Singular Toric Varieties

by

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Contents

Introduction

The study of algebraic geometry has had tremendous success by defining many geometrical concepts generally and abstractly. Many theoretical results could not have been proved without this focus. However this tendency for making theoretical definitions sometimes makes it difficult to find obvious examples or being able to make specific calculations. The theory of toric varieties is a part of algebraic geometry for which, due to its relation with combinatorics, many easily computable examples exist.

A toric variety X is a variety which contains an algebraic torus T as an open dense subset, thus much of the structure of X will be decided by what happens on the torus. The key idea is that the sets $M = \mathrm{Hom}(T, \mathbb{C}^*)$ and $N = \mathrm{Hom}(\mathbb{C}^*, T)$ turn out to be free abelian groups of finite order (lattices), and thus have a combinatorical descripion. Geometrical concepts, for instance smoothness, completeness, properness, the theory of divisors and cohomology (and more), can be described in terms of these lattices, and thus are often much easier to compute than for general varieties.

Given a projective space $\mathbb{P}^N$, one has that the set of hyperplanes form a new projective space $(\mathbb{P}^N)^\vee$. Given any variety $X \subset \mathbb{P}^N$ one can define the corresponding dual variety $X^\vee \subset \mathbb{P}^{N^\vee}$ which typically will be a hypersurface. Finding the equation for this is generally very difficult, but there are results which describe the degree. Gelfand, Kapranov and Zelevinsky showed in [GKZ94] that for a smooth toric variety X_P associated with a polytope P the degree is given by

$$\deg X_P^\vee = \sum_{Q \preceq P} (-1)^{\operatorname{codim} Q} (\dim Q + 1) \operatorname{Vol}(Q) \tag{1}$$

Our main examples of study will be the weighted projective spaces, a generalization of the usual projective space where each coordinate gets assigned an integer weight. These are toric varieties, however the weighted projective spaces are singular, so the formula above does not apply. Following chapter 5 of [Mor11], we will use generalizations of the formula above proved by Matsui and Takeuchi [MT11] for singular toric varieties, to calculate the degree for weighted projective planes. Mork considered only planes of the form $\mathbb{P}(1, m, n)$, while here we consider the more general $\mathbb{P}(k, m, n)$.

The theory of dual varieties, though interesting in itself, also relates to that of discriminantal varieties. Given a general polynomial p of a fixed degree, one can assoicate another polynomial in the coefficients of p, the discriminant Δ, with the property that $\Delta = 0$ whenever p has a double root. The easiest example of this is a quadratic polynomial $p(x) = ax^2 + bx + c$, which gives the discriminant $\Delta_p = b^2 - 4ac$. This notion can be generalized to polynomials in several variables or to sets of polynomials, and we can define discriminant polynomials which have analogous properties, we will use the following: Given a set of monomials A , let $\mathbb{C}^A$ be the space of all polynomials which are linear combinations of the monomials in A. Then the discriminant $\Delta_A(f)$ is an irreducible polynomial in the coefficents of $f \in \mathbb{C}^A$ which vanishes when f has a double root.

Now, choosing a polytope P giving a toric variety X_P corresponds to choosing a set of Laurent monomials A. Then the dual variety will be exactly the set

$$\{f \in \mathbb{C}^A | \Delta_A(f) = 0\}$$

Thus we see that descrbining the dual variety can be interpreted as describing a discriminantal variety of certain Laurent monomials.

Also the degree of the dual variety can be interpreted another way: As the number of singular curves of a certain type on the variety, called the Severi degree, hence we can tie this to the subject of enumerative geometry. In the smooth case the Severi degrees are described as polynomials in the four topological numbers $K \cdot L$, L^2, K^2,c_2. The first Severi degree $N^{L,1}$ equals exactly the degree of the dual variety, and in the singular case c_2 is replaced by the sum of Euler obstructions of the vertices. In the singular case one would hope to find corrections to the other numbers which give higher Severi degrees.

The problem of computing the dual degree of singular toric surfaces has been the motivating problem behind most of this work. This, it turns out, is closely related to resolving singularities, weighted blow-ups, continued fractions and intersection theory, so we give quite a lot of room to these topics.

In Chapter 1 we go through basic definitions and examples from the theory of toric varieties. The choice of material is largely motivated by what we will, in some sense or another, need in later chapters. We also introduce dual varieties, the formula for computing its degree and the Euler obstruction. We show how to compute the Euler obstruction in the surface case.

In Chapter 2 we study in detail the weighted projective spaces from some different angles. We study their singularities, the class and Picard groups, and consider intersection theory on the varieties. We prove a Bezout type theorem for weigthed projective spaces:

Theorem Given n torus-invariant divisors $D_1, ..., D_n$ on $\mathbb{P}(q_0, ..., q_n)$, we have

$$D_1 \cdots D_n = \frac{\Pi_{i=1}^n \deg D_i}{q_0 \cdots q_n}$$

We then specialize to the surface case, consider a polytope giving $\mathbb{P}(k, m, n)$, and use this to compute the degree of the dual variety in some special cases. However we realize we need more machinery for general k, m, n.

In Chapter 3 we start with a diversion into the world of continued fractions. We see how this relates to both the Euler obstruction and the minimal resolution of singularities for the singular surface. We show that the Euler obstruction of a vertex is 0 if and only if the corresponding singularity is Gorenstein. We give our own toric proofs of the previously known results that the resolution of singularities is given by a sequence of weighted blowups, that the self-intersections of the exceptional divisors is described by HJ-fractions and describe intersection theory on the blown-up surface. We show a general formula for the dual degree of $\mathbb{P}(k, m, n)$ in terms of HJ-fractions, which can be algorithmically computed:

Theorem Given $\mathbb{P}(k, m, n)$, find minimal natural numbers a, b, c such that

$$k + am \equiv 0 \pmod{n}$$

$$n + bk \equiv 0 \pmod{m}$$

$$m + cn \equiv 0 \pmod{k}$$

Let $\frac{n}{n-a} = a_1 - \frac{1}{a_2 - \frac{1}{\cdots - \frac{1}{a_r}}}$, $\frac{m}{m-b} = b_1 - \frac{1}{b_2 - \frac{1}{\cdots - \frac{1}{b_s}}}$, $\frac{k}{k-c} = c_1 - \frac{1}{c_2 - \frac{1}{\cdots - \frac{1}{c_t}}}$.

Then $\deg \mathbb{P}(k, m, n)^\vee$ equals

$$3kmn - 2(k + n + m) + \sum_{i=1}^{r}(2 - a_i) + \sum_{i=1}^{s}(2 - b_i) + \sum_{i=1}^{t}(2 - c_i)$$

We then do a small attempt at going to 3 dimensions, where we find examples of isolated singularities which have Euler obstruction 1.

In Chapter 4 we see how this relates to curve counting, where we relate our general toric description to existing counting forumlas which only works for a subclass of toric varieties, those coming from h-transverse polytopes. We classify the weighted projective planes which come from h-transverse polytopes. We compute the first and second Severi degree for the h-transverse varieties, hoping to see new candidates for invariants in the singular case. No obvious results were found. We conclude with some remarks about possible further directions one could try.

Throughout we will assume familiarity with basic algebraic geometry, for instance Hartshorne's Algebraic Geometry chapter I and II [Har77]. For a commutative ring R we will write $\operatorname{Spec} R$ even though we only ever use closed points, i.e. we consider the associated variety. This slight abuse of notation is justified by noting that varieties are a full subcategory of schemes, and made because much literature are written in the language of schemes.

We work over $\mathbb{C}$, however much of this could be generalized to other fields, but we do not go into any details here.

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I would like to thank my supervisor Ragni Piene for introducing me to the subject of this thesis. She has always been very helpful by explaining things in intuitive ways, listening to my mathematical ramblings and pointing me in the right directions.

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Chapter 1

Toric Varieties

1.1 Definitions and examples

Most of the definitions, claims and propositions in this chapter come from [CLS11] and [Ful93], most proofs are omitted.

$(\mathbb{C}^*)^n = \mathrm{Spec}(\mathbb{C}[x_1, x_1^{-1}, x_2, x_2^{-1}, ..., x_n, x_n^{-1}])$ is an affine variety which is a group under componentwise multiplication. An algebraic torus is a variety isomorphic to $(\mathbb{C}^*)^n$. A torus has two associated lattices:

A character of a torus $T = (\mathbb{C}^*)^n$ is a group homomorphism $\chi : T \to \mathbb{C}^*$. One can show that the set of all characters forms a group isomorphic to $M = \mathbb{Z}^n$, given by, for any $m = (m_1, ..., m_n)$:

$$\chi^m(t_1, ..., t_n) = t_1^{m_1} \cdots t_n^{m_n}$$

Thus we see that a character determines a monomial in n variables which is allowed to have arbitrary integer exponents. This is called a Laurent monomial.

A one-parameter subgroup of a torus T is a group homomorphism $\lambda : \mathbb{C}^* \to T$. The set of all one-parameter subgroups will also be isomorphic to $\mathbb{Z}^n$, denote this lattice by N, given by, for any $l = (l_1, ..., l_n)$:

$$\lambda^l(t) = (t^{l_1}, ..., t^{l_n})$$

One can define a bilinear pairing $M \times N \to \mathbb{Z}$ defined explicitly by the dot product, for $m \in M$ and $l \in N$ as above,

$$\langle m, l \rangle = \sum_{i=1}^{n} l_i m_i$$

this translates to, for χ^m and λ^l, we have $\chi^m \circ \lambda^l$ is a group homomorphism $\mathbb{C}^* \to \mathbb{C}^*$ and thus has to be of the form $z \mapsto z^n$. Then $\langle \chi^m, \lambda^l \rangle = n$. This pairing identifies $M \simeq \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ thus showing they are dual lattices (some useful facts about lattices are collected in Appendix A).

Also $N \otimes \mathbb{C}^* \cong T$ via $l \otimes t \mapsto \lambda^l(t)$, leading to the common notation of T_N for the torus.

Definition 1.1.1. A toric variety is an irreducible variety X containing a torus $T_N = (\mathbb{C}^*)^n$ as a Zariski open subset such that the action of T_N on itself extends to a morphism $T_N \times X \to X$.

Example 1.1.2. $\mathbb{P}^n$ is a toric variety with torus

$$T_{\mathbb{P}^n} = \mathbb{P}^n \setminus V(x_0 \cdots x_n) = \{(1, t_1, ..., t_n) \in \mathbb{P}^n | t_1, ..., t_n \in \mathbb{C}^*\} \cong (\mathbb{C}^*)^n$$

Example 1.1.3. $X = V(x^3 - y^2) \subset \mathbb{C}^2$ is a toric variety with torus

$$X \cap (\mathbb{C}^*)^2 = \{(t^2, t^3) | t \in \mathbb{C}^*\} \cong \mathbb{C}^*$$

Example 1.1.4. $Y = V(xy - zw) \subset \mathbb{C}^4$ is a toric variety with torus

$$Y \cap (\mathbb{C}^*)^4 = \{(t_1, t_2, t_3, t_1 t_2 t_3^{-1}) | t_i \in \mathbb{C}^*\} \cong (\mathbb{C}^*)^3$$

Given a torus T with character lattice $M \cong \mathbb{Z}^n$ and a finite subset $\mathscr{A} = \{m_1, ..., m_s\} \subset M$ we can define the associated affine toric variety $Y_{\mathscr{A}}$ by defining the map

$$\Phi_{\mathscr{A}} : T_N \to \mathbb{C}^s$$
$$\Phi_{\mathscr{A}}(t_1, ..., t_n) = (\chi^{m_1}(t_1, ..., t_n), ..., \chi^{m_s}(t_1, ..., t_n))$$

and letting $Y_{\mathscr{A}}$ be the closure of the image of the above map. This will be an affine toric variety with character lattice $\mathbb{Z}\mathscr{A}$.

We can also obtain a projective variety from $\mathscr{A}$ by a similar construction. Let

$$\Psi_{\mathscr{A}} : T_N \to \mathbb{P}^{s-1}$$
$$\Psi_{\mathscr{A}}(t_1, ..., t_n) = (\chi^{m_1}(t_1, ..., t_n), ..., \chi^{m_s}(t_1, ..., t_n))$$

The closure of $\mathrm{im}(\Psi(\mathscr{A}))$ will be a projective variety denoted by $X_{\mathscr{A}}$. The character lattice of this variety will be

$$\mathbb{Z}'\mathscr{A} = \{\sum_{i=1}^{s} a_i m_i | a_i \in \mathbb{Z},\ \sum_{i=1}^{s} a_i = 0\}$$

Example 1.1.5. Let $\mathscr{A} = \{(0,0),(1,0),(2,0),(0,1)\} \subset \mathbb{Z}^2$. Then the induced map is

$$\Psi_{\mathscr{A}} : (\mathbb{C}^*)^2 \to \mathbb{P}^3$$

$$\Psi_{\mathscr{A}}(s,t) = (1 : s : s^2 : t)$$

This corresponds to an affine open subset $\mathrm{Spec}(\mathbb{C}[x,y,z]/(x^2 - y))$ which after homogenizing gives the homogenous coordinate ring $\mathbb{C}[x,y,z,w]/(x^2 - yw)$.

1.2 Cones and toric varieties

We will now see how to construct affine toric varieties in a systematic way. Fix dual lattices $N \simeq M \simeq \mathbb{Z}^n$, which in turn give dual vector spaces $N_{\mathbb{R}} = N \otimes \mathbb{R} \simeq \mathbb{R}^n$ and $M_{\mathbb{R}} = M \otimes \mathbb{R} \simeq \mathbb{R}^n$.

Definition 1.2.1. A convex polyhedral cone in $N_{\mathbb{R}}$ is a set of the form

$$\sigma = \mathrm{Cone}(S) = \{\sum_{u \in S} \lambda_u u | \lambda_u \geq 0\} \subset N_{\mathbb{R}}$$

where $S \subset N_{\mathbb{R}}$ is finite. A convex polyhedral cone is rational if $\sigma = \mathrm{Cone}(S)$ for some $S \subset N$.

Given $m \in M_{\mathbb{R}}$ we can define

$$H_m = \{u \in N_{\mathbb{R}} | \langle m, u \rangle = 0\} \subset N_{\mathbb{R}}$$

$$H_m^+ = \{u \in N_{\mathbb{R}} | \langle m, u \rangle \geq 0\} \subset N_{\mathbb{R}}$$

Given a convex polyhedral cone σ we define H_m to be a supporting hyperplane if $\sigma \subset H_m^+$. If this is the case we call H_m^+ a supporting half-space.

Definition 1.2.2. Given a convex polyhedral cone $\sigma \subset N_{\mathbb{R}}$ we define its dual cone by

$$\sigma^\vee = \{m \in M_{\mathbb{R}} | \langle m, u \rangle \geq 0 \, \forall u \in \sigma\}$$

Remark 1.2.3. From [Ful93, p.11] we have a practical procedure for finding generators of the dual cone of σ: For each set of $n-1$ linearly independent generators of σ, find a vector u annihilating the set. If u or $-u$ is nonnegative on all generators of σ, it is part of a generating set of $\sigma^\vee$, otherwise it is discarded. We will freely use this without further reference.

Definition 1.2.4. A face of a cone σ is a set $\tau = \sigma \cap H_m$ for some $m \in \sigma^\vee$. We write this as $\tau \preceq \sigma$.

A face of a cone is itself a cone. Faces of dimension 0 are called vertices, of dimension 1 edges and of codimension 1 facets.

The dual cone will itself be a convex polyhedral cone in $M_{\mathbb{R}}$. There is a one-to-one inclusion reversing correspondence between faces of σ and faces of $\sigma^{\vee}$. Now, given such a cone σ, the lattice points $S_\sigma = \sigma^{\vee} \cap M \subset M$ form a semigroup. These semigroups will be used to construct toric varieties.

Definition 1.2.5. A convex polyhedral cone σ is strongly convex if $\{0\}$ is a face of σ.

Definition 1.2.6. A semigroup is a set S with an associative binary operation and an identity element.

An affine semigroup is a semigroup such that:

- The binary operation is commutative. We write the operation as $+$ and the identity element as 0. Then a finite set $\mathscr{A} \subset S$ gives
  $\mathbb{N}\mathscr{A} = \{\sum_{m \in \mathscr{A}} a_m m | a_m \in \mathbb{N}\} \subset S$
- The semigroup is finitely generated, meaning there exists a finite $\mathscr{A} \subset S$ such that $\mathbb{N}\mathscr{A} = S$
- The semigroup can be embedded in a lattice M

The key result which will give us toric varieties from cones is the following.

Proposition 1.2.7. *(Gordan's Lemma) For σ a rational polyhedral cone, $S_\sigma = \sigma^{\vee} \cap M$ is finitely generated. Hence S_σ is an affine semigroup.*

Given an affine semigroup $S \subset M$ we can construct an affine toric variety as follows: Let the semigroup algebra $\mathbb{C}[S]$ be defined by

$$\mathbb{C}[S] = \{\sum_{m \in S} c_m \chi^m | c_m \in \mathbb{C},\, c_m = 0 \text{ for all but finitely many } m\}$$

Note that choosing $S = M$ we get the algebra of all Laurent monomials in n variables, thus all such semigroup algebras will be subalgebras of $\mathbb{C}[M]$.

Let $\mathrm{Spec}(\mathbb{C}[S])$ be the affine variety with coordinate ring $\mathbb{C}[S]$. Then [CLS11] shows that

Proposition 1.2.8. $\mathrm{Spec}(\mathbb{C}[S])$ *is an affine toric variety with character lattice $\mathbb{Z}S$. If $S = \mathbb{N}\mathscr{A}$ for a finite set $\mathscr{A} \subset M$, then* $\mathrm{Spec}(\mathbb{C}[S]) = Y_{\mathscr{A}}$

It follows that rational polyhedral cones gives affine toric varieties by $\sigma \mapsto U_\sigma = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M])$. If we also require that σ is strongly convex we get that the torus of U_σ is T_N, or equivalently, that $\dim U_\sigma = n$. Since we are only interested in these cones, we will from now on always mean a strongly convex rational polyhedral cone when we say cone.

Example 1.2.9. If $\sigma = \mathrm{Cone}(\{0\})$ then $\sigma^\vee = \mathrm{Cone}(\pm e_1, ..., \pm e_n)$ which gives $U_\sigma \cong (\mathbb{C}^*)^n$.

Example 1.2.10. If $\sigma = \mathrm{Cone}(e_1, ..., e_n)$ then $\sigma^\vee = \sigma$ so $U_\sigma = \mathbb{C}^n$.

One of the reasons for studying toric geometry is that many properties of varieties can be checked combinatorially in the lattices M or N.

Definition 1.2.11. Given an edge of a cone $\sigma \subset N_\mathbb{R}$, the semigroup $N \cap \sigma$ is generated by a unique element called the minimal generator of the edge.

A cone σ is called smooth if the minimal generators of its edges form a subset of a $\mathbb{Z}$-basis for N.

For a n-dimensional cone being smooth is, by Remark A.0.4, equivalent to the determinant of the minimal generators being 1, and this generalizes to arbitrary cones, were we take the determinant in the lattice spanned by $\sigma \cap N$. We say that a cone has multiplicity k if the determinant of its minimal generators equals k. Hence σ is smooth if and only if its multiplicity equals 1.

Not surprisingly this definition is chosen to obtain the following characterization.

Proposition 1.2.12. *Given any cone σ, the associated toric variety U_σ is smooth if and only if σ is smooth.*

The Hilbert basis $\mathcal{H}(S_\sigma)$ of the affine semigroup S_σ is the unique minimal set of generators for S_σ as a semigroup. Thus $\mathbb{C}(S_\sigma)$ will be generated by $\mathcal{H}(S_\sigma) = \{m_1, ..., m_s\}$ as a $\mathbb{C}$-algebra. Define

$$\mathbb{Z}^s \to M$$

$$e_i \mapsto m_i,$$

this map will have a kernel K, which records all linear relations among $\{m_1, ..., m_s\}$. Define the ideal $I_K \subset \mathbb{C}[x_1, ..., x_s]$ by

$$I_K = \langle x_1^{a_1} \cdots x_s^{a_s} - x_1^{b_1} \cdots x_s^{b_s} | a = (a_1, ..., a_s), b = (b_1, ..., b_s) \in \mathbb{N}^s, a - b \in K \rangle$$

Then $U_\sigma = \mathrm{Spec}\, \mathbb{C}(S_\sigma) = \mathrm{Spec}\, \mathbb{C}[x_1, ..., x_s]/I_K$. In other words, the ideal of a toric variety is generated by binomials.

Definition 1.2.13. A cone is simplicial if its generators are linearly independent over $\mathbb{R}$.

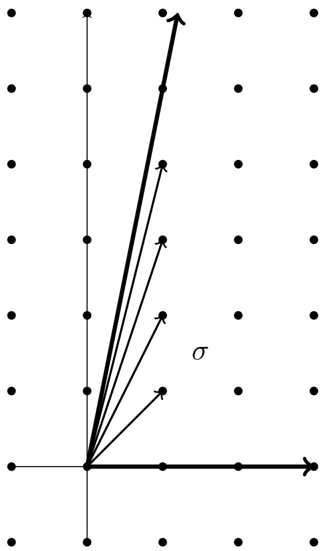

Figure 1.1: Hilbert basis for $\sigma = \text{Cone}((1,0),(1,5))$. $\sigma \cap N$ is generated by 6 elements as a semigroup

1.3 Fans and toric varieties

Definition 1.3.1. A fan Σ in a vector space $N_{\mathbb{R}}$ is a finite collection of cones satisfying:

For all $\sigma \in \Sigma$ each face of σ is also in Σ.

For all $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of each.

Given a fan Σ denote by $\Sigma(d)$ the set of d-dimensional cones in Σ.

We will show that from a fan one can construct a general, not necessarily affine, toric variety, but first we need some more results from semigroup theory.

Proposition 1.3.2. *Take σ a cone and $u \in S_\sigma = \sigma^\vee \cap M$. Then $\tau = \sigma \cap u^\perp = \{v \in \sigma | \langle u, v\rangle = 0\}$ is a rational convex polyhedral cone. All faces of σ have this form, and $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-u)$.*

Proposition 1.3.3. *If σ and τ are cones which intersect in a common face $\sigma \cap \tau$, then $S_{\sigma\cap\tau} = S_\sigma + S_\tau$.*

Using this we get the key to constucting our toric varieties. Recall (see for instance [Har77, II.2]that any affine scheme $\text{Spec}(A)$ has a basis for its topology consisting of the sets $D(f) = \text{Spec}(A) \setminus V(f)$, $f \in A$. These are called principal open subsets.

Proposition 1.3.4. *If τ is a face of σ then we get an inlcusion $U_\tau \to U_\sigma$ which embeds U_τ as a principal open subset of U_σ.*

Proof. By Proposition 1.3.2 any basis element of $C[S_\tau]$ is of the form $\chi^{w-nu} = \frac{\chi^w}{\chi^{un}}$ for $w \in S_\sigma$ and $u \in S_\sigma$ with $\tau = \sigma \cap u^\perp$. Thus $C[S_\tau] = C[(S_\sigma)]_{\chi^u}$ which corresponds to an embedding of the principal open subset $D(\chi^u)$ by applying the Spec functor. □

Now given a fan Σ we can construct an associated toric variety X_Σ. Take the disjoint union of the affine varieties U_σ for all $\sigma \in \Sigma$. Glue them along all common intersections, the above ensures the glueing conditions are satsified. By Proposition 1.3.3 we can show that X_Σ is separated. In fact all normal, separated toric varieties are of this form. In the literatue one often requires a toric variety to be normal and separated, and since all varieties we will study are of this form, we will adopt this convention. Hence any toric variety is isomorphic to X_Σ for some fan Σ.

Proposition 1.3.5. *X_Σ is smooth if and only if each cone $\sigma \in \Sigma$ is smooth.*

Proof. This follows from Proposition 1.2.12 and the fact that smoothness is defined locally. □

Example 1.3.6. Let $N = \mathbb{Z}^n$ with standard basis $e_1, ..., e_n$. Let $e_0 = -e_1 - e_2 - ... - e_n$. Let Σ be the fan consisting of all proper subsets of $\{e_0, ..., e_n\}$. The maximal cones are $\sigma_i = \mathrm{Cone}(e_0, ..., \hat{e_i}, ...e_n)$. Calculating the dual cones we get

$$\sigma_0^\vee = \mathrm{Cone}(e_1, ..., e_n)$$

$$\sigma_i^\vee = \mathrm{Cone}(e_1 - e_i, e_2 - e_i, ..., -e_i, ..., e_n - e_i), i \neq 0$$

$$U_{\sigma_0} = \mathrm{Spec}\,\mathbb{C}[x_1, ..., x_n]$$

$$U_{\sigma_i} = \mathrm{Spec}\,\mathbb{C}[\frac{x_1}{x_i}, ..., \frac{1}{x_i}, ..., \frac{x_n}{x_i}]$$

For homogenous coordinates $(t_0 : ... : t_n)$ on $\mathbb{P}^n$, set $x_j = \frac{t_j}{t_0}$ we see that the U_{σ_i} corresponds to the normal open affine cover of $\mathbb{P}^n$ by copies of $\mathbb{A}^n$. Thus $X_\Sigma \cong \mathbb{P}^n$.

Example 1.3.7. Given natural numbers $q_0, ..., q_n$ with $\gcd(q_0, ..., q_n) = 1$, consider the quotient lattice $\mathbb{Z}^{n+1}$ by the subgroup generated by $(q_0, ..., q_n)$, we write $N = \mathbb{Z}^{n+1}/\mathbb{Z}(q_0, ..., q_n)$. Let u_i for $i = 0, ..., n$ be the images in N of the standard basis vectors of $\mathbb{Z}^{n+1}$. This means that in N we have

$$q_0 u_0 + ... + q_n u_n = 0$$

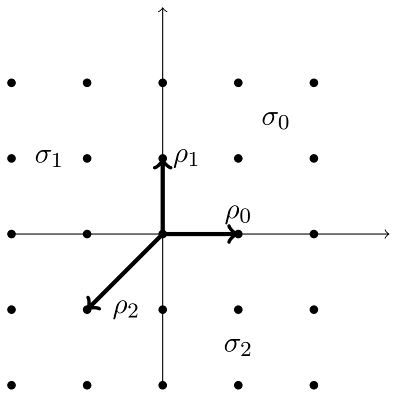

Figure 1.2: Fan for $\mathbb{P}^2$. The 1-dimensional cones are generated by ρ_i. The two-dimensional cones are σ_i.

Let Σ be the fan consisting of all cones generated by proper subsets of $\{u_0, ..., u_n\}$. We call X_Σ a weighted projective space with respect to the weights $(q_0, ..., q_n)$, we write this $\mathbb{P}(q_0, ..., q_n)$. Observe that $\mathbb{P}^n \simeq \mathbb{P}(1, 1, ..., 1)$. These will be important examples for us.

A variety is said to be complete if it is compact in the Euclidean topology. In the toric case we have very nice criterion for checking if a variety is complete. For a fan Σ let its support, $|\Sigma|$, be the union (in $N_\mathbb{R}$) of all cones in Σ. Then we have:

Proposition 1.3.8. *[Ful93, chp. 2.4] A toric variety X_Σ is complete if and only if $|\Sigma| = N_\mathbb{R}$.*

In that case we say that Σ is a complete fan.

Definition 1.3.9. A fan Σ is simplicial if every cone $\sigma \in \Sigma$ is simplicial. We say that X_Σ is simplicial if Σ is simplicial.

It turns out being simplicial is euqivalent to having at most finite quotient singularities. This notion will appear later.

1.4 Polytopes and toric varieties

Now that we have constructed general toric varieties from fans, we will consider another way to get a toric variety, via polytopes. This will only be the varieties $X_\mathscr{A}$ we have seen before, where $\mathscr{A}$ are all lattice points contained in a polytope.

Definition 1.4.1. A polytope in $M_{\mathbb{R}}$ is a set of the form

$$P = \mathrm{Conv}(S) = \{\sum_{u \in S} \lambda_u u | \lambda_u \geq 0, \sum_{u \in S} \lambda_u = 1\} \subset M_{\mathbb{R}}$$

where $S \subset M_{\mathbb{R}}$ is finite.

A polytope is a lattice polytope if it equals $\mathrm{Conv}(S)$ for some $S \subset M$. We will only be interested in lattice polytopes, so we adopt the convention that whenever we write polytope, we mean a lattice polytope.

The dimension of a polytope is the dimension of the smallest affine subspace of $M_{\mathbb{R}}$ containing P.

Given a nonzero vector $u \in N_{\mathbb{R}}$ and $b \in \mathbb{R}$ we can define the affine hyperplane $H_{u,b}$ and closed half-space $H_{u,b}^{+}$ by

$$H_{u,b} = \{m \in M_{\mathbb{R}} | \langle m, u \rangle = b\}$$

$$H_{u,b}^{+} = \{m \in M_{\mathbb{R}} | \langle m, u \rangle \geq b\}$$

Definition 1.4.2. A subset $Q \subset P$ is a face of P if there is $u \in N_{\mathbb{R}} \setminus \{0\}$ and $b \in \mathbb{R}$ such that

$$Q = H_{u,b} \cap P$$

$$P \subset H_{u,b}^{+}$$

We write $Q \preceq P$ and say that $H_{u,b}$ is a supporting hyperplane of P. The dimension of Q is the dimension of the smallest affine subspace of $N_{\mathbb{R}}$ containing Q.

Vertices of a polytope P are faces of dimension 0, edges of dimension 1 and facets of codimension 1.

Any polytope may be written as a finite intersection of closed half-spaces. When it is full-dimensional we get a unique half-space for each facet F of P,

$$H_F^{+} = \{m \in M_{\mathbb{R}} | \langle m, u_F \rangle \geq -a_F\},$$

where $(u_F, a_F) \in N_{\mathbb{R}} \times \mathbb{R}$ is unique up to multiplication by a positive real number. If we choose u_F to be the unique minimal generator of the facet normal, we get a unique facet presentation.

Now given a polytope P we get an associated toric variety $X_{\mathscr{A}}$ by letting $\mathscr{A}$ be the points contained in $P \cap M$. This is not necessarily normal (meaning all local rings are integrally closed), which we usually want, so we define the following.

Definition 1.4.3. An affine semigroup $S \subset M$ is saturated if for all $k \in \mathbb{N} \setminus \{0\}$ and $m \in M$, $km \in S$ implies $m \in S$.

