x_nx_1 \\ 2x_nx_2 \\ \vdots \\ 2x_n^2 - 1 \end{pmatrix}$$

The fiber of e_n is easily seen to be $(0, 0, \cdots, \pm 1)$. The Jacobian of pg is

$$J(pg) = \begin{pmatrix} 2x_n & 0 & \cdots & 0 & 2x_1 \\ 0 & 2x_n & \cdots & 0 & 2x_2 \\ \vdots & & \ddots & & \\ 0 & 0 & \cdots & & 4x_n \end{pmatrix}$$

$$\det(J(pg)) = 2^{n+1}x_n^n$$

From this it follows that

$$\deg(pg) = (1 + (-1)^n)$$

Since $p_*[f] = 2p_*([g]) = 2(1 + (-1)^n)$ this shows (once again) that $[f]$ is not nullhomotopic for n even.

If I once again assume the tables 2.1 and 2.2 it's clear that $p_* = 0$ for n odd simply because it will always be a map from $\mathbf{Z}$ to a sum of torsion groups (or the trivial group). So (once again) it's impossible to conclude that $[g] = 0$ from $p_*[g] = 0$ in the odd case.

Chapter 3

The Euler class

I start this chapter with a quick recap of the Euler class of an oriented vector bundle and some of its key properties. This exposition is very close to what [12] does.

I proceed with a series of implications, each guaranteeing the existence of orientation reversing gauge transformations. I discuss to what extent the implications are equivalences. I then state some results from obstruction theory (postponing any discussion of obstruction theory to the next chapter) which give definite answers in some situations.

I end the chapter with a partial result for non-orientable bundles and an inconclusive discussion of bundle splitting induced by an automorphism.

Definition of Euler class

Let $E \xrightarrow{\pi} X$ be a real vector bundle of rank n. Denote by E^0 the fiber sub-bundle with fibers $E_x \smallsetminus \{0\} \cong \mathbb{R}^n \smallsetminus \{0\}$. By excision we know that $H^n(E_x, E_x \smallsetminus \{0\}; \mathbf{Z}) \cong \tilde{H}^n(\mathbb{S}^n; Z) = \mathbf{Z}$ and an orientation of E is a choice of generators $u_x \in H^n(E_x, E_x \smallsetminus \{0\}; \mathbf{Z})$ in such a way that that there are trivialising neighbourhoods U with cohomology classes $u \in H^n(\pi^{-1}(U), \pi^{-1}(U)^0; \mathbf{Z})$ such that $u_{|(E_x, E_x \smallsetminus \{0\})} = u_x \in H^n(E_x, E_x \smallsetminus \{0\}; \mathbf{Z})$ for each $x \in U$. I will state a theorem about this situation which is theorem 9.1 on page 97 in [12]. It is proven in the same book as theorem 10.4 on page 110.

Theorem 3.1 (Thom Isomorphism - No proof)**.** *Let $E \to X$ be a rank n oriented vector bundle. Then $H^i(E, E^0; \mathbf{Z}) = 0$ for $i < n$ and $H^n(E, E^0; \mathbf{Z})$ contains one and only one cohomology class u whose restriction $u_{|(E_x, E_x \smallsetminus \{0\})}$ equals the chosen generator u_x for all $x \in X$.*

Furthermore, the map $H^k(E; \mathbf{Z}) \xrightarrow{\cup u} H^{k+n}(E, E^0; \mathbf{Z})$ is an isomorphism for every k and is called the Thom isomorphism. The element u is called a Thom class.

The inclusion $(E, \varnothing) \xrightarrow{i} (E, E^0)$ gives a map $H^*(E, E^0; \mathbf{Z}) \to H^*(E; Z)$, written $y \mapsto y_{|E}$. For $u \in H^n(E, E^0; \mathbf{Z})$ the unique element restricting to the orientation class we define $e(E) = u_{|E} \in H^n(E; \mathbf{Z}) \cong H^n(X; \mathbf{Z})$ as the Euler class of E.

Properties of the Euler class

- Changing the orientation of E changes the sign of $e(E)$, since $e(E)$ is the restriction of the global orientation class.
- The Euler class is natural. If $f : E \to E'$ is a bundle map then $f^*(e(E')) = e(f^*(E'))$.
- The Whitney-sum axiom holds. $e(E' \oplus E) = e(E')e(E)$.

The naturality implies that $e(\epsilon^1) = 0$ by pullback with a constant map

$$\begin{array}{ccc} \epsilon^1 \cong f^*(\epsilon^1) & \longrightarrow & \epsilon^1 \\ \pi \big\downarrow & & \swarrow \pi \\ X & \xrightarrow{f} & * \end{array}$$

Let me furthermore state two more results without proof. The first is prop. 9.5 on page 99 in [12]. I omit the proofs because I haven't taken the time to properly introduce the Stiefel-Whitney classes.

Proposition 3.1 (No proof). *Let $E \to X$ be an oriented rank n vector bundle with Euler class $e(E) \in H^n(X;\mathbf{Z})$. Then the natural homomorphism $H^n(X;\mathbf{Z}) \to H^n(X;\mathbf{Z}/2)$ carries $e(E)$ to $w_n(E)$.*

Proposition 3.2 (No proof). *Let M be a compact and smooth $n-$dimensional manifold. If M is oriented and $\iota \in H^n(M;\mathbf{Z})$ is a generator then $e(TM) = \chi(M)\iota$. If M is not orientable and $\iota \in H^n(M;\mathbf{Z}/2)$ is a generator then $w_n(TM) = \chi(M)\iota$ where this equality is of course modulo 2.*

This is corollary 11.12 on page 130 in [12].

Interlude on orientability

The Euler class of a k-plane bundle is only defined for oriented vector bundles, so in this chapter I will eventually focus on that case. Examples include of course the tangent bundle of an oriented manifold.

There is a very neat criterion for orientability which I shall be making use of several times. I have taken the proof from [4], but it is formulated as an exercise for CW complexes in [12], so it's definitely well-known.

Lemma 3.1. *Assume $E \to X$ is a real vector bundle. Then E is orientable if and only if $w_1(E) \in H^1(X;\mathbf{Z}/2)$ vanishes.*

Proof. Let $E \to X$ be a vector bundle and define $\Phi : \pi_1(X) \to \mathbf{Z}/2$ by $\Phi(\gamma) = \pm 1$ depending on whether $E_{\gamma(0)}$ has the same orientation as $E_{\gamma(1)}$ or not. $\mathbf{Z}/2$ is abelian, so this passes to a map $H_1(X) \to \mathbf{Z}/2$, i.e. an element of $H^1(X;\mathbf{Z}/2)$. This then has to be $w_1(E)$ as it satisfies the same axioms. □

Corollary 3.1.1. *Let $E \to X$ be a real vector bundle and assume $H^1(X;\mathbf{Z}/2) = 0$ (which for instance happens if $\pi_1(X) = 0$). Then E is an orientable vector bundle.*

Corollary 3.1.2. *There are no non-orientable vector bundles over $\mathbb{S}^n$ for $n \geq 2$.*

I will use the following lemma when talking about oriented bundles.

Lemma 3.2. *An oriented real vector bundle $E \to X$ can not be written as the sum of 2 vector bundles $E \cong E_1 \oplus E_2$ where E_1 is orientable and E_2 is not.*

Proof. Assume $E \cong E_1 \oplus E_2$ with E_1 orientable and E_2 not orientable. Consider the first Stiefel-Whitney class:

$$0 = w_1(E) = w_1(E_1) + w_1(E_2) = w_1(E_2) \neq 0$$

This is a contradiction. □

Note that this does not preclude an orientable bundle as being the sum of non-orientable bundles.

Several implications

We have the list of implications, all of which may be used to determine whether or not a bundle admits an orientation reversal. In what follows I have counter examples for all reverse implications except for $iv) \implies iii)$. I will have more to say about this towards the end of the chapter.

Theorem 3.2. *Let $E \to X$ be a rank k vector bundle over a paracompact Hausdorff space X. We then have the following list of implications ($i) \implies ii)$ etc.).*

i) E is trivial; $E \cong \epsilon^k$.

ii) E splits off a line-bundle; $E \cong E' \oplus L$, rk(L)=1.

iii) E splits off an odd-rank bundle; $E \cong E' \oplus E''$, rk(E') odd.

iv) E admits an orientation reversing automorphism.

Furthermore, if E is an oriented bundle with euler class $e(E) \in H^k(X;\mathbf{Z})$, then we have the additional implications

- $E \cong E' \oplus \epsilon^1 \implies e(E) = 0$
- *iii),iv)* $\implies 2e(E) = 0$.

Proof. $i) \implies ii) \implies iii)$ is clear. $iii) \implies iv)$ follows from my discussion in chapter 1. Prop. 1.6 in particular.

If $E \cong E' \oplus \epsilon^1$ is oriented with Euler class $e(E)$ then the Whitney-sum property tells us that $e(E) = e(\epsilon^1)e(E') = 0$ since $e(\epsilon^1) = 0$.