A polytope $P \subset M_{\mathbb{R}}$ is very ample if for every vertex $m \in P$, the semigroup $\mathbb{N}(P \cap M - m)$ is saturated in M.

If the polytope is very ample, it turns out that the variety is normal. It is shown in [EW91] that any full dimensional polytope has an integer multiple which is very ample. Then we define the toric variety associated to a polytope P as $X_{\mathscr{A}}$ where $\mathscr{A} = kP \cap M$ for any k such that kP is very ample. We will see later that this relates to a certain divisor being very ample. Denote this variety by X_P.

Example 1.4.4. Consider in $M = \mathbb{Z}^2$ the polytope $\Delta_2 = \mathrm{Conv}(0, e_1, e_2)$. This gives the affine map $(x, y) \mapsto (1, x, y)$, hence the closure X_{Δ_2} will be $\mathbb{P}^2$. If we instead consider $k\Delta_2 = \mathrm{Conv}(0, ke_1, ke_2)$ we will again obtain $\mathbb{P}^2$, but embedded differently into a bigger space by the Veronese-embedding of degree k.

In general, the standard n-simplex $\Delta_n = \mathrm{Conv}(0, e_1, ..., e_n) \subset \mathbb{Z}^n$ will give $X_{\Delta_n} = \mathbb{P}^n$, while multiplying the polytope with an integer corresponds to different embeddings of $\mathbb{P}^n$ into bigger projective spaces. The same phenomena happens for any very ample polytope.

We can also construct a fan associated to a full dimensional polytope P, called the normal fan of P. Let the facet presentation of P be given as

$$\{m \in M_{\mathbb{R}} | \langle m, u_F \rangle \geq -a_F \ F \text{ is a facet of } P\}$$

To each vertex $v \in P$ ve can define the cone $C_v = \mathrm{Cone}(P \cap M - v) \subset M_{\mathbb{R}}$. This gives a dual cone $\sigma_v = C_v^\vee \subset N_{\mathbb{R}}$. For a face $Q \preceq P$ containing v, we get a cone $Q_v \subset C_v$. This is in fact a bijective inclusion preserving correspondence via the maps

$$Q \mapsto Q_v = \mathrm{Cone}(Q \cap M - v)$$

$$Q_v \mapsto Q = (Q_v + v) \cap P$$

In particular we have the equality $\sigma_v = \mathrm{Cone}(u_F|$ facets F containing $v)$.

Generalising this to any face $Q \preceq P$, set $\sigma_Q = \mathrm{Cone}(u_F|$ facets F containing $Q)$. The collection $\{\sigma_Q | Q \preceq P\}$ turns out be our desired fan Σ_P. When P is very ample we have $X_P = X_{\Sigma_P}$.

Example 1.4.5. Consider again $\Delta_2 = \mathrm{Conv}(0, e_1, e_2) \in \mathbb{Z}^2$. We see that $C_0 = \mathrm{Cone}(e_1, e_2)$, $C_{e_1} = \mathrm{Cone}(e_2, -e_1 - e_2)$ and $C_{e_2} = \mathrm{Cone}(e_1, e_1 - e_2)$.

Calculating the dual cones we get $\sigma_0 = \text{Cone}(e_1, e_2)$, $\sigma_{e_1} = \text{Cone}(e_2, -e_1 - e_2)$ and $\sigma_{e_2} = \text{Cone}(e_1, -e_1 - e_2)$. We recognize this as the fan from Example 1.3.6 as expected.

Definition 1.4.6. Let $P \subset M_\mathbb{R}$ be a polytope. Given a vertex, consider the set of all minimal generators of the edges emanating from v. If these form a subset of a $\mathbb{Z}$-basis for M then the corresponding vertex is smooth. P is smooth if all vertices are smooth.

Again this definition fits with the other ones.

Proposition 1.4.7. *For a full dimensional polytope P, the toric variety X_P is smooth if and only if P is a smooth polytope.*

Proof. The normal fan of P has maximal cones generated by, for each vertex v, the minimal generators emanating from v. Thus, for each vertex v we need the cone C_v to be smooth. But C_v is smooth if and only if its dual σ_v is smooth, since if a maximal cone σ is smooth, we can choose a basis for the lattice $e_1, ...e_n$ such that $\sigma = \text{Cone}(e_1, ...e_n)$. But then it is self-dual, so the dual is smooth as well. But C_v we know to be smooth if and only if the generators are subset of a $\mathbb{Z}$-basis. $\square$

1.5 Toric morphisms

Assume we have a $\mathbb{Z}$-linear map of lattices $\bar{\phi} : N_1 \to N_2$ and cones $\sigma_1 \in (N_1)_\mathbb{R}$, $\sigma_2 \in (N_2)_\mathbb{R}$ such that $\bar{\phi}_\mathbb{R}(\sigma_1) \subset \sigma_2$. Then we get an induced morphism

$$\bar{\phi}^\vee : M_2 \to M_1$$

which in turn induces a morphism

$$\mathbb{C}[\sigma_2^\vee \cap M_2] \to \mathbb{C}[\sigma_1^\vee \cap M_1]$$

$$\sum c_i \chi^{m_i} \mapsto \sum c_i \chi^{\bar{\phi}^\vee(m_i)}$$

that induces a map

$$\text{Spec}(\mathbb{C}[\sigma_1^\vee \cap M_1]) = U_{\sigma_1} \to U_{\sigma_2} = \text{Spec}(\mathbb{C}[\sigma_2^\vee \cap M_2])$$

Definition 1.5.1. Let N_1, N_2 be lattices, Σ_1 be a fan in $(N_1)_\mathbb{R}$, Σ_2 a fan in $(N_2)_\mathbb{R}$. A morphism $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ is toric if it maps the torus T_{N_1} into the torus T_{N_2} and $\bar{\phi}|_{T_{N_1}}$ is a group homomorphism.

Definition 1.5.2. A $\mathbb{Z}$-linear map $\bar{\phi} : N_1 \to N_2$ is compatible with the fans Σ_1 and Σ_2 if for every cone $\sigma_1 \in \Sigma_1$ there exists $\sigma_2 \in \Sigma_2$ such that $\phi_{\mathbb{R}}(\sigma_1) \subset \sigma_2$, where $\phi_{\mathbb{R}}$ is the induced map $N_1 \otimes \mathbb{R} \to N_2 \otimes \mathbb{R}$.

By the remarks above, a compatible $\bar{\phi}$ induces maps $U_{\sigma_1} \to U_{\sigma_2}$ for all $\sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2$. It turns out these glue to a morphism $X_{\Sigma_1} \to X_{\Sigma_2}$. In fact we have the following characterization:

Theorem 1.5.3 (Thm 3.3.4 [CLS11])**.** *A $\mathbb{Z}$-linear map $\bar{\phi} : N_1 \to N_2$ compatible with the fans Σ_1 and Σ_2 induces a toric morphism $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$.*

Conversely a toric morphism $X_{\Sigma_1} \to X_{\Sigma_2}$ induces a $\mathbb{Z}$-linear map $\bar{\phi} : N_1 \to N_2$ which is compatible with Σ_1 and Σ_2.

Example 1.5.4. The map $\mathbb{A}^2 \to \mathbb{P}^2$ given by $(x, y) \mapsto (1, x, y)$ is a toric morphism induced by the identity map $\mathbb{Z}^2 \to \mathbb{Z}^2$.

1.6 The orbit-cone correspondence

Another well known fact about toric varieties is that one has a bijective dimension-reversing correspondence between the cones $\sigma \in \Sigma$ and the orbits under the torus action. More precisely:

Theorem 1.6.1. *[CLS11, Thm. 3.2.6] Given a toric variety X_Σ coming from a fan Σ in $N_{\mathbb{R}}$ we have the following:*

There is a $1-1$-correspondence between cones $\sigma \in \Sigma$ and orbits under the group action by T_N given by

$$\sigma \mapsto O(\sigma) = T_{N(\sigma)}$$

where $N(\sigma) = N/N_\sigma$ and N_σ is the lattice spanned by $\sigma \cap N$.

Let $n = \dim N$. Then $\dim(O(\sigma)) = n - \dim(\sigma)$.

For a cone $\sigma \in \Sigma$ we have

$$U_\sigma = \cup_{\tau \preceq \sigma} O(\tau)$$

The closure $\overline{O(\tau)}$ of an orbit is given by

$$\overline{O(\tau)} = \cup_{\tau \preceq \sigma} O(\sigma)$$

Example 1.6.2. Consider $\mathbb{P}^2$ with coordinates $(t_0 : t_1 : t_2)$. The torus $(\mathbb{C}^*)^2$ are the points $(1, a, b)$, with $a, b \neq 0$. Under this action there are 7 orbits: $O_i = \{t_i \neq 0, t_j = 0, j \neq i\}$, $O_{ij} = \{t_i, t_j \neq 0, t_k = 0\}$, $O_{012} = \{t_0, t_1, t_2 \neq 0\}$. Consider the fan for $\mathbb{P}^2$ with cones generated by proper subsets of $\{e_1, e_2, e_0 = -e_1 - e_2\}$. With the notation as in Example 1.3.6 we get the correspondence

$$O_0 \leftrightarrow \mathrm{Cone}(e_1, e_2)$$

$$O_1 \leftrightarrow \mathrm{Cone}(e_0, e_2)$$

$$O_2 \leftrightarrow \mathrm{Cone}(e_0, e_1)$$

$$O_{01} \leftrightarrow \mathrm{Cone}(e_2)$$

$$O_{02} \leftrightarrow \mathrm{Cone}(e_1)$$

$$O_{12} \leftrightarrow \mathrm{Cone}(e_0)$$

$$O_{012} \leftrightarrow \mathrm{Cone}(\{0\})$$

Remark 1.6.3. It turns out the orbit closures $\overline{O(\tau)}$ are themselves toric varieties, constructed from a fan the following way: For a cone σ containing τ consider its image $\overline{\sigma}$ in $N(\tau)_\mathbb{R}$. Then

$$\mathrm{Star}(\tau) = \{\overline{\sigma} \subset N(\tau)_\mathbb{R} | \tau \preceq \sigma \in \Sigma\}$$

is a fan in $N(\tau)_\mathbb{R}$ and $X_{\mathrm{Star}(\tau)} \cong \overline{O(\tau)}$.

Example 1.6.4. Consider the fan Σ_1 with 2-dimensional cones $\mathrm{Cone}(e_1, e_1 + e_2)$ and $\mathrm{Cone}(e_2, e_1 + e_2)$ and their faces. Let Σ_2 be the fan for $\mathbb{C}^2$ given by $\mathrm{Cone}(e_1, e_2)$ and its faces. The identity mapping $\mathbb{Z} \to \mathbb{Z}$ is compatible with the fans, hence it induces a map $X_{\Sigma_1} \to X_{\Sigma_2} = \mathbb{C}^2$.

By the orbit-cone correspondence the 1-dimensional cone σ_1 generated by $e_1 + e_2$ corresponds to an orbit, whose closure is a divisor D isomorphic to $\mathrm{Star}(\sigma_1)$. This is the fan of $\mathbb{P}^1$: For instance choose $v_1 = (1, 0), v_2 = (1, 1)$ as basis for $\mathbb{Z}^2$ In this basis, the cones containing σ_1 will be $\mathrm{Cone}(v_1, v_2)$, $\mathrm{Cone}(v_2 - v_1, v_2)$ and $\mathrm{Cone}(v_2)$. The quotient lattice N_{σ_1} is generated by v_1, so the images of these cones will be $\mathrm{Cone}(v_1)$, $\mathrm{Cone}(-v_1)$ and $\mathrm{Cone}(\{0\})$ which we recongize from Example 1.3.6 as the fan for $\mathbb{P}^1$.

By removing all cones containing σ_1 from Σ_1 we see that $X_{\Sigma_1} \setminus D$ is isomorphic to $\mathbb{C}^2 \setminus \{0\}$. Hence X_{Σ_1} is the classical blowup of $\mathbb{C}^2$ at 0, which can also be checked by considering coordinate rings of affine charts.

In general the blowup $\mathrm{Bl}_0(\mathbb{C}^n)$ is the subvariety of $\mathbb{P}^{n-1} \times \mathbb{C}^n$ defined by $V(x_i y_j - x_j y_i | 1 \leq i < j \leq n)$ for coordinates $x_1, ..., x_n$ on $\mathbb{P}^n$ and $y_1, ..., y_n$

on $\mathbb{C}^n$. In the toric case we can generalize this as above, the fan for $\mathbb{C}^n$ is $\text{Cone}(e_1, ..., e_n)$ and its faces. Create a new fan Σ by adding the 1-dimensional cone $e_0 = e_1 + e_2 + ... + e_n$ and let Σ consist of all cones generated by proper subsets of $\{e_0, e_1, ..., e_n\}$. By checking on coordinate rings we get that X_Σ equals $\text{Bl}_0(\mathbb{C}^n)$.

1.7 Divisors on toric varieties

We will look at the concepts of divisors on toric varieties. Let $\text{Div}(X)$ be the group of Weil divisors on X and let $\text{Div}_0(X)$ be the set of principal divisors, that is divisors of the form $\text{div}(f)$ for some $f \in \mathbb{C}(X)^*$. The class group of X is defined as $\text{Cl}(X) = \text{Div}(X)/\text{Div}_0(X)$. We define Cartier divisors as follows.

Definition 1.7.1. A Weil divisor D on X is called Cartier if there exists an open cover $\{U_i\}$ and $f_i \in \mathbb{C}(U_i)$ such that $D|_{U_i} = \text{div}(f_i)$. The set of Cartier divisors will be denoted by $\text{CDiv}(X)$.

The Picard group of X is defined as $\text{Pic}(X) = \text{CDiv}(X)/\text{Div}_0(X)$.

Now let X_Σ be the toric variety associated to a fan Σ in $N_\mathbb{R}$. The $n-k$-dimensional orbits of the torus action correspond to k-dimensional cones of Σ. Thus for each 1-dimensional cone $\rho \in \Sigma$ we get a corresponding codimension 1 orbit, whose closure is a divisor invariant under the torus action, denoted by D_ρ. Letting $u_\rho \in N_\mathbb{R}$ be a minimal generator of ρ , one can compute that for any character χ^m, its divisor is given by

$$\text{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho$$

Using this we can compute the class and Picard groups by the following exact sequences.

Proposition 1.7.2. *Let* $\text{Div}_{T_N}(X_\Sigma) = \bigoplus \mathbb{Z} D_\rho \subset \text{Div}(X_\Sigma)$. *Then the following sequence is exact*

$$M \to \text{Div}_{T_N}(X_\Sigma) \to \text{Cl}(X_\Sigma) \to 0$$

where the first map is $m \mapsto \text{div}(\chi^m)$ *and the second sends an element of* $\text{Div}_{T_N}(X_\Sigma)$ *to its equivalence class in* $\text{Cl}(X_\Sigma)$. *The sequence is left exact if and only if* $\{u_\rho\}$ *spans* $N_\mathbb{R}$.

For Cartier divisors one obtains a similar exact sequence

$$M \to \text{CDiv}_{T_N}(X_\Sigma) \to \text{Pic}(X_\Sigma) \to 0$$

where $\text{CDiv}_{T_N}(X_\Sigma)$ *is the group of* T_N*-invariant Cartier divisors.*

Thus we see that the divisors invariant under the torus action determine these important groups.

Proposition 1.7.3. *[CLS11, Prop. 4.2.2] Let σ be a cone. Then any T_N-invariant Cartier divisor on U_σ is the divisor of a character $\chi^u \in M$.*

One is often interested in when a Weil divisor is Cartier. We present an example followed by a more general characterization.

Example 1.7.4. Take $\sigma = \text{Cone}((2,-1),(-1,2))$. Then a Weil divisor $aD_1 + bD_2$ is Cartier if and only if it equals $\text{div}(\chi^u)$ for some $u \in M$. This amounts to there existing $u = (p,q)$ such that

$$\text{div}(\chi^u) = (2p-q)D_1 + (2q-p)D_2$$

Solving for p and q we get

$$p = \frac{2a+b}{3} \text{ and } q = \frac{a+2b}{3},$$

which have solutions if and only if $a \equiv b \pmod 3$.

Proposition 1.7.5. *[Ful93, Exc. Ch. 3.3] Let $D = \sum_\rho a_\rho D_\rho$. Then D is Cartier if and only if for each maximal cone $\sigma \in \Sigma$ there is $m_\sigma \in M$ with $\langle m_\sigma, v_\rho \rangle = -a_\rho$ for all $\rho \in \sigma(1)$, where v_ρ is the minimal generator of ρ. We call $\{m_\sigma\}$ the Cartier data of D.*

Proof. We proceed exactly as in the example above. D is Cartier on a maximal cone σ if and only if it equals $\text{div}(\chi^u)$ for some $u \in M$. That is if

$$\text{div}(\chi^u) = \sum_{i=1}^{n} \langle v_\rho, u \rangle D_\rho = \sum_\rho a_\rho D_\rho$$

In other words, if $\langle v_\rho, u \rangle = a_\rho$ for all $\rho \in \sigma(1)$. To be consistent with the literature we pick $m_\sigma = -u$ to get the minus sign. □

Given a full dimensional polytope $P \subset M_{\mathbb{R}}$ we get an induced divisor D_P defined as follows. Let the facet presentation of P be given as

$$\{m \in M_{\mathbb{R}} | \langle m, u_F \rangle \geq -a_F \text{ —} F \text{ is a facet of } P\}$$

A facet F of the polytope correponds to a $n-1$-dimensional face of a cone $\sigma^\vee$ which in turns corresponds to a 1-dimensional cone σ, which gives the divisor D_σ, here denoted by D_F. Define $D_P = \sum a_F D_F$. This will always be an ample Cartier divisor. We have

Theorem 1.7.6. *[CLS11, Thm. 6.2.1]*

There is a one-to-one correspondence between the following sets

$$\{P \subset M_{\mathbb{R}} | P \text{ is a full dimensional polytope}\}$$

$$\{(X_\Sigma, D) | \Sigma \text{ complete fan} \subset N_{\mathbb{R}} \text{ , } D \text{ is a torus-invariant ample divisor}\}$$

The first map sends P to (X_{Σ_P}, D_P).

The second map sends X_Σ and $D = \sum a_\rho D_\rho$ to

$$P_D = \{m \in M_{\mathbb{R}} | \langle m, u_\rho \rangle \geq -a_\rho \text{ for all } \rho \in \Sigma(1)\}$$

P is a very ample polytope if and only if D_P is a very ample divisor. Different multiples lP correspond to different divisors lD_P which in turn gives different embeddings of the variety in projective spaces.

1.8 Intersections of divisors

Given a divisor D on X_Σ one can associate a sheaf $\mathcal{O}_{X_\Sigma}(D)$ defined by

$$\mathcal{O}_{X_\Sigma}(D)(U) = \{f \in \mathbb{C}(X_\Sigma)^* | \operatorname{div}(f)|_U + D|_U \geq 0\} \cup \{0\}$$

The global sections of this sheaf is described in terms of the lattice as follows:

$$\Gamma(X_\Sigma, \mathcal{O}_{X_\Sigma}(D)) = \bigoplus_{\operatorname{div}(\chi^m)+D\geq 0} \mathbb{C} \cdot \chi^m$$

We now wish to define an intersection product on our varieties, we follow the presentation in [CLS11, ch. 6]. Given a smooth complete irreducible curve C on a variety X, one has that any divisor D on C is a weighted sum of points $D = \sum a_i P_i$, $a_i \in \mathbb{Z}$, $P_i \in C$. Thus we can define the degree of D as $\deg D = \sum a_i$.

For general, non-smooth curves C we do not necesarily have this property, however we will consider the normalization $\overline{C}$ of the curve C which is a map

$$\phi : \overline{C} \to C$$

such that $\overline{C}$ is normal. It turns out $\overline{C}$ is smooth, hence we can define the degree of a divisor: For a divisor D on X, consider the composed map $f : \overline{C} \to X$. Define $C \cdot D = \deg(f^* D)$.

In nice cases this behaves as one would expect of an interesection product, i.e. if D and C intersect transversally, we have $C \cdot D = |C \cap D|$. We also have that the intersection product has the following properties:

$$C \cdot (D + E) = C \cdot D + C \cdot E$$

$$C \cdot D = C \cdot E \text{ when } D \text{ is linearly equivalent to } E$$

Repeatedly applying the first also shows that

$$(kC) \cdot D = k(C \cdot D) \text{ when } k \in \mathbb{Z}$$

As usual, in the toric case there are quite explicit ways of computing intersection products. In particular we will use the following result

Proposition 1.8.1. *[CLS11, Prop. 6.3.8] Let $C = \overline{O(\tau)}$ be a complete torus-invariant curve in X_Σ, where $\tau = \sigma \cap \sigma' \in \Sigma(n-1)$ for $\sigma, \sigma' \in \Sigma(n)$. Let D be a Cartier divisor and let $m_\sigma, m_{\sigma'}$ be Cartier data corresponding to σ, σ'. Pick $u \in \sigma' \cap N$ which maps to the minimal generator of the quotient $(N/\operatorname{Span}(\tau) \cap N)_\mathbb{R}$. Then*

$$D \cdot C = \langle m_\sigma - m_{\sigma'}, u \rangle$$

For simplicial toric varieties, every Weil-divisor has an integer multiple which is Cartier (they are called $\mathbb{Q}$-Cartier). Any toric surface will by simplicial, hence we have that for any Weil divisor D and curve C one can define $D \cdot C = \frac{1}{l}(lD) \cdot C \in \mathbb{Q}$. One can check that the propositions above generalizes to $\mathbb{Q} - Cartier$ divisors, i.e. one obtains Cartier data $m_\sigma \in M_\mathbb{Q}$. The concept of pullbacks of divisors also generalizes to $\mathbb{Q}$-Cartier divisor, and by reformulating [CLS11, Prop 6.2.7] we get the following result.

Proposition 1.8.2. *Given a toric morphism of simplicial toric vareties $\phi : X_{\Sigma'} \to X_\Sigma$, let $\Sigma(1) = \{\sigma_1, ..., \sigma_s\}$ and $\Sigma'(1) = \{\tau_1, ..., \tau_r\}$ and let $D_1, ..., D_s$, $E_1, ..., E_r$ be the corresponding torus-invariant divisors. Let $u_1, ..., u_r$ be the minimal generators of $\tau_1, ..., \tau_r$. Then*

$$\phi^*(\sum_{i=1}^{s} a_i D_i) = \sum_{j=1}^{r} -\langle m_{\sigma_j}, \phi(u_j) \rangle E_j$$

where m_{σ_j} is $\mathbb{Q}$-Cartier data of the maximal cone σ_j such that $\phi(\tau_j) \subset \sigma_j$.

Inspired by the calculations in the appendix of [LO14] we get the following result.

Proposition 1.8.3. *Given a two-dimensional toric variety, let $\rho_0, ..., \rho_{n-1}$ be the 1-dimensional cones of the normal fan, and $D_0, ..., D_{n-1}$ be the prime torus-invariant divisors. Let $d_{i,i+1}$ be the determinant of the matrix with columns minimal generators of ρ_i, ρ_{i+1}. Let d_i be determinant of the matrix formed by ρ_{i-1}, ρ_{i+1} (take indices modulo n). Then*

$$D_i \cdot D_j = \begin{cases} -\frac{d_i}{d_{i-1,i}d_{i,i+1}} & \text{if } j = i \\ \frac{1}{d_{i,j}} & \text{if } j = i+1 \\ \frac{1}{d_{j,i}} & \text{if } j = i-1 \\ 0 & \text{else} \end{cases}$$

Proof. Let $\sigma_i = \text{Cone}(\rho_i, \rho_{i+1})$ be the maximal cones of Σ. Let u_i be the minimal generator of ρ_i. Assume without loss of generality that $\rho_1 = \text{Cone}(e_1)$. We wish to find the intersections for D_1. To find $D_1 \cdot D_1$, observe that there exists Cartier data $m_{\sigma_0}, m_{\sigma_1} \in M_{\mathbb{Q}}$ such that

$$\langle m_{\sigma_0}, u_0 \rangle = 0$$

$$\langle m_{\sigma_0}, u_1 \rangle = -1$$

$$\langle m_{\sigma_1}, u_1 \rangle = -1$$

$$\langle m_{\sigma_1}, u_2 \rangle = 0$$

Letting $m_{\sigma_0} = (x, y), m_{\sigma_1} = (u, v), u_0 = (a, b), u_2 = (c, d)$ we get

$$ax + by = 0$$

$$x = -1$$

$$u = -1$$

$$uc + vd = 0$$

Solving we get $y = \frac{a}{b}$, $v = \frac{c}{d}$

Now since $N_{\rho_1} = N/(\rho_1 \cap N)$ are just the lattice points on the y-axis, a point of σ_1 mapping to the minimal generator of N_{ρ_1} will be of the form $(l, 1)$ for some l. We have that $m_{\sigma_1} - m_{\sigma_2} = (0, \frac{a}{b} - \frac{c}{d})$, so we get $D_1^2 = \langle (0, \frac{a}{b} - \frac{c}{d}), (l, 1) \rangle = \frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd} = -\frac{d_1}{d_{0,1}d_{1,2}}$.

For D_2 there also exist Cartier data corresponding to σ_0, σ_1, let these by abuse of notation be denoted $(x, y), (u, v)$. Then one gets the equations

$$ax + by = 0$$

$$x = 0$$

$$u = 0$$
$$uc + vd = -1$$

Solving yields $v = -\frac{1}{d}$

Then $D_1 \cdot D_2 = \langle (0, \frac{1}{d}), (l, 1) \rangle = \frac{1}{d} = \frac{1}{d_{1,2}}$

Similarly $D_1 \cdot D_0 = \frac{1}{-b} = \frac{1}{d_{0,1}}$

For any $i \neq 0, 1, 2$ we get $x = y = u = v = 0$, hence $D_1 \cdot D_i = 0$. Doing this computation for all D_i yields the result.

□

Given any normal variety X, there is an associated canonical sheaf, constructed as $w_X = \widehat{\Omega}^n$, that is the n-th exterior product of the pushforward of the sheaf of Kähler differentials on the smooth locus of X. This sheaf will be isomorphic to $\mathcal{O}(K_X)$ for some Weil divisor K_X. In the toric case one can choose $K_{X_\Sigma} = \sum_\rho -D_\rho$ where D_ρ are all torus-invariant prime divisors. As a corollary of the above we obtain:

Corollary 1.8.4. *Given a two-dimensional toric variety, let $\rho_0, ..., \rho_{n-1}$ be the 1-dimensional cones of the normal fan. Let $d_{i,i+1}$ be the determinant of the matrix with columns minimal generators of ρ_i, ρ_{i+1}. Let d_i be determinant of the matrix formed by ρ_{i-1}, ρ_{i+1} (take indices modulo n). Then*

$$K_{X_\Sigma}^2 = K_{X_\Sigma} \cdot K_{X_\Sigma} = \sum_{i=0}^{n-1} \left(\frac{1}{d_{i-1,i}} + \frac{1}{d_{i,i+1}} - \frac{d_i}{d_{i-1,i} d_{i,i+1}} \right)$$

1.9 Ehrhart polynomials

Given a full dimensional lattice polytope $P \subset M_\mathbb{R}$ one can define the functions

$$L(P) = |P \cap M|$$
$$L^*(P) = |\operatorname{Int}(P) \cap M|$$

which counts the lattice points of the polytope and interior lattice points.

Using sheaf cohomology on the sheaves $\mathcal{O}(lD_P)$ one shows the well-known fact:

Proposition 1.9.1. *Let $P \subset M_\mathbb{R}$ be a full dimensional lattice polytope. Then there exists a polynomial $E_P(x) \in \mathbb{Q}[x]$ such that for $l \in \mathbb{N}$*

$$E_P(l) = L(lP)$$

If l is positive, we also have

$$E_P(-l) = (-1)^n L^*(lP)$$

This coincides with the Hilbert polynomial $\chi(\mathcal{O}(lD_P))$.

Example 1.9.2. Consider the polytope $P = \mathrm{Conv}(0, e_1, e_2, ..., e_n) \subset \mathbb{Z}^n$ which gives $\mathbb{P}^n$.

The set $lP \cap M$ corresponds bijectively to $(m_1, ..., m_n) \in M$ such that $\sum_{i=1}^n m_i \leq l$, $m_i \geq 0$. This easily corresponds bijectively to all monomials in n variables of degree $\leq l$ which in turn corresponds bijectively to monomials of degree l in $n+1$ variables. By a well-known combinatorical argument this is $\binom{n+l}{n}$. Thus

$$|lP \cap M| = \binom{n+l}{n}$$

Now, the interior lattice points can be described as the $(m_1, ..., m_n)$ such that $\sum_{i=1}^n m_i < l$, $m_i > 0$. Setting $(k_1, ..., k_n) = (m_1 - 1, ..., m_n - 1)$ we get a bijective correspondence to $(k_1, ..., k_n) \in M$ such that $\sum_{i=1}^n k_i \leq l - n - 1$, $k_i \geq 0$. This is exactly the lattice points of $(l-n-1)P$, where this is empty if $l - n - 1 < 0$. Thus

$$|\mathrm{Int}(lP) \cap M| = \binom{l-1}{n}$$

Picking $E_P(x) = \frac{(x+n)(x+n-1)\cdots(x+1)}{n!}$ we can verify that $E_P(x)$ satisfies the required properties.

Let P have dimension n. The normalized volume $\mathrm{Vol}(P)$ is the Euclidean volume scaled such that $\mathrm{Vol}(\mathrm{Conv}(0, e_1, e_2, ..., e_n)) = 1$. It can be shown (for instance in [BR07, Lemma 3.19]) that

$$\frac{\mathrm{Vol}(P)}{n!} = lim_{l \to \infty} \frac{L(lP)}{l^n}$$

This shows that $E_P(l)$ has degree n and the leading coefficient is $\frac{Vol(P)}{n!}$.

If we now are in dimension 2 one can be more specific: By the remarks above the leading coefficient is $\frac{\mathrm{Vol}(P)}{2}$ which equals the Euclidean area of P, denoted $\mathrm{Area}(P)$. The constant term has to be 1 since $L(0) = 1$. Inserting $l = 1$ and $l = -1$ we get

$$\mathrm{Area}(P) + b + 1 = L(P)$$

$$\mathrm{Area}(P) - b + 1 = (-1)^2 L^*(P) = L^*(P)$$

Solving for b we obtain $\frac{b}{2} = L(P) - L^*(P) = |\partial P \cap M|$. Thus

$$E_P(x) = \text{Area}(x) + \frac{1}{2}|\partial P \cap M|x + 1$$

Also, as a corollary of this, solving for the area we obtain the famous Pick's formula.