Assume next that E admits an orientation reversing automorphism $\phi : E \to E$. Let E_+ and E_- denote E with the two different choices of orientation. Using the diagram below shows that $e(E_+) = e(\mathrm{id}^* E_-) = e(E_-)$. But $e(E_+) = -e(E_-)$, so $2e(E) = 0$.

$$\begin{array}{ccc} E_+ & \xrightarrow{\ \phi\ } & E_- \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & \xrightarrow{\ \mathrm{id}\ } & X \end{array}$$

□

Remark: *ii)* $\implies$ *i)* fails to hold. Consider for instance any non-trivial line-bundle (the tautological line bundle over $\mathbb{RP}^n$ for instance).

Remark: *iii)* $\implies$ *ii)* fails to hold by the following example, which I found when leafing through [14]. I do not know if they are the original discoverers.

Example 3.1. Consider oriented rank 3 bundles over $\mathbb{S}^4$. The topological classification theorem says that the isomorphism classes of such bundles are in bijection with $\pi_3(SO(3)) \cong \pi_3(\mathbb{S}^3) \cong \mathbf{Z}$. So let E be a non-trivial rank 3 bundle over $\mathbb{S}^4$. Since there are no non-trivial 2-bundles or line bundles over $\mathbb{S}^4$, $\pi_3(O(2) = \pi_3(O(1)) = 0$ it's impossible for a non-trivial 3-bundle over $\mathbb{S}^3$ to split off a line bundle.

Note in addition that $H^3(\mathbb{S}^4; \mathbf{Z}) = 0$ so $e(E) = 0$. This show that $e(E) = 0$ does not imply that E splits off a line bundle for an arbitrary rank k bundle over an n-dimensional CW-complex.

I have a counter example to the implication $2e(E) = 0 \implies$ splits off an odd bundle (orientable or otherwise), but I will first present some results on when we do have the implication $e(E) = 0 \implies E = E' \cong \epsilon^1$. I will only state the results without proof in this chapter, and instead get back to obstruction theory and some of the background of the results in the next chapter.

Theorem 3.3 (Proven in chapter 4)**.** *Let $E \to X$ be an oriented rank n vector bundle over an $n-$dimensional CW-complex X. Then $E \cong E' \oplus \epsilon^1$ if and only if $e(E) = 0$.*

This is a small rewriting of Corollary 14.4 in [8]. The above example from [14] shows that the fact that the rank of the bundle matches the dimension of the base space is essential.

Example 3.2. The converse to E splits off an odd orientable bundle $\implies$ $2e(E) = 0$ is false. Consider the embedding of $i : \mathbb{RP}^2 \to \mathbb{RP}^4$. By naturality of the Stieffel-Whitney classes, the pullback of the tangent bundle of $\mathbb{RP}^4$ has total SW class $w(i^*(T\mathbb{RP}^4)) = i^* w(T\mathbb{RP}^4) = i^*(1 + a + a^4) = 1 + a$ where, by abuse of notation, a denotes the generator of both $H^1(\mathbb{RP}^2; \mathbf{Z}/2)$ and $H^1(\mathbb{RP}^4; \mathbf{Z}/2)$. Consider the normal bundle N of this embedding. Since $w(N)w(T(\mathbb{RP}^2)) = w(N \oplus T\mathbb{RP}^2) = w(i^*(T\mathbb{RP}^4)) = 1 + a$ and $w(T(\mathbb{RP}^2)) = 1 + a + a^2$ this can be solved to give $w(N) = 1 + a^2$. Since $w_1(N) = 0$, N is an orientable rank 2 bundle over $\mathbb{RP}^2$. Since $0 \neq a^2 = w_2(N) = e(N) \bmod 2$ this also tells us that $e(N) \neq 0$.

What this shows is an example of an orientable vector bundle with $2e(E) = 0$ but which does not split off any orientable odd bundles, since the only orientable bundles it can split off are trivial line bundles, which is impossible as $e(E) \neq 0$.

There is a stronger example still which I found by experimenting with Stiefel-Whitney classes.

Example 3.3. This example is intended to show that an oriented n-bundle over an n-dimensional CW-complex does not have to split off an odd bundle (orientable or not) when $2e(E) = 0$.

Consider the tangent bundle $T\mathbb{RP}^5 \to \mathbb{RP}^5$. It has total Stiefel-Whitney class $w(T\mathbb{RP}^5) = 1 + a^2 + a^4$ and is an orientable rank 5 bundle over $\mathbb{RP}^5$. By corollary 3.4.2 for instance (since $\chi(\mathbb{RP}^5) = 0$ as $\mathbb{RP}^5$ is an odd dimensional manifold) or theorem 3.3 it splits as $T\mathbb{RP}^5 \cong E_4 \oplus \epsilon^1$ with $w(E_4) = 1 + a^2 + a^4$. E_4 is an orientable 4-bundle over $\mathbb{RP}^5$ with Euler class $e(E) = a^4$, so $2e(E) = 0 \in H^4(\mathbb{RP}^5; \mathbf{Z}) \cong \mathbf{Z}/2$, but E_4 does not split off an odd bundle. This would have to have been an unorientable line bundle which is impossible from the Stiefel-Whitney classes. Indeed, assume $w(L) = 1 + a$ and $w(E_3) = 1 + a + c_2 a^2 + a^3$ with $E_4 \cong E_3 \oplus L$. The equation to solve would be $1 + (1 + c_2)a^2 + (1 + c_2)a^3 + a^4 = 1 + a^2 + a^4$ which is impossible.

This is not changed if we pull back E_4 to $\mathbb{RP}^4$ via the inclusion as $i^*(w(E_4)) = w(i^*(E_4)) = 1 + a^2 + a^4$ (with my usual abuse of notation $i^*(a) = a$). This is my example of an orientable 4-bundle over a 4-dimensional CW complex with $2e(E_4) = 0 \in H^4(\mathbb{RP}^4; \mathbf{Z}) \cong \mathbf{Z}/2$ but with no odd bundles splitting off.

Definite answers in some scenarios

Theorem 3.4. *Assume X is an n-dimensional CW-complex with $H^n(X; \mathbf{Z})$ torsion-free and $E \to X$ is a rank n oriented bundle over X. Then X admits an orientation reversal if and only if $e(E) = 0$.*

Proof. Assume an orientation reversal exists. Then $2e(E) = 0$ but per assumption on $H^n(X; \mathbf{Z})$ this can only happen if $e(E) = 0$.

Conversely, if $e(E) = 0$ theorem 3.3 says that $E \cong E' \oplus \epsilon^1$ and thus admits an orientation reversal. □

Corollary 3.4.1. *Assume X is an n-dimensional compact oriented manifold and $E \to X$ an oriented rank n vector bundle. Then E admits and orientation reversal if and only if $e(E) = 0$.*

Proof. By Poincaré duality $H^n(X; \mathbf{Z}) \cong H_0(X) = \mathbf{Z}^k$ where k is the number of connected components (which I explicitly or implicitly assume is 1 most places) and as such is torsion free. □

Corollary 3.4.2. *Assume X is an n-dimensional compact manifold. Then the tangent bundle $TX \to X$ admits an orientation reversing gauge transformation if and only if $\chi(X) = 0$.*

Proof. This follows from the previous corollary and proposition 3.2. □

Remark 3.1. Corollary 3.4.2 of course covers the sphere computations in chapter 2 where $\chi(\mathbb{S}^{2n}) = 2 \neq 0$ as well as any odd dimensional orientable manifold as $\chi(M) = 0$ by Poincaré duality.

The fact that I'm only talking about compact manifolds above is not because they are simpler. It's because they are the only interesting case in the following sense.

Proposition 3.3. *Let $E \to M$ be an oriented rank n bundle over the non-compact, oriented n-manifold M. Then E admits an orientation reversal.*

Proof. Note that M is a finite dimensional CW complex and as such satisfies the criteria of theorem 3.3. It's a known fact of topological manifolds that $H_{n-1}(M)$ has trivial torsion when M is orientable. It's also known that $H_n(M) = 0$ if M is non-compact. These results are corollary 3.28 and proposition 3.29 respectively in [3]. By the universal coefficient theorem we get $0 \to$

$\mathrm{Ext}(H_{n-1}(M), \mathbf{Z}) \to H^n(M; \mathbf{Z}) \to \mathrm{Hom}(H_n(M), \mathbf{Z}) \to 0$. Here $\mathrm{Ext}(H_{n-1}, \mathbf{Z}) = 0$ as $H_{n-1}(M)$ is free Abelian per assumption of orientability and $H_n(M) = 0 \implies \mathrm{Hom}(H_n(M), \mathbf{Z}) = 0$. So $H^n(M; \mathbf{Z}) = 0$ and consequently $e(E) = 0$. □

Remark 3.2. Proposition 3.3 covers the tangent bundle of an orientable non-compact manifold.

Almost complex structure

There are some translations perhaps worth making when the rank $2k$ real vector bundle comes from an underlying rank k complex bundle. We have the following fact, which is actually a definition in [12].

Proposition 3.4 (No proof)**.** *Assume $E_{\mathbb{C}} \to X$ is a rank k complex vector bundle with associated rank $2k$ real bundle $E_{\mathbb{R}}$. Then $E_{\mathbb{R}}$ is an orientable vector bundle and if it is given the induced orientation from the complex structure, we have the following identity: $c_k(E_{\mathbb{C}}) = e(E_{\mathbb{R}}) \in H^{2k}(X; \mathbf{Z})$.*

With the same hypothesis as above, there are relations between Chern classes and Pontryagin classes.