Proposition 1.9.3. *(Pick's Formula) The area of a 2-dimensional lattice polytope is given by*

$$\text{Area}(P) = |\operatorname{Int}(P) \cap M| + \frac{1}{2}|\partial P \cap M| - 1$$

We can give another interpretation of the Ehrhart polynomial in the 2-dimensional case in terms of intersections of divisors.

Proposition 1.9.4. *(Riemann-Roch for surfaces) [CLS11, Prop. 10.5.2] Let D be a divisor on a smooth projective surface X with canonical divisor K_X. Then*

$$\chi(\mathcal{O}(D)) = \frac{D \cdot D - D \cdot K_X}{2} + \chi(\mathcal{O}_X)$$

For a smooth polytope one then obtains, since $\chi(\mathcal{O}_X) = 1$ for a smooth complete toric surface, that

$$E_P(l) = \chi(\mathcal{O}(lD_P)) = l^2 \frac{D_P \cdot D_P}{2} - l \frac{D_P \cdot K_X}{2} + 1$$

For a general, not necessarily smooth polytope, one can pick a resolution of singularities X and pull the divisor D_P back to a divisor $\phi^* D_P$. Using sheaf cohomology one obtains that $\chi(\mathcal{O}(l\phi^* D_P)) = E_P(l)$. From Riemann-Roch one then obtains:

$$E_P(l) = \frac{1}{2}(\phi^* D_P \cdot \phi^* D_P) l^2 - \frac{1}{2}(\phi^* D_P \cdot K_{X_{\Sigma_P}}) l + 1,$$

We also have that $D_P^2 = \phi^* D_P^2$ and $K_X \cdot \phi^* D_P = K_{X_{\Sigma_P}} \cdot D_P$, this will be shown later, see the remarks following Proposition 3.4.5. As a consequence one obtains by combining with the description above:

Proposition 1.9.5. *Let P be a 2-dimensional polytope. Then*

$$D_P \cdot D_P = \text{Vol}(P)$$

$$-D_P \cdot K_{X_{\Sigma_P}} = |\partial P \cap M|$$

1.10 Dual Varieties

We will now define and look at some examples of dual varieties. We will follow the presentation used in [GKZ94].

For a finite dimensional vector space V let $\mathbb{P}(V)$ be the set of 1-dimensional subspaces of V. Then $\mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$.

If $W \subset V$ is a vector subspace then $\mathbb{P}(W)$ is a subset of $\mathbb{P}(V)$, these are called projective subspaces. Projective subspaces of dimension 1 are called lines, of dimension 2 planes and of codimension 1 hyperplanes.

Now consider $\mathbb{P}(V)$ for a vector space V. Hyperplanes in $V^\vee$, the dual vector space, form a new projective space $\mathbb{P}(V)^\vee = \mathbb{P}(V^\vee)$. Conversely, to a point $p \in \mathbb{P}(V)$ one can associate a hyperplane $p^\vee$ in $\mathbb{P}(V)^\vee$; the set of all hyperplanes in $\mathbb{P}(V)$ containing p. Thus $\mathbb{P}(V)^\vee)^\vee$ is isomorphic to $\mathbb{P}(V)$. Set $\mathbb{P} = \mathbb{P}(V)$.

Now let $X \subset \mathbb{P}$ be a closed irreducible subvariety. A hyperplane $H \subset \mathbb{P}$ is said to be tangent to X if there exists a smooth $x \in X$ such that $x \in H$ and the tangent space to H at x contains the tangent space to X at x. Denote by $X^\vee \subset \mathbb{P}^\vee$ the closure of the set of all hyperplanes tangent to X. This is the dual variety to X.

When X is smooth and does not lie in any hyperplane the definition of dual variety has a geometric interpretation: $H \in X^\vee$ if and only if $H \cap X$ is singular.

In the general case we can consider the set $I_X^0 \subset \mathbb{P} \times \mathbb{P}^\vee$ of pairs (x, H) where $x \in X_{sm}$ (the smooth locus of X) and H is the hyperplane tangent to X at x. The projection $pr_1 : I_X^0 \to X_{\text{sm}}$ makes I_X^0 a projective bundle over X_{sm} of dim $n - \dim X - 1$. Hence I_X^0 and its closure I_X are irreducible varieties of dim $n-1$.

From this we expect the dimension of $X^\vee$ to be $n-1$. The number codim $X^\vee - 1$ is called the defect of X, typically this is 0, in which case $X^\vee$ is defined by a single polynomial, which we will call Δ_X.

Example 1.10.1. Consider the Veronese embedding X of $\mathbb{P}^1$ in $\mathbb{P}^d = \mathbb{P}(V^\vee)$ given by

$(x, y) \mapsto (x^d : x^{d-1}y : x^{d-2}y^2 : ... : xy^{d-1} : y^d)$

Let $z_0, ..., z_d$ be coordinates on $\mathbb{P}^d$. Any linear form $l = \sum_{i=0}^d a_i z_i$ is uniquiely determined by its values on X which is a binary form $f(x, y) = \sum_{i=0}^d a_i x^i y^{d-i}$. De-homoginizing we get $f(x) = \sum_{i=0}^d a_i x^i$. The condition that $l \in X^\vee$ translates to $f(x)$ having a double root. Hence Δ_X is the normal discriminant of a polynomial in one variable.

To justify calling these notions dual we have:

Theorem 1.10.2. *[GKZ94] For any projective variety $X \subset \mathbb{P}$, we have $(X^\vee)^\vee = X$. More precisely, if z is a smooth point of X and H a smooth point of $X^\vee$, then H is tangent to X at z if and only if z, regarded as a hyperplane in $\mathbb{P}^\vee$, is tangent to $X^\vee$ at H.*

The case we will be primarily interested in is a toric variety coming from a polytope P. For smooth polytopes [GKZ94] shows, by considering the discriminant variety of the associated Laurent monomials, a formula for the degree of the dual variety:

$$\deg X_P^\vee = \sum_{Q \preceq P} (-1)^{\operatorname{codim} Q} (\dim Q + 1) \operatorname{Vol}(Q) \tag{1.1}$$

In the singular case this doesn't work, however [MT11] shows a similar formula involving Euler-obstructions as correction terms.

Proposition 1.10.3. *For any lattice polytope P we have*

$$\deg(X_P)^\vee = \sum_{Q \preceq P} (-1)^{\operatorname{codim} Q} (\dim Q + 1) \operatorname{Vol}(Q) \operatorname{Eu}(Q)$$

Again the volume is normalized with respect to the lattice. Unless explicitly stated otherwise, we will always by $\operatorname{Vol}(P)$ mean the volume normalized with respect to the lattice spanned by lattice points in P (sometimes in dimension 1/2 we write length/area instead).

This degree is 0 if and only if the variety is defect. To be able to compute this we must now consider the Euler obstruction.

1.11 Euler obstruction of toric varieties

The local Euler obstruction was introduced in [Mac74] as a way of constructing Chern classes for singular varieties. On the smooth locus of a variety it is constantly equal to 1. To calculate it we will use a formula for the Euler obstruction of toric varieties proved in [MT11, Ch. 4].

Let $N \cong \mathbb{Z}^n$ be a lattice of rank n, and let σ be a cone in $N_\mathbb{R}$. One can describe the Euler obstruction combinatorically by induction on the codimension of the faces of $\sigma^\vee$. Given two faces Δ_α and Δ_β of $\sigma^\vee$ such that $\Delta_\beta \preceq \Delta_\alpha$ consider the following:

Let $L(\Delta_\beta)$ be the smallest linear subspace in $M_\mathbb{R}$ containing Δ_β. This will have dimension the same as $\dim \Delta_\beta$. Now let $L(\Delta_\beta)' = M_\mathbb{R}/L(\Delta_\beta)$ and let $p_\beta : M_\mathbb{R} \to L(\Delta_\beta)'$ be the projection. Then $M'_\beta = p_\beta(M) \subset L(\Delta_\beta)'$ is a lattice of rank $n - \dim \Delta_\beta$. Also $K_{\alpha,\beta} = p_\beta(\Delta_\alpha) \subset L(\Delta_\beta)'$ is a convex cone with apex 0.

Definition 1.11.1. Given Δ_α and Δ_β of $\sigma^\vee$ such that $\Delta_\beta \preceq \Delta_\alpha$ we define the normalized relative subdiagram volume $RSV_\mathbb{Z}(\Delta_\alpha, \Delta_\beta)$ of Δ_α along Δ_β by

$$\mathrm{RSV}_\mathbb{Z}(\Delta_\alpha, \Delta_\beta) = \mathrm{Vol}(K_{\alpha,\beta} \setminus \Theta_{\alpha,\beta})$$

where $\Theta_{\alpha,\beta}$ is the convex hull of $K_{\alpha,\beta} \cap M'_\beta \setminus \{0\}$ in $L(\Delta_\beta)'$. $\mathrm{Vol}(K_{\alpha,\beta} \setminus \Theta_{\alpha,\beta})$ is the normalized $\dim\ \Delta_\alpha - \Delta_\beta$-dimensional volume with respect to the lattice $M'_\beta \cap L(K_{\alpha,\beta})$. If $\Delta_\alpha = \Delta_\beta$ we set $\mathrm{RSV}_\mathbb{Z}(\Delta_\alpha, \Delta_\beta) = 1$.

Using this we have that the values of the Euler-obstruction on the faces of $\sigma^\vee$ are determined by this function.

Proposition 1.11.2. *[MT11, Cor 4.4] The values of* $\mathrm{Eu}(\Delta_\alpha)$ *are determined by induction on the codimension of the faces of* $\sigma^\vee$ *by the following:*

$\mathrm{Eu}(\sigma^\vee) = 1$

$\mathrm{Eu}(\Delta_\beta) = \sum_{\Delta_\beta \lneqq \Delta_\alpha} (-1)^{\dim \Delta_\alpha - \dim \Delta_\beta - 1} \mathrm{RSV}_\mathbb{Z}(\Delta_\alpha, \Delta_\beta)\, \mathrm{Eu}(\Delta_\alpha)$

The case we are interested in is the Euler-obstruction of the vertices of a toric variety coming from a n-dimensional polytope P. By the definition of the normal fan, we have that given a vertex v the corresponding cone $C_v = \mathrm{Cone}(P \cap M - v)$ is dual to a cone σ in the normal fan. Thus we get a $1-1$ inclusion preserving correspondence between faces of P and faces $\sigma^\vee = C_v$. Hence we can describe the Euler-obstruction on the codimension of the faces of P by inheriting the above. In other words:

Corollary 1.11.3. *The values of* $\mathrm{Eu}(\Delta_\alpha)$ *for a face* Δ_α *of* P *are determined by induction on the codimension of the faces of* P *by the following:*

$\mathrm{Eu}(P) = 1$

$\mathrm{Eu}(\Delta_\beta) = \sum_{\Delta_\beta \lneqq \Delta_\alpha} (-1)^{\dim \Delta_\alpha - \dim \Delta_\beta - 1} \mathrm{RSV}_\mathbb{Z}(\Delta_\alpha, \Delta_\beta)\, \mathrm{Eu}(\Delta_\alpha)$

To simplify calculations, we observe the following:

Proposition 1.11.4. *If* $\dim \Delta_\alpha - \dim \Delta_\beta = 1$ *then* $\mathrm{RSV}_\mathbb{Z}(\Delta_\alpha, \Delta_\beta) = 1$.

Proof. This follows almost by construction: The quotient lattice M'_β will be a 1-dimensional lattice isomorphic to $\mathbb{Z}$. Then the projection of Δ_α must be either Cone(1) or Cone(−1) (since we assume cones are strongly convex), thus it follows $\mathrm{RSV}_\mathbb{Z}(\Delta_\alpha, \Delta_\beta) = 1$. □

Setting $\Delta_\alpha = P$ in the above, we get $\mathrm{Eu}(\Delta_\alpha) = \mathrm{RSV}_\mathbb{Z}(P, \Delta_\alpha) = 1$, thus we deduce:

Corollary 1.11.5. *Given a polytope P, let $\dim P = n$. Then for any $(n-1)$-dimensional face $\Delta \subsetneq P$ we have $\mathrm{Eu}(\Delta) = 1$.*

Remark 1.11.6. This could also be deduced from the known fact that normal toric varieties are smooth in codimension 1.

We are mainly interested in the Euler obstruction of the vertices of a 2-dimensional polytope $P \subset M_\mathbb{R}$. By the Corollary 1.11.3 we get for a vertex v, letting e_1, e_2 be the edges of P containing v:

$$\mathrm{Eu}(v) = \mathrm{RSV}_\mathbb{Z}(e_1, v)\,\mathrm{Eu}(e_1) + \mathrm{RSV}_\mathbb{Z}(e_2, v)\,\mathrm{Eu}(e_2) - \mathrm{RSV}_\mathbb{Z}(P, v),$$

By Proposition 1.11.4 $\mathrm{Eu}(e_i) = \mathrm{RSV}_\mathbb{Z}(P, e_i) = 1$ and $\mathrm{RSV}_\mathbb{Z}(e_i, v) = 1$ for $i = 1, 2$, thus we reduce calculations to:

$$\mathrm{Eu}(v) = 2 - \mathrm{RSV}_\mathbb{Z}(P, v)$$

To calculate $\mathrm{RSV}_\mathbb{Z}(P, v)$ we get that M'_v will equal $M_\mathbb{R}$. Hence $K_{P,v}$ will just be the cone generated by the polytope P with apex v. Then $\mathrm{Vol}_\mathbb{Z}(K_{P,v} \backslash \Theta_{P,v})$ will be the area removed, if we instead of P consider the convex hull of the points of $(P \setminus \{v\}) \cap M$. Hence we obtain

Proposition 1.11.7.

$$\mathrm{Eu}(v) = 2 - \mathrm{Vol}(P \setminus \mathrm{Conv}((P \setminus v) \cap M))$$

where $\mathrm{Conv}((P \setminus v) \cap M)$ is the convex hull of the lattice points of P with the point v removed.

Remark 1.11.8. Since we define RSV for polytopes via its definition for cones, one can also get a formula for the Euler-obstruction of a vertex in terms of cones. In that case one would get analogously

$$\mathrm{Eu}(v) = 2 - \mathrm{Vol}(\sigma^\vee \setminus K(\sigma^\vee)),$$

where σ is the cone corresponding to v and $K(\sigma^\vee) = \mathrm{Conv}(\sigma^\vee \cap (M \setminus \{0\}))$.

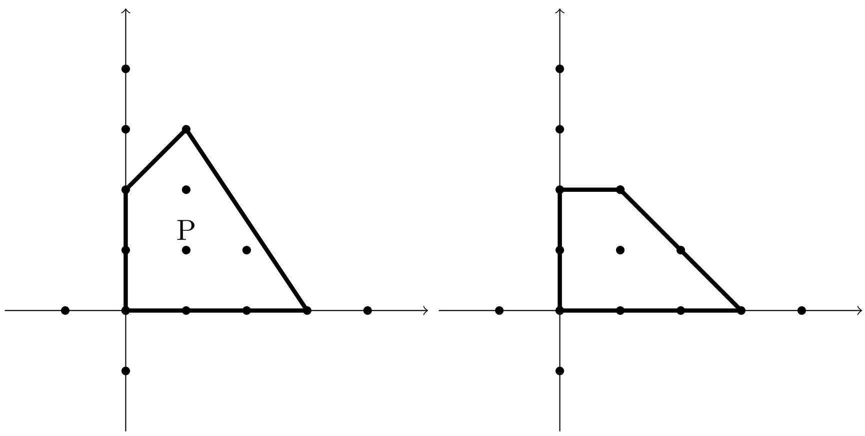

Figure 1.3: The polytope $P = \mathrm{Conv}((0,0),(0,2),(1,3),(3,0))$. Removing the vertex $(1,3)$ we get the right figure. $\mathrm{Vol}_{\mathbb{Z}}(P) = 11$ while the volume of the new polytope is 8. Hence $\mathrm{Eu}(1,3) = 2 - 11 + 8 = -1$.

Chapter 2

Weighted Projective Spaces

2.1 Definition and examples

Definition 2.1.1. Let $q_0, ..., q_n \in \mathbb{N}$ satisfy $\gcd(q_0, ..., q_n) = 1$. Define $\mathbb{P}(q_0, ..., q_n) = \mathbb{C}^{n+1} \setminus \{0\}/\sim$ where $\sim$ is the equivalence relation:

$(a_0, ...a_n) \sim (b_0, ..., b_n) \Leftrightarrow a_i = \lambda^{q_i} b_i \ \forall i$ for some $\lambda \in \mathbb{C}^*$

We call $\mathbb{P}(q_0, ..., q_n)$ weighted projective space corresponding to $q_0, ..., q_n$.

We observe that $\mathbb{P}(1, ..., 1) = \mathbb{P}^n$. Also we see that if we consider the polynomial ring $S = \mathbb{C}[x_0, ..., x_n]$ where the grading is given by deg $x_i = q_i$ we can define varieties the following way: Call a monomial $\Pi x_i^{\alpha_i}$ weighted homogeneous of degree d if $\Sigma \alpha_i q_i = d$. Then zerosets of polynomials are well defined on $\mathbb{P}(q_0, ..., q_n)$ for weighted homogeneous polynomials, hence we can define varieties the usual way.

Example 2.1.2. We can embed $\mathbb{P}(1, 1, 2)$ in $\mathbb{P}^3$ by the map:

$(a_0, a_1, a_2) \mapsto (a_0^2, a_0 a_1, a_1^2, a_2)$

By considering affine patches it is easy to see this is injective. We will show that the image is exactly $V(y_0 y_2 - y_1^2)$ where y_i are homogenous coordinates of $\mathbb{P}^3$:

One inclusion is obvious, so assume (y_0, y_1, y_2, y_3) satisfies $y_0 y_2 = y_1^2$.

If $y_0 = 0$ then $y_1 = 0$ hence either we are in $(0 : 0 : 0 : 1)$ or $(0 : 0 : 1 : y_3)$ which obviously is in the image.

If $y_0 \neq 0$ we can set $y_0 = 1 \Rightarrow y_2 = y_1^2$ hence we have the point $(1 : y_1 : y_1^2 :$

y_3) which also is in the image.

On the affine set where $y_3 \neq 0$ we see (by differentiating) that $(0:0:0:1)$ is a singular point.

One can also view $\mathbb{P}(1,1,2)$ from a different perspective. Consider the polytope $P = \text{Conv}(0, 2e_1, e_2) \subset \mathbb{R}^2$. This induces the map $(\mathbb{C}^*)^2 \to \mathbb{P}^3$ given by:

$$(s,t) \mapsto (1 : s : s^2 : t)$$

The toric variety corresponding to the polytope, X_P, will be the Zariski closure of the image. We see that affinely this is $V(x_1^2 - x_2)$. Homogenizing we get $V(x_1^2 - x_2x_0)$. We see that this is the same as we had before, hence $X_P \simeq \mathbb{P}(1,1,2)$.

Now we can show that $\mathbb{P}(1,1,2)$ is singular in a different way: We know X_P is smooth if and only if P is a smooth polytope. $(0,1)$ is not smooth, since the vectors $(0,-1)$ and $(2,-1)$ do not generate $\mathbb{Z}^2$, for instance $(1,0)$ is not in their span.

Given a ring R and a group G acting on it, one gets a subring

$$R^G = \{x \in R \mid gx = x \ \forall g \in G\}$$

In [CLS11, Ch. 5.1] it is shown that the fan Σ from Example 1.3.7 in fact gives the same variety as in the definition above. The defining equivalence relation can also be described as a group action by $\mathbb{C}^*$ on $\mathbb{C}^{n+1}\backslash\{0\}$. We have an open affine cover of the form $\operatorname{Spec}\mathbb{C}[x_0, ..., \hat{x}_i, ..., x_n]^{\mu_{q_i}}$ for μ_{q_i} induced by the global action by $\mathbb{C}^*$ as follows: Let $(t_0 : t_1 : ... : t_n)$ be coordinates on $\mathbb{P}(q_0, ..., q_n)$. Then we get an open cover by the sets $X_i = \{t_i \neq 0\}$. On X_i we can set $t_i = 1$ which forces $\lambda \in \mathbb{C}^*$ to satisfy $\lambda^{q_i} = 1$, hence we get that X_i is isomorphic to the orbits of the action

$$\mu_{q_i} \times X_i \to X_i$$
$$(\zeta, (1, t_1, ..., t_n)) \mapsto (1, \zeta^{q_1}t_1, ..., \zeta^{q_n}t_n) \tag{2.1}$$

where μ_{q_i} is the set of q_i roots of unity, and ζ is a primitive q_i-th root of unity.

On coordinate rings this is exactly the equality $X_i = \operatorname{Spec}\mathbb{C}[x_0, ..., \hat{x}_i, ..., x_n]^{\mu_{q_i}}$. From this one gets that

$$\mathbb{C}[x_0, ..., \hat{x}_i, ..., x_n]^{\mu_{q_i}} = \mathbb{C}[x_0^{m_0} \cdots x_n^{m_n} | \sum_{j \neq i} m_j q_j \equiv 0 \pmod{q_i}]$$

Example 2.1.3. Consider $\mathbb{P}(2,3,5)$. Then we have the open affine cover

$$X_0 = \operatorname{Spec} \mathbb{C}[x_1, x_2]^{\mu_2} = \operatorname{Spec} \mathbb{C}[x_1^2, x_1x_2, x_2^2]$$

$$X_1 = \operatorname{Spec} \mathbb{C}[x_0, x_2]^{\mu_3} = \operatorname{Spec} \mathbb{C}[x_0^3, x_0^2x_2, x_0x_2^2, x_2^3]$$

$$X_2 = \operatorname{Spec} \mathbb{C}[x_0, x_1]^{\mu_5} = \operatorname{Spec} \mathbb{C}[x_0^5, x_0x_1, x_1^5]$$

As in [CAMMOG14] one can also describe the points of $\mathbb{P}(q_0, ..., q_n)$ as the orbits of the action of $G = \mu_{q_0} \times \mu_{q_1} \times ... \times \mu_{q_n}$ on $\mathbb{P}^n$ given by

$$G \times \mathbb{P}^n \to \mathbb{P}^n$$

$$(\zeta_{\mu_0}, ..., \zeta_{\mu_0}), (t_0 : ... : t_n) \mapsto (\zeta_{\mu_0}^{q_0} t_0 : ... : \zeta_{\mu_n}^{q_n} t_n)$$

This is induced by the branched covering map

$$\mathbb{P}^n \to \mathbb{P}(q_0, ..., q_n)$$

$$(t_0 : ... : t_n) \mapsto (t_0^{q_0} : ... : t_n^{q_n}) \tag{2.2}$$

which has degree $q_0 \cdots q_n$ and is unramfied where all coordinates are nonzero. The fiber over a point $p = (1 : t_1^{q_1} : ... : t_n^{q_n})$ consists of the following points: Let $\zeta_{q_0}, ..., \zeta_{q_n}$ be primitive q_i-th roots of unity. Then the points of $\mathbb{P}^n$ of the form $(\zeta_{q_0}^{l_0} : \zeta_{q_1}^{l_1} t_1 : ..., \zeta_{q_n}^{l_n} t_n)$, $0 \le l_i < q_i$, all map to p. If any of these points are equivalent under the equivalence relation defining $\mathbb{P}^n$, one needs to have $c \in \mathbb{C}^*, c \neq 1$, such that $c = \zeta_{q_i}^{l_i - l_i'}$ for all i. If for some i, $l_i = l_i'$, then $c = 1$. Otherwise, c has to simultaneously be a q_i-th root of unity, for all i. But the set of simultaneous $q_0, ..., q_n$-th roots of unity are exactly the $\gcd(q_0, ..., q_n)$-th roots of unity. Since $\gcd(q_0, ..., q_n) = 1$, we have $c = 1$, and all points are different. Hence we have $q_0 \cdots q_n$ points in the fiber on the torus.

In general set $Y_{i_1,...,i_s} = \{t_{i_1}, ..., t_{i_s} \neq 0, t_j = 0, j \neq i_s\}$. Then on $Y_{i_1,...,i_s}$ we, by the same argument as above, have $q_{i_1} \cdots q_{i_s}$ elements in the fiber, however now they are not necessarily all different. One checks that, in $\mathbb{P}^n$, $\mu_{\gcd(q_{i_1},...,i_s)}$ acts on the fiber by multiplication with a primitive element, making every point equivalent to $\gcd(q_{i_1}, ..., q_{i_s})$ other points. Thus the fiber of a point in $Y_{i_1,...,i_s}$ has size $\frac{q_{i_1},...,q_{i_s}}{\gcd(q_{i_1},...,q_{i_s})}$.

This map turns out to be a toric morphism described as follows:

Recall that the fan Σ_1 for $\mathbb{P}^n$ consists of all cones generated by proper subsets of the basis elements $\{e_0, ..., e_n\}$ in the lattice $N_1 = \mathbb{Z}^{n+1}/\mathbb{Z}(1, ..., 1)$. The fan Σ_2 for $\mathbb{P}(q_0, ..., q_n)$ consist of all cones generated by proper subsets of the basis elements $\{v_0, ..., v_n\}$ in $N_2 = \mathbb{Z}^{n+1}/Z(q_0, ..., q_n)$. Consider the map

$$\bar{\phi} : N_1 \to N_2$$

$$e_i \mapsto q_i v_i$$

Then $\bar{\phi}(\mathrm{Cone}(e_j|\ j \in I)) \subset \mathrm{Cone}(v_j|\ j \in I)$, hence the mapping is compatible with the fans, so it induces a toric morphism $\mathbb{P}^n \to \mathbb{P}(q_0, ..., q_n)$. We will check that this is the same map as above.

The dual lattices are $M_1 = \{(m_0, ..., m_n) \in \mathbb{Z}^{n+1} |\ \sum_{i=0}^n m_i = 0\}$ and $M_2 = \{(m_0, ..., m_n) \in \mathbb{Z}^{n+1} |\ \sum_{i=0}^n m_i q_i = 0\}$.The induced map on these are

$$\bar{\phi}^\vee : M_2 \to M_1$$

$$(m_0, ..., m_n) \mapsto (m_0 q_0, ..., m_n q_n)$$

Meaning that the associated map $\mathbb{C}[\sigma_2^\vee \cap M_2] \to \mathbb{C}[\sigma_1^\vee \cap M_1]$ sends a monomial $x_0^{m_0} \cdots x_n^{m_n}$ to $y_0^{m_0 q_0} \cdots y_n^{m_n q_n}$.

Writing this as a map of polynomial rings on the coordinate rings of the affine sets corresponding to $\mathrm{Cone}(e_1, ..., e_n) \in \Sigma_1$ and $\mathrm{Cone}(v_1, ..., v_n) \in \Sigma_2$ we get

$$\mathbb{C}[x_1, ..., x_n]^{\mu_{q_0}} \to \mathbb{C}[y_1, ..., y_n]$$

$$x_1^{m_1} \cdots x_n^{m_n} \mapsto y_1^{m_1 q_1} \cdots y_n^{m_n q_n}$$

where $\sum_{i=1}^n m_i q_i \equiv 0 \pmod{q_0}$. By exercise 3.2.P [Vak] we get that the map induced by the Spec-functor looks like

$$\mathrm{Spec}\,\mathbb{C}[y_1, ..., y_n] \to \mathrm{Spec}\,\mathbb{C}[x_1, ..., x_n]^{\mu_{q_0}}$$

$$(a_1, ..., a_n) \mapsto (a_1^{q_1}, ..., a_n^{q_n}),$$

which we recognize as an affine patch of the map (2.2). By doing this for all maximal cones we get that the two maps are the same.

There are characterizations of when $\mathbb{P}(q_0, ..., q_n) \simeq \mathbb{P}(s_0, ..., s_n)$ in terms of the weights, see for instance [RT11]. For a given set of weights $(q_0, ..., q_n)$ we will describe its reduction $(q'_0, ..., q'_n)$. Set:

$$d_i = \gcd(q_0, ..., \hat{q}_i, ..., q_n)$$

$$a_i = \mathrm{lcm}(d_0, ..., \hat{d}_i, ..., d_n)$$

Setting $q'_i = \frac{q_i}{a_i}$ we obtained the reduced weights $(q'_0, ..., q'_n)$. We have:

Proposition 2.1.4. *[RT11, Prop 1.26] There is an isomorphism*

$$\mathbb{P}(q_0, ..., q_n) \simeq \mathbb{P}(q'_0, ..., q'_n)$$

.

One upshot is that one can always assume the weights are reduced, i.e. that $\gcd(q_0, ..., \overline{q_i}, ...q_n) = 1$ for all i, we will always do this. In particular, in the surface case $\mathbb{P}(k, m, n)$, we can always assume that $\gcd(k, m) = \gcd(k, n) = \gcd(m, n) = 1$.

As noted before, $\mathbb{P}(q_0, ..., q_n)$ is a singular variety, we will describe this in more detail.

For a fan Σ, [CLS11, Thm. 11.4.8] shows that Σ is simplicial if and only if X_Σ has only finite quotient singularities, i.e., for every point p there exists a finite subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$ such that p is analytically equivalent to $0 \in \mathbb{C}^n/G$. Thus $\mathbb{P}(q_0, ..., q_n)$ has only finite quotient singularities.

Proposition 2.1.5. *[CLS11, Prop 11.1.2] The singular locus of X_Σ equals,*

$$(X_\Sigma)_{\mathrm{sing}} = \cup_{\sigma \text{ singular}} \overline{O(\sigma)}$$

Proposition 2.1.6. *[CLS11, Prop. 3.3.11] Let $N' \subset N$ be a sublattice, with $\dim N_\mathbb{R} = n$, $\dim N'_\mathbb{R} = k$. Let Σ' be a fan in $N'_\mathbb{R}$, via the inclusion this is also a fan in $N_\mathbb{R}$. Extend a basis for N' to a basis for a sublattice $N'' \subset N$ of finite index. Set $G = N/N''$. Then we have*

$$X_{\Sigma',N} \simeq (X_{\Sigma',N'} \times (\mathbb{C}^*)^{n-k})/G$$

where $X_{\Sigma',N}$ is the variety associated with Σ', considered as a fan in N.