Proposition 3.5 (No proof)**.** *With the same assumptions and notation as the foregoing statement, the Pontryagin classes of $E_{\mathbb{R}}$ can be read out from the formula*

$$1 - p_1 + p_2 - \cdots \pm p_n = (1 - c_1 + c_2 - \cdots \pm c_n)(1 + c_1 + c_2 + \cdots c_n)$$

where $p_k = p_k(E_{\mathbb{R}})$ and $c_k = c_k(E_{\mathbb{C}})$.

The above is Corollary 15.5 on page 177 in [12].

Proposition 3.4 allows for the following definite answer to my main question

Theorem 3.5. *Assume X is a real manifold of dimension $2n$ whose tangent bundle has an almost complex structure. Then TX admits an orientation reversing automorphism if and only if $c_n(X) = c_n(TX) = 0$.*

Proof. This is a direct consequence of theorem 3.4.1 and prop. 3.4. □

Example 3.4 (Complex Projective Space)**.** $T\mathbb{CP}^n \to \mathbb{CP}^n$ does not allow orientation reversing automorphisms for any $n \geq 1$ since $c_n(\mathbb{CP}^n) = (n+1)b^n$ where $b \in H^2(\mathbb{CP}^n; \mathbf{Z})$ is a generator. This is of course also covered by corollary 3.4.2 as $\chi(\mathbb{CP}^n) = (n+1) \neq 0$.

$\mathbb{S}^4$ revisited

Recall that I showed that a real oriented rank 4 vector bundle E over $\mathbb{S}^4$ admitted an orientation reversal if and only if $e(E) = 0$ but the Pontryagin class was not required to vanish. Assume now in addition that that E comes from a rank 2 complex bundle $E_{\mathbb{C}}$. By proposition 3.5 we can then write $p_1(E) = c_1(E_{\mathbb{C}})^2 - 2c_2(E_{\mathbb{C}})$, but $c_1(E_{\mathbb{C}}) \in H^2(\mathbb{S}^4; \mathbf{Z}) = 0$ and $c_2(E_{\mathbb{C}}) = e(E) = 0$ so $p_1(E) = 0$[1].

[1] The fact that the Pontryagin class vanishes for complex n-bundles over $\mathbb{S}^{2n}$ admitting orientation reversal actually holds for all n, not just $n = 2$. This is simply because the Pontryagin classes for complex bundles can be expressed as polynomials in the Chern classes. For $\mathbb{S}^{2n}$, only c_2 has a chance of being non-zero, but the existence of an orientation reversal kills this as well.

This I suppose can be traced back to the fact that $e(E) = 0 \implies E \cong E' \oplus \epsilon^1$ as a real bundle. If this is supposed to be a complex bundle we need to have the further splitting $E \cong E'' \oplus \epsilon^2$, still as a real bundle. But $\pi_3(U(1)) = 0$, i.e. there are no non-trivial rank 1 complex bundles over $\mathbb{S}^4$ so $E'' \cong \epsilon^2$ and $E \cong \epsilon^4$. This example goes to show that some subtleties of the real variant of the problem are lost if one imposes the assumption that the real bundle has an almost complex structure.

Orientation reversal versus splitting

This entire chapter revolves around the implication $iii) \implies iv)$, that $E \to X$ admits an orientation reversal if E splits of an odd (not neccessarily trivial) bundle. This really begs the question whether the converse holds, i.e. does an orientation reversal induce a splitting, and I honestly do not know. I suspect it to be false, but I am unable to prove it. Here is what I can prove.

Proposition 3.6. *Assume $E \to X$ is a rank 2 real bundle. The E admits an orientation reversal if and only if E splits off a line bundle (possibly non-trivial).*

Proof. Assume E has an orientation-reversing gauge-transformation $\phi : E \to E$. Then $\phi_x \in O(2)_-$ for all $x \in X$, hence it has precisely 1 positive and 1 negative eigenvalue. As such one can find a continuous map $S : X \to GL(2)$ such that $\tilde{\phi}_x = S(x)\phi_x S(x)^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Or in other words, ϕ can be diagonalized at each point. This induces a splitting $E \cong E_+ \oplus E_-$ where $(\phi_x)_{|E_\pm} = \pm 1$. Technically speaking, this induces a splitting of an isomorphic bundle since diagonalising ϕ is equivalent to changing the transition functions of the bundle via a similarity transformation; if $g\phi_x g^{-1} = \phi_x$ at each point then $(S^{-1}gS)\phi_x(S^{-1}g^{-1}S) = \phi_x \Leftrightarrow g(S\phi_x S^{-1})g^{-1} = (S\phi_x S^{-1})$. Then recall proposition 1.5 which said that two bundles are isomorphic if their transition functions can be related via a similarity transformation.

The converse is covered above. □

Remark 3.3. The above could be used for my computation of bundles over $\mathbb{S}^2$, since an orientation reversal would imply $E \cong L_1 \times L_2$ but $\mathbb{S}^2$ does not admit non-orientable bundles and $L_1 \cong L_2 \cong \epsilon^1$.

rank $2n$-bundles

The main problem I have in generalising the above procedure is that for a rank $2n$-bundle and orientation reversal can be block-diagonalised over each point to be on the form

$$\phi_x \sim \begin{pmatrix} R_1 & & & & \\ & R_2 & & & \\ & & \ddots & & \\ & & & -1 & \\ & & & & 1 \end{pmatrix}$$

$R_i \in SO(2)$. For ranks 4 and above, a rotation can easily go from $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ to $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$ so the dimension of the subspace in each fiber where $\phi_x = \pm 1$ might change from fiber to fiber and as such not give a vector bundle. This seems even more acute for rank 6 and above, where one could imagine $\phi_x \sim \mathrm{diag}(1,1,1,-1-,1,-1)$ and $\phi_y \sim \mathrm{diag}(1,-1,-1,-1,-1,-1)$ via rotations. I.e. one fiber seemingly splits as $3+3$ whereas another splits as $1+5$. This does not patch together to form two vector bundles which per axiom have constant fiber dimension.

I have tried disproving the statement that an orientation reversal of an even bundle implies that a line bundle splits off by looking for a 6-bundle[2] $E_6 \to X$ which is such that $E_6 \cong E_3 \oplus E_3'$ but it does not split off a line bundle. My attempt at finding such a bundle came to a halt at a hurdle. Assume one can find 3-bundles over $\mathbb{RP}^N$, $N \geq 6$ with $w(E_3) = 1 + a^2 + a^3$ and $w(E_3') = 1 + a + a^3$. Then the 6-bundle $E_3 \oplus E_3'$ will have Stiefel-Whitney class $(1+a^2+a^3)(1+a+a^3) = 1 + a + a^2 + a^3 + a^4 + a^5 + a^6$, and this can not be written as the sum of a line bundle and a 5-bundle; $(1+a)(1+a^2+c_3a^3+c_4a^4+a^5) = \sum_{i=0}^6 a^i$ does not have a solution. The problem is finding E_3 and E_3'. As a matter of fact, Wu's formula says that E_3' cannot be the tangent bundles of any smooth manifold. This isn't too hard to see. The formula[3] I'm thinking of says that if $w_i = w_i(TX)$ for some smooth manifold X then there are relations coming from Steenrod squares:

$$Sq^i(w_j) = \sum_{t=0}^{i} \binom{j+t-i-1}{t} w_{i-t} w_{j+t}$$

It's not important for my purposes how one computes the left hand side. What I need is that $Sq^i(0) = 0$. As $w_2(E_3') = 0$ the formula reads

$$0 = Sq^1(w_2) = \sum_{t=0}^{1} \binom{t}{t} w_{t-1} w_{2+t} = w_1 w_2 + w_0 w_3 = w_3$$

which is supposed to be false for E_3'. This does not a priori preclude the existence of the desired E_3', but it does make finding it (if it exists) a bit harder.

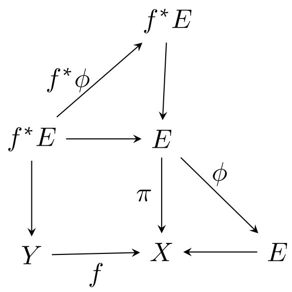
Non-orientable bundles

I have a partial result for non-orientable bundles. Let me start by formulating a lemma which applies more broadly.

Lemma 3.3. *Let $E \xrightarrow{\pi} X$ be a rank n vector bundle and let $Y \xrightarrow{f} X$ be some map. Assume $\phi : E \to E$ is a bundle automorphism. Then there is an induced automorphism $f^*\phi : f^*E \to f^*E$.*

[2] Any even bundle of rank 6 or above would do as long as it can be shown to be the sum of 2 odd bundles but which does not split off a line bundle.

[3] For instance given on page 197 in [17]



Furthermore, the relation between ϕ_x and $f^*\phi_y$ is $f^*\phi_y = \phi_{f(y)}$. In particular this means that $\det(\phi_x) = -1$ for all x implies $\det(f^*\phi_y) = -1$ for all y. So orientation reversing gauge transformations pull back to orientation reversals.