Recall again the fan from Example 1.3.7: Take $\mathbb{Z}^{n+1}$ with basis $e_0, ..., e_n$ and let u_i be the image of e_i in the quotient lattice $N = \mathbb{Z}^{n+1}/(q_0, ..., q_n)$. Let Σ be the collection of cones $\mathrm{Cone}(u_j | j \in J)$ for all proper subsets $J \subset \{0, ..., n\}$. Set $\sigma_{j_1,...,j_s} = \mathrm{Cone}(u_{j_1}, ..., u_{j_s})$. Let $(t_0 : ... : t_n)$ be coordinates on $\mathbb{P}(q_0, ..., q_n)$, and set $X_{j_1,...,j_s} = \{(t_0 : ... : t_n) \in \mathbb{P}(q_0, ..., q_n) | t_{j_1} = ... = t_{j_s} = 0, t_i \neq 0, \text{ for } i \neq j_s\}$. Then we have $O(\sigma_{j_1,...,j_s}) = X_{j_1,...,j_s}$.

For a cone $\sigma \in \Sigma$ we will use Proposition 2.1.6 to describe the group actions, where N' is the sublattice of N spanned by the generators of σ, and Σ' the fan of all subcones of σ, thus $X_{\Sigma',N} = U_\sigma$, and $X_{\Sigma',N'} \simeq \mathbb{C}^{\dim \sigma}$ by construction of the lattice N'.

Since constructing an explicit basis for N is some work, we will instead use a trick for computing the indices of sublattices: By Proposition A.0.5 the vector $w_0 = (q_0, ..., q_n) \in \mathbb{Z}^{n+1}$ can be extended to a basis $\{w_0, ..., w_n\}$ for $\mathbb{Z}^{n+1}$. Letting $\{v_1, ..., v_s\}$ be the generators of σ, considered as vectors in $\mathbb{Z}^{n+1}$, extend the set $\{w_0, v_1, ..., v_s\}$ to a basis for a sublattice of finite index l in $\mathbb{Z}^{n+1}$. Taking the quotients by the basis vector w_0 we obtain that N' has index l in N as well.

As an example, take $\sigma_{1,...,n} = \mathrm{Cone}(u_1, ..., u_n)$. Then $\det(w_0, u_1, ..., u_n) = q_0$, thus N' has index q_0 in N. The corresponding orbit closure $\overline{O(\sigma)}$ will be the point $(1 : 0 : ... : 0)$, thus this is a singular point. We also get a n open neighbourhood of the point

$$U_{\sigma_{1,...,n}} \simeq X_{\Sigma',N} \simeq \mathbb{C}^n \times \{pt\}/\mathbb{Z}_{q_0},$$

and we recognize the above as exactly the action 2.1 on the set $\{t_0 \neq 0\}$. We see that σ has multiplicity q_0, which is singular if $q_0 > 1$. Similarly for other maximal cones $\sigma_{0,...,\hat{i},...,n}$, the corresponding multiplicity is q_i.

Next we take $\sigma_{2,...,n} = \text{Cone}(u_2, ..., u_n)$. The corresponding orbit closure will, by the orbit-cone correspondence, be the the points $(1 : t : 0 : ... : 0)$, for $t \neq 0$. We want to expand the set $\{w_0, u_2, ..., u_n)$ to a sublattice of $\mathbb{Z}^{n+1}$, so let $(x_0, ..., x_n)$ be any vector in $\mathbb{Z}^{n+1}$. Taking determinants of the $n+1$ vectors, we get $q_0x_1 - q_1x_0$. The minimal value this can obtain by choosing $x_0, x_1 \in \mathbb{Z}$ is $\gcd(q_0, q_1)$. Thus the multiplicity of $\sigma_{2,...,n}$ is $\gcd(q_0, q_1)$. If this multiplicity is greater than 1, wee see that the entire orbit closure will be singular, thus we do not have isolated singularities. We also have

$$U_{\sigma_{2,...,n}} \simeq (\mathbb{C}^{n-1} \times \mathbb{C}^*)/\mathbb{Z}_{\gcd(q_0,q_1)}$$

which is the induced action by 2.1 on the set $\{t_0, t_1 \neq 0\}$.

Generally for any cone $\sigma_{j_1,...,j_s}$, let $I = \{i_0, ..., i_{n-s}\} = \{0, ..., n\} \setminus \{j_1, ..., j_s\}$. Extending the set $\{w_0, e_{j_1}, ..., e_{j_s}\}$ to a basis for a full dimensional sublattice of $\mathbb{Z}^{n+1}$ we see, by taking determinants, is equivalent to expanding the vector $(q_{i_0}, ..., q_{i_{n-s}})$ to a basis for a sublattice of $\mathbb{Z}^{n-s+1}$. By Proposition A.0.5 we can always extend $\frac{1}{\gcd(q_{i_0},...,q_{i_{n-s}})}(q_{i_0}, ..., q_{i_{n-s}})$ to a basis for $\mathbb{Z}^{n-s+1}$. Using this, we obtain that the multiplicity of $\sigma_{j_1,...,j_s}$ will be $\gcd(q_{i_0}, ..., q_{i_{n-s}})$. Then we have

$$U_{\sigma j_1,...,j_s} \simeq (\mathbb{C}^s \times (\mathbb{C}^*)^{n-s})/\mathbb{Z}_{\gcd(q_{i_0},...,q_{i_{n-s}})}$$

which is the set $\{t_{i_0}, ..., t_{i_{n-s}} \neq 0\}$.

Hence we have that the orbit closure $\overline{O(\sigma_{j_1,...,j_s})}$ will be singular if and only if $\gcd(q_{i_0}, ..., q_{i_{n-s}}) > 1$.

Note also that the orbit closure $O(\sigma_{j_1,...,j_s})$ by Remark 1.6.3 is isomorphic to $\mathbb{P}(q_{i_0}, ..., q_{i_{n-s}})$, where now the weights aren't necessarily reduced. Hence we obtain that all orbit closures are themselves weighted projective spaces.

Summing up, we obtain:

Proposition 2.1.7. *$\mathbb{P}(q_0, ..., q_n)$ is nonsingular in codimension k if for all $\{j_1, ..., j_k\}$, the corresponding $\gcd(q_{i_0}, ..., q_{i_{n-k}}) = 1$. In particular:*

$\mathbb{P}(q_0, ..., q_n)$ is nonsingular in codimension 1 .

$\mathbb{P}(q_0, ..., q_n)$ has isolated singularities if and only if it is nonsingular in codimension $n-1$ if and only if $\gcd(q_i, q_j) = 1$ for all i, j.

For surfaces we will always have isolated singularities, but in larger dimensions we might have larger singular locus, for instance $\mathbb{P}(2,2,3,3)$ does not have isolated singularities.

2.2 Divisors on Weighted Projective Space

We will describe $\mathrm{Cl}(\mathbb{P}(q_0, ..., q_n))$ and $\mathrm{Pic}(\mathbb{P}(q_0, ..., q_n))$ (cf. [CLS11, Ex. 4.1.5 and 4.2.11]).

Let $N \cong \mathbb{Z}^{n+1}/\mathbb{Z}(q_0, ..., q_n)$ and M be the dual lattice:

$$M = \{(a_0, ..., a_n) \in \mathbb{Z}^{n+1} | a_0 q_0 + ... + a_n q_n = 0\}$$

Let $u_0, ... u_n \in \mathbb{N}$ be images in N of the standard basis $e_0, ..., e_n$ of $\mathbb{Z}^{n+1}$. Define maps

$$M \to \mathbb{Z}^{n+1} : m \mapsto (\langle m, u_0 \rangle, ..., \langle m, u_n \rangle)$$

$$\mathbb{Z}^{n+1} \to \mathbb{Z} : (a_0, ..., a_n) \mapsto a_0 q_0 + ... + a_n q_n$$

If we can show that these maps form an exact sequence:

$$0 \to M \to \mathbb{Z}^{n+1} \to \mathbb{Z} \to 0$$

we have by Proposition 1.7.2 that $\mathrm{Cl}(\mathbb{P}(q_0, ..., q_n) \cong \mathbb{Z}$.

That the first map is injective follows from the properties of the dual pairing: If $m, m' \in M$ has the same image we have $\langle m - m', u_i \rangle = 0$ for all i hence $m = m'$. Since $\gcd(q_0, ..., q_n) = 1$ we can find $(a_0, ..., a_n)$ such that $a_0 q_0 + ... + a_n q_n = 1$. Thus we see that the last map is surjective.

That the sequence is exact in the middle follows from the definition of M and u_i, hence we are done.

For the Picard group we use Proposition 1.7.5 to determine when a general Weil divisor $D = \sum b_i D_i$ is Cartier. Assuming D is Cartier we know that for each maximal cone there exist Cartier-data $m_\sigma \in M$. As before let $e_0, ..., e_n$ be a basis for $\mathbb{Z}^{n+1}$ such that in N the relation $\sum_{i=0}^{n} q_i e_i = 0$ holds. Let σ be a maximal cone, assume without loss of generality $\sigma = \mathrm{Cone}(e_1, ..., e_n)$. Then $m_\sigma = (m_0, ..., m_n)$ has to satisfy, for $i = 1, ..., n$,

$$\langle m_\sigma, e_i \rangle = m_i = -b_i$$

Since $m_\sigma \in M$, it must satisfy $\sum_{i=0}^n m_i q_i = 0$, so we must have

$$m_0 q_0 - \sum_{i=1}^n b_i q_i = 0$$

This implies that $q_0 | \sum_{i=0}^n b_i q_i$. Similarly for the other maximal cones we get that for all i, $q_i | \sum_{i=0}^n q_i b_i$. Thus any Picard-divisor maps to a multiple of $\operatorname{lcm}(q_0, ..., q_n) \in \operatorname{Cl}(\mathbb{P}(q_0, ..., q_n)) \equiv \mathbb{Z}$.

By much linear algebra [RT11, thm 1.19] show that, in the reduced case, the Picard group actually equals the subgroup generated by $\operatorname{lcm}(q_0, ..., q_n)$.

Since $\operatorname{Cl}(\mathbb{P}(q_0, ..., q_n)) \equiv \mathbb{Z}$, we can define a degree function $\deg(\sum_{i=0}^n a_i D_i) = \sum_{i=0}^n a_i q_i$.

The Cox ring associated to a toric variety X_Σ is the graded polynomial ring $S = \mathbb{C}[x_\rho | \rho \in \Sigma(1)]$ where $\deg x_\rho = \deg D_\rho$. In our case we get

$$S = \mathbb{C}[x_0, ..., x_n] \text{ , } \deg x_i = q_i$$

In [CLS11, Ch. 5.3] it is shown that if $\deg D = \deg E$, then $\mathcal{O}(D) \equiv \mathcal{O}(E)$. Thus all sheaves associated to divisors of a given degree d are isomorphic, denote this isomorphism class by $\mathcal{O}(d)$. Let S_d be the d-th graded piece of S. Then we have

Proposition 2.2.1.

$$\Gamma(X_\Sigma, \mathcal{O}(d)) \equiv S_d$$

Thus the global sections of the sheaf $\mathcal{O}(d)$ corresponds to all weighted homogenouos polynomials of degree d in $n+1$ variables.

2.3 Intersection theory on Weighted Projective Space

We now wish to look at intersection theory on our varieties. For any n-dimensional variety X let $Z_k(X)$ be the free abelian group generated by the set of irreducible closed subvarieties of dimension k on X. Note that $Z_{n-1}(X) = \operatorname{Div}(X)$. As in the case of divisors we define rational equivalence: Let $\alpha \in Z_k(X)$ be equivalent to zero if there exists finitely many $(k+1)$-dimensional subvarieties $V_i \subset X$ such that α is the divisor of a rational function on V_i for all i. Then the k-th Chow group $A_k(X)$ is $Z_k(X)$ modulo rational equivalence. In the toric case this behaves very well as a generalization of divisors:

Proposition 2.3.1. *[Ful93, Ch.5.1] For a toric variety X_Σ, $A_k(X_\Sigma)$ is generated by the classes of the orbit closures $\overline{O(\sigma)}$ of the cones $\sigma \in \Sigma(n-k)$.*

In the toric case, if Σ is complete and simplicial, setting $A^k(X_\Sigma) = A_{n-k}(X_\Sigma)$, one can define a product

$$A^k(X) \otimes \mathbb{Q} \times A^l(X) \otimes \mathbb{Q} \to A^{k+l}(X) \otimes \mathbb{Q}$$

which agrees with geometric intersection in nice cases. This makes the groups of cycles into a graded ring $A^\bullet(X_\Sigma)_\mathbb{Q}$.

To compute intersections we will also consider the Chow ring of a toric variety, as defined in [CLS11, Ch. 12.5].

Given a fan Σ, let $\Sigma(1) = \{\rho_1, ..., \rho_r\}$. Denote by u_i the minimal generator of ρ_i. We will consider two ideals $\mathscr{I}$, $\mathscr{J}$ in the polynomial ring $\mathbb{Q}[x_1, ..., x_r]$. Let

$$\mathscr{I} = \langle x_{i_1} \cdots x_{i_s} | \text{ all } i_j \text{ distinct and } \rho_{i_1} + \cdots + \rho_{i_s} \text{ is not a cone in } \Sigma \rangle$$

$$\mathscr{J} = \langle \sum_{i=1}^{r} \langle m, u_i \rangle x_i | \text{ where } m \text{ ranges over a basis of } M \rangle$$

$\mathscr{I}$ is called the Stanley–Reisner ideal. The Chow ring $R_\mathbb{Q}(\Sigma)$ is defined as

$$R_\mathbb{Q}(\Sigma) = \mathbb{Q}[x_1, ..., x_r]/\mathscr{I} + \mathscr{J}$$

For completeness we also note that there is a third algebraic object one could consider, the singular cohomology ring $H^\bullet(X_\Sigma, \mathbb{Q})$. Then we have:

Theorem 2.3.2. *[CLS11, Thm 12.5.3] If X_Σ is complete and simplicial, then*

$$R_\mathbb{Q}(\Sigma)_\mathbb{Q} \cong A^\bullet(X_\Sigma)_\mathbb{Q} \cong H^\bullet(X_\Sigma, \mathbb{Q}).$$

The weighted projective space is both complete and simplicial , so the theorem applies. Letting Σ be the normal fan for $\mathbb{P}(q_0, ..., q_n)$ we see that

$$\mathscr{I} = \langle x_0 \cdots x_n \rangle$$

Since we are now over $\mathbb{Q}$, a basis for $M = \{m \in \mathbb{Z}^{n+1} | \sum q_i m_i = 0\}$ will be $(q_i, ..., -q_0, ...0)$ for $i = 1, ...n$. This gives the ideal

$$\mathscr{J} = \langle q_i x_0 - q_0 x_i | i = 1, ..., n \rangle$$

Doing the computations, we can eliminate $x_1, ..., x_n$ since $x_i = \frac{q_i}{q_0} x_0$., so the Chow ring will be

$$R_\mathbb{Q}(\Sigma) \cong \mathbb{Q}[x_0]/x_0^{n+1}$$

The 1-graded part of $R_{\mathbb{Q}}(\Sigma)$ corresponds to divisors, with x_0 corresponding to D_0, thus we can compute generalized intersections of divisors from this. Taking any torus-invariant divisor $D = \sum_{i=0}^n a_i D_i$, let $d = \deg D = \sum_{i=0}^n a_i q_i$. Then in the Chow ring, D gets mapped to $\sum_{i=0}^n a_i x_i = \sum_{i=0}^n a_i \frac{q_i}{q_0} x_0 = \frac{x_0}{q_0} \sum_{i=0}^n a_i q_i = \frac{x_0}{q_0} \deg D$.

Taking n different divisors $D_1, ..., D_n$ with $\deg D_j = d_j$, it then follows,

$$D_1 \cdots D_n = \frac{\Pi_{j=1}^n d_j}{q_0^n} D_0^n$$

thus we have determined intersections of divisors modulo D_0^n. To obtain actual numbers for these intersections, we need to normalize, which amounts to finding a natural candidate for the self-intersection D_0^n. This is possible by generalizing Proposition 1.9.5, saying that for a 2-dimensional polytope P, the associated divisor D_P has self-intersection equal to $\mathrm{Vol}(P)$. This can be generalized as follows (reformulating the statement a bit for our needs, to avoid having to introduce too many definitions):

Theorem 2.3.3. *[CLS11, Thm 13.4.3] Let P be a very ample polytope giving the variety X_{Σ_P} embedded in $\mathbb{P}^s$, where $s = |P \cap M|$. Define $D_P^n = \deg(X_{\Sigma_P} \subset \mathbb{P}^s)$. Then*

$$D_P^n = \mathrm{Vol}(P_D)$$

To apply this, we need to make a diversion to describe a polytope giving $\mathbb{P}(q_0, ..., q_n)$. However this will be useful anyway, since we need the polytope to compute Euler-obstructions of our varieties.

From [RT11, Remark 1.24 and Cor 1.25] we have the following polytope:

Given $(q_0, ..., q_n)$ and $M \cong \mathbb{Z}^{n+1}$, let $\delta = \mathrm{lcm}(q_0, ..., q_n)$. Consider the $n+1$ points of $M_{\mathbb{R}} \cong \mathbb{R}^{n+1}$:

$$v_i = (0, ..., \frac{\delta}{q_i}, ...0)$$

Let Δ be the convex hull of 0 and all v_i. Intersecting Δ with the hyperplane $H = \{(x_0, ..., x_n) | \sum_{i=0}^n x_i q_i = \delta\}$, we get a n-dimensional polytope P. Then $X_P \cong \mathbb{P}(q_0, ..., q_n)$ and the associated divisor D_P will be $\frac{\delta}{q_0} D_0$ (to see that Proposition 1.7.6 is still fulfilled, note that P is only full-dimensional in the lattice generated by H. Getting $D_P = \frac{\delta}{q_0} D_0$ really corresponds to choosing $(q_1 \cdots q_n, 0..., 0)$ as the origin of the lattice generated by H, while a different choice of origin would result in a different, although linearly equivalent, divisor).

If we then can determine the volume of P, we have a way of naturally determining D_0^n, since one then would have

$$\mathrm{Vol}(P_D) = D_P^n = \frac{\delta^n}{q_0^n} D_0^n$$

implying that $D_0^n = \mathrm{Vol}(P_D)\frac{q_0^n}{\delta^n}$.

To determine the volume of P, we will use the generalized cross product (see [Mas83]). For n vectors $v_1, ..., v_n \in \mathbb{R}^{n+1}$, let A be the matrix with i-th row v_i. We can define the cross product $v_1 \times \cdots \times v_n \in \mathbb{R}^{n+1}$ by having the k-th coordinate be $(-1)^k$ times the $n \times n$ minor of A obtained by removing the k-th column. This cross product is orthogonal to all v_i and satisfies

$$|v_1 \times \cdots \times v_n| = \mathrm{Vol}(v_1, ..., v_n)$$

where $\mathrm{Vol}(v_1, ..., v_n)$ is the n-dimensional volume of the parallelotope spanned by $v_1, ..., v_n$. (For the more algebraically inclinced, this product can be expressed by exterior algebra operations as the Hodge dual $*(v_1 \wedge \cdots \wedge v_n)$.)

To determine the volume, we first need to normalize with respect to the lattice, i.e. we need to determine the volume spanned by a basis. To find a basis for the lattice spanned by H, we need to cleverly choose vectors. First we choose an edge of the polytope P, say the edge $v_0 v_1$, which is generated by $(-\frac{\delta}{q_0}, \frac{\delta}{q_1}, 0, ..., 0)$. For simpler notation set $q_{i_1,...,i_s} = \gcd(q_{i_1}, ..., q_{i_s})$. The primitive generator of the edge $v_0 v_1$ will be $e_1 = (-\frac{q_1}{q_{01}}, \frac{q_0}{q_{01}}, 0, ..., 0)$. Now, choose any lattice point of H of the form

$$(x_{20}, x_{21}, \frac{q_{01}}{q_{012}}, 0, ..., 0),$$

this exists since the numbers obtained as integral linear combination of q_0, q_1 are exactly all multiples of q_{01}, and $\delta - q_2\frac{q_{01}}{q_{012}}$ is such a multiple (the subscripts are chosen for notational purposes which will become clear) . Set e_2 as the difference between this point and v_0, in other words

$$e_2 = (x_{20} - \frac{\delta}{q_0}, x_{21}, \frac{q_{01}}{q_{012}}, 0, ..., 0)$$

In general, for all $2 \leq s \leq n$ find a lattice point of the form

$$(x_{i0}, x_{i1}, ..., x_{i(s-1)}, \frac{q_{0...s-1}}{q_{0...s}}, 0, ..., 0).$$

This is equivalent to saying

$$x_{i0}q_0 + x_{i1}q_1 + \cdots + x_{i(s-1)}q_{s-1} + \frac{q_{0...s-1}}{q_{0...s}}q_s = \delta,$$

and set

$$e_s = (x_{i0} - \frac{\delta}{q_0}, x_{i1}, ..., x_{i(s-1)}, \frac{q_{0...s-1}}{q_{0...s}}, 0, ..., 0).$$

Then we have

Proposition 2.3.4. *The n vectors $\{e_1, ..., e_n\}$ constructed above, are a basis for the lattice spanned by H.*

Proof. We will use Lemma A.0.2 to show this. Assume we have a lattice point $l = \sum_{i=1}^{n} c_i e_i$, where $0 \leq c_i < 1$ for all i. Then it suffices to show that all $c_i = 0$. We will show this by descending induction on c_n. Let $l = (y_0, ..., y_n)$. Then we have, by definition of H,

$$\sum_{i=0}^{n} q_i y_i = \delta \tag{2.3}$$

Consider the $(n+1)$-the coordinate. Since the basis is constructed in such a way that the only vector having nonzero $(n+1)$-th coordinate is e_n, we must have $y_n = c_n \frac{q_{0,\ldots,n-1}}{q_{0,\ldots,n}}$. When we defined weighted projective space we assumed $q_{0,\ldots,n} = 1$. Thus we must have $y_n = c_n q_{0,\ldots,n-1}$. Now consider (2.3) modulo $(q_{0,\ldots,n-1})$: The righthand side is 0 and the first terms $q_0 y_0 + ... + q_{n-1} y_{n-1}$ will be zero, since, in general integral linear combinations of a set of integers are exactly the multiples of their greatest common divisor. Thus we must have

$$q_n y_n \equiv q_n c_n q_{0,\ldots,n-1} \equiv 0 \pmod{q_{0,\ldots,n-1}}.$$

Now since $c_n < 1$, we have $c_n q_{0,\ldots,n-1} < q_{0,\ldots,n-1}$, and if $0 < c_n$ there must be some prime power p^r dividing $q_{0,\ldots,n-1}$ which does not appear in $c_n q_{0,\ldots,n-1}$. But then we must have that p divides q_n, which implies $q_{0,\ldots,n} > 1$ which is a contradiction. Thus $c_n = 0$.

Assume in general we have proved that $c_n = c_{n-1} = ... = c_{s+1} = 0$. We will show that $c_s = 0$. We will use the same method as above: Since $c_{s+1} = ... = c_n = 0$, we have a linear combination $l = \sum_{i=0}^{s} c_i e_i$. In the set $\{e_1, ..., e_s\}$, the only vector with $(s+1)$-th coordinate nonzero will be e_s. Thus we must have $y_s = c_s \frac{q_{0,\ldots,s-1}}{q_{0,\ldots,s}}$. Considering (2.3) modulo $q_{0,\ldots,s-1}$ we get

$$q_s y_s \equiv q_s c_s \frac{q_{0,\ldots,s-1}}{q_{0,\ldots,s}} \equiv 0 \pmod{q_{0,\ldots,s-1}}.$$

Now, since l is a lattice point, $c_s \frac{q_{0,\ldots,s-1}}{q_{0,\ldots,s}}$ is an integer $k < \frac{q_{0,\ldots,s-1}}{q_{0,\ldots,s}}$. Rewriting the above we get

$$\frac{q_s}{q_{0,\ldots,s}} k q_{0,\ldots,s} \equiv 0 \pmod{q_{0,\ldots,s-1}} \tag{2.4}$$

since $k q_{0,\ldots,s} = c_s q_{0,\ldots,s-1} < q_{0,\ldots,s-1}$, we must have, if $0 < c_s$, that there is a prime power p^r in the prime factorization of $q_{0,\ldots,s-1}$, which appears to a smaller degree in the prime factorization of $c_s q_{0,\ldots,s-1}$. By the previous equality, the highest power of p which can appear in $q_{0,\ldots,s}$ will also be smaller than r, say it is $(r-t)$. But to satisfy (2.4) we must also have that p divides $\frac{q_s}{q_{0,\ldots,s}}$, which implies that p^{r-t+1} divides q_s, but then p^{r-t+1} will divide $q_{0,\ldots,s}$ which is a contradiction. Thus we must have $c_s = 0$.

The last case is an exception. If $s = 0$ we have $l = c_0 e_0$, but by construction of e_0 as a primitive vector we must have $c_0 = 0$. Hence we are done. $\square$

Now we can use this to calculate the normalization of the volume.

Proposition 2.3.5. *The volume of the parallelotope spanned by* $e_1, ..., e_n$ *is* $\sqrt{q_0^2 + ... + q_n^2}$.

Proof. The coordinates of $z = e_1 \times \cdots \times e_n$ will be (modulo a sign) the $n \times n$ minors of the matrix A with row i equal to e_i.

$$A = \begin{bmatrix} -\frac{q_1}{q_{01}} & \frac{q_0}{q_{01}} & 0 & 0 & \cdots & 0 \\ x_{20} - \frac{\delta}{q_0} & x_{21} & \frac{q_{01}}{q_{012}} & 0 & \cdots & 0 \\ x_{30} - \frac{\delta}{q_0} & x_{31} & x_{32} & \frac{q_{012}}{q_{0123}} & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ x_{n0} - \frac{\delta}{q_0} & x_{n1} & x_{n2} & x_{n3} & \cdots & \frac{q_{0,...,n-1}}{q_{0,...,n}} \end{bmatrix}$$

Set $z = (z_0, ..., z_n)$. We see immediately that $z_0 = q_0$ and $z_1 = q_1$, since the corresponding minors are lower triangular and $q_{0,...,n} = 1$. To calculate z_s we get, by expanding along the columns from the right, that $z_s = (-1)^s q_{0,...,s} \det(D_s)$ where D_s is the $s \times s$ submatrix from the upper left of A. Consider such a D_s:

$$D_s = \begin{bmatrix} -\frac{q_1}{q_{01}} & \frac{q_0}{q_{01}} & 0 & 0 & \cdots & 0 \\ x_{20} - \frac{\delta}{q_0} & x_{21} & \frac{q_{01}}{q_{012}} & 0 & \cdots & 0 \\ x_{30} - \frac{\delta}{q_0} & x_{31} & x_{32} & \frac{q_{012}}{q_{0123}} & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \frac{q_{0,...,s-2}}{q_{0,...,s-1}} \\ x_{s0} - \frac{\delta}{q_0} & x_{s1} & x_{s2} & x_{s3} & \cdots & x_{s(s-1)} \end{bmatrix}$$

Enumerating the columns $0, ..., s-1$, after multiplying column i by q_i (thus changing the determinant by a factor of $q_0 \cdots q_{s-1}$) for all i, observe that, by the construction of e_i, the sum of all rows except the last one are 0. For $i = 0, ..., s-2$ do successively the column operation: add column i to column $i+1$. This will not change the determinant, and observe that by the remark about the row sums, the new matrix will be lower triangular. Thus the determinant will be the product of the diagonal elements.

Diagonal entry number r will be equal to $x_{r0}q_0 - \delta + x_{r1}q_1 + ... + x_{r(r-1)}q_{r-1}$, which by construction equals $-\frac{q_{0,...,r-1}}{q_{0,...,r}}$. So we get

$$\frac{1}{q_0 \cdots q_{s-1}} \det(D_s) = (-1)^s \frac{q_0 \cdots q_s}{q_{0,...,s}}$$

implying that $z_s = q_s$.

The result now follows from the fact that $|z|^2 = q_0^2 + \cdots q_n^2$. $\square$

By this result, we have that a Euclidean volume of $\frac{\sqrt{q_0^2+\cdots q_n^2}}{n!}$ will be normalized to 1 in the lattice spanned by H. Using this we have:

Proposition 2.3.6. *The volume of P is $\frac{\delta^n}{q_0 \cdots q_n}$.*

Proof. The edges emanating from v_0 are spanned by the vectors

$$w_i = (-\frac{\delta}{q_0}, 0, ..., \frac{\delta}{q_i}, 0, ..., 0),$$

for $i = 1, ..., n$. The Euclidean volume of P will be $\frac{|w_1 \times \cdots \times w_n|}{n!}$. The corresponding matrix is

$$\begin{bmatrix} -\frac{\delta}{q_0} & \frac{\delta}{q_1} & 0 & 0 & \cdots & 0 \\ -\frac{\delta}{q_0} & 0 & \frac{\delta}{q_2} & 0 & \cdots & 0 \\ -\frac{\delta}{q_0} & 0 & 0 & \frac{\delta}{q_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ -\frac{\delta}{q_0} & 0 & 0 & 0 & \cdots & \frac{\delta}{q_n} \end{bmatrix}$$

We see that $w_1 \times \cdots \times w_n = (\frac{\delta^n}{q_1 \cdots q_n}, \frac{\delta^n}{q_0 q_2 \cdots q_n}, ..., \frac{\delta^n}{q_0 \cdots \hat{q_i} \cdots q_n}, \cdots, \frac{\delta^n}{q_0 \cdots q_{n-1}})$. This implies that

$$|w_1 \times \cdots w_n|^2 = \frac{\delta^{2n} q_0^2 + \delta^{2n} q_1^2 + ... + \delta^{2n} q_n^2}{q_0^2 \cdots q_n^2} =$$

giving

$$|w_1 \times \cdots w_n| = \frac{\delta^n}{q_0 \cdots q_n} \sqrt{q_0^2 + ... + q_n^2}.$$

Combinining this with the normalization yields the result.