Proof. Per definition of f^*E, a point $\tilde{e} \in f^*E$ is of the form $\tilde{e} = (y, e)$ with $f(y) = \pi(e)$. Define $f^*\phi(\tilde{e}) = (y, \phi(e))$. This is fine as $f(y) = \pi(e) = \pi(\phi(e))$.

For the statement $(f^*\phi)_y = \phi_{f(y)}$, recall that for $U \subset X$ a trivialising neighbourhood for E, $E_{|U} \xrightarrow{t} U \times \mathbb{R}^n$ one defines $f^*E_{|f^{-1}(U)} \xrightarrow{\tilde{t}} f^{-1}(U) \times \mathbb{R}^n$ by $\tilde{t}(\tilde{e}) = \tilde{t}(y, e) = (y, \pi_2(t(e))$ where $\pi_2 : U \times \mathbb{R}^n$ is the projection onto the second factor. The inverse is given by $\tilde{t}^{-1}(y, v) = (y, e)$ where $\pi_2(t(e)) = v$ and $\pi_1(t(e)) = f(y)$. Computing $(f^*\phi)_y$ from the definition can look like this.

$$\tilde{t} \circ (f^*\phi) \circ \tilde{t}^{-1}(y, v) = \tilde{t} \circ (f^*\phi)(y, e) = \tilde{t}(y, \phi(e)) = (y, \pi_2(t(\phi(e))))$$

$$= (y, \pi_2(t\phi t^{-1})(x, v)) = (y, \phi_x v) = (y, \phi_{f(y)} v)$$

□

The promised result for non-orientable bundles is this.

Proposition 3.7. *Assume $E \to X$ is a non-orientable bundle and assume X has a universal covering space $\tilde{X} \xrightarrow{f} X$. The $f^*E \to \tilde{X}$ is an orientable bundle. If $f^*E \to \tilde{X}$ does not admit an orientation reversal, then neither can $E \to X$.*

Proof. $f^*E \to \tilde{X}$ is orientable as $\pi_1(\tilde{X}) = 0$ and corollary 3.1.1. If $E \to X$ admitted an orientation reversal it would pull back to an orientation reversal of $f^*E \to \tilde{X}$ by lemma 3.3. □

Tangent bundles of $\mathbb{RP}^n$

Corollary 3.5.1. *The tangent bundle $T\mathbb{RP}^n \to \mathbb{RP}^n$ does not allow any orientation reversing automorphism when n is even.*

Proof. $\mathbb{S}^n \xrightarrow{A} \mathbb{RP}^n$ is the universal cover where A is the antipodal map. $T\mathbb{S}^n$ is the pullback of $T\mathbb{RP}^{2n}$ via this map which can be seen directly. A point in $A^*(T\mathbb{RP}^n)$ is of the form (x, e) where $A(x) = \{\pm x\}$ and $e = (\{\pm x\}, v)$ with $x \cdot v = 0$ for $x \in \mathbb{S}^n$ and $v \in \mathbb{R}^{n+1}$. I.e. the set of all pairs $(x, v) \in \mathbb{S}^n \times \mathbb{R}^{n+1}$ with $x \cdot v = 0$, also known as $T\mathbb{S}^n$. Thus $T\mathbb{RP}^n \to \mathbb{RP}^n$ cannot admit an orientation reversal as this would give an orientation reversal on $T\mathbb{S}^{2n} \to \mathbb{S}^{2n}$ which I have demonstrated numerous times is impossible. □

Remark 3.4. The case of $\mathbb{RP}^2$ can be done more directly, since an orientation reversal for this rank 2 bundle would mean it splits as (possibly non-trivial) line bundles. Using the Stiefel-Whitney classes this can be seen to not hold. Let $a \in H^1(\mathbb{RP}^2, \mathbf{Z}/2)$ be the generator. It's known that $w(T\mathbb{RP}2) = 1 + a + a^2$. Assume $T\mathbb{RP}^2 \cong L_1 \oplus L_2$. Then $w(L_i) = 1 + c_i a$ with $c_i \in \mathbf{Z}/2$ and $w(T\mathbb{RP}2) = 1 + a + a^2 = w(L_1)w(L_2) = 1 + (c_1 + c_2)a + c_1 c_2 a^2$. This has no solution, hence $T\mathbb{RP}^2$ can not be written as the sum of two line bundles.

Tangent bundles of non-orientable manifolds

The argument for $\mathbb{RP}^n$ in the foregoing section generalise readily to arbitrary manifolds. Any non-orientable manifold M admits an orientable 2-1 cover $\tilde{M} \to M$.

Lemma 3.4. *Let M be a non-orientable manifold. Then $TM \to M$ does not admit an orientation reversal unless $T\tilde{M} \to \tilde{M}$ does.*

Proof. $T\tilde{M}$ is the pullback of TM for instance by the same argument as for $\mathbb{S}^{2n} \to \mathbb{RP}^{2n}$. □

I need a lemma before I proceed with tangent bundles of non-orientable manifolds. The lemma and the proof is proposition 13.5 on page 216 in [8]

Lemma 3.5. *If $X \to Y$ be a k-sheeted covering space ($k < \infty$) and Y is a finite $\Leftrightarrow$ compact CW complex then X is a finite CW complex as well and $\chi(X) = k\chi(Y)$.*

Proof. I will assume it is known that the Euler characteristic of a CW complex can be expressed as the alternating sum of the number of cells.

The characteristic maps $\Phi_\alpha : e^i_\alpha \to Y$ map from contractible spaces and lift to X in exactly k ways. This gives X the structure of a CW complex with the number of i-cells of X being exactly k times the number of i cells of Y. So $\chi(X) = \sum_{i=0}^n (-1)^i \# e_i(X) = \sum_{i=0}^n (-1)^i k \# e_i(Y) = k\chi(Y)$. □

Proposition 3.8. *Let M be a non-orientable compact manifold M. Then $TM \to M$ does not admit an orientation reversing automorphism unless $\chi(M) = 0$.*

Proof. Let $\tilde{M} \to M$ be the orientable cover of M. If $TM \to M$ admits an orientation reversal then so does $T\tilde{M} \to \tilde{M}$. By corollary 3.4.2 and proposition 3 this is impossible unless $\chi(\tilde{M}) = 0$. By the previous lemma and the fact that the orientable cover is 2-1 we have $\chi(\tilde{M}) = 2\chi(M)$ which is 0 if and only if $\chi(M) = 0$. □

Corollary 3.5.2. *Let M be a compact manifold. A necessary (but not in general sufficient) condition for $TM \to M$ to admit an orientation reversal is that $w_n(TM) = 0$.*

Proof. Proposition 3.8 and corollary 3.4.2 combined says that $\chi(M) = 0$ is a necessary requirement, and proposition 3.1 says that $w_n(TM) = 0$ if and only if $\chi(M) = 0$ modulo 2.

Any manifold with even Euler characteristic is a counter example showing why $w_n(M) = 0$ is not sufficient. □

Remark 3.5. The reason I include this last corollary is that it does not assume any orientability. It is also a sufficient criterion to rule out orientation reversal whenever $\chi(M)$ is odd (so in particular M has to be non-orientable). This covers non-orientable surfaces and $\mathbb{RP}^{2n}$, since both both have odd Euler characteristic.

Universal bundle automorphisms

The examples of $T\mathbb{S}^{2n} \to \mathbb{S}^{2n}$ and $T\mathbb{RP}^{2n} \to \mathbb{RP}^{2n}$ provide examples of respectively orientable and non-orientable rank $2n$ bundles for $n \geq 1$ which do not admit orientation reversing gauge transformations. They provide counter examples in each possible (i.e. even) dimension.

Universal bundle as example

This idea belongs to John Rognes. Since there are even rank bundles without orientation reversals for all even numbers, there can be no universal orientation reversal. By this is meant the following. Any real rank n bundle (possible non-orientable) over a paracompact Hausdorff space is the pullback of the universal bundle $\gamma_n \to BO(n)$. The counter examples for all even ranks allow me to state this result.

Proposition 3.9. *Let $\gamma_n \to BO(n)$ be the universal rank n bundle. Then it admits an orientation reversing gauge transformation if and only if n is odd.*

Proof. The n odd case should be clear by now. Assume there is an orientation reversal $\tilde{\phi} : \gamma_n \to \gamma_n$ and that n is even. This pulls back to an orientation reversal of any rank n bundle

$$\begin{array}{ccc} \gamma_n & \xrightarrow{\tilde{\phi}} & \gamma_n \\ \tilde{f} \uparrow & & \uparrow \tilde{f} \\ E & \xrightarrow[\phi]{} & E \end{array}$$

Explicitly with $E = f^*(\gamma_n)$, a point $e = (x, \sigma) \in X \times \gamma_n$ with $f(x) = \pi(\sigma) \in BO(n)$ gets sent to $\phi(e) = (x, \tilde{\phi}(\sigma))$. This is fine as $f(x) = \pi(\sigma) = \pi(\tilde{\phi}(\sigma))$ by $\tilde{\phi}$ being a bundle automorphism. This is impossible as there are rank $n = 2k$ bundles without orientation reversing automorphisms for any k. □

Proposition 3.10. *Let $\tilde{\gamma}_n \to BSO(n)$ be the universal oriented rank n bundle. Then it admits an orientaion reversing gauge transformation if and only if n is odd.*

Proof. Same proof as above since $TS^{2k} \to \mathbb{S}^{2k}$ is an orientable vector bundle with no orientation reversals. □

Chapter 4

Some Obstruction Theory

In this chapter I will start by sketching some highlights of non-equivariant obstruction theory in order to give some justification for theorem 3.3 and another way of viewing the Euler class. I first deal in quite some detail with the non-equivariant extension theorem, whose equivariant version could have been used to attack the real or complex version of my problem. I then move on to sketch the lifting problem which is more relevant for proving the main propositions in chapter 3.

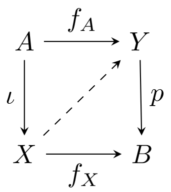
I end by saying a few words about equivariant obstruction theory and how I could have proceeded in this thesis, which is closer to how Riddervold proceeded in his master's thesis [1]. I believe this is better suited to the complex version since a lot of the interesting cases for real bundles are covered by my various criteria in chapter 3.

Obstruction theory in general

Assume X is a CW complex and A is a subcomplex. Assume a map $A \to Y$ is given for some space Y. A question is whether or not this map extends to $X \to Y$. A related question is if given a fibration $Y \to B$ does a given map $X \to B$ factor as a map $X \to Y \to B$?

What I aim to explain but not give a complete proof of is a special case of theorem 13.11 on page 507 of [8].

Theorem 4.1 (Only partial proof)**.** *Let (X, A) be a relative CW-complex and $p : Y \to B$ a simple map with homotopy fiber F. For the lifting problem*



there exists a sequence of obstructions $c_f^{n+1} \in H^{n+1}(X, A; \pi_n(Y))$ where all the previous obstructions must be 0 before the next one is defined, and where different choices of previous liftings may lead to different obstructions, such that there *is a complete sequence of obstructions of which all are 0 $\Leftrightarrow$ there is a solution to the lifting problem. Also, if f_0 and f_1 are solutions and if $p \circ f_0 \cong p \circ f_1$ relative to A via $G : X \times I \to B$, then there exists a sequence of obstructions $d^n_G(f_0, f_1) \in H^n(X, A; \pi_n(F))$ to lifting the homotopy G.*

A map $f : Y \to B$ is simple if $[\pi_1(B), \pi_1(B)] \subset f_*(\pi_1(Y))$.

I will follow [5] with minor insertions and notational differences for my exposition. A slicker, but more technologically advanced, approach can be found in [8] and to some extent in [3]. As detailed above I will for clarity start with the extension problem corresponding to $B = *$ above.

The Obstruction cocycle

Let $X^{(k)}$ denote the k-skeleton of X. Assume a map $f : A \to Y$ is given. This always extends to a map $f_0 : A \cup X^{(0)} \to Y$ by arbitrarily prescribing values at points in $X^{(0)} \setminus \{X^{(0)} \cap A\}$. This extends to a map $f_1 : X^{(1)} \cup A \to Y$ if and only if it extends to the 1-cells of X, which is the same as asking whether 2 points in $f_0(X^{(0)}) \subset Y$ can be joined by a path. I will therefore assume Y is path connected. Assume one has got all the way to the k-skeleton of X with a map $f_k : X^{(k)} \to Y$. The question is if this extends to the $(k+1)$-skeleton of X. Assume furthermore that $\pi_1(Y)$ acts trivially on $\pi_i(Y)$ for $1 \le i \le n$. Let $\{e^{k+1}_\alpha\}_\alpha$ be the $(k+1)$-cells of X with characteristic maps $\Phi_\alpha : (D^{k+1}_\alpha, \partial D^{k+1}_\alpha) \to (X^{(k+1)}, X^{(k)})$ and $(\Phi_\alpha)_{|\partial D^{k+1}_\alpha} = \phi_\alpha : \partial D^{k+1}_\alpha \to X^{(k)}$ as attaching map. We get a map $f_k \circ \phi_\alpha : \partial D^{k+1}_\alpha \to Y$, and this represents an element of $\pi_k(Y)$ which I will call $c_k = c_k(f, e^{k+1}_\alpha)$. With f_k fixed we have defined an assignment of $e^{k+1}_\alpha \mapsto c_{k+1}(f)(e^{k+1}_\alpha) = c_{k+1}(f, e^{k+1}_\alpha) \in \pi_k(Y)$. Since this is defined on the basis of the chain group $\Gamma_{k+1}(X, A)$ it extends to a homomorphism $\Gamma_{k+1}(X, A) \to \pi_k(Y)$, also known as a cochain $c_{k+1}(f) \in \Gamma^{n+1}(X, A; \pi_k(Y))$.

There are some important but easy facts about the cochain c_{k+1}, and I'll list them here.

Lemma 4.1. *The following hold in the situation described above.*

i) For each $(k+1)$-cell e^{k+1}_α of (X, A), $f_{|\partial e^{k+1}_\alpha}$ extends over e^{k+1}_α if and only if $c^{k+1}(f)(e^{k+1}_\alpha) = 0$.

ii) The map $f : X^{(k)} \to Y$ can be extended to a map $f : X^{(k+1)} \to Y$ if and only if $c^{k+1}(f) = 0$.

iii) If (X', A') is another relative CW-complex and $g : (X', A') \to (X, A)$ is a cellular map, then $c^{k+1}(f \circ (g_{|X'^{(k)}})) = g^\sharp c^{k+1}(f)$.

iv) If Y' is k-simple and $h : Y \to Y'$ is some map, then $c^{k+1}(h \circ f) = h_* \circ c^{k+1}(f)$.

v) $f_0 \cong f_1$ (homotopic), then $c^{k+1}(f_0) = c^{k+1}(f_1)$.

Proof. $i)$ is the important fact that a map $g : \partial D^n \to Y$ extends to a map $G : D^n \to Y$ if and only if g is nullhomotopic.

$ii)$ is the fact that a map extends from the k-skeleton to the $k+1$-skeleton if and only if it extends across all the $(k+1)$-cells, whereupon $f_{|\partial D^{k+1}_\alpha} \to Y$ has to be nullhomotopic for all α.

$iii)$ Is more or less per definition.

iv) Is also pretty much per definition. For a basis element of $\Gamma_{k+1}(X,A)$ we have $c^{k+1}(h \circ f)(e_\alpha^{k+1}) = (h \circ f)_{|\partial e_\alpha^{k+1}} = h_*(f)_{\partial e_\alpha^{k+1}}$.

v) is per definition of $\pi_k(Y)$. □

A key step towards theorem 4.1 is the next proposition.

Proposition 4.1. *The obstruction cochain $c^{k+1}(f)$ is a cocycle.*

Proof. We have this diagram.

$$\begin{array}{ccccccc}
H_{k+2}(X^{(k+2)}, X^{(k+1)}) & \xleftarrow{\rho_1} & \pi_{k+2}(X^{(k+2)}, X^{(k+1)}) & & & & \\
\downarrow \partial_1 & & \downarrow \partial_2 & & & & \\
H_{k+1}(X^{(k+1)}) & \xleftarrow{\rho} & \pi_{k+1}(X^{(k+1)}) & & & & \\
\downarrow i_1 & & \downarrow i_2 & & & & \\
H_{k+1}(X^{(k+1)}, X^{(k)}) & \xleftarrow[\rho_2]{} & \pi_{k+1}(X^{(k+1)}, X^{(k)}) & \xrightarrow{\partial_3} & \pi_k(X^{(k)}) & \xrightarrow{f_*} & \pi_k(Y)
\end{array}$$

The maps ∂ are connecting homomorphisms in the long exact sequences associated to the pair $(X^{(k+2)}, X^{(k+1)})$ or $(X^{(k+1)}, X^{(k)})$. The i maps are induced by inclusions. The ρ maps are Hurewicz homomorphisms. Start out with $\delta c^{k+1}(\sigma) = \pm c^{k+1}(\partial\sigma)$ for some $(k+2)$-chain $\sigma \in \Gamma_{k+2}(X) \cong H_{k+2}(X^{(k+2)}, X^{(k+1)})$ and $\partial\sigma \in \Gamma_{k+1}(X) \cong H_{k+1}(X^{(k+1)}, X^{(k)})$. It's also known that $\partial\sigma = (i_1 \circ \partial_1)(\sigma)$. This establishes the starting point of the computation in [5].

$$\begin{aligned}
\pm(\delta c^{k+1}) \circ \rho_1 &= (c^{k+1} \circ i_1 \circ \partial_1) \circ \rho_1 \\
&= f_* \circ \partial_3 \circ \rho_2^{-1} \circ i_1 \circ \partial_1 \circ \rho_1 \\
&= f_* \circ \partial_3 \circ \rho_2^{-1} \circ \rho_2 \circ i_2 \circ \partial_2 \\
&= f_* \circ \partial_3 \circ i_2 \circ \partial_2
\end{aligned}$$

Remarks worth making about the calculations are that ρ_2 is an isomorphism as $(X^{(k+1)}, X^{(k)})$ is k-connected and the replacement of c^{k+1} by $f_* \circ \partial_3 \circ \rho_2^{-1}$ is per definition. The other transitions are commutativity of the diagram. $\partial_3 \circ i_2 = 0$ since these are two consequetive maps in the long exact sequence of the homotopy groups of the pair $(X^{(k+1)}, X^{(k)})$, showing that $(\delta c^{k+1}) \circ \rho_1 = 0$. $(X^{(k+2)}, X^{(k+1)})$ is $(k+1)$-connected so ρ_1 is an isomorphism and we conclude that $\delta c^{(k+1)} = 0$. □