$\square$

Finally we can return to intersection theory on $\mathbb{P}(q_0, ..., q_n)$. Recall that we wanted to have $D_0^n = \mathrm{Vol}(P_D) \frac{q_0^n}{\delta^n}$. Inserting the above gives $D_0^n = \frac{q_0^n}{q_0 \cdots q_n}$. Combining this with the previous calculations, we obtain a Bezout type theorem for weighted projective space:

Theorem 2.3.7 (Bézout's Theorem)**.** *Given n torus-invariant divisors $D_1, ..., D_n$ on $\mathbb{P}(q_0, ..., q_n)$, we have*

$$D_1 \cdots D_n = \frac{\Pi_{i=1}^n \deg D_i}{q_0 \cdots q_n}$$

2.4 Weighted projective plane

Specializing to the surface case, we can now determine some things about the divisors on the weighted projective plane.

Proposition 2.4.1.

$$K^2_{\mathbb{P}(k,m,n)} = \frac{(k+m+n)^2}{kmn}$$

Proof. This follows from Theorem 2.3.7, but to illustrate how to compute intersections for general singular toric varieties, we will instead prove this by using the formula from Corollary 1.8.4.

Example 1.3.7 describes the fan of $\mathbb{P}(k,m,n)$, the one-dimensional cones are $\mathrm{Cone}(e_1), \mathrm{Cone}(e_2), \mathrm{Cone}(e_3)$ in $N = \mathbb{Z}^3/\mathbb{Z}(k,m,n)$. We will describe this more explicitly: Choose $e, f \in \mathbb{Z}$ such that $me + nf = 1$. Then a $\mathbb{Z}$-basis for $\mathbb{Z}^3$ will by Remark A.0.4 be

$$v_1 = \begin{pmatrix} 0 \\ -f \\ e \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} k \\ m \\ n \end{pmatrix}$$

thus the quotient $N = \mathbb{Z}^3/v_3\mathbb{Z}$ is generated by v_1, v_2. Expressing the e_i in this basis we get

$$e_1 = v_2, \qquad e_2 = -nv_1 - kev_2 + ev_3, \qquad e_3 = mv_1 - kfv_2 + fv_3$$

So the images of the e_i in N will be

$$\rho_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \rho_2 = \begin{pmatrix} -n \\ -ke \end{pmatrix}, \rho_3 = \begin{pmatrix} m \\ -kf \end{pmatrix}$$

Now by the notation from Corollary 1.8.4, $d_i = d_{i-1,i+1} = -d_{i+1,i+2}$. Thus

$$K^2_{X_\Sigma} = \sum_{i=0}^{2} \left(\frac{1}{d_{i-1,i}} + \frac{1}{d_{i,i+1}} - \frac{d_i}{d_{i-1,i}d_{i,i+1}}\right)$$

$$= \sum_{i=0}^{2} \left(\frac{d_{i-1,i}d_{i+1,i+2} + d_{i,i+1}d_{i+1,i+2} + d^2_{i+1,i+2}}{d_{i-1,i}d_{i,i+1}d_{i+1,i+2}}\right)$$

$$= \frac{(d_{0,1} + d_{1,2} + d_{2,0})^2}{d_{0,1}d_{1,2}d_{2,0}}$$

Then

$$d_{0,1} = \det((0,1), (-n, -ke)) = n$$

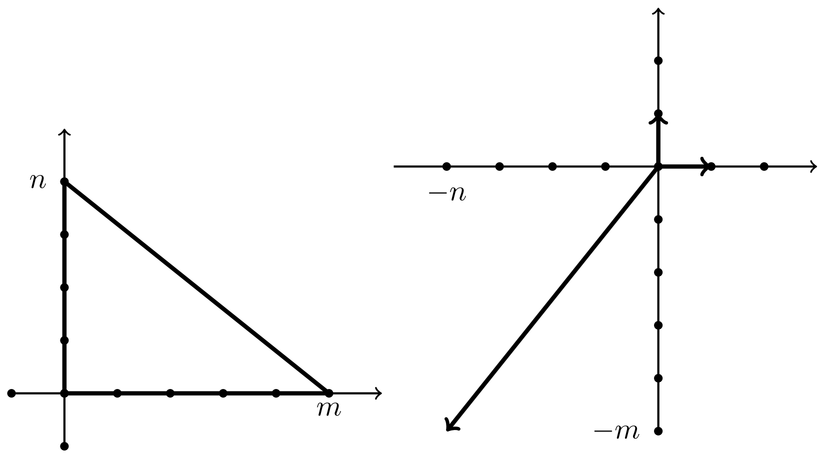

Figure 2.1: Left: Polytope giving $\mathbb{P}(1, m, n)$ Right: Its normal fan

$$d_{1,2} = \det((-n, -ke), (m, -kf)) = k$$
$$d_{2,0} = \det((m, -kf), (0, 1)) = m$$
$$K^2_{X_\Sigma} = \frac{(k+m+n)^2}{kmn}$$

□

If one would use our Bezout's theorem, we would easily obtain the same as above, since it implies that $C \cdot D = \frac{\deg C \deg D}{kmn}$. Since $\deg K_{\mathbb{P}(k,m,n)} = -k - m - n$ this gives $K^2_{\mathbb{P}(k,m,n)} = \frac{(k+m+n)^2}{kmn}$.

We now consider in more detail the polytope P giving $X_P = \mathbb{P}(k, m, n)$. Again we assume that k, m, n pairwise have no common factors.

From [Mor11] we have that $\mathrm{Conv}(0, me_1, ne_2)$ will be a polytope giving $\mathbb{P}(1, m, n)$. We want the more general polytope, but when we are in this special case, we will use this instead.

As in section 2.3 we get the following: In $\mathbb{R}^3$ consider the cone generated by the points $(0, 0, 0)$, $(mn, 0, 0)$, $(0, nk, 0)$, $(0, 0, mk)$. Intersecting this with the plane $xk + ym + zn = kmn$ gives a well-defined 2-dimensional polytope P with $X_P = \mathbb{P}(k, m, n)$.

Describing this more explicitly, consider the map:

$\phi : \mathbb{P}(k, m, n) \to \mathbb{P}^N$

defined by sending coordinates (x, y, z) on $\mathbb{P}(k, m, n)$, for each natural number solution (r, s, t) of the equation

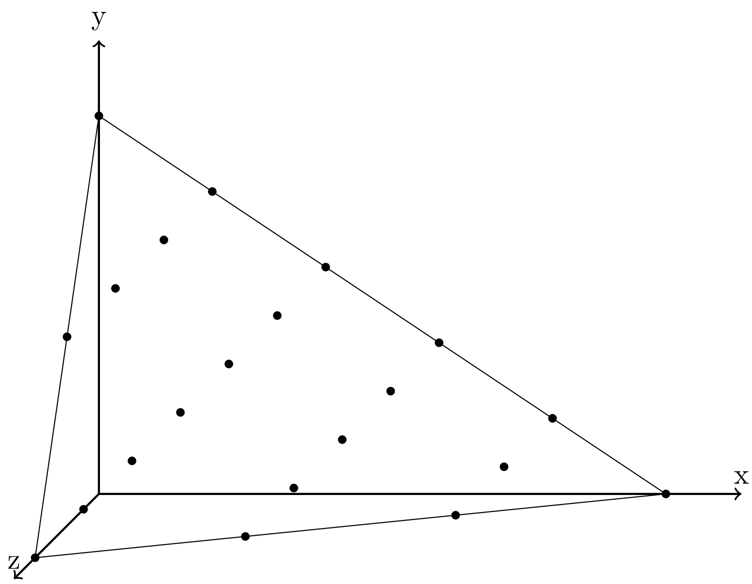

Figure 2.2: The polytope for $\mathbb{P}(2,3,5)$

$$kr + ms + nt = kmn \tag{2.5}$$

(N is the number of such solutions -1), to the coordinate $(x^r y^s z^t)$. Note that such (r,s,t) are in a one-to-one correspondence with lattice points in P. Also, by construction, this map is well defined. One can also consider the same map as going from $(\mathbb{C}^*)^3$, and in that case the closure of the image is exactly the toric variety X_P.

We now wish to find how many solutions we have, i.e., the number of lattice points in P.

The lattice points along the edges of P will be needed several times, and they are easy to describe, so we collect them in the following lemma.

Lemma 2.4.2. *The lattice points along the edges of P are the following:*

Points on the edge where $x = 0$ are $(0, nk - ln, lm)$ where $l = 0, ..., k$
Points on the edge where $y = 0$ are $(mn - jn, 0, jk)$ where $j = 0, ..., m$
Points on the edge where $z = 0$ are $(mn - im, ik, 0)$ where $i = 0, ..., n$

Proof. We only do the $x = 0$ case.

We wish to find integral solutions to $my + nz = kmn \Leftrightarrow my = n(km - z)$ with y, z positive. All solutions listed above obviously works. Since $\gcd(m,n) = 1$ n has to divide y. Letting l run as above we see that this is in fact all solutions.

$\square$

Proposition 2.4.3. *The number of lattice points of P is $\frac{kmn+k+m+n}{2} + 1$.*

For now, let the number of solutions be $f(m, n, k)$. To show the proposition we will consider the number of solutions (x, y) of $mx + ny = kmn - kj$ for j ranging from 0 to mn.

Lemma 2.4.4. *As j ranges from 0 to $mn - 1$, $kmn - kj$ ranges over all classes modulo mn.*

Proof. Assume $kmn - kj \equiv kmn - ki \pmod{mn}$. Then,

$$kj \equiv ki \pmod{mn}$$

hence $i \equiv j \pmod{mn}$ since $gcd(mn, k) = 1$. $\square$

If we now consider the general equation $mx + ny = s$ for any $s \in \mathbb{N}$. Let s_0 be the reduction of s modulo mn.

Lemma 2.4.5. *The number of solutions positive integral solutions to $mx + ny = s$ is $\frac{s-s_0}{mn} + 1$ or $\frac{s-s_0}{mn}$.*

Proof. First if $s \equiv 0 \pmod{mn}$ it is easy to see that there are $\frac{s}{mn} + 1$ solutions: Let $s = mnl$. Then $(nj, m(l - j))$ for $0 \leq j \leq l$ are all solutions since $\gcd(m, n,) = 1$. Then we have two cases:

If s_0 can be written as a linear combination $s_0 = am + bn$ where a, b are nonnegative integers, then our equation is equivalent to $m(x-a)+n(y-b) = s - s_0$ which by the above has $\frac{s-s_0}{mn} + 1$ solutions for $(x - a, y - b)$. If there were solutions with $0 < x < a$, then n has to divide $a - x$, so $a - x = nt$ giving $s_0 = (nt + x)m + bn > mn$, which is a contradiction.

Else, $s_0 + mn$ can be written as such a linear combination (since all numbers $\geq nm$ can be written this way), hence our equation is $m(x - a) + n(y - b) = s - mn - s_0$, which by the above has $\frac{s-s_0}{mn}$ solutions. $\square$

Then by combining Lemmas 2.4.4 and 2.4.5 we get that the s_0 will vary through all numbers less than mn, hence we get

$$f(m, n, k) = 1 + \sum_{j=0}^{mn-1} \frac{kmn - kj}{mn} - \sum_{j=0}^{mn-1} \frac{j}{mn} + g(m, n),$$

where $g(m, n)$ is the number of $s_0 \leq mn$ which cannot be written as a linear combination as in the proof above. The extra 1 corresponds to the single solution corresponding to $j = mn$. Writing out the sums we get

$$f(m, n, k) = \frac{kmn}{2} + \frac{k}{2} - \frac{mn}{2} + \frac{3}{2} + g(m, n) \tag{2.6}$$

The polytope $\text{Conv}(0, me_1, ne_2)$ giving $\mathbb{P}(1, m, n)$ has lattice points corresponding to all solutions (x, y) such that $nx + my \leq mn$. We see that these are in one to one correspondence with solutions (x, y, z) of $nx+my+z = mn$. Lemma 5.2.4 [Mor11] (or an easy counting argument) counts the number of these, yielding $f(m, n, 1) = \frac{(m+1)(n+1)}{2} + 1$.

Inserting this into (2.6) with $k = 1$, we get $g(m, n) = \frac{mn+m+n-1}{2}$. Inserting this back in the general (2.6) we get the result

$$f(m, n, k) = \frac{kmn + k + m + n}{2} + 1$$

We could also obtain this easier, using the extended machinery of Ehrhart polynomials:

Proof. The Ehrhart polynomial is given by $E_P(x) = Area(P)x^2 + \frac{1}{2}|\partial P \cap M|x + 1$. We know that the number of lattice points equals $E_P(1)$. By Lemma 2.4.2 $|\partial P \cap M| = k + m + n$. In the next section we compute the volume to be kmn. Combining these yields the result. $\square$

The lattice points in the plane $kx + my + nz = kmn$ form a 2-dimensional lattice L which, after choosing a point of origin, say $(mn, 0, 0)$, is isomorphic to the lattice $M = \{(x, y, z) \in \mathbb{Z}^3 |\ kx + my + nz = 0\}$ under $(x, y, z) \mapsto (x - mn, y, z)$. Thus for $N = \mathbb{Z}^3/(k, m, n)\mathbb{Z}$ the dual pairing $L \times N$ sends $(x, y, z), (r, s, t)$ to $r(x - mn) + sy + tz$.

Using this we can determine the associated divisor D_P. This is determined by the facet presentation, i.e. we want to find a_i such that P is given by

$$\langle m, u_i \rangle \geq -a_i$$

where the u_i are basis vectors for N. We can determine a_i by choosing a point on the corresponding facet, corresponding to where each of the coordinates are 0. Choosing

$$m_0 = (0, 0, km)$$

$$m_1 = (mn, 0, 0)$$

$$m_2 = (0, kn, 0)$$

we get

$$\langle m_0, u_0 \rangle = -mn$$
$$\langle m_1, u_1 \rangle = 0$$
$$\langle m_2, u_2 \rangle = 0$$

Thus $D_P = mnD_0$, which was what we expected.

If we now wish to find the normal fan of P, we wish to find vectors u_i in N orthogonal to the edges of P. The edges are generated by

$$v_0 = (0, -n, m)$$
$$v_1 = (-n, 0, k)$$
$$v_2 = (-m, k, 0),$$

giving the equalities in the quotient N,

$$u_0 = (0, -m, -n) = (k, 0, 0)$$
$$u_1 = (-k, 0, -n) = (0, m, 0)$$
$$u_2 = (-k, -m, 0) = (0, 0, n),$$

which are exactly the 1-dimensional cones from Example 1.3.7, so we recover the normal fan as expected.

2.5 Degree of duals

We now wish to calculate the degree of the dual variety of $\mathbb{P}(k, m, n)$. We have from Proposition 1.10.3

$$\deg X^\vee_{\mathbb{P}(k,m,n)} = 3\operatorname{Vol}(P) - 2E(P) + \sum_{v \text{ vertex } \in P} \operatorname{Eu}(v)$$

By Proposition 2.1.7 the singularities of $\mathbb{P}(k, m, n)$ will always be isolated, at the points corresponding to the vertices. Since the Euler-obstruction equals 1 on the smooth locus of a variety, we need to determine the Euler obstruction of the vertices. Recall that this was given by Proposition 1.11.7:

$$\operatorname{Eu}(v) = 2 - \operatorname{Vol}(P) + \operatorname{Vol}(\operatorname{Conv}(P \setminus v))$$

Without loss of generality we still consider the vertex $(mn, 0, 0)$ as the origin of our plane. We will find two different bases for the lattice, each containing a vector generating one of the edges.

Lemma 2.5.1. *There exists a solution of* (2.5) *of the form* $(r, s, 1)$

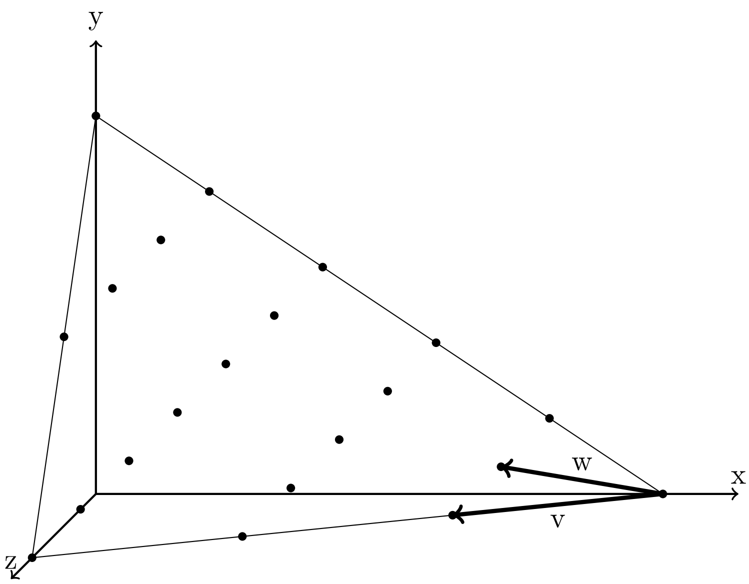

Figure 2.3: $\mathbb{P}(2,3,5)$ with the basis $\{v, w\}$

Proof. We wish to find a solution to $kr + ms + n = kmn$. If we consider this modulo k we see that for any $0 \leq s_0, s_1 \leq k-1$,

$$ms_0 + n \equiv ms_1 + n \pmod{k}$$

Thus $s_0 \equiv s_1 \pmod{k}$.

Hence letting s vary from 0 to $k-1$ we see that all modulo classes will appear, in particular there is a s such that $ms + n \equiv 0 \pmod{k}$. Therefore $ms + n = kv$. Then choosing $r = mn - v$ proves the lemma. $\square$

From Lemma 2.5.1 we obtain there exist solutions of (2.5) of the form $(a, 1, b)$ and $(c, d, 1)$. Pick these such that b and d are the least possible. Then we can consider the lattice vectors

$v = (-n, 0, k)$ (along the edge $y = 0$)
$w = (a - mn, 1, b)$

Lemma 2.5.2. *The vectors v, w form a basis for the lattice spanned by the lattice points of the plane* (2.5)*.*

Proof. By Lemma A.0.2 it is enough to show that $T(v, w) \cap M = 0$. Assume that $sv + tw = l$ is a lattice point, where $0 \leq s, t < 1$. Then by considering the y-coordinate we see that $t = 0$. But then $l = sv$, and by Lemma 2.4.2 we see that $s = 0$. $\square$

Remark 2.5.3. Note that this is only the 2-dimensional case of Proposition 2.3.4.

Similarly $v' = (-m, k, 0)$ and $w' = (c-mn, d, 1)$ will be a basis corresponding to the other edge lying next to our chosen vertex.

From Proposition 2.3.5 a Euclidean area of $\frac{\sqrt{k^2+m^2+n^2}}{2}$ will have normalized area of 1.

From Lemma 2.4.2 we get that the length of the edges of P is $k + m + n$. Also the volume of our polytope P is mnk, by calculationing the area of P the triangle (for instance $|(0, -nk, mk) \times (-nm, 0, km)|$). To summarize

$$\mathrm{Vol}(P) = kmn$$

$$E(P) = k + m + n$$

What remains is finding the Euler-obstruction of the vertices, we will try to calculate this as well.

Call the polytope we get when we remove a vertex P'. Consider the line l through $(mn - n, 0, k)$ spanned by the vector $w - v = (a + n - nm, 1, b - k)$ (Alternatively this is the line through the points $(a, 1, b)$ and $(nm - n, 0, k)$. This will by definition be a supporting halfspace of P', since P' is the convex hull of the remaining lattice points. Similarly the line l' through $(mn - m, k, 0)$ spanned by $w' - v'$ will also be a supporting halfspace. If these lines intersect in a lattice point (or are the same line), then P' is defined by these and we can calculate the new area. In general there can be any number of edges to the new polytope, and we will need more general methods to compute this.

Now we can find some Euler-obstructions:

Proposition 2.5.4. *Consider* $\mathbb{P}(k, m, n)$. *Then* $\mathrm{Eu}(0, 0, mk) = 0$ *if and only if* $m + k \equiv 0 \pmod n$

Proof. We wish to find solutions of the form $(1, b, a), (d, 1, c)$ with b, d minimal. That is

$$k + bm \equiv 0 \pmod n$$

$$dk + m \equiv 0 \pmod n$$

We see that if $k + m \equiv 0 \pmod n$ then we can choose $b = d = 1$, and we will remove two triangles spanned by basis vectors, so the Euler obstruction is zero by the above. Conversely if the removed area is two, then the points $(1, b, a), (d, 1, c)$ coincide, so $b = d = 1$, hence $k + m \equiv 0 \pmod n$. □

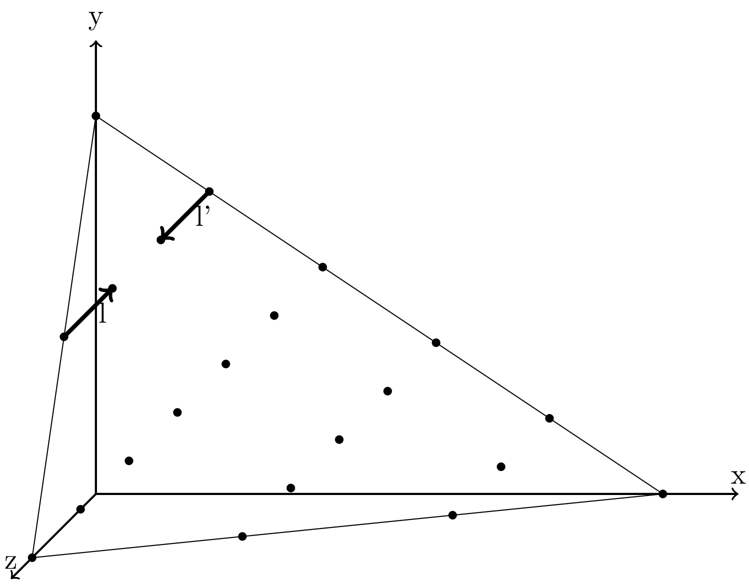

Figure 2.4: For the vertex $(0, 10, 0)$ in $\mathbb{P}(2, 3, 5)$ we see that the lines l, l' are the same line. We see that the removed area consists of 3 triangles spanned by basis vectors, hence $\mathrm{Eu}(0, 10, 0) = 2 - 3 = -1$

Remark 2.5.5. Of course similar results also holds for the other vertices, by cyclicly permuting k, m, n.

In general one cannot find a closed formula for the Euler-obstruction, as it is realated to the behaviour of continued fractions, for which there is no closed formula, we will see this in detail in the next chapter. However in special cases, where there are relations between the numbers k, m, n, it is possible to find a formula, as the following proposition shows.

Proposition 2.5.6. $\deg \mathbb{P}(m, n, m+n)^\vee = 3mn(m+n) - 5(m+n) + 4$

Proof. Consider $\mathbb{P}(m, n, m+n)$, we will find the degree of $\mathbb{P}(m, n, m+n)^\vee$.

From Proposition 1.10.3 we have

$$\deg \mathbb{P}(m, n, m+n)^\vee = 3mn(m+n) - 2(2m+2n) + \sum_{v \in P} \mathrm{Eu}(v)$$

Now we proceed as described above.

From Lemma 2.5.4 we get

$$\mathrm{Eu}(0, 0, mn) = 0$$

Consider now $(n(m+n),0,0)$. As above we need to find minimal b,d (for lattice points $(a,1,b)$ and $(c,d,1)$) such that

$$n + b(n+m) \equiv 0 \pmod{m}$$

$$dn + m + n \equiv 0 \pmod{m}$$

Reducing, this amounts to

$$(b+1)n \equiv 0 \pmod{m}$$

$$(d+1)n \equiv 0 \pmod{m}$$

So $b = m-1 = d$.

Solving for a and c we obtain $a = n(m+n) - (m+n) + 1$ and $c = n(m+n) - n - 1$. A calculation now shows that the two lines l, l' we get when we remove this vertex will be the line through $\begin{pmatrix} n(m+n)-m-n \\ 0 \\ m \end{pmatrix}$ spanned by $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and the line through $\begin{pmatrix} n(m+n)-n \\ m \\ 0 \end{pmatrix}$ spanned by $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$.

Now we have that

$$\begin{pmatrix} n(m+n)-m-n \\ 0 \\ m \end{pmatrix} + m \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} n(m+n)-n \\ m \\ 0 \end{pmatrix}.$$

Thus the two lines are really the same, hence it defines the new polytope P', so the total area removed will be the area of the triangle spanned by the vectors

$$\begin{pmatrix} -m-n \\ 0 \\ m \end{pmatrix} \qquad \begin{pmatrix} -n \\ m \\ 0 \end{pmatrix}.$$

A calculation shows that the area is given by

$$4A^2 = m^2(m^2 + n^2 + (m+n)^2$$

Hence the normalized volume is m, giving a total Euler obstruction of

$$\mathrm{Eu}(n(m+n),0,0) = 2 - m$$

A similiar calculation for $(0, m(m+n), 0)$ yields

$$\mathrm{Eu}((0, m(m+n), 0) = 2 - n$$

Hence the degree we are looking for is

$$\begin{aligned}\deg \mathbb{P}(m,n,m+n)^\vee &= 3mn(m+n) - 2(2m+2n) - (m+n) + 4 \\ &= 3mn(m+n) - 5(m+n) + 4\end{aligned}$$

$\square$

Proposition 2.5.7. *For odd $m > 1$,*

$$\deg \mathbb{P}(m-2, m, m+2)^\vee = 3m^3 - 19m + 3$$

Proof. Consider $\mathbb{P}(m-2,m,m+2)$. Again we will find the Euler obstruction of the vertices.

For $(0,0,m(m-2))$ we wish to find lattice points $(1,b,a)$ and $(d,1,c)$ with minimal b,d. This gives

$$m - 2 + bm \equiv 0 \pmod{m+2}$$

$$d(m-2) + m \equiv 0 \pmod{m+2}$$

which gives:

$$-2(b+2) \equiv 0 \pmod{m+2}$$

$$-2(2d+1) \equiv 0 \pmod{m+2},$$

resulting in $b = m$ and $d = \frac{m+1}{2}$.

One calculates that $a = m^2 - 3m + 1$ and $c = m^2 - \frac{5}{2}m + \frac{1}{2}$. Then we get a basis consisting of $(0,-m-2,m), (1,-2,1)$ and

$$(1,b,a) + \frac{m-1}{2}(1,-2,1) =$$

$$(1,m,m^2-3m+1) + \frac{m-1}{2}(1,-2,1) = (\frac{m+1}{2}, 1, m^2 - \frac{5}{2}m + \frac{1}{2}) = (d,1,c),$$

so that these lines define P'. Then we get:

$$\mathrm{Eu}(0,0,m(m-2)) = 2 - \frac{m+3}{2} = \frac{-m+1}{2}$$

Similarly solving for lattice points $(a,1,b)$ and $(d,1,c)$ with minimal b,c one gets $b = \frac{m-3}{2}, c = m-4$, implying that $a = m^2 + \frac{3m}{2} - \frac{3}{2}, d = m^2 + m + 1$, so we obtain a basis consisting of $(m, 2-m, 0), (1-2, 1)$. Then since

$$(m^2+m, m-2, 0) + \frac{m-3}{2}(1,-2,1) = (m^2 + \frac{3m}{2} - \frac{3}{2}, 1, \frac{m-3}{2})$$

we can again calculate the area yielding

$$\mathrm{Eu}(m(m+2),0,0) = 2 - \frac{m-1}{2} = \frac{-m+5}{2}$$

By Remark 2.5.5,

$$\mathrm{Eu}(0,(m-2)(m+2),0) = 0.$$

So $\deg \mathbb{P}(m-2,m,m+2)^\vee = 3(m-2)(m+2)m - 2(3m) - m + 3 = 3m^3 - 19m + 3$

$\square$

Chapter 3

Resolution of singularities

3.1 Continued fractions and resolution of singularities

The presentation in this section mainly follows [PP07], but with a view towards [CLS11] and [Dai06]. We refer some propositions and write out some proofs.

Given a rational number λ, we can consider two different expansions as a continued fraction:

$$\lambda = b_1 - \frac{1}{b_2 - \frac{1}{...-\frac{1}{b_r}}} = a_1 + \frac{1}{a_2 + \frac{1}{...+\frac{1}{a_s}}}$$

the first is called the Hirzebruch-Jung continued fraction and will be denoted by $\lambda = [b_1, ..., b_r]^-$. The second is called the Euclidean continued fraction and is denoted by $\lambda = [a_1, ..., a_s]^+$.

We will say that the length of the continued fraction $\lambda = [b_1, ..., b_r]^\pm$ is r.

Given λ, one can calculate the Euclidean continued fraction by the Euclidean algorithm(hence the name), while one can calculate the HJ fraction by doing a modified Euclidean algorithm:

Let $\lambda = \frac{d}{k}$. Set $r_0 = k$. Find $r_1, q_1 \in \mathbb{N}$ such that $d = r_0q_1 - r_1$ where $0 < r_1 < r_0$. Then find r_2, q_2 such that $r_0 = r_1q_2 - r_2$ with $0 < r_2 < q_2$. Proceed in general to find r_i, q_i such that $r_{i-1} = r_iq_{i+1} - r_{i+1}$, where $0 < r_i < q_i$. Then $\frac{d}{k} = [q_1, ..., q_s]^-$.

For theoretical reasons we will also construct these another way. Define two sequences of polynomials with integer coefficients inductively by

$$Z^{\pm}(\emptyset) = 1$$

$$Z^{\pm}(x) = x$$

$$Z^{\pm}(x_1,...,x_n) = x_1 Z^{\pm}(x_2,...,x_n) \pm Z^{\pm}(x_3,...,x_n) \text{ when } n \geq 2$$

Proposition 3.1.1. $[x_1,...,x_n]^{\pm} = \frac{Z^{\pm}(x_1,...,x_n)}{Z^{\pm}(x_2,...,x_n)}$ *for* $n \geq 1$

Proof. We prove this by induction. The case $n = 1$ is clear. Assume the proposition is true for $n-1$. Then

$$\begin{aligned} \frac{Z^{\pm}(x_1,...,x_n)}{Z^{\pm}(x_2,...,x_n)} &= \\ \frac{x_1 Z^{\pm}(x_2,...,x_n) \pm Z^{\pm}(x_3,...,x_n)}{Z^{\pm}(x_2,...,x_n)} &= \\ x_1 \pm \frac{Z^{\pm}(x_3,...,x_n)}{Z^{\pm}(x_2,...,x_n)} &= \\ x_1 \pm \frac{1}{\frac{Z^{\pm}(x_2,...,x_n)}{Z^{\pm}(x_3,...,x_n)}} &= \\ x_1 \pm \frac{1}{[x_2,...,x_n]^{\pm}} &= \\ [x_1,...,x_n]^{\pm} & \end{aligned}$$

$\square$

Proposition 3.1.2. $Z^{\pm}(x_1,...,x_n) = Z^{\pm}(x_1,...,x_{n-1})x_n \pm Z^{\pm}(x_1,...,x_{n-2})$ *for* $n \geq 2$.

Proof. Again we proceed by induction. The case $n = 2$ is obvious. Assume it holds for all $k \leq n$. Then

$$\begin{aligned} Z^{\pm}(x_1,...,x_{n+1}) &= \\ x_1 Z^{\pm}(x_2,...,x_{n+1}) \pm Z(x_3,...,x_{n+1}) &= \\ x_1(Z^{\pm}(x_2,...,x_n)x_{n+1} \pm Z^{\pm}(x_2,...,x_{n-1}))\pm & \\ (x_{n+1}Z^{\pm}(x_3,...,x_n) \pm Z^{\pm}(x_3,...,x_{n-1})) &= \\ x_{n+1}(x_1 Z^{\pm}(x_2,...,x_n) \pm Z^{\pm}(x_3,...,x_n))\pm & \\ (x_1 Z^{\pm}(x_2,...,x_{n-1}) \pm Z^{\pm}(x_3,...,x_{n-1})) &= \\ Z^{\pm}(x_1,...,x_n)x_{n+1} \pm Z^{\pm}(x_1,...,x_{n-1}) & \end{aligned}$$

where we use the induction hypothesis in the second equality. $\square$

Taking a short break from the general theory, we will also need the following result, a proof can be found in [PP07].

Lemma 3.1.3. *Assume $\lambda > 1 \in \mathbb{Q}$ has HJ-fraction $\lambda = [(2)^{m_1}, n_1 + 3, (2)^{m_2}, n_2 + 3, ..., (2)^{m_{s+1}}]^-$ where $n_i \geq 0$ and $(2)^{m_i}$ denotes m_i consecutive 2's, where $m_i \geq 0$ (i.e. an empty string of 2's also gives a m_i).*

Then $\frac{\lambda}{\lambda-1} = [m_1+2, (2)^{n_1}, m_2+3, (2)^{n_2}, m_3+3, ..., m_s+3, (2)^{n_s}, m_{s+1}+2]^-$.

Example 3.1.4. If $\lambda = [2, 2, 2, 3, 4, 2, 2, 3]^-$ then

$$m_1 = 3,\ m_2 = 0,\ m_3 = 2,\ m_4 = 0$$

$$n_1 = 0,\ n_2 = 1,\ n_3 = 0$$

so $\frac{\lambda}{\lambda-1} = [5, 3, 2, 5, 2]$.

Using the above, we prove the following result on lengths of HJ-fractions which will be needed later.

Proposition 3.1.5. *Let $\frac{d}{k} = [b_1, ..., b_s]^-$. Then*

$$s = 1 + \sum_{i=1}^{r} (c_i - 2)$$

where $\frac{d}{d-k} = [c_1, ..., c_r]^-$.

Proof. Setting $\lambda = \frac{d}{k}$, we have that $\frac{\lambda}{\lambda-1} = \frac{d}{d-k}$, so the continued fractions are related as in Lemma 3.1.3. Consider all $c_i \neq 2$ (i.e., all $c_i \geq 3$). Assume there are t of these. From Lemma 3.1.3, each of these contribute $c_i - 3$ to the length of the HJ-fraction of $\frac{d}{k}$. Also, each of the $t+1$ (possibly empty) strings of 2's each contribute one to the length of the HJ fraction of $\frac{d}{k}$. From the lemma one thus sees that the total length $s = t + 1 + \sum_{i,c_i \neq 2}(c_i - 3) = 1 + \sum_{i,c_i \neq 2}(c_i - 2)$. Since $\sum_{i,c_i=2}(c_i - 2) = 0$ we can add this, yielding $s = 1 + \sum_i (c_i - 2)$, which was what we wanted to show. $\square$

It turns out that this result is also in [Oda88, Lemma 1.22].

Back to the general theory, given a 2-dimensional lattice $L \cong \mathbb{Z}^2$ and a line l through the origin of $L_{\mathbb{R}}$ with slope $\lambda \in \mathbb{Q}$, one can consider the cone generated by the positive x-axis and this line. In fact all 2-dimensional strongly convex rational polyhedral cones are of this form:

Proposition 3.1.6. *Given any 2-dimensional cone σ one can choose a basis $\{e_1, e_2\}$ for the lattice L such that in this basis $\sigma = Cone(e_1, ke_1 + de_2)$ where $d > k > 0$ and gcd(d, k)= 1.*

Proof. By Proposition A.0.5 we can always choose a primitive generator of an edge of σ, v, as the first basis vector of our lattice. Let $(e_1 = v, e_2')$ be a basis for the lattice. The other facet of the cone will in this basis be generated by a vector $w = ae_1 + be_2'$. Now let $d = |b|$ and $k = a \mod d$, where $0 < k < d$.

Then $w = (a-k+k)e_1 + \operatorname{sign}(b)de_2' = ke_1 + d(\operatorname{sign}(b)e_2' + \frac{a-k}{d}e_1)$. Thus we see that in the new basis $\{e_1, e_2 = \operatorname{sign}(b)e_2' + \frac{a-k}{d}e_1\}$, $w = ke_1 + de_2$. □

Definition 3.1.7. We say that a cone σ is of type (d, k) if it can be written as in Proposition 3.1.6 with parameters d, k. We will use the method from the proof above to turn a cone into a (d, k)-cone.

Note also that some literature, notably [CLS11] and [Ful93], use a different convention for a (d, k)-cone, so that results sometimes look a bit different.

Now assume that the lattice we are in is the familiar character lattice M with basis e_1, e_2, we also have its dual N with induced dual basis e_1^*, e_2^*.

Proposition 3.1.8. *Assume $\sigma^\vee$ is a (d, k)-cone in $M_\mathbb{R}$ with respect to $\{e_1, e_2\}$. Then σ is a $(d, d-k)$-cone in $N_\mathbb{R}$ with respect to the basis $\{e_2^*, e_1^* - e_2^*\}$.*

Proof. Recall that the dual is defined as $\sigma^\vee = \{m \in M_\mathbb{R} | \langle m, u\rangle \geq 0\, \forall u \in \sigma\}$. Since $\sigma = \operatorname{Cone}(e_1, ke_1 + de_2)$, we see that $xe_1^* + ye_2^* \in M$ is in $\sigma^\vee$ if

$$x \geq 0$$

$$dy + kd \geq 0$$

This is exactly $\operatorname{Cone}(e_2^*, de_1^* - ke_2^*) = \operatorname{Cone}(e_2^*, (d-k)e_2^* + d(e_1^* - e_2^*))$ □

Now given a (d, k)-cone $\sigma \subset N_\mathbb{R}$, we can consider the supplementary cone σ_0 which is $\operatorname{Cone}(-e_1, (d, k))$. That is $\sigma \cup \sigma_0$ is the halfplane $y \geq 0$. Rotating the coordinate system 90 degrees clockwise turns σ_0 into $\operatorname{Cone}((0, 1), (d, -k))$ which is isomorphic to the dual cone $\sigma^\vee$, by Proposition 3.1.8. Thus the dual will be isomorphic to the supplementary cone.

Define $K(\sigma) = \operatorname{Conv}(\sigma \cap (N \setminus \{0\}))$. Let $P(\sigma)$ be the boundary of $K(\sigma)$, $V(\sigma)$ the set of vertices and $E(\sigma)$ the set of edges. $P(\sigma)$ is a connected polygonal line with endpoints coinciding with the generators of σ. We index the edges such that the first edge E_1 is the edge bordering the x-axis and then clockwise along the boundary.

Let $A_0 = (1, 0)$. Define A_i, $i \geq 0$ as the sequence of integral points as one goes along the enumerated edges of $P(\sigma)$. Since λ is rational this is a finite sequence, the last point we denote by A_{r+1}. [CLS11, Thm 10.2.8] shows that the primitive generators of OA_i is the Hilbert basis of the semigroup $\sigma \cap N$.

By construction and Lemma A.0.2, we see that each pair (OA_i, OA_{i+1}) is a basis for N. Also the slopes of the set $\{OA_i\}$ have to be increasing with increasing i, since A_i are on the boundary of a convex set. Thus we have relations:

$$rOA_{i-1} + sOA_i = OA_{i+1}$$

$$tOA_i + uOA_{i+1} = OA_{i-1}$$

$$\Rightarrow (rt + s)OA_i + (ru - 1)A_{i+1} = 0$$

$$\Rightarrow rt + s = 0,\ ru = 1$$

If $r = u = 1$ we get $s = -t$ and

$$sOA_i + OA_{i-1} = OA_{i+1}$$

But this contradicts the increasing of the slopes. Thus we must have $r = u = -1$ and $s = t$ resulting in the relation

$$OA_{i-1} + OA_{i+1} = b_i OA_i \tag{3.1}$$

By convexity we must have $b_i \geq 2$.

Proposition 3.1.9. *$OA_i = Z^-(b_1, ..., b_{i-1})OA_1 - Z^-(b_2, ..., b_{i-1})OA_0$ for $i \geq 2$. In particular the slope of $OA_{r+1} = \lambda$ in the basis $(-OA_0, OA_1)$ equals $[b_1, ..., b_r]^-$.*

Proof. The first assertion is proved by induction on i. For $i = 2$ this is just the relation above. For general i we have

$$\begin{aligned} OA_{i+1} &= b_i OA_i - OA_{i-1} \\ = b_i(Z^-(b_1, ..., b_{i-1})OA_1 &- Z^-(b_2, ..., b_{i-1})OA_0) \\ -Z^-(b_1, ..., b_{i-2})OA_1 &+ Z^-(b_2, ..., b_{i-2})OA_0 \\ = OA_1(b_i Z^-(b_1, ..., b_{i-1}) &- Z^-(b_1, ..., b_{i-2})) \\ -OA_0(b_i Z^-(b_2, ..., b_{i-1}) &- Z^-(b_2, ..., b_{i-2})) \\ = OA_1 Z^-(b_1, ..., b_i) &- OA_0 Z^-(b_2, ..., b_i), \end{aligned}$$

where the last equality is by Proposition 3.1.2

That $[b_1, ..., b_r]^-$ is the slope in the chosen basis follows directly from Proposition 3.1.1. $\square$

Observation 3.1.10. *If now* $[b_1, ..., b_r]^- = \frac{e}{f}$ *for some* e, f*, then we see that the line* OA_{r+1} *is generated by both* (k, d) *and* $-fOA_0 + eOA_1$*. Since* $d > k$*,* $OA_1 = (1, 1)$ *in the standard basis, hence* $(k, d) = (e - f, e)$*, which results in* $[b_1..., b_r]^- = \frac{d}{d-k}$*.*

Now we finally can relate this to toric varieties. Given a singular affine toric surface U_σ we will describe how to resolve its singularity.

Definition 3.1.11. Given a singular variety X, a resolution of singularities is a smooth variety Y with a proper morphism $\phi : Y \to X$ which induces an isomorphism on the smooth locus: $Y \setminus \phi^{-1}(X_{sing}) \cong X \setminus X_{sing}$.

A resolution of singularities for X is called minimal if for every other resolution of singularities $\psi : Z \to X$ there exists a $\rho : Z \to Y$ such that the diagram is commutative:

$$\begin{tikzcd} & Y \arrow[d, "\phi"] \\ Z \arrow[ur, dotted, "\rho"] \arrow[r, "\psi"] & X \end{tikzcd}$$

For surfaces, being a minimal resolution of singularities turns out to be equivalent to no component of the exceptional divisor $E = \phi^{-1}(O)$ having self-intersection -1.

In general resolutions of singularities exist in characteristic 0. In the toric surface case this can be constructed explicitly using the above. Given σ construct the points A_i as above. Let $\sigma_i = \text{Cone}(OA_i)$. Let Σ be the fan with 2-dimensional cones $\text{Cone}(\sigma_i, \sigma_{i-1})$ for $i = 0, ..., r$. The identity map on the lattice N induces toric morphisms $U_{\sigma_i} \to U_\sigma$ which glue to a morphism $\phi : X_\Sigma \to U_\sigma$.

Proposition 3.1.12. *The morphism* ϕ *is a resolution of singularities for* U_σ*.*

Proof. As remarked above, each pair OA_i, OA_{i+1} is a basis for the lattice, hence each cone in the fan Σ is smooth. Thus X_Σ is smooth. It is the identity, except at its singular point, thus it is a resolution of singularities. $\square$

In fact it turns out that this is the minimal resolution.

Proposition 3.1.13. *The exceptional divisor* $\phi^{-1}(0)$ *has* r *components* $D_1, ..., D_r$ *and the self-intersection of* D_i *equals* $-b_i$*. Hence* ϕ *is the minimal resolution.*

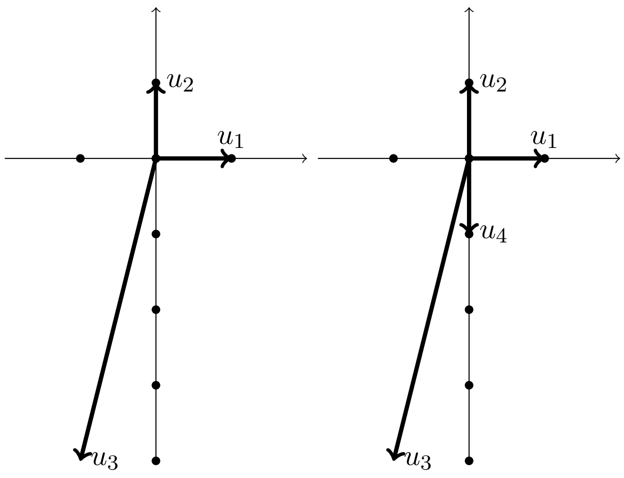

Figure 3.1: Left: Fan for $\mathbb{P}(1,1,4)$ Right: Fan for $\mathcal{H}_4$.

Proof. From Proposition 1.8.3 we see that $D_i \cdot D_i = -\frac{\det(OA_{i-1}, OA_{i+1})}{\det(OA_{i-1}, OA_i)\det(OA_i, OA_{i+1})}$.

$\det(OA_{i-1}, OA_i) = \det(OA_i, OA_{i+1}) = 1$, since both pairs are bases of the lattice. Since $OA_{i-1} + OA_{i+1} = b_i OA_i$ we get that

$\det(OA_{i-1}, OA_{i+1}) = \det(OA_{i-1}, b_i OA_i - OA_{i-1}) = \det(OA_{i-1}, b_i OA_i) = b_i \Rightarrow D_i^2 = -b_i$

That ϕ is minimal follows from the fact that $b_i \geq 2$. $\square$

Example 3.1.14. Consider $\mathbb{P}(1,1,m)$. Its normal fan of this will have 1-dimensional cones generated by $u_1 = e_1, u_2 = e_2, u_3 = -e_1 - ne_2$, where e_1, e_2 are the standard basis vectors of the plane.

Cone(u_1, u_3) and Cone(u_1, u_2) are smooth, so the only place one does anything will be Cone(u_1, u_3) =Cone$(e_2, -e_1 - ne_2)$. We see we have to add $u_4 = Cone(-e_2)$ to get a smooth fan. The resulting smooth variety is called the Hirzeburch surface $\mathcal{H}_n$.

Turning Cone(u_1, u_3) into the form of a (d,k)-cone we use the proof of Proposition 3.1.6. Choose new basis $v_1 = e_1, v_2 = -e_1 - e_2$. Then the singular cone will ba a $(n, n-1)$-cone with respect to this basis. Picking $m = 4$, we get a $(4,3)$-cone. We see that

$$\begin{pmatrix} -1 \\ -4 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 4 \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

which comes from the fact that $\frac{4}{4-3} = 4 = [4]^-$.

Euclidean(ordinary) continued fractions also appear in this setting, giving a sort of duality property. Again we must first do some work.

Assume $\gcd(d,k) = 1$. Let $\frac{d}{k} = [a_1, ..., a_r]^+$ and define associated integer sequences P_i and Q_i for $0 \leq i \leq r$ by

$P_0 = 1$, $P_1 = a_1$, $P_i = a_i P_{i-1} + P_{i-2}$
$Q_0 = 0$, $Q_1 = 1$, $Q_i = a_i Q_{i-1} + Q_{i-2}$

Proposition 3.1.15. *The P_i and Q_i are increasing sequences of natural numbers satisfying $[a_1, ..., a_i]^+ = \frac{P_i}{Q_i}$ and $P_{i-1}Q_i - P_iQ_{i-1} = (-1)^i$ for $1 \leq i \leq r$*

Proof. The first equality is proved by induction on the length i. Observe that continued fractions are well-defined for all rational numbers a_i, hence assuming the equality for n we get $[a_1, ..., a_{n+1}]^+ = [a_1, ..., a_n + \frac{1}{a_{n+1}}]^+ = \frac{(a_n+\frac{1}{a_{n+1}})P_{n-1}+P_{n-2}}{(a_n+\frac{1}{a_{n+1}})Q_{n-1}+Q_{n-2}} = \frac{P_n+\frac{P_{n-1}}{a_{n+1}}}{Q_n+\frac{Q_{n-1}}{a_{n+1}}} = \frac{a_{n+1}P_n+P_{n-1}}{a_{n+1}Q_n+Q_{n-1}} = \frac{P_{n+1}}{Q_{n+1}}$.

The second equality follows directly by induction since

$P_iQ_{i+1} - P_{i+1}Q_i = P_i(a_{i+1}Q_i + Q_{i-1}) - Q_i(a_{i+1}P_i + P_{i-1}) = P_iQ_{i-1} - Q_iP_{i-1} = (-1)^{i+1}$ □

Now given a type (d,k) cone $\sigma \subset N$ defined by a line l one can consider also the cone $\sigma' = Cone((0,1),(k,d))$. Let $\Theta = Conv(\sigma \cap (N \setminus \{0\})$ and $\Theta' = Conv(\sigma' \cap (N \setminus \{0\})$. Then construct vectors as follows:

Let $u_{-1} = (1,0), u_0 = (0,1), u_i = Q_iu_{-1} + P_iu_0$ for $i = 1, ..., r$. Then we have:

Proposition 3.1.16. *Θ has vertex set $\{u_r\} \cup \{u_j | j$ odd $\}$ while Θ' has vertex set $\{u_r\} \cup \{u_j | j$ even $\}$. For all $1 \leq i \leq r$ $u_{i-2}u_i$ is and edge of the respective convex hull containing $a_i + 1$ lattice points.*

Proof. We prove this by induction on r. Since $u_r = ku_{-1} + du_0$ and $d > k$, the ray starting at u_{-1} and going through $u_1 = u_{-1} + a_1u_0$ intersects the line l at a point between $u_{-1} + \lfloor \frac{d}{k} \rfloor u_0$ and $u_{-1} + (\lfloor \frac{d}{k} \rfloor + 1)u_0$. Since $\lfloor \frac{d}{k} \rfloor = a_1$ we see that the segment $u_{-1}u_1$ is an edge of Θ.

By Proposition 3.1.15 (u_{i-1}, u_i) is a basis for all $0 \leq i \leq r$. In particular $u_0 = (0,1), u_1 = (1, a_1)$ is a basis. Now

$$\begin{pmatrix} k \\ d \end{pmatrix} = k \begin{pmatrix} 1 \\ a_1 \end{pmatrix} + (d - ka_1) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Then we can repeat the entire process above in this new basis. Then we want the continued fraction of $\frac{k}{d-ka_1} = \frac{1}{\frac{d}{k}-a_1} = [a_2, ..., a_r]^+$. We get the same vectors u_i, thus by induction the vertices of Θ, Θ' are given by u_i.

Now $u_i - u_{i-2} = a_i u_{i-1}$ hence the edge has $a_i + 1$ lattice points. $\square$

From this we see that ordinary continued fractions give the vertices of the convex hulls of both a cone and its dual, while HJ-fractions give all the lattice points on the edges of one of them.

Example 3.1.17. Consider $\mathbb{P}(1, m, n)$. The normal fan will have 1-dimensional cones generated by $v_1 = e_1, v_2 = e_2, v_3 = -ne_1 - me_2$ where e_1, e_2 are the standard basis vectors of the plane.

Assume without loss of generality that $m > n$. Then $\mathrm{Cone}(v_1, v_3)$ is a $(m, m-n)$-cone with respect to the basis $\{v_1 = e_1, v_2 = -e_1 - e_2\}$. By considering the continued fraction of $\frac{m}{n-n}$ we can apply the previous result to the third quadrant to obtain vertices u_i on both sides of the vector v_3. Consider the set of all lattice points $\{A_j\}$ on edges $u_{i-2}u_i$. Refine the fan by adding $\mathrm{Cone}(OA_j)$ for all j. Then the fan is smooth, hence this will be the minimal resolution of singularities as constructed above.

For explicit calculations, pick $m = 7, n = 4$. Then $\frac{7}{7-4} = \frac{7}{3} = [2, 3]^+$. Doing the procedure above we get

$$P_0 = 1 \quad P_1 = 2 \quad P_2 = 7$$

$$Q_0 = 0 \quad Q_1 = 1 \quad Q_2 = 3$$

$$u_{-1} = v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad u_0 = v_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$u_1 = v_1 + 2v_2 \begin{pmatrix} -1 \\ -2 \end{pmatrix} \quad u_2 = 3v_1 + 7v_2 = \begin{pmatrix} -4 \\ -7 \end{pmatrix},$$

since this is with respect to the basis $\{v_1 = e_1, v_2 = -e_1 - e_2\}$. To get a smooth fan we have to add all the u_i as well as cones generated by lattice points on the interior of edges $u_{i-2}u_i$. The edge $u_{-1}u_1$ has $a_1 + 1 = 3$ lattice points and the edge u_0u_2 has $a_2 + 1 = 4$ lattice points, thus we will also have to add cones generated by the additional vectors, i.e.

$$\begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad \begin{pmatrix} -2 \\ -3 \end{pmatrix} \quad \begin{pmatrix} -3 \\ -5 \end{pmatrix},$$

adding all these cones, produces a smooth fan, see Figure 3.1.

Of course we could also do this cone by cone using HJ-fractions, consider our $(7, 3)$-cone, then $\frac{7}{7-3} = \frac{7}{4} = [2, 4]^-$. The first added 1-dimensional cone

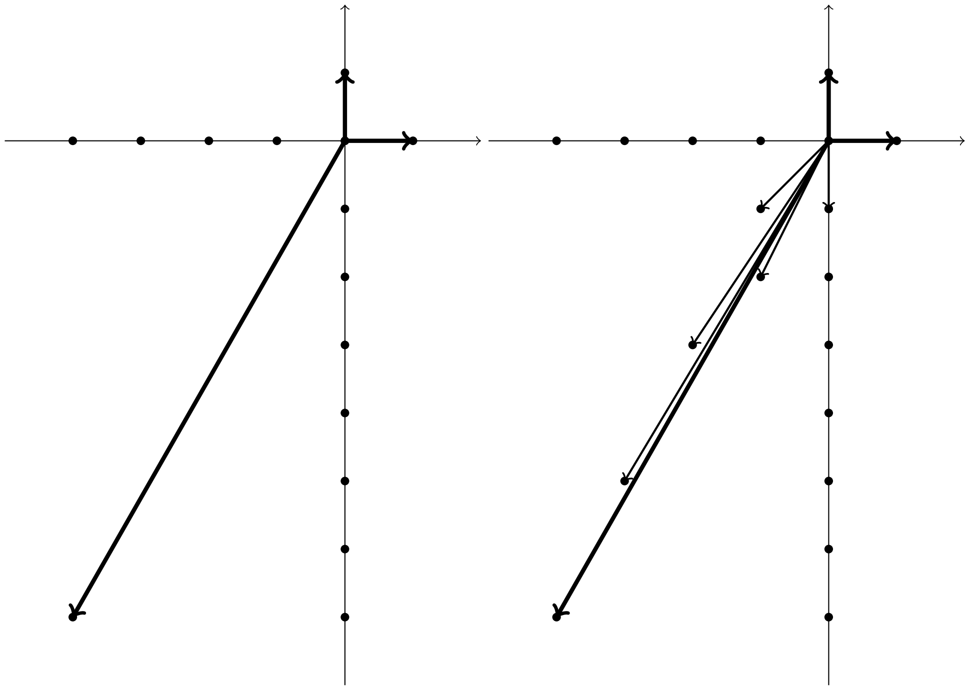

Figure 3.2: Left: fan for $\mathbb{P}(1,4,7)$ Right: Resolution of singularities

will be generated by $w_1 = v_1 + v_2 = (0,-1)$ the rest have to satisfy the recursion relation 3.2.2, so $w_2 = 2w_1 - w_0 = (-1,-2)$, $w_3 = 4w_2 - w_1 = (-4,-7)$. The other cone Cone$(v_2, v_3)$ is a $(4,1)$-cone with respect to the basis $f_0 = (0,1), f_1 = (-1,-2)$. Then, since $\frac{4}{4-1} = \frac{4}{3} = [2,2,2]^-$, we must have cones generated by $z_0 = f_0 = (0,1), z_1 = f_0 + f_1 = (-1,-1), z_2 = 2z_1 - z_0 = (-2,-3), z_3 = 2z_2 - z_1 = (-3,-5), z_4 = 2z_3 - z_2 = (-4,-7)$. This is exactly the cones we found using Euclidean continued fractions.

3.2 Euler-obstructions from HJ-fractions

Now we can return to task of calculating the Euler-obstruction of the vertices of the polytope P giving $\mathbb{P}(k,m,n)$. We have the following proposition.

Proposition 3.2.1. *[GS82] Let $p \in S$ be a normal cyclic surface singularity, and $X \to S$ a minimal resolution of p with exceptional curves E_i. Then*

$\mathrm{Eu}_p(S) = \sum_i (2 + E_i \cdot E_i)$

We will prove this for toric surface singularities.

Proof. Given any toric surface, consider a singular vertex v . From Remark 1.11.8 we have that $\mathrm{Eu}(v) = 2 - \mathrm{Vol}(\sigma^\vee \setminus K(\sigma^\vee))$ where σ is the cone corresponding to v and $K(\sigma^\vee) = \mathrm{Conv}(\sigma^\vee \cap (M \setminus \{0\}))$.

Assume $\sigma^\vee$ is a (d,k)-cone, then σ is a $(d, d-k)$-cone. From Proposition 3.1.13 $E_i \cdot E_i = -b_i$ where $\frac{d}{k} = [b_1, ..., b_r]^-$. Using the construction of $K(\sigma^\vee)$ from above, and that each OA_i, OA_{i+1} is a basis for the lattice, so that each such pair will contribute a triangle of normalized area 1, $\mathrm{Vol}(\sigma^\vee \setminus K(\sigma^\vee)) = 1 + s$ where s is the length of the HJ-fraction of $\frac{d}{d-k}$. Thus what we wish to show is that $2 - (1+s) = 1 - s = \sum_i (2 + E_i \cdot E_i) = \sum_i (2 - b_i)$. But this is just Proposition 3.1.5. $\square$

Combining this with Proposition 3.1.13, we get the following corollary.

Corollary 3.2.2. *Given a (d,k)-cone in $M_\mathbb{R}$ (equivalently a $(d, d-k)$-cone in $N_\mathbb{R}$) and let v be the singular point, write*

$$\frac{d}{k} = b_1 - \cfrac{1}{b_2 - \frac{1}{\dots - \frac{1}{b_r}}}$$

Then $Eu(v) = \sum_{i=1}^r (2 - b_i)$.

We can apply this to make our earlier calculations easier. Consider again the case $\mathbb{P}(m, n, m+n)$ from Proposition 2.5.6. In this case we have the familiar polytope P. As remarked before for each vertex v, $\mathrm{Cone}(P - v) = \sigma^\vee$, where σ is the cone of the normal fan corresponding to v . From the proof of Proposition 2.5.6 we see that for $v = (n(n+m), 0, 0)$ we have basis $e_1 = (-m-n, 0, -m), e_2 = (1, 1, -1)$ for the lattice. Then the other edge emanating from the vertex is generated by $(-n, m, 0) = e_1 + me_2$, hence it is a $(m, 1)$-cone. Since $\frac{m}{1} = [m]^-$, by the remarks above $\mathrm{Eu}(n(n+m), 0, 0) = 2 - m$.

More generally we consider $\mathbb{P}(k, m, n)$ for arbitrary k, m, n. Look at the vertex $v = (0, 0, mk)$ of the polytope P. From Lemma 2.4.2 one sees that the edges emanating from v are generated by $(0, -n, m)$ and $(-n, 0, k)$. Now we wish to find a basis for the lattice containing one of these vectors. Without loss of generality pick $e_1 = (0, n, -m)$. Now we wish to find a second basis vector, one way to do this is as before: let $(1, a, d)$ be the point in P with minimal a. Then $e_2 = (1, a, d - mk)$ will be a second basis vector (this generates the line through $(1, a, d)$ and $(0, n, mk - m)$, the first lattice point along the edge generated by e_1). Now

$$(-n, 0, k) = -ae_1 + ne_2$$

Hence it is a $(n, n-a)$-cone. Thus if $\frac{n}{n-a} = [a_1, ..., a_r]^-$ then $\mathrm{Eu}(v) = \sum_i (2 - a_i)$.

Similarly for the vertex $(0, kn, 0)$ pick lattice point $(b, e, 1)$ with b minimal to obtain the basis $e_1 = (m, -k, 0)$, $e_2 = (b, e - kn, 1)$. Then the vector generating the second edge is $(0, -n, m) = -be_1 + me_2$. Thus it is a $(m, m - b)$-cone. Letting $\frac{m}{m-b} = [b_1, ..., b_s]^-$, then $\mathrm{Eu}(v) = \sum_i (2 - b_i)$.

For $(mn, 0, 0)$ pick lattice point $(f, 1, c)$ with c minimal, obtaining basis $e_1 = (-n, 0, k)$, $e_2 = (f - mn, 1, c)$. Then $(-m, k, 0) = -ce_1 + ke_2$. So it is a $(k, k - c)$-cone. Letting $\frac{k}{k-c} = [c_1, ..., c_t]^-$ then $\mathrm{Eu}(v) = \sum_i (2 - c_i)$

Observe that finding lattice points of the polytope $(1, a, d)$, $(b, e, 1)$, $(f, 1, c)$ with minimal a, b, c corresponds to finding minimal a, b, c such that

$$k + am \equiv 0 \pmod{n}$$

$$n + bk \equiv 0 \pmod{m}$$

$$m + cn \equiv 0 \pmod{k}$$

Collecting this together we get the following way of determining the degree of the dual variety of a weighted projective space.