The difference cochain

The next result I need is a tool for analysing when a homotopy class of maps may be extended. Let $(\hat{X}, \hat{A}) = I \times (X, A)$. This is a relative CW complex with $\hat{X}^{(k)} = I \times X^{(k-1)} \cup \partial I \times X^{(k)}$. A map $F : \hat{X}^{(k)} \to Y$ consists of a pair of maps $f_0, f_1 : X^{(k)} \to Y$ together with a homotopy $G : I \times X^{(k-1)} \to Y$ between $f_{0|X^{(k-1)}}$ and $f_{1|X^{(k-1)}}$. The difference cochain of (f_0, f_1) with respect to G is defined to be the cochain $d^k = c^k(f_0, G, f_1) = d^k(F) \in \Gamma^n(X, A; \pi_k(Y))$ such that $d^k(c) = (-1)^k c^{k+1}(F)(e_1 \times \sigma)$ for any k-chain $\sigma \in \Gamma_k(X, A)$ where e_1 is the

1-cell of I. When f_0 and f_1 agree on $X^{(k-1)}$ and G is stationary it is customary to write $d^k(f_0, f_1)$.

The difference cochain has several properties analogous to the obstruction cocycle.

Lemma 4.2 (No proof). *The situation is as described above. The following hold.*

i) For each k-cell e^k_α of (X, A), $F_{|\partial(I\times e^k_\alpha)}$ can be extended over $I \times e^k_\alpha$ if and only if $d^k(e^k_\alpha) = 0$.

ii) There is a homotopy of f_0 to f_1 extending G if and only if $d^k = 0$.

iii) If (X', A') is a relative CW-complex, $g : (X', A') \to (X, A)$ a cellular map, and if $g' : \hat{X}'^{(k)} \to \hat{X}^{(k)}$ is the restriction of $1\times g$, then $d^k(F\times g') = g^\sharp d^k(F)$.

iv) If Y' is a k-simple space and $h : Y \to Y'$ is a map, then $h_* \circ d^k(F) = d^k(h \circ F)$.

v) If $F \cong F' : \hat{X}^{(k)} \to Y$, then $d^k(F) = d^k(F')$.

The next proposition gives a formula for the coboundary of d^k.

Proposition 4.2. *The coboundary of the difference cochain is given by*

$$\delta d^k(f_0, G, f_1) = c^{k+1}(f_1) - c^{k+1}(f_0)$$

Proof. For $\sigma \in \Gamma_{k+1}(X, A)$ we have

$$\delta d^k(\sigma) = (-1)^k d^k(\partial\sigma) = c^{k+1}(F)(e_i \times \partial\sigma)$$

By proposition 4.1 we have

$$0 = (-1)^{k+1}\delta c^{k+1}(F)(e_1\times\sigma) = c^{k+1}(F)(\partial(e_1\times\sigma)) = c^{k+1}(F)(\{1\}\times\sigma-\{0\}\times\sigma+e_1\times\partial\sigma)$$

Per construction we have that $c^{k+1}(F)(\{t\} \times \sigma) = c^k(f_t)(\sigma)$. Putting this together gives the formula. □

The difference cochain is involved in the proof of the next theorem.

Theorem 4.2 (Partially proven). *Let $f : X^{(k)} \to Y$. Then $f_{|X^{(k-1)}} \to Y$ can be extended over $X^{(k+1)}$ if and only if $[c^{k+1}(f)] = 0 \in H^{k+1}(X, A; \pi_k(Y))$.*

Proof. Assume $f_{|X^{(k-1)}}$ has an extension $\tilde{f} : X^{(k+1)} \to Y$. Let $F : \hat{X}^{(k)} \to Y$, $F(0, x) = \tilde{f}(x)$, $F(1, x) = f(x)$ for $x \in X^{(k)}$ and $F(t, x) = f(x) = \tilde{f}(x)$ for $0 < t < 1$ and $x \in X^{(k-1)}$. The formula for the difference cochain, proposition 4, says that

$$\delta d^k(F) = c^{k+1}(F(1, x)) - c^{k+1}(F(0, x) = c^{k+1}(f) - c^{k+1}(\tilde{f}_{|X^{(k)}})$$

By lemma 4.1 *ii*), $c^{k+1}(\tilde{f}_{X^{(k)}}) = 0$ since $\tilde{f}_{|X^{(k)}}$ clearly extends from $X^{(k)}$ to $X^{(k+1)}$. So $c^{k+1}(f) = \delta d^k(F)$ and $[c^{k+1}(f)] = 0$.

The converse is left unproven. The interested reader may consult Theorem 5.14 on page 233 of [5] and his preceding lemmas. □

Obstruction to splitting off a trivial bundle

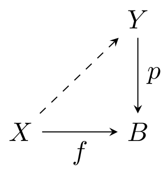
Let $E \to X$ be a rank k real bundle over X and assume X is a CW-complex. I do not assume $\dim(X) = k$ yet. The real version of the problem was easily seen to have a solution if E splits off a trivial line bundle. This is equivalent to asking for a non-zero section.

Lemma 4.3. *Let $E \to X$ be as above. Then $E \cong E' \oplus \epsilon^1$ if and only if there is a nowhere zero section $s : X \to E$.*

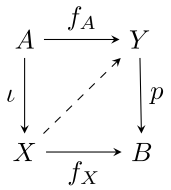
Proof. Assume a section exists. Define a map $E \to \epsilon^1 \oplus E'$ by $e \mapsto (s(\pi(e)), e^{\perp}) \in \epsilon^1 \oplus E'$. Here it's crucial that $s \neq 0$ to ensure that the rank of E' is constant. Conversely, assume $E \cong E' \times \epsilon^1$. Define $s(x) = (\vec{0}, 1) \in E'_x \times \mathbb{R}$. □

The idea inductively defining a section over the skeleton of X will involve many of the same ideas as before. I will try to make the discussion less detailed.

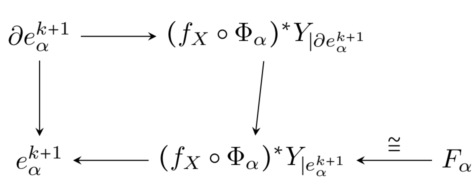
First up is the observation that solutions of the lifting problem



are in 1-1 correspondence with sections of f^*Y. The idea is therefore to solve a lifting problem as a way of finding a section. Let (X, A) be a relative CW complex and assume $f_A : A \to Y$ and $f_X : X \to B$ are given with $p : Y \to B$ fibration with simple fiber. That is to say we are in the situation of theorem 4.1



Assume a partial lifting $g_k : X^{(k)} \to Y$ is given. Let e_α^{k+1} be a (k+1)-cell of (X, A) with $\Phi_\alpha : e_\alpha^{k+1} \to X$ as characteristic map. The composition $\phi_\alpha \circ g_k : \partial e_\alpha^{k+1} \to Y$ corresponds bijectively to a section in $(f \circ \Phi_\alpha)^* Y_{|\partial e_\alpha^{k+1}}$. Part of the situation is this diagram.



F_α is the fiber and $\cong$ means homotopy equivalent. The homotopy class of a map I am interested in is the composition $\partial e_\alpha^{k+1} \to F_\alpha$. What has been established so far is that once a partial lifting $g_k : X^{(k)} \to Y$ is chosen one can associate an element of $\pi_k(F_\alpha)$ to each $(k+1)$-cell e_α^{k+1}. I.e. a partial lifting g_k gives an element $\mathfrak{o}_k(f_k) \in \Gamma^{k+1}(X, A; \underline{\pi}_k(F))$. In general there are some issues in choosing a homeomorphism for each fiber, and one ends up with a local coefficient system, hence the underscore. Let me just state where this is headed.

Theorem 4.3 (No proof)**.** *The obstruction cochain $\mathfrak{o}_{k+1}(g)$ has (amongst others) the following properties.*

- *g can be extended to a partial lifting over $X^{(k)} \cup e_\alpha^{k+1}$ if and only if $\mathfrak{o}_{k+1}(g)(e_\alpha^{k+1}) = 0$.*
- *The map g can be extended to a partial lifting over $X^{(k+1)}$ if and only if $\mathfrak{o}(g) = 0$.*
- *$\mathfrak{o}_{k+1}$ only depends on the homotopy class.*
- *$\mathfrak{o}_{k+1}$ is a cocycle and determines a class in $H^{k+1}(X, A; \underline{\pi}_k(F))$.*
- *The cochain $\mathfrak{o}_{k+1}(g)$ is a coboundary if and only if the map $g_{|X^{(k-1)}}$ can be extended to a partial lifting $g_1 : X^{(k+1)} \to Y$.*

The above is a mixture of theorem 5.5 and corollary 5.7 on pages 294-297 in [5].

Remark 4.1. I have for the occasion changed from the notation $c^{k+1}(g)$ to $\mathfrak{o}_{k+1}(g)$. This is because I will compare with [12] in the next paragraph and they use the latter. It's also to avoid having to write $c^{k+1}(E)$ and potentially make the reader think this is a Chern class or some such.

Let me finally get on to describing how this pertains to my situation. Let $E \to X$ be a rank n vector bundle. The fiber bundle is $Y = E^0 \to X$ with fiber $F \cong \mathbb{S}^{n-1}$ satisfies the criterion having simple fibers. Furthermore, there are canonical identifications of the interesting coefficient groups. $\pi_k(F) = 0$ when $k < n-1$ and $\pi_{n-1}(F_x) = \pi_{n-1}(E_x \setminus \{0\}) \cong H_{n-1}(E_x \setminus \{0\}) \cong H_n(E_x, E_x \setminus \{0\}) \cong \mathbf{Z}$ canonically according to [12]. The conclusion is this

Proposition 4.3. *Let $E \to X$ be a rank n vector bundle. Then the primary obstruction to defining a non-zero section is an element of $\mathfrak{o}_n(E) \in H^n(X; \mathbf{Z})$.*

I am not saying anything about higher obstructions at this point, and that's sort of what complicates matters when $\mathfrak{o}_n(E) = 0$ but there are higher homotopy groups of X.

This is just about how much I intend to say about the general setup of obstruction theory and how one can go about proving something like theorem 4.1. Let me instead move on to a central interpretation of $\mathfrak{o}_n(E)$.

The Euler class as obstruction

There is a well-known interpretation of the Euler class as an obstruction, and I thought to relay that here. This material can be found in (amongst others) [4], [12], [10], and [8]. I will soon need the Gysin sequence, so I thought I might as well start formulating that. I decided to include a proof seeing how I already postulate the Thom isomorphism as theorem 3.1 when I defined the Euler class.

Theorem 4.4 (The Gysin sequence)**.** *Let $E \xrightarrow{\pi} X$ be an oriented rank n bundle with Euler class $e(E) = e$. Denote by $E^0 \xrightarrow{\pi_0} X$ the fiber bundle associated to E where each fiber is homeomorphic to $\mathbb{R}^n \setminus \{0\}$. Then we have the long exact sequence known as the Gysin sequence*

$$\cdots \to H^i(X;\mathbf{Z}) \xrightarrow{\cup e} H^{i+n}(X;\mathbf{Z}) \xrightarrow{\pi_0^*} H^{i+n}(E^0;\mathbf{Z}) \to H^{i+1}(X;\mathbf{Z}) \xrightarrow{\cup e} \dots$$

Proof. All cohomology in this proof is with $\mathbf{Z}$ coefficients. From the inclusions $E^0 \to E \to (E, E^0)$ there is a long-exact sequence in cohomology

$$\cdots \to H^i(E,E^0) \to H^i(E) \to H^i(E^0) \to H^{i+1}(E,E^0) \to \cdots$$

The Thom isomorphism - theorem 3.1 - stated that there is a class $u \in H^n(E,E^0)$ with the property that $H^i(E) \xrightarrow{\cup u} H^{i+n}(E,E^0)$ is an isomorphism. Replacing $H^i(E,E^0)$ in the long exact sequence results in

$$\cdots \to H^{i-n}(E) \xrightarrow{g} H^i(E) \to H^i(E) \to H^i(E_0) \to H^{i-n+1}(E) \to \cdots$$

where $g(x) = (x \cup u)_{|E} = x \cup (u_{|E})$. Using the isomorphism $H^i(E) \cong H^i(X)$ and the definition $u_{|E} \cong e(E) \in H^n(X)$ we get the Gysin sequence. □

The eponymous proposition of this section is this.

Proposition 4.4 (Euler class as obstruction)**.** *Let $E \to X$ be an oriented rank n bundle over a CW complex X. Then $\mathfrak{o}_n(E) = e(E)$.*

I follow [12] for the proof, inserting some details of my own.

Proof. Consider the pullback bundle $\pi_0^*(E) \to E^0$

$$\begin{array}{ccc} \pi_0^*(E) & \longrightarrow & E \\ \downarrow & & \downarrow \pi \\ E^0 & \xrightarrow[\pi_0]{} & X \end{array}$$

By definition of the pullback $\pi_0^*(E) = \{(e_0, e) \in E^0 \times E \big| \pi(e) = \pi_0(e_0)\}$ there is clearly a non-zero section, $e_0 \mapsto (e_0, e_0)$. This cause the obstruction to vanish and by naturality we conclude that $\pi_0^*(\mathfrak{o}_n(E)) = \mathfrak{o}_n(\pi_0^* E) = 0$ so $\mathfrak{o}_n(E) \in \ker(\pi_0^*)$. Now I need the Gysin sequence:

$$\cdots \to H^0(X;\mathbf{Z}) \xrightarrow{\cup e} H^n(X;\mathbf{Z}) \xrightarrow{\pi_0^*} H^n(E^0) \to \cdots$$

The exactness of the Gysin sequence says $\ker(\pi_0^*) = \operatorname{im}(\cup e)$ so $\mathfrak{o}_n(E) = \lambda e(E)$ for some $\lambda \in \mathbf{Z}$.

The bundle $E \to X$ is arbitrary, so this also holds for the universal oriented bundle $\tilde{\gamma}_n \to \tilde{Gr}_n$ or $\tilde{\gamma}_n \to BSO(n)$, and by universality[1] it follows that $\lambda = \lambda_n$,

[1] The complete argument runs as follows. If $E \to X$ is some rank n oriented bundle, there is a map $f : X \to BSO(n)$ with $E \cong f^*(\tilde{\gamma}_n)$. The relation between the Euler class and obstruction class is then expressed as $\mathfrak{o}_n(E) = \mathfrak{o}_n(f^*\tilde{\gamma}_n) = f^*\mathfrak{o}_n(\tilde{\gamma}_n) = f^*\lambda_n e(\tilde{\gamma}_n) = \lambda_n e(f^*\tilde{\gamma}_n) = \lambda_n e(E)$.

which is to say that it only depends on the rank of the bundle, not the actual bundle.

The rest of the proof is about establishing $\lambda_n = 1$ for all n. For odd n I proved as part of theorem 3.2 that $2e(E) = 0$ when $\text{rk}(E)$ is odd, so $\lambda_n = 1$ for odd n. For even n [12] suggests looking at tangent bundles of spheres, and take for granted that $e(T\mathbb{S}^n) = \chi(\mathbb{S}^n)\iota = 2\iota$ with $\iota \in H^n(\mathbb{S}^n;\mathbf{Z})$ the orientation class.

To see that $\mathfrak{o}_n(T\mathbb{S}^n) = 2\iota$ when n is even, look at $E = T\mathbb{S}^n$ trivialised over the two disc $D^n_\pm$ and pick non-zero sections $s_\pm : D^n_\pm \to E^0_{|D^n_\pm} \cong D^n_\pm \times \mathbb{S}^{n-1}$. The sections' restrictions to the equator $\mathbb{S}^{n-1} \subset D^n_- \cap D^n_+$ can be arranged to look like $s_-(x) = (x, e_n)$ and $s_+(x) = (x, g(x)e_n)$ for $g : \mathbb{S}^{n-1} \to SO(n)$ the clutching function of the frame bundle of the sphere. Look at the projection $(x, v) \mapsto v \in \mathbb{S}^{n-1}$, meaning that $s_+(x) \mapsto g(x)e_n$. In the notation of chapter 2, this is $pg(x)$ with p being the projection in the fiber bundle $SO(n-1) \to SO(n) \xrightarrow{p} \mathbb{S}^{n-1}$. In chapter 2 I computed that $g(x)e_n$ has degree $1 + (-1)^n = 2$ when n is even. Armed with this knowledge I evaluate $\mathfrak{o}_n$ on the orientation class of $\mathbb{S}^n$, written as $[D^n_+] - [D^n_-]$. $\mathfrak{o}_n(TS^n)([\mathbb{S}^n]) = \mathfrak{o}_n(TS^n)([D^n_+] - [D^n_-]) = \mathfrak{o}_n([D^n_+]) - \mathfrak{o}_n([D^n_-]) = 2 - 0 = 2$, so $\mathfrak{o}_n(T\mathbb{S}^n) = 2\iota \in H^n(\mathbb{S}^n;\mathbf{Z})$. To conclude: $2\iota = \mathfrak{o}_n(T\mathbb{S}^n) = \lambda_n e(T\mathbb{S}^n) = 2\lambda_n\iota \implies \lambda_n = 1$.

□

I am finally in a position to prove the main result I used in chapter 3.

Theorem 4.5. *Let $E \to X$ be an oriented rank n bundle over an n-dimensional CW-complex. Then $E \cong E' \oplus \epsilon^1$ if and only if $e(E) = 0$.*

Proof. Assume $E \cong E' \oplus \epsilon^1$. Then $e(E) = e(E')e(\epsilon^1) = 0$ by a property of the Euler class which I proved in chapter 3.

Conversely, assume $e(E) = 0$. By theorem 4.1 the obstructions to extending a non-zero section from the k-skeleton of X to the $k+1$-skeleton of X is an element of $H^{k+1}(X; \pi_k(\mathbb{S}^{n-1})) = 0$ unless $k \geq n-1$. For $k = n-1$ the obstruction is $\mathfrak{o}_n(E) = e(E) = 0 \in H^n(X;\mathbf{Z})$ per proposition 4.4 and assumption. So a non-zero section can be defined on $X^{(n)} = X$. □

I also promised a proof of proposition 1.7 and here it is.