Theorem 3.2.3. *Given $\mathbb{P}(k, m, n)$, find minimal natural numbers a, b, c such that*

$$k + am \equiv 0 \pmod{n}$$

$$n + bk \equiv 0 \pmod{m}$$

$$m + cn \equiv 0 \pmod{k}$$

Let $\frac{n}{n-a} = [a_1, ..., a_r]^-$, $\frac{m}{m-b} = [b_1, ..., b_s]^-$, $\frac{k}{k-c} = [c_1, ..., c_t]^-$.

Then $\deg \mathbb{P}(k, m, n)^\vee$ equals

$$3kmn - 2(k + n + m) + \sum_{i=1}^{r}(2 - a_i) + \sum_{i=1}^{s}(2 - b_i) + \sum_{i=1}^{t}(2 - c_i)$$

Remark 3.2.4. As we only wish to get the singularity into the form of a (d, k)-cone, any basis will suffice for doing this. Our choice of using a vector with one coordinate equal to 1 is just one choice which always will work.

Proposition 2.1.6 implies that a (d, k)-cone gives an action of a finite abelian group on $\mathbb{C}^2$. By the dicussion in [CLS11, Prop 10.1.2] this will be of the form:

$$U_\sigma \cong \mathbb{C}^2/\mu_d$$

$$\zeta_d \cdot (x, y) = (\zeta_d x, \zeta_d^{-k} y),$$

where μ_d are the d-th roots of unity, and ζ_d is a choice of primitive root. In the case of $\mathbb{P}(k, m, n)$ with coordinates $(x_0 : x_1 : x_2)$, consider $X_0 =$

$\{x_0 \neq 0\}$. By the above this set comes from a (k,c)-cone, where $m + cn \equiv 0 \pmod k$, thus the action is given by

$$\zeta_k \cdot (x_1, x_2) = (\zeta_k x_1, \zeta_k^{-c} x_2),$$

By applying the above n times, we have that the orbit of (x_1, x_2) also can be decribed as :

$$(\zeta_k^n x_1, \zeta_k^{-cn} x_2) = (\zeta_k^n x_1, \zeta_k^m x_2)$$

Thus we recover the action on affine coordinate rings we described in 2.1 (with switched coordinates, this corresponds to choosing minimal c such that $m + cn \equiv 0 \pmod k$. If we instead chose c such that $cm + n \equiv 0 \pmod k$ we would have coordinates ordered normally).

Using Theorem 3.2.3 it is easier to find closed formulas in special cases.

Proposition 3.2.5. *For* $k \geq 1$, $\deg \mathbb{P}(2k-1, 2k, 2k+1)^\vee = 24k^3 - 20k + 3$

Proof. We wish to find minimal a, b, c satisfying

$$\begin{aligned} 2k - 1 + a2d &\equiv 0 \pmod{2k+1} \\ b(2k-1) + 2k + 1 &\equiv 0 \pmod{2k} \\ c(2k+1) + 2d &\equiv 0 \pmod{2k-1} \end{aligned}$$

Some easy algebra shows that a, b, c must satisfy

$$\begin{aligned} a &\equiv -2 \pmod{2k+1} \\ b &\equiv 1 \pmod{2k} \\ 2c &\equiv -1 \pmod{2k-1} \end{aligned}$$

Resulting in $a = 2k-1, b = 1, c = k-1$. Now

$$\frac{2k+1}{2k+1-(2k-1)} = \frac{2k+1}{2} = [k+1, 2]^-$$

$$\frac{2k}{2k-1} = [2, ..., 2]^-$$

$$\frac{2k-1}{2k-1-(k-1)} = \frac{2k-1}{k} = [2, k]^-$$

Combining these yields the formula. $\square$

Proposition 3.2.6. $\deg \mathbb{P}(m,n,m+2n)^\vee = 6mn^2 + 3m^2n - 7n - \frac{9}{2}m + \frac{5}{2}$

Proof. Following Theorem 3.2.3 we want minimal a, b, c such that

$$m + an \equiv 0 \pmod{m+2n}$$
$$mb + m + 2n \equiv 0 \pmod{n}$$
$$n + (m+2n)c \equiv 0 \pmod{m}$$

One sees that $a = 2, b = n-1, c = \frac{m-1}{2}$ (m has to be odd, if not then $\gcd(m, m+2n) \neq 1$). Now $\frac{m+2n}{m+2n-2} = 2 - \frac{m+2n-4}{m+2n-2} = 2 - \frac{1}{\frac{m+2n-2}{m+2n-4}} = [2, ..., 2, 3]^-$ where the 3 is by induction, since $\frac{3}{1} = [3]^-$. The HJ-fraction $\frac{n}{n-(n-1)} = \frac{n}{1} = [n]^-$. Also $\frac{m}{m - \frac{m-1}{2}} = \frac{m}{\frac{m+1}{2}} = [2, \frac{m+1}{2}]^-$. Combining these yields the formula. □

The following Python code will for any given k, m, n, calculate the dual degree.

```
from math import *

def deg(k,m,n):                    #Calulate the degree
    a=minsol(k,m,n)
    b=minsol(n,k,m)
    c=minsol(m,n,k)
    A=fracsum(n,n-a)
    B=fracsum(m,m-b)
    C=fracsum(k,k-c)
    deg=3*k*m*n-2*(k+m+n)+A+B+C
    return deg

def minsol(x,y,z):                 #Find a,b,c
    for i in range(1,z):
        if (x+y*i) % z == 0:
            return i

def fracsum(d,k):                  #Calulate sum  2-b_i
    sum =0
    HJ=frac(d,k)
    for i in range(len(HJ)):
        sum += 2-HJ[i]
    return sum
```

```
def frac(d,k):                    #Find HJ-fraction of d/k
    HJ= []
    while k != 0:
        q=int(ceil(d/float(k)))
        HJ.append(q)
        r=q*k-d
        d=k
        k=r
    return HJ
```

3.3 Gorenstein singularities

Definition 3.3.1. A variety is called Gorenstein if the canonical divisor is Cartier.

For an affine toric surfaces, it is easy to classify which are Gorenstein.

Proposition 3.3.2. *[CLS11, Exc. 8.2.13] An affine toric surface U_σ is Gorenstein if and only if σ is a $(d,1)$-cone.*

Proof. Assume σ is a (d,k)-cone and the canonical divisor is Cartier. Then there exists Cartier-data $m_\sigma = (x,y)$ such that

$$\langle m_\sigma, (1,0)\rangle = 1$$

$$\langle m_\sigma, (k,d)\rangle = 1$$

The first equation gives $x = 1$. Since $\gcd(k,d) = 1$ the second equation can only be true if either x or y is 0. Thus $y = 0$, which forces $k = 1$.

Conversely, if we have a $(d,1)$-cone, $(1,0)$ will be Cartier-data for the canonical divisor, hence it is Cartier. □

We say that a singularity of a surface is Gorenstein if it is contained in an affine open neigbourhood which is Gorenstein.

Proposition 3.3.3. *[CLS11, Exc. 8.3.2] A weighted projective space $\mathbb{P}(q_0,...,q_n)$ is Gorenstein if and only if $q_i|q_0 + ... + q_n$ for all i.*

Proof. This follows directly from the facts that the canonical divisor has degree $-q_0 - ... - q_n$ and that the Picard group is the subgroup of the classgroup generated by $\mathrm{lcm}(q_0,...,q_n)$. □

From this one can easily find the Gorenstein planes.

Corollary 3.3.4. *The only weighted projective planes which are Gorenstein are* $\mathbb{P}(1,2,3)$, $\mathbb{P}(1,1,2)$ *and* $\mathbb{P}(1,1,1)$.

Also we can generalize Proposition 2.5.4 to arbitrary surface singularities.

Proposition 3.3.5. *A toric surface singularity has Euler-obstruction* 0 *if and only if it is Gorenstein.*

Proof. Let the singularity be given as a (d,k)-cone in $N_{\mathbb{R}}$. Let $\frac{d}{d-k} = [b_1,...,b_r]$. By Corollary 3.2.2 the Euler-obstruction is 0 if and only if all $b_i = 2$. Now if the singularity is Gorenstein, then $k=1$, so $\frac{d}{d-k} = \frac{d}{d-1}$. It is easy to check that the HJ-fraction of $\frac{d}{d-1}$ is a chain of $d-1$ 2's.

Conversely if the singularity has Euler-obstruction 0, then all b_i's are 0, but by the above this implies that in $M_{\mathbb{R}}$ it is a $(d,d-1)$-cone, so it is a $(d,1)$-cone in $N_{\mathbb{R}}$. □

3.4 Weighted blow up

In example 1.6.4 we defined the classical blow up $\mathrm{Bl}_0(\mathbb{C}^n)$ as a subvariety of $\mathbb{P}^{n-1}\times\mathbb{C}^n$ and saw that it can be realized at the level of cones. We now wish to define a weighted blowup, and relate this to the resolution of singularities presented earlier. The idea of this comes from [ABMMOG14], where this is done in coordinates. Here we will translate into the toric language of cones and fans lying in vector spaces coming from lattices.

Definition 3.4.1. Given a fan $\Sigma \in N_{\mathbb{R}}$ and a cone $\sigma \in \Sigma$ with $\dim(\sigma) = \dim(N)$. We define the weighted blowup of X_Σ with respect to the weights $(q_1,...,q_n)$ in the point corresponding to σ as $X_{\Sigma'}$ where Σ' is defined as follows: Let $\sigma = \mathrm{Cone}(e_1,...,e_n)$ and set $e_0 = \sum_{i=1}^n q_i e_i$. Then Σ' is the fan consisting of all proper subsets of $\{e_0, e_1, ..., e_n\}$.

Remark 3.4.2. This is a special case of what [CLS11] calls the star subdivision, which they use to construct a general resolution of singularities for toric varieties of any dimension. Since we here are interested in an explicit resolution for the 2-dimensional case, we will only need the weighted blowup.

Example 3.4.3. To motivate this definition, do a weighted $(q_1,...,q_n)$ blowup of $\mathbb{C}^n$ at its maximal cone, i.e., at 0, to obtain the variety $X_{\Sigma'}$. Then by Remark 1.6.3 we see that the divisor corresponding to the new 1-dimensional cone $\sigma = \mathrm{Cone}(e_0) = \mathrm{Cone}(\sum_{i=1}^n q_i e_i)$ is $\mathrm{Star}(\sigma)$. This will be the fan in $\mathbb{Z}/\mathbb{Z}(q_1,...,q_n)$ with cones generated by the image of all proper subsets of $\{e_1, ..., e_n\}$, which is exactly the fan for $\mathbb{P}(q_1, ..., q_n)$ by example 1.3.7. Also $X_{\Sigma'} \setminus D_\sigma = \mathbb{C}^n \setminus \{0\}$, so we have a morphism $\phi : X_{\Sigma'} \to \mathbb{C}^n$ such that $\phi^{-1}(0) = \mathbb{P}(q_1, ..., q_n)$, which is an isomorphism away from 0.

The following is proved in [ABMMOG14], here we do our own proof using toric methods.

Proposition 3.4.4. *Given a (d, k)-cone $\sigma \subset N_\mathbb{R}$. Then the resolution of singularities $\phi : X_\Sigma \to U_\sigma$ constructed in Proposition 3.1.12 is obtained by a sequence of weighted blowups.*

Proof. Recall that one way of constructing the Hirzebruch-Jung continued fraction of $\frac{d}{d-k}$ was the following: Set $r_{-1} = d, r_0 = d - k$. Inductively find r_i, q_i such that $r_{i-1} = r_i q_{i+1} - r_{i+1}$, where $0 < r_i < q_i$. Then $\frac{d}{d-k} = [q_1, ..., q_s]^-$.

The blowups resulting in the resolution of singularities will be $(r_i, 1)$-blowups for $i = 0, ..., s-1$. Starting with Cone$((1, 0), (k, d))$ we first do a $(r_0, 1) = (d - k, 1)$-blowup, giving

$$\widehat{u_1} = d \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Then we perform a $(r_i, 1)$-blowup on the new Cone$(\hat{u}_i, (k, d))$ giving the next cones

$$\widehat{u_2} = (d - k) \begin{pmatrix} q_1 - 1 \\ q_1 \end{pmatrix}$$

$$\widehat{u_3} = ((d - k)q_1 - d) \begin{pmatrix} q_1 q_2 - q_2 - 1 \\ q_1 q_2 - 1 \end{pmatrix}$$

We see that $\widehat{u_i}$ isn't a primitive vector. Recall that the resolution of singularities constructs 1-dimensional cones $v_0, v_1, ..., v_s, v_{s+1}$ that satisify the relation $v_{i-1} + v_{i+1} = q_i v_i$. We will show that $\widehat{u_i} = z_i u_i$ for a natural number z_i, such that the u_i's satisfies $u_{i-1} + u_{i+1} = q_i u_i$. This will imply $u_i = v_i$, thus the sequence of weighted blowups gives the same fan as the resolution of singularities.

We will prove this by induction on the length s of the HJ-fraction of $\frac{d}{d-k}$. Based on our calculations of the first cones, our induction hypothesis will be that $z_i = r_{i-2}$, for $i \geq 1$, in other words that $\widehat{u_i} = r_{i-2} u_i$ for some u_i satisfying $u_{i-2} + u_i = q_{i-1} u_{i-1}$. Assume this holds for i. Then

$$\widehat{u_{i+1}} = \begin{pmatrix} k \\ d \end{pmatrix} + r_i u_i = r_{i-2} u_i - r_{i-1} u_{i-1} + r_i u_i$$

by the induction hypothesis. Now by definition $r_i + r_{i-2} = q_i r_{i-1}$, so $r_i u_i + r_{i-2} u_i = q_i u_i r_{i-1}$, giving $\widehat{u_{i+1}} = q_i u_i r_{i-1} - r_{i-1} u_{i-1} = r_{i-1}(q_i u_i - u_{i-1})$, which is exactly what wanted to show.

□

We will now give formulas for intersection of divisors on the weighted blowup. This is also proved in [ABMMOG14], but again we give our own proof by toric methods.

Take a 2-dimensional simplicial toric variety X_Σ. Let $\Sigma(1) = \{\rho_1, ..., \rho_n\}$, with minimal generators $u_1, ..., u_n$ and associated divisors be $D_1, ..., D_n$. Perform a weighted (p, q)-blowup at a maximal cone σ, without loss of generality let $\sigma = \text{Cone}(\rho_1, \rho_2)$, to obtain a new fan Σ' with a new 1-dimensional cone τ with associated divisor E, and a morphism $\phi : X_{\Sigma'} \to X_\Sigma$. Let the divisor on $X_{\Sigma'}$ associated with ρ_i be denoted D'_i. For a divisor $D = \sum_{i=1}^n a_i D_i$ on X_Σ, let $D' = \sum_{i=1} a_i D'_i$ be the corresponding divisor on $X_{\Sigma'}$, this is called the strict transform of D. Assume we have written σ as a (d, k)-cone, with ρ_1 as the first basis vector. Set $e = \gcd(dq, p + kq)$. Then we have:

Proposition 3.4.5. *In the above setup, let $D = \sum_{i=1}^n a_i D_i$, $C = \sum_{i=1}^n b_i D_i$ be any torus-invariant divisors on X_Σ. Then*

$$\begin{aligned}
\phi^* D &= D' + \frac{a_1 p + a_2 q}{e} E \\
\phi^* D \cdot E &= 0 \\
E^2 &= -\frac{e^2}{dpq} \\
D' \cdot E &= \frac{a_1 e}{dq} + \frac{a_2 e}{dp} \\
D' \cdot C' &= D \cdot C - \frac{a_1 b_1 p}{dq} - \frac{a_1 b_2 + a_2 b_1}{d} - \frac{a_2 b_2 q}{dp} \\
\phi^* D \cdot \phi^* C &= D \cdot C
\end{aligned}$$

Proof. After performing a (p, q)-blowup at $\text{Cone}((1, 0), (k, d))$ we get that τ is generated by $(p + kq, dq)$. However this isn't necessarily primitive, so a primitive generator will be $v = \frac{1}{e}(p + kq, dq)$. From Proposition 1.8.3 we have

$$D'_1 \cdot E = \frac{1}{\det(u_1, v)} = \frac{e}{dq}$$

$$D'_2 \cdot E = \frac{1}{\det(v, u_2)} = \frac{e}{pd}$$

$$D'_i \cdot E = 0 \text{ when } i = 3, ..., n$$

$$E^2 = -\frac{\det(u_1, u_2)}{\det(u_1, v)\det(v, u_2)} = -\frac{e^2}{dpq}$$

From Proposition 1.8.2 we have that

$$\phi^* D = \sum_{i=1}^{n} a_i D_i' - \langle m_\sigma, v \rangle E$$

where $m_\sigma = (x, y)$ is $\mathbb{Q}$-Cartier data of σ. By definition this has to satisfy:

$$\langle m_\sigma, (1,0) \rangle = -a_1$$

$$\langle m_\sigma, (k,d) \rangle = -a_2$$

implying that $x = -a_1$, $y = \frac{a_1 k - a_2}{d}$. Then we get

$$\phi^* D = \sum_{i=1}^{n} a_i D_i' + \frac{a_1 p + a_2 q}{e} E$$

Using the intersection numbers listed above we get

$$\phi^* D \cdot E = \sum_{i=1}^{n} a_i D_i' \cdot E + \frac{a_1 p + a_2 q}{e} E^2 = \frac{a_1 e}{dq} + \frac{a_2 e}{dp} + \frac{a_1 p + a_2 q}{e}(-\frac{e^2}{dpq}) = 0$$

Next we use that

$$D' = \phi^* D - \frac{a_1 p + a_2 q}{e} E$$

$$C' = \phi^* C - \frac{b_1 p + b_2 q}{e} E$$

thus

$$D' \cdot C' = D \cdot C + (\frac{a_1 p + a_2 q}{e})(\frac{b_1 p + b_2 q}{e}) E^2$$

combining this with the above calculations gives the desired result. The last claim is proved by applying the previous ones, we have:

$$\begin{aligned} \phi^* D \cdot \phi^* C &= \\ \phi^* D \cdot (C' + \frac{b_1 p + b_2 q}{e} E) &= \\ \phi^* D \cdot C' &= \\ D' \cdot C' + \frac{a_1 p + a_2 q}{e} E \cdot C' &= \end{aligned}$$

$$D \cdot C - \frac{a_1 b_1 p}{dq} - \frac{a_1 b_2 + a_2 b_1}{d} - \frac{a_2 b_2 q}{dp} + \frac{a_1 p + a_2 q}{e}(\frac{b_1 e}{dq} + \frac{b_2 e}{dp}) = D \cdot C$$

□

Now we can prove the the claims made before Proposition 1.9.5. Take a surface X_Σ, and let $X_{\Sigma'}$ be the minimal resolution of singularities. By the above, this is obtained by a sequence of weighted blowups, let the exceptional divisors be $E_1, ..., E_s$. Then we have that

$$\mathscr{K}_{X_\Sigma} \cdot D_P = \phi^* \mathscr{K}_{X_\Sigma} \cdot \phi^* D_P$$

now, $\phi^* \mathscr{K}_{X_\Sigma} = -\sum_\rho D'_\rho + \sum_{i=1}^s c_i E_i$ for some coefficients $c_i \in \mathbb{Q}$. By Proposition 3.4.5 a pulled back divisor intersects all exceptional divisors trivially, so we get

$$(-\sum_\rho D'_\rho + \sum_{i=1}^s c_i E_i) \cdot \phi^* D_P = -\sum_\rho D'_\rho \cdot \phi^* D_P =$$

$$-\sum_\rho D'_\rho \cdot \phi^* D_P - \sum_{i=1}^s E_i \cdot \phi^* D_P = \mathscr{K}_{X_{\Sigma'}} \cdot \phi^* D_P$$

which shows that $\mathscr{K}_{X_\Sigma} \cdot D_P = \mathscr{K}_{X_{\Sigma'}} \cdot \phi^* D_P$, which was what we wanted.

From the above we could also recover the self-intersections of the minimal resolution of singularities. For a (d, k)-cone in $N_\mathbb{R}$, the first blowup is a $(d-k, 1)$-blowup. Then $e = \gcd(d, d) = d, p = d-k, q = 1$, so $E^2 = -\frac{d}{d-k}$. Now if this continued fraction has length 1, say $\frac{d}{d-k} = [l]^-$, we get that $E^2 = -l$ as before. Otherwise we could do further blowups, and check that we would get the same self-intersections, however we will not do the details here.

3.5 Going to 3 dimensions

We will attempt to look at the Euler obstructions of some singularities of 3-dimensional varieties.

From 1.10.3 we get the formula for the degree of the dual variety:

$$\deg X^\vee = 4\,\mathrm{Vol}(P) - 3A(P) + 2\sum_{e \lneq P} \mathrm{Vol}(e)\,\mathrm{Eu}(e) - \sum_{v \in P} \mathrm{Eu}(v),$$

where $A(P)$ is the sum of the normalized areas of the faces of P while e the is collection of all edges of P, and the last sum is over all vertices v of P.

Again the hard part is finding the Euler-obstruction of the vertices. From Corollary 1.11.3 we get

$$\mathrm{Eu}(v) = \mathrm{Eu}(P)\,\mathrm{RSV}_\mathbb{Z}(P, v) - \sum_i \mathrm{Eu}(f_i)\,\mathrm{RSV}_\mathbb{Z}(f_i, v) + \sum_j \mathrm{Eu}(e_j)\,\mathrm{RSV}_\mathbb{Z}(e_j, v),$$

where f_i loops over all facets of P containing v, while e_j over all edges of P containing v.

By Proposition 1.11.4 $\mathrm{RSV}_{\mathbb{Z}}(e_j, v) = 1$ for all j and by Corollary 1.11.5 $\mathrm{Eu}(f_i) = 1$ for all i. $\mathrm{RSV}_{\mathbb{Z}}(f_i, v)$ will as before equal $\mathrm{Vol}(f_i \setminus \mathrm{Conv}(f_i \setminus v))$.

If we now assume that the singularities are isolated, we get that $\mathrm{Eu}(e_j) = 1$ for all edges e_j. Combining all this, we have reduced the calculations to

$$\mathrm{Eu}(v) = \mathrm{RSV}_{\mathbb{Z}}(P, v) - \sum_i \mathrm{RSV}_{\mathbb{Z}}(f_i, v) + \#\text{edges}$$

Combining Proposition 3.2.1 with Proposition 1.11.7 we obtain that $RSV(f_i, v) = 2 + \sum_{i=1}^{s}(b_i - 2) = 2 - \mathrm{Eu}_{f_i}(v)$ where the edges emanating from f_i form a (d, k)-cone and $\frac{d}{k} = [b_1, ..., b_s]^-$. By $\mathrm{Eu}_{f_i}(v)$ we mean the Euler-obstruction of v considered as a point of the affine variety associated with the cone generated by f_i, which is not the same as $\mathrm{Eu}(v)$.

The only remaining term turns out to be problematic, since $\mathrm{RSV}_{\mathbb{Z}}(P, v) = \mathrm{Vol}(\mathrm{Conv}((P \setminus \{v\}) \cap M))$ is difficult to calculate in general. There isn't even a unique minimal desingularization of 3-dimensional singularities, as the following example from [CLS11] shows:

Example 3.5.1. Let $N \cong \mathbb{Z}^3$ with basis e_1, e_2, e_3 and take $\sigma = \mathrm{Cone}(e_1, e_2, e_1 + e_3, e_2 + e_3)$. Then we have the de-singularizations:

$$\Sigma_1 = \{\mathrm{Cone}(e_1, e_2, e_2 + e_3), \mathrm{Cone}(e_1, e_1 + e_3, e_2 + e_3) \text{ and their subcones}\}$$

$$\Sigma_2 = \{\mathrm{Cone}(e_1, e_2, e_1 + e_3), \mathrm{Cone}(e_2, e_1 + e_3, e_2 + e_3) \text{ and their subcones}\}$$

Both Σ_1 and Σ_2 are desingularizations, but there is no map between them. If we also let $\tau = e_1 + e_2 + e_3$, then

$\Sigma_3 = \{\mathrm{Cone}(e_1, e_2, \tau), \mathrm{Cone}(e_1, e_1 + e_3, \tau), \mathrm{Cone}(e_2, e_2 + e_3, \tau), \mathrm{Cone}(e_1 + e_3, e_2 + e_3, \tau) \text{ and their subcones}\}$ is a common desingularization.

There are different ways of computing resolutions which are canonical in some form or other, see [Dai02], but, as far as we know, none will yield formulas for the Euler-obstruction as we want.

However if we restrict ourselves to simple cases, we may compute this.

Consider a polytope $P = \mathrm{Conv}(p_1, ..., p_s) \in \mathbb{Z}^2$. Let $\sigma^\vee \subset M = \mathbb{Z}^3$ be $\mathrm{Cone}((1, p_1), ..., (1, p_s))$. Assuming the cone has an isolated singularity, we can compute the Euler-obstruction, by the formula above.

Since the polytope P lives in height 1, the term $\mathrm{RSV}_{\mathbb{Z}}(\sigma^{\vee}, v)$ will equal the area of P, which we can compute by Pick's formula Proposition 1.9.3

$$\mathrm{RSV}(\sigma^{\vee}, v) = 2|\operatorname{Int}(P) \cap \mathbb{Z}^2| + |\partial P \cap \mathbb{Z}^2| - 2$$

The sum $-\sum_{i=1}^{s} \mathrm{RSV}_{\mathbb{Z}}(f_i, v)$, where f_i are the 2-dimensional faces of $\sigma^{\vee}$ equals the sum of the areas of f_i. This equals $-|\partial P \cap \mathbb{Z}^2|$. Thus we obtain:

Proposition 3.5.2. *For an isolated singularity of the form described above, we have*

$$\mathrm{Eu}(v) = 2|\operatorname{Int}(P) \cap \mathbb{Z}^2| - 2 + s$$

We could also have calculated Euler-obstructions for non-isolated singularities, but calculations get pretty messy and our primary example (see the next section) has only isolated singularities.

3.6 $\mathbb{P}(1, k, m, n)$

Going back to the case of $X = \mathbb{P}(1, k, m, n)$. From [RT11, Prop 1.22] we have that a polytope P for X will be the convex hull of the points $(0, 0, 0), (mn, 0, 0), (0, kn, 0), (0, 0, km) \subset M_{\mathbb{R}}$ for a 3-dimensional lattice M.

We find the first terms needed in the formula.

$$\mathrm{Vol}_{\mathbb{Z}}(P) = k^2 m^2 n^2$$

$$A(P) = kmn + k^2 mn + km^2 n + kmn^2 = kmn(1 + k + m + n)$$

$$E(P) = k + m + n + mn + kn + km$$

By Proposition 2.1.7 the singularities are isolated if and only if $\gcd(k, m) = \gcd(k, n) = \gcd(m, n) = 1$. Since we assume the weights are reduced, this will always happen. By the discussion in the previous section, we have that:

$$\mathrm{Eu}(v) = \mathrm{RSV}_{\mathbb{Z}}(P, v) - \mathrm{RSV}_{\mathbb{Z}}(f_1, v) - \mathrm{RSV}_{\mathbb{Z}}(f_2, v) - \mathrm{RSV}_{\mathbb{Z}}(f_3, v) + 3.$$

This can also be formulated as

$$\mathrm{Eu}(v) = \mathrm{RSV}_{\mathbb{Z}}(P, v) + \mathrm{Eu}_{f_1}(v) + \mathrm{Eu}_{f_2}(v) + \mathrm{Eu}_{f_3}(v) - 3.$$

Choose $v_1 = (mn, 0, 0)$. Then we have 3 edges emanating from v_1, $e_1 = (-1, 0, 0)$, $e_2 = (-n, 0, k)$ and $e_3 = (-m, k, 0)$. $\mathrm{Cone}(e_2, e_3)$ will as before be a $(k, k - c)$ cone where $m + cn \equiv 0 \pmod{k}$. Since

$$(-n, 0, k) = n(-1, 0, 0) + k(0, 0, 1),$$

we get that $\mathrm{Cone}(e_1, e_2)$ is a (k, n')-cone where $0 < n' < k$ and $n \equiv n' \pmod{k}$. Similarly since

$$(-m, k, 0) = m(-1, 0, 0) + k(0, 1, 0),$$

$\mathrm{Cone}(e_1, e_3)$ will be a (k, m')-cone where $0 < m' < k$ and $m \equiv m' \pmod{k}$. As usual we get similar results for the other vertices by cyclicly permuting k, m, n.

In specific cases we can use computer programs, here matlab, to calculate the missing term $\mathrm{RSV}_{\mathbb{Z}}(P, v)$, as in the following example.

Example 3.6.1. Consider $\mathbb{P}(1, 2, 3, 5)$. The polytope P has vertices $v_0 = (0, 0, 0), v_1 = (15, 0, 0), v_2 = (0, 10, 0), v_3 = (0, 0, 6)$. Using matlab we calculate that

$$RSV(P, v_1) = 4$$
$$RSV(P, v_2) = 5$$
$$RSV(P, v_3) = 6$$

We have $e_1 = (-1, 0, 0), e_2 = (-5, 0, 2), e_3 = (-3, 2, 0)$. Using the above we get $c = 1$ thus $\mathrm{Cone}(e_2, e_3)$ is a $(2, 1)$-cone. Also since

$$5 \equiv 1 \pmod{2}$$

$$3 \equiv 1 \pmod{2}$$

$\mathrm{Cone}(e_1, e_2)$ and $\mathrm{Cone}(e_1, e_3)$ will be also be $(2, 1)$-cones. A $(2, 1)$-cone has Euler-obstruction $2 - 2 = 0$, thus for all facets f_i we have $Eu_{f_i}(v_1) = 0$.

Summing up we get that $\mathrm{Eu}(v_1) = 4 + 0 + 0 + 0 - 3 = 1$.

For v_2 we have edges generated by $(0, -1, 0), (3, -2, 0), (0, -5, 3)$ that

$$5 \equiv 2 \pmod{3}$$

thus both facets involving $(0, -1, 0)$ are $(3, 2)$-cones, which have Euler-obstruction 0. Solving $5 + 2b \equiv 0 \pmod{3}$ for minimal b gives $b = 2$, thus the third facet is a $(3, 1)$-cone, which has Euler-obstruction -1. Summing up we get $\mathrm{Eu}(v_2) = 5 + 0 + 0 - 1 - 3 = 1$.