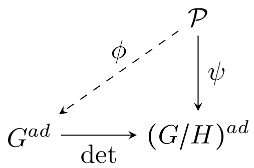
Proposition 4.5. *Assume $E \to X$ is a rank k oriented vector bundle over a CW complex X of dimension $n < k$. Then there exists a rank n vector bundle $E' \to X$ such that $E \cong E' \oplus \epsilon^{n-k}$.*

Proof. The obstruction to extending a non-zero section from the m-skeleton to the $(m+1)$-skeleton of X lies in $H^{m+1}(X; \pi_m(\mathbb{S}^{k-1})) = 0$ unless $m \geq k-1$. For $m \geq k-1$ the cohomology group $H^m(X; \pi_m(\mathbb{S}^{k-1})) = 0$ as $k > n$. Hence there can be no obstructions to extending a non-zero section, and $E \cong E' \oplus \epsilon^1$. Repeat this process using E' until the rank of E' is n. □

Equivariant obstruction approach

There is a more general approach to the equivariant formulation of the problem which I have not pursued (nor do I intend to in this thesis), namely to try building the desired map step by step.

The equivariant formulation was as follows. Let P be a principal G-bundle and let $H \subset G$ be a closed subgroup such that there is a determinant map $G \to G/H$. Given a G-map $\psi : P \to G/H$, does there exist a G-map $\phi : P \to G^{ad}$ which makes the following diagram commute



An idea would be to build P inductively, for instance as a G-CW-complex, and try to build the desired map ϕ stepwise and ask if a given G-map $\phi_{(k)} : P_{(k)} \to G^{ad}$ extends to a G-map $P_{(k+1)} \to G^{ad}$. Since P is supposed to be a G-space it's natural to look for a G-equivariant CW-structure where an n-cell D^n_α is replaced by $D^n_\alpha \times G/H_\alpha$ for some suitable $H_\alpha \in G$ a closed subgroup and where G acts on G/H_α via the quotient map. The attaching maps are also assumed to be G-equivariant.

As in the non-equivariant setting, a map $\phi_{(0)} : P_{(0)} \to G^{ad}$ always exists by choosing a G-map which is adjoint to the constant map. Any map $\phi_{(0)} \to G^{ad}$ can be lifted to a map $\phi_{(1)} : P_{(1)} \to G^{ad}$ if and only if G is connected. Since $G = O(n)$ is not connected, this might have caused a problem. But I'm not interested in lifting any map, only an equivariantly constant map hence I can assume $\phi_{(0)}$ has values in a single connected component of G, and thus lifts to a map $\phi_{(1)} : P_{(1)} \to G^{ad}$. For higher k, some heavy machinery is needed.

Assume P is n-simple for all n, i.e. that $\pi_1(P)$ acts trivially on $\pi_n(P)$ for all P. For instance, for the n-sphere we had that $P = O(n+1)$ which is simple because it is a Lie group. Then Theorem 5.1 in [22] applies which is an the equivariant version of theorem 3.3 and the content (in my setting) is as follows:

Theorem 4.6 (No proof)**.** *Let $\phi : P_{(k)} \to G^{ad}$ be given. The restriction $\phi|_{P_{(k-1)}} \to G^{ad}$ extends to a map $\phi : P_{(k+1)} \to G^{ad}$ if and only if $[c_\phi] \in H^{k+1}_G(P; \underline{\pi}_k(G))$ vanishes.*

There are a couple of points I want to underscore here. The obstruction cocycle c_f determines a class in the Bredon cohomology of P, which can usually be quite hard to compute. Secondly, the notation $\underline{\pi}_k(G)$ means MacKay functor and not, as I had originally hoped, local coefficients. The bright side is that P is not just any space. It is a free G-space, and this hypothesis should simplify matters.

Closing Remarks

I feel I have made some headway on determining when the real variant of the problem has a solution. Nevertheless, there are several areas where I think there is more to be said. I thought I would take the time to indicate what these are.

Outstanding Questions

Real case

Let $E \to X$ be a non-orientable real vector bundle. Are there sufficient and necessary condition for E to admit an orientation reversing gauge transformation?

I found some conditions here, namely that the answer is no if the orientable cover of E does not admit an orientation reversal since an orientation reversal can be pulled back. It's not in general possible to push a bundle automorphism forward, so the converse does not a priori have to hold.

If X is an n-dimensional CW complex and E is a rank k bundle with $k < n$, are there good criteria for determining whether or not E admits an orientation reversal? Here there is very little hope of using characteristic classes, since the cohomology of X might be concentrated in the range $k+1, \cdots, n$ (as is the case for $\mathbb{S}^n$).

My biggest unanswered question is what the precise relationship between orientation reversals and odd bundles splitting off?[2] Are they equivalent? If not, what is a counter example? And if orientation reversals and odd bundle splitting is not equivalent, are there reasonable conditions for them to be equivalent?

Finally, is the equivariant obstruction theory angle tractable and does it yield anything new?

Complex case

A main puzzle is whether or not there are complex bundles $E \to X$ where X is a finite dimensional CW complex and a map $\psi : X \to U(1)$ such that there is no bundle automorphism $\phi : E \to E$ with $\det(\phi) = \psi$. [1] has an infinite dimensional counter example, but nothing in finite dimension.

Another question could be to look for a complex bundle variant of the Euler class (and I don't mean the Euler class of the underlying real bundle), but rather some complex version of what I do in chapter 3. If such a thing doesn't exist it could still be interesting to know *why* it doesn't.

[2] If the reader knows or has an idea here I should very much like to hear about it!

Pseudo-Riemannian versions

I barely sketched version of the problem with $O(p,q)$ or $U(p,q)$ bundles in chapter 1. A first approach here could be to look for a pseudo-Riemannian version of the Stiefel-Whitney classes for $O(p,q)$-bundles. In particular a criterion for when an $O(p,q)$-bundle admits a reduction of the structure group to $SO(p,q)_0$. Then it would be nice to know if there is an Euler class for such bundles and whether or not it serves a purpose analoguous to my chapter 3.

Is equivariant obstruction theory viable here or does the combined non-connectivity and non-compactness of $O(p,q)$ create more hassle than it is worth?

Bibliography

[1] Hans Jørgen Riddervold, *Å realiserer avbildninger som determinanter av buntautomorfier*, Thesis for the degree Master of Mathematics, University of Oslo, 2002

[2] Franc Forstneric, Erik Løw, Nils Øvrelid, *Solving the d and $\overline{\partial}$-equation in thin tubes and applications to mappings*, arXiv:math/0003149v2 [math.CV], 2001

[3] Allan Hatcher, *Algebraic Topology*, Cambridge, 2001 (Ninth printing 2008)

[4] Allan Hatcher, *Vector Bundles and K-theory*, Version 2.1, 2009

[5] George W. Whitehead, *Elements of Homotopy Theory*, Springer, 1978

[6] Victor W. Guillemin, Shlomo Sternberg, *Supersymmetry and Equivariant de Rham Theory*, Springer, 1999

[7] Clifford Henry Taubes, *Differential Geometry*, Oxford University Press, 2011

[8] Glen E. Bredon, *Topology and Geometry*, Springer, 1993

[9] Daniel Huybrechts, *Complex Geometry*, Springer, 2005

[10] Norman Steenrod, *The Topology of Fibre Bundles*, Princeton University Press, 1951 (Seventh printing 1999)

[11] Dale Husemoller, *Fibre Bundles*, Springer, 1996 (3rd edition)

[12] John W. Milnor, James D. Stasheff, *Characteristic Classes*, Princeton University Press, 1974

[13] John W. Milnor, *On Manifolds Homeomorphic to the 7-sphere*, Annals of Mathematics, Vol 64, No. 2, 1956

[14] Raoul Bott, Loring W. Tu, *Differential Forms in Algebraic Topology*, Springer, 1982

[15] William Browder, *Surgery on Simply-Connected Manifolds*, Springer, 1972

[16] Michel A. Kervaire, *Some Nonstable Homotopy Groups of Lie Groups*, 1958

[17] J. Peter May, *A Concise Course in Algebraic Topology*, 1999

[18] Joseph A. Wolf, *Spaces of Constant Curvature* McGraw-Hill, 1967

[19] Giovanni Giachetta, Luigi Mangiarotti, Gennadi Sardanashvily, *Advanced Classical Field Theory*, World Scientific, 2009

[20] Kenkichi Iwasawa, *On Some Types of Topological Groups*, Annals of Mathematics, Second Series, Vol. 50, No. 3, 1949

[21] John W. Milnor, *Morse Theory* Princeton University Press, 1963

[22] J. Peter May et al., *Equivariant homotopy and cohomology theory*, AMS CBMS 91, 1996

[23] Anthony W. Knapp, *Lie Groups Beyond an Introduction*, Birkhauser, 2005 (2nd edition, 3. printing)

[24] Jason DeVito, *Homotopy groups of Lie groups*, 15.12.09, MathOverflow, Retrieved 15.05.16, http://mathoverflow.net/questions/8957/homotopy-groups-of-lie-groups