The edges generated by v_3 are generated by $(0, 0, -1), (0, 5, -3), (5, 0, -2)$. Then we have a $(5, 2)$-cone, giving Euler-obstruction -1 and a $(5, 3)$-cone also giving Euler-obstruction -1. Solving $2 + 3a \equiv 0 \pmod{5}$, we get $a = 1$ and Euler-obstruction 0. Summing up we get $\mathrm{Eu}(v_3) = 6 - 1 - 1 + 0 - 3 = 1$. Thus

$$\deg \mathbb{P}(1, 2, 3, 5)^\vee = 4 * 900 - 3 * 330 + 2 * 41 - 4 = 2688.$$

This example is very interesting, as it shows a variety with only isolated singularities which has Euler-obstruction constantly equal to 1. In dimension two [MT11] shows that for toric varieties coming from polytopes, the Euler-obstruction is constantly equal to 1 if and only if the variety is smooth. They conjecture it will also hold for dimensions $n \geq 3$. Assuming our calculations are correct, this is a counterexample to this conjecture.

Example 3.6.2. Consider $\mathbb{P}(1, 2k-1, 2k, 2k+1)$ for $k \geq 1$. By using the calculations of Proposition 3.2.5 and the above methods to get in general

$$\mathrm{Eu}(v_1) = RSV(P, v_1) - 4k + 4$$

$$\mathrm{Eu}(v_2) = RSV(P, v_2) - 2k - 1$$

$$\mathrm{Eu}(v_3) = RSV(P, v_3) - k - 3$$

Using matlab to calculate $RSV_{\mathbb{Z}}(P, v_i)$ for $k = 1, ..., 6$ we get the following candidates for the Euler obstructions

$$\mathrm{Eu}(v_1) = 2k^2 - 6k + 5$$

$$\mathrm{Eu}(v_2) = 1$$

$$\mathrm{Eu}(v_3) = 1$$

To prove this in general, one would have to describe the 3-dimensional convex hull in some systematic way.

Matlab-code computing the convex hulls: Note that this is a brute force method, we find all lattice points of the polytope and compute the convex hull when we remove a vertex, already for pretty small values computations takes time. Since we only are interested in what happens close to the vertices, we could restrict ourselves to each of the regions containing a vertex, and compute using only a selection of lattice points, thus speeding computations a lot.

```
function [w_1,w_2,w_3] = chull(k,m,n)
A=novertices(k,m,n);
A=[A;0 0 k*m; 0 k*n 0;m*n 0 0];
[K_1 v_1] = convhull(A(1:end-1,:));
[K_2 v_2] = convhull(A([1:end-2,end],:));
[K_3 v_3] = convhull(A([1:end-3,end-1:end],:));
M=k^2*m^2*n^2;
w_1 =M-6*v_1;
w_2 =M-6*v_2;
w_3 =M-6*v_3;
end
```

```
% find lattice points except the vertices:
function A = novertices(k,m,n)
N=n*m*k;
A = [];
for x = 0:m*n-1
    for y = 0:n*k-1
        for z=0:m*k-1
            if k*x+m*y+n*z<=N
                A=[A;x y z];
            end
        end
    end
end
end
```

Chapter 4

Counting curves on weighted projective planes

4.1 h-transverse polytopes

There has been a lot of work in recent years in enumerative geometry, trying to answer the following question. For an irreducible surface, in a given linear system L, how many curves with δ nodes pass through a sufficient number of general points? This is called the Severi-degree $N^{L,\delta}$. For smooth surfaces this question has been solved, with the result that the Severi degree is a polynomial in the four Chern numbers $K^2, L^2, L \cdot K, c_2$, where K is the canonical divisor, L is a divisor in the linear system and c_2 is the second Chern class. In the case of singular surfaces there is not much data yet, but one would hope to obtain similar results. This relates to our interests by the following characterization of the Severi degree $N^{L,1}$.

Consider a (non-defect) variety X embedded in a projective space $\mathbb{P}$ via L. Then the dual variety is a hypersurface in the dual space $(\mathbb{P})^\vee$. Intersecting $X^\vee$ with a general line in the dual space gives a number of intersection points equal to $\deg X^\vee$ by Bezout's Theorem. A line $L \subset \mathbb{P}^\vee$ corresponds to a 1-dimensional family of hyperplanes in $\mathbb{P}$. The hyperplanes $H \subset \mathbb{P}$ corresponding to a point in $L \cap X^\vee$ are exactly those which contain the tangent space of a point $x \in X$, in other words those which intersect X in a singular curve. By choosing L sufficiently general, one gets that all singularities will be nodes and that L intersect $X^\vee$ transversally, hence $N^{L,1} = \deg X^\vee$.

In [AB13] Ardila and Block, using tropical geometry, showed how to calculate severi degrees for singular toric varieties coming from polytopes which are h-transverse, meaning all the slopes of 1-dimensional cones of the normal fan has integral (or infinite) slope. Their answer is a polynomial in the slopes and lengths of the polytope, however this isn't as satisfying as one would hope. In [LO14] Liu and Osserman improves on this for what they call strongly h-transverse polytopes (meaning h-transverse and Gorenstein, see below), giving polynomials in the number of vertices with a fixed determinant. They also have correction terms which gives formulas for the general h-transverse case.

Translating this into our language, we will see that the h-transverse condition constitutes quite a restriction on the toric variety. Given a polytope P, it either has a unique top vertex v_t, or there is a horizontal edge at the top and similarly for the bottom vertex v_b (if it exists). All vertices except those at the top or bottom are called internal vertices, and they are quite special. Choose an internal vertex. The edges emanating from the vertex must be generated by

$$\{\begin{pmatrix} a \\ 1 \end{pmatrix}, \begin{pmatrix} b \\ -1 \end{pmatrix}\}$$

for a, b not both 0. By turning this into a (d, k)-cones, we get that they are all Gorenstein singularities or smooth.

If there is a unique top (or bottom) vertex then it is generated by

$$\{\begin{pmatrix} -a \\ -1 \end{pmatrix}, \begin{pmatrix} b \\ -1 \end{pmatrix}\}$$

Since $\begin{pmatrix} -a \\ -1 \end{pmatrix} = \begin{pmatrix} b \\ -1 \end{pmatrix} + (a+b)\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ this is a $(a+b, 1)$-cone in $M_{\mathbb{R}}$, which is a $(a+b, a+b-1)$-cone in $N_{\mathbb{R}}$.

If there isn't a unique vertex on the top, there are two vertices along the horizontal edge. The rightmost will be generated by

$$\{\begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} a \\ -1 \end{pmatrix}\}$$

which is smooth. Similarly the other vertex will also be smooth.

Thus we can conclude that a h-transverse polytope has at most 2-non Gorenstein singularities, which have to correspond to $(l, 1)$ and $(k, 1)$-cones in $M_{\mathbb{R}}$ for some l, k. The strongly h-transverse condition mentioned above is simply requiring all singularities to be Gorenstein.

Using this we can classify the weighted projective planes which are h-transverse.

Proposition 4.1.1. *The only weighted projective planes which come from a h-transverse polytope are* $\mathbb{P}(m, n, m+n)$ *and* $\mathbb{P}(1, 1, n)$ *for* $m, n \geq 1$.

Proof. Consider $\mathbb{P}(k, m, n)$. We will split into 3 cases. First assume that $k, m, n > 1$, so there are 3 singular points. Then one singularity has to be Gorenstein, assume it is the vertex with determinant n, while the other 2 have to be $(l, 1)$-cones in $M_\mathbb{R}$. Using Theorem 3.2.3 we must have $a = 1, b = m-1, c = k-1$, in other words

$$k + m \equiv 0 \pmod{n}$$

$$n + (m-1)k \equiv 0 \pmod{m}$$
$$m + (k-1)n \equiv 0 \pmod{k}$$

The bottom two can be reformulated as

$$n \equiv k \pmod{m}$$

$$m \equiv n \pmod{k}$$

The planes $\mathbb{P}(m, n, m+n)$ obviously satisfies this. Checking if there are other cases, we get $k + m = ns$ for $s \geq 1$. Inserting this into the the other equations we get $s = kt+1 = mr+1$ for some $t, r \geq 0$. Since $\gcd(k, m) = 1$ we get $s = mkl + 1$. Inserting this back in the original equation we get $k + m = n(mkl + 1)$, for which the only integral solution is $l = 0$. Thus we have found all that could possibly be h-transverse. Choosing the basis $v = (1, 1, -1), w = (n, -m, 0)$ one checks that $\mathbb{P}(m, n, m+n)$ is in fact h-transverse:

The edges of the polytope are generated by $(n, -m, 0) = w, (m+n, 0, -m) = mv + w, (0, m+n, -n) = nv - w$. The normal directions of these will have slopes $0, m, n$, so it is h-transverse.

Assume then we have one smooth vertex and two singular ones. If the internal vertex is the smooth one, we get the same restrictions as before with $n = 1$. So we have

$$k - 1 \equiv 1 \pmod{m}$$

$$m - 1 \equiv 1 \pmod{k},$$

giving $k - 1 = ms$ and $m - 1 = kt$ for $s, t \geq 1$. Inserting one into the other gives $m(st-1) = -1-t$. The righthand side is negative, while the lefthand is ≥ 0, thus we have no solutions.

If instead the smooth vertex is at the top or bottom, assume here top, we have $m = 1$. Then we get

$$k + 1 \equiv 0 \pmod{n}$$

$$1 \equiv n \pmod{k},$$

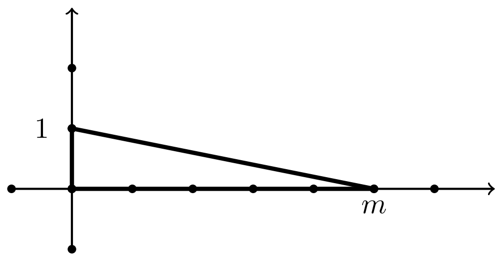

Figure 4.1: h-transverse polytope giving $\mathbb{P}(1, 1, m)$.

giving $k + 1 = ns$ and $n - 1 = kt$ for $t, s \geq 0$. Inserting one into the other yields $n(st - 1) = t - 1$. This can have solutions if and only if $t = s = 1$. Thus $\mathbb{P}(n-1, 1, n)$ is the only possible solution. Choosing the same basis as the previous example works also in this case.

If there are 2 smooth vertices, it is easy to find a 2-dimensional polytope which is h-transverse which gives $\mathbb{P}(1, 1, n)$ (see figure 4.1). $\square$

4.2 The number of curves

The Severi degree $N^{L,\delta}$ can be computed in terms of coefficients $Q^{L,i}$, where $N^{L,1} = Q^{L,1}$ and $N^{L,2} = \frac{{Q^{L,1}}^2 + Q^{L,2}}{2}$, where $d = \deg L \in \mathrm{Cl}(X)$. There are formulas for $Q^{L,\delta}$ for larger δ, and polynomials in them giving larger Severi degrees , but the combinatorical calculations get very messy so we will not consider that here.

In the smooth case one has from [KP99]

$$Q^{L,1} = 3L^2 + 2L \cdot K + c_2$$

$$Q^{L,2} = -42L^2 - 39L \cdot K - 6K^2 - 7c_2$$

For a h-transverse polytope one can calculate the coefficients $Q^{D_P,\delta}$ for sufficiently large polytopes P, meaning that the lengths of the edges of P is at least δ.

By combining examples 8.2 and 8.3 [LO14] we have

$$Q^{L,1} = 3L^2 + 2L \cdot K + 4 - \mathrm{tdet}(P) - \mathrm{bdet}(P) + v_1'$$

$$Q^{L,2} = -42L^2 - 39L \cdot K + 8\,\mathrm{idet}(P) + C(\mathrm{tdet}(P)) + C(\mathrm{bdet}(P)) - 9v_1' + 2v_2' - 76$$

where $\mathrm{tdet}(P)$ ($\mathrm{bdet}(P)$) is the determinant of the unique top (bottom) vertex if it exists, or 0 if not, $\mathrm{idet}(P)$ are the sum of the internal determinants and v'_i are the number of internal vertices of determinant i. $C(0) = 0$ and $C(p) = 19p - 18$ for $p > 0$. The intersections can be calculated by computing lengths and areas by Proposition 1.9.5.

Going to the case of the weighted projective plane $\mathbb{P}(k, m, n)$, recall that the polytope P gives the divisor D_P with $\deg D_P = kmn$. Hence we can calculate the Severi degrees for $d = lkmn, l \in \mathbb{N}$, where multiplying by l corresponds to multiplying the polytope with l, or equivalently multiplying the divisor D_P by l. Also note that intersections are more convieniently computed by Bézout's Theorem 2.3.7.

For $\mathbb{P}(1, 1, m)$ we get $d = lm$ ($l > 1$ for $\delta = 2$ by the restriction that the length of the edges must be at least δ) so

$$\begin{aligned}
D_P^2 &= \frac{(lm)^2}{m} = l^2 m \\
D_P \cdot K &= \frac{lm(-m-2)}{m} = -lm - 2l \\
\operatorname{bdet}(P) &= 0 \\
\operatorname{tdet}(P) &= m \\
\operatorname{idet}(P) &= 0 \\
v'_1 &= v'_2 = 0
\end{aligned}$$

This gives:

Proposition 4.2.1. *For $\mathbb{P}(1, 1, m)$ we have*

$$Q^{lD_P,1} = 3l^2m - 2lm - 4l - m + 4$$

$$Q^{lD_P,2} = -42l^2m + 39lm + 78l + 19m - 94$$

giving the Severi degrees

$$N^{lD_P,1} = 3l^2m - 2lm - 4l - m + 4$$

$$\begin{aligned}
N^{lD_p,2} = \frac{1}{2}(&9l^4m^2 - 12l^3m^2 - 24l^3m - 2l^2m^2 - 2l^2m \\
&+16l^2 + 4lm^2 + 31lm + 46l + m^2 + 11m - 78)
\end{aligned}$$

Note that setting $m = 1$ in the above correctly reduces to the smooth case.

For $\mathbb{P}(m, n, m+n)$ we similarly get

$$\begin{aligned}
D_P^2 &= mn(m+n)l^2 \\
D_P \cdot K &= -2l(m+n) \\
\mathrm{bdet}(P) &= n \\
\mathrm{tdet}(P) &= m \\
\mathrm{idet}(P) &= m+n \\
v_1' &= 0 \\
v_2' &= 0 \text{ unless } m = n = 1\text{, in which case } v_2' = 1
\end{aligned}$$

yielding

Proposition 4.2.2. *For $\mathbb{P}(m, n, m+n)$ not equal to $\mathbb{P}(1, 1, 2)$ we have*

$$\begin{aligned}
Q^{lD_P,1} &= 3mn(m+n)l^2 - 4l(m+n) - m - n + 4 \\
Q^{lD_P,2} &= -42mn(m+n)l^2 + 78l(m+n) + 27(m+n) - 112.
\end{aligned}$$

If $m = n = 1$ we get almost the same formula, but the constant term of $Q^{lD_P,2}$ is -110 instead of -112.

Note that setting $l = 1$ in the above gives back the dual degree we calculated before, as expected, see for instance Proposition 2.5.6. Larger l corresponds to the dual degree for the variety X_{lP}.

Note also that the formula for $Q^{D_P,1}$ above is easily deduced from our old formula for the dual degree. As noted before one may have a number of Gorenstein singularities, each contributing an Euler obstruction of 0, one may have a number of smooth vertices each contributing 1 and the top and bottom contribute $2 - \mathrm{tdet}(P)$ and $2 - \mathrm{bdet}(P)$. Thus

$$\sum \mathrm{Eu}(v_i) = 4 - \mathrm{tdet}(P) - \mathrm{bdet}(P) + v_1'$$

as expected.

Then we can ask if we can find new topological numbers to replace the ones that appear in the smooth case? For $Q^{D_P,1}$, c_2 is replaced by the sum of Euler-obstructions of the vertices. Trying this in the formula for $Q^{D_P,2}$ gives no satisfactory candidate for K^2. So it seems difficult without more data to make qualified guesses, since the singularities of the h-transverse varieties aren't very general.

4.3 Resolution of singularitites revisited

Given the singular surface $\mathbb{P}(k, m, n)$, one can as before construct the resolution of singularities, here denoted X. This is a smooth surface, so we can calculate the Severi degrees by the ordinary formula. The topological numbers needed are calculated by well-known results.

Using the facts about Chern classes from [CLS11, Ch. 13.1] one has that for a smooth complete toric surface X_Σ the Euler-characteristic $e(X_\Sigma) = |\Sigma(2)|$, the number of 2-dimensional cones. This also equals the second Chern class c_2 by [CLS11, Prop. 13.1.2].

Theorem 4.3.1. *[CLS11, Thm 10.5.3, Noether's Theorem] Let X be a smooth complete projective variety with canonical divisor K_X. Then*

$$K_X^2 = 12 - e(X)$$

Thus for a smooth toric surface $K_X^2 = 12 - c_2$. The computation of $Q^{L,1}, Q^{L,2}$ then reduces to

$$Q^{L,1} = 3L^2 + 2L \cdot K + c_2$$

$$Q^{L,2} = -42L^2 - 39L \cdot K - 6(12 - c_2) - 7c_2 = -42L^2 - 39L \cdot K - c_2 - 72$$

Now given the polytope P for $\mathbb{P}(k, m, n)$, let X_Σ be the minimal resolution of singularities. By the remarks at the end of section 3.4 we have that $D_P^2 = D^2$ and $D \cdot K_X = D_P \cdot K_{X_P}$. So these numbers will be equal for both surfaces. What remains is to describe c_2 in terms of k, m, n.

What we need to determine is the number of 2-dimensional cones in the fan Σ, this equals the number of 1-dimensional cones. By the construction of Σ and Proposition 3.1.13 this will be the original 3 plus the number of exceptional divisors in the resolution of each singularity, i.e. the length of the appropriate HJ-fraction. Using the notation of Theorem 3.2.3 we see that we will get

$$c_2 = 3 + r + s + t$$

where as before

$$\frac{n}{n-a} = [a_1, ..., a_r]^- \qquad \frac{m}{m-b} = [b_1, ..., b_s]^- \qquad \frac{k}{k-c} = [c_1, ..., c_t]^-$$

Alternatively this can be formulated in terms of the continued fractions of the form $\frac{\lambda}{\lambda-1}$: By Proposition 3.1.5 we have $r = 1 - \sum_{i=1}^{u}(2 - d_i)$ where $\frac{n}{a} = [d_1, ..., d_u]$ and similar results for s and t. Summing up we have the following.

Proposition 4.3.2. *Given $\mathbb{P}(k, m, n)$, find minimal natural numbers a, b, c such that*

$$k + am \equiv 0 \pmod{n}$$

$$n + bk \equiv 0 \pmod{m}$$

$$m + cn \equiv 0 \pmod{k}$$

Let $\frac{n}{a} = [d_1, ..., a_u]^-$, $\frac{m}{b} = [e_1, ..., e_v]^-$, $\frac{k}{c} = [f_1, ..., f_w]^-$ *and* X_Σ *be the minimal desingularization of* $\mathbb{P}(k, m, n)$ *and* c_2 *its second chern class. Then*

$$c_2 = 6 - \sum_{i=1}^{u}(2 - d_i) - \sum_{i=1}^{v}(2 - e_i) - \sum_{i=1}^{w}(2 - f_i)$$

Remark 4.3.3. It is not clear how to interpret this formula if one or more of the vertices of $\mathbb{P}(k, m, n)$ already are smooth, but if we by convention set the corresponding continued fraction equal to $[1]^-$ we obtain a consistent formula, for instance for $\mathbb{P}(1, 1, 1)$ we set $d_1 = e_1 = f_1 = 1$ giving the correct answer $c_2 = 3$.

Applying this to our h-transverse polytopes, we get for $\mathbb{P}(1, 1, m)$ the desingularization $\mathcal{H}_m$, see example 3.1.14. $\mathbb{P}(1, 1, m)$ has two smooth vertices and the last a $(m, m-1)$-cone in $N_\mathbb{R}$. Since $\frac{m}{m-1} = [2, ..., 2]^-$ we get $\sum(2-2) = 0$, so $c_2 = 4$. Thus for $\mathcal{H}_m$ with D being the pullback of D_P we have

$$Q^{lD,1} = 3l^2m - 2lm - 4l + 4$$

$$Q^{lD,2} = -42l^2m + 39lm + 78l - 76$$

For $\mathbb{P}(m, n, m + n)$ we have $a = 1$, $b = n - 1$ and $c = m - 1$. Thus $c_2 = 6 - (2 - m - n) + 0 + 0 = 4 + m + n$ which gives

$$Q^{lD,1} = 3mn(m + n)l^2 - 4l(m + n) + 4 + m + n$$

$$Q^{lD,2} = -42mn(m + n)l^2 + 78l(m + n) - m - n - 76$$

4.4 Further research

Some directions could be investigated further.

One could try to do more computations in the 3-dimensional case, at least in the case of $\mathbb{P}(1, k, m, n)$, or for other 3-dimensional singular varieties. The general 3-dimensional weighted projective space seems difficult to handle, since its polytope will be embedded in a 4-dimensional vector space the same way the polytope giving the weighted plane is embedded in $\mathbb{R}^3$. To handle 3-dimensional polytopes one would need a systematic way of handling 3-dimensional convex hulls. The numerical data suggest that it should be possible to find closed formulas in at least some cases, for instance $\mathbb{P}(1, 2k - 1, 2k, 2k + 1)$.

In the surface case one could continue to do calculations on other families of singular varieties or one could try to say more about curve counting on the weighted projective planes, as we tried in the last chapter. Ardila and Block [AB13] write that they suspect Severi degrees of all large toric surfaces are polynomial, however for the time being, using tropical geometry to count curves only works in the h-transverse case. The results of Ardila and Block give polynomials in lengths and directions of the polytopes, while [LO14] have polynomials in the Gorenstein case involving determinants of vertices. Since all singularities of h-transverse polytopes only depend on one variable (i.e., either are $(d, 1)$-or $(d, d-1)$-cones) we suspect that a formulas for general toric varieties might have two parameters for each singularity (instead of 1, direction/determinant), as well as some glueing parameters describing how the different cones are related.

The success of the Euler-obstruction as a modified c_2 in the formula for $Q^{L,1}$ leads one to hope that it should work for higher $Q^{L,i}$ as well. Under current knowledge this seems problematic, at least if we still want only 4 topological numbers, since no suitable candidate for K^2 exists. For instance, consider $Q^{D_P,2}$ in the case $\mathbb{P}(1, 1, m)$, letting c_2 be the sum of Euler-obstructions force $K^2 = 11 - 2m$, but this doesn't fit in the formulas for $Q^{D_P,3}$ and $Q^{D_P,4}$, computed by Florian Block (private correspondence).

In [Dai06] there is proved a Noether's theorem for singular surfaces

Proposition 4.4.1. *[Dai06, Proposition 4.9] The self intersection of the canoncial divisor on a singular toric surface X_Σ is*

$$K^2_{X_\Sigma} = 12 - \Sigma(2) + \sum_{\sigma_i} \frac{d_i - k_1 + 1}{d_i} + \frac{d_i - \hat{k}_i + 1}{d_i} - 2 + \sum_{j=1}^{s_i} (b_j - 3),$$

where the sum is over all singular cones σ_i, σ_i is a (d_i, k_i)-cone, $\frac{d_i}{d_i - k_i} = [b_1, ..., b_{s_i}]^-$ and $\hat{k}_i$ is the unique integer $0 \leq \hat{k}_i < d_i$ such that $k_i\hat{k}_i \equiv 1 \pmod{d_i}$.

One might hope that this could hint at a suitable candidate for a modified $\mathscr{K}^2$. Since we are counting curves, we need an integer value, while on the general surface intersection products take values in $\mathbb{Q}$. Other invariants that are integer valued might be what we need.

One could also do more computations and experimentation on the resolution of singularities, hoping that the formulas for the smooth resolved surface might be related to the singular case. As seen in Proposition 4.3.2 c_2 for the resolved surface is related to the Euler obstructions of the duals of the cones in the fan. Maybe this invariant may appear in a general formula.

Another possible approach to work more on this, is to consider the covering map (2.2)

$$\mathbb{P}^2 \to \mathbb{P}(k, m, n)$$
$$(X : Y : Z) \mapsto (X^k : Y^m : Z^n)$$

Using the results for the smooth $\mathbb{P}^2$, one might attempt to use this map to say something about the Severi degrees of $\mathbb{P}(k, m, n)$.

Appendix A

Lattices

We will give some well known results on lattices and bases of them. Given a lattice M of dimension n in a vecor space, we wish to know when a given set of linear independent vectors $b_1, ..., b_n \in M_{\mathbb{R}} = M \otimes \mathbb{R}$ is a basis for the lattice. Consider the set of points in $M_{\mathbb{R}}$ given by $T(b_1, ..., b_n) = \{\sum_{i=1}^{n} c_i b_i | 0 \leq c_i < 1\}$. We have that:

Lemma A.0.2. *$b_1, ..., b_n$ is a basis for the lattice M if and only if $T(b_1, ..., b_n) \cap M = \{0\}$*

Proof. Assume $(b_1, ..., b_n)$ is a basis. Let $x \in T(b_1, ..., b_n) \cap M$. Then $x = \sum_{i=1}^{n} c_i b_i = \sum_{i=i}^{n} n_i b_i$ for $0 \leq c_i < 0$, $n_i \in \mathbb{Z}$. Thus $0 = \sum_{i=1}^{n}(c_i - n_i)b_i$. Since the b_is are linearly independent this implies that $c_i = n_i$, hence $c_i = 0$.

Assume $T(b_1, ..., b_n) \cap M = \{0\}$. Pick a lattice point $x \in M$. Since $b_1, ..., b_n$ is a basis for the vector space $M_{\mathbb{R}}$ we can find $d_i \in \mathbb{R}$ such that $x = \sum_{i=1}^{n} d_i b_i$. Let $d_i = n_i + c_i$ where $n_i \in \mathbb{Z}$ and $0 \leq c_i < 1$. Then $x - \sum_{i=1}^{n} n_i b_i \in T(b_1, ..., b_n) \cap M = \{0\}$, hence $c_i = 0$ for all i. Thus $b_1, ..., b_n$ is a basis for M. □

Sometimes we are also interested in different bases for the same lattice. Given n linearly independent vectors $\mathcal{B} = \{b_1, ..., b_n\}$ define the lattice generated by $\mathcal{B}$ as $\mathcal{L}(\mathcal{B}) = \{\sum_{i=1}^{n} \mathbb{Z} b_i\} = \{Bx | x \in \mathbb{Z}^n\}$ where B is the matrix with columns b_i.

Lemma A.0.3. *Two bases $\mathcal{B}$, $\mathcal{C}$ for $M_{\mathbb{R}}$ generate the same lattice L if and only if $B = CU$ for a matrix U with integral coefficients and determinant $= \pm 1$.*

Proof. Assume that $\mathcal{B}$ and $\mathcal{C}$ generate the same lattice. Then we have equations $b_i = \sum_{i=1}^{j} a_{ij} c_i$ which is equivalent to $B = CU$ for a matrix U with integral coefficients. By switching the roles of $\mathcal{B}$ and $\mathcal{C}$ we get $C = BV$, thus $B = BVU$. Taking determinants we get $1 = det(V)det(U)$ which implies that $det(U) = \pm 1$ since U and V has integral coeffiecients, and therefore also integral determinant.

Now assume that $B = CU$ for a matrix U with integral coefficients and determinant $= \pm 1$. Using Cramer's rule on the equation $Ux = e_i$ one shows that each column of U^{-1} also has integral coefficents, and the determinant also equals ± 1. Hence we have $B = CU$ and $C = BU^{-1}$. Thus $\mathcal{L}(\mathcal{B}) \subset \mathcal{L}(\mathcal{C})$ and $\mathcal{L}(\mathcal{C}) \subset \mathcal{L}(\mathcal{B})$, hence they are equal. $\square$

Remark A.0.4. In particular n vectors generate $\mathbb{Z}^n$ if and only if their determinant equals ± 1.

We can define the determinant of a lattice as the determinant of a basis. By the lemma above this will be independant of choice of basis. This will also be the volume of any fundamental domain $T(b_1, ..., b_n)$ where $\{b_1, ..., b_n\}$ is a basis. For our purposes we usually want to normalize the lattice-volume such that the volume spanned by a simplex is 1, and this equals $\frac{\det(b_1,...,b_n)}{n!}$. We will also need the following result:

Proposition A.0.5. *[Cas97, Cor.3 p. 14] Any lattice vector* $v = (v_1, ..., v_n) \in \mathbb{Z}^n$ *with* $\gcd(v_1, ..., v_n) = 1$ *can be extended to a basis for* $\mathbb{Z}^n$*.*

Definition A.0.6. Given a lattice $L \cong \mathbb{Z}^n$, we define its dual lattice $L^\vee$ as the following set:

$$L^\vee = \{x \in L_\mathbb{R} | \langle x, y \rangle \in \mathbb{Z} \ \forall y \in L\}$$

where we use the normal inner product on $L_\mathbb{R} \cong \mathbb{R}^n$.

From the inner product on $L_\mathbb{R}$ we inherit a pairing

$$L \times L^\vee \to \mathbb{Z}$$

which induces isomorphisms

$$L^\vee \simeq \mathrm{Hom}_\mathbb{Z}(L, \mathbb{Z})$$

$$L \simeq \mathrm{Hom}_\mathbb{Z}(L^\vee, \mathbb{Z})$$

Proposition A.0.7. *The dual of the lattice* $N = \mathbb{Z}^{n+1}/(q_0, ..., q_n)$ *is*

$$M = \{(m_0, ..., m_n) \in \mathbb{Z}^{n+1} | \sum_{i=0}^{n} m_i q_i = 0\}$$

Proof. $N^\vee = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{n+1}/(q_0, ..., q_n), \mathbb{Z})$. This amounts to, for each basiselement e_i of $\mathbb{Z}^{n+1}$, assigning a value $m_i \in \mathbb{Z}$. However, since the element $(q_0, ..., q_n)$ must map to zero, we must have $\sum_{i=0}^{n} m_i q_i = 0$. $\square$

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