

On dominated extensions in
linear subspaces of $\mathcal{C}_0(X)$.

E.M. Alfsen and B. Hirsberg

Introduction.

The aim of this paper is to study extensions within a given linear subspace A of $\mathcal{C}_0(X)$ of functions defined on a compact subset of the Choquet boundary $\partial_A X$, in such a way that the extended function remains dominated by a given A -superharmonic function ψ . (Precise definitions follow). Our main result is the possibility of such extensions for all functions in $A|_F$ provided F satisfies the crucial requirement that the restriction to F of every orthogonal boundary measure shall remain orthogonal (Theorem 4.5). Taking $\psi \equiv 1$ in this theorem we obtain that F has the norm preserving extension property (Corollary 4.6). This was first stated by Bjørk [5] for a real linear subspace A of $\mathcal{C}_{\mathbb{R}}(X)$ and for a metrizable X . A geometric proof of the latter result was given by Bai Andersen [3]. In fact, he derived it from a general property of split faces of compact convex sets, which he proved by a modification of an inductive construction devised by Pelczynski for the study of simultaneous extensions within $\mathcal{C}_{\mathbb{R}}(X)$ [12]. Our treatment of the more general extension property proceeds along the same lines as Bai Andersen's work. It depends strongly upon the geometry of the state space of A , and Bai Andersen's construction is applied at an essential point in the proof. Note however, that this is no mere translation of real arguments. The presence of complex orthogonal measures seems to present a basically new situation. Applying arguments similar to those indicated above, we obtain a general peak set - and peak point criterion (Theorem 5.4 and Corollary 5.5) of which the latter has been proved for real spaces by Bjørk [6]. In section 6 (Theorem 6.1) it is shown how the Bishop - Rudin - Carleson Theorem follows from the general extension theorem mentioned above. In section 7 we assume that A is a sup-norm algebra over X and study the interrelationship between our conditions on F and a condition introduced by Gamelin and Glicksberg [9], [10]. Finally we should like to point out that some related investigations have been carried out recently by Briem [7]. However, his methods are rather different. The geometry of the state space is not invoked, but instead he applies in an essential way a measurable selection theorem of Rao [14].

We want to thank Bai Andersen for many stimulating discussions of the problems of the present paper. Also we are indebted to A.M. Davie for the counterexample at the end of section 7.

1. Preliminaries and notation.

In this note X shall denote a compact Hausdorff space and A a closed, linear subspace of $\mathcal{C}_\mathbb{C}(X)$, which separates the points of X and contains the constant functions.

The state space of A , i.e.

$$S = \{p \in A^* \mid p(1) = \|p\| = 1\},$$

is convex and compact in the w^* -topology.

Since A separates the points of X , we have a homeomorphic embedding Φ of X into S , defined by

$$\Phi(x)(a) = a(x), \quad \text{all } a \in A.$$

Similary we have an embedding Ψ of A into the space $A_\mathbb{C}(S)$ of all complex valued w^* -continuous affine functions on S ; namely

$$\Psi(a)(p) = p(a), \quad \text{all } p \in S.$$

By taking real parts of the functions $\Psi(a)$ we obtain the linear space of those real valued w^* -continuous affine functions on S , which can be extended to real valued w^* -continuous linear functionals on A^* , and this space $A_\mathbb{R}(S, A^*)$ is dense in the space $A_\mathbb{R}(S)$ of all real valued affine w^* -continuous functions on S , [1, Cor.I.1.5].

We shall denote by $M(X)$, resp. $M(S)$, the Banach space of all complex Radon measures on X , resp. S ; by $M^+(X)$ resp. $M^+(S)$ the cone of positive (real) measures, and by $M_1^+(X)$ resp. $M_1^+(S)$ the w^* -compact convex set of probability measures. The set of extreme points of S will be denoted by $\partial_e S$, and the Choquet boundary of X with respect to A is defined as the set

$$\partial_A X = \{x \in X \mid \varphi(x) \in \partial_e S\}.$$

From [13, p.38] it follows that $\partial_e S \subset \varphi(X)$ so that φ maps $\partial_A X$ homeomorphically onto $\partial_e S$.

A measure $\mu \in M(S)$ is said to be a boundary measure on S if the total variation $|\mu|$ is a maximal measure in Choquet's ordering of positive measures [1, ch.I, §3], [13, p.24]. A boundary measure is supported by $\partial_e S$ [1, Prop.I.4.6]. For a metrizable X (and S) a measure $\mu \in M(S)$ is a boundary measure if and only if $|\mu|(S \setminus \partial_e S) = 0$. We shall denote by $M(\partial_e S)$ the set of boundary measures on S (abuse of language). Observe that if $\mu \in M(\partial_e S)$, then the real and imaginary parts of μ are both boundary measures. The set of boundary measures on X is defined by

$$M(\partial_A X) = \{\mu \in M(X) \mid \varphi\mu \in M(\partial_e S)\},$$

where $\varphi\mu$ denotes the transport of the measure μ on X to a measure on S . For a metrizable X a measure μ on X belongs to $M(\partial_A X)$ if and only if $|\mu|(X \setminus \partial_A X) = 0$.

For every $\mu \in M_1^+(S)$ we shall use the symbol $r(\mu)$ to denote the barycenter of μ , i.e. the unique point in S such that $a(r(\mu)) = \mu(a)$ for all $a \in A_{\mathbb{R}}(S)$. The Choquet-Bishop-de Leeuw Theorem states that each point in S is the barycenter of a maximal (boundary) probability measure [1, Th.I.4.8]. Accordingly we shall denote by $M_p^+(\partial_e S)$ the non-empty set of maximal (boundary) probability measures on S with barycenter $p \in S$. For $x \in X$ we define $M_x^+(\partial_A X)$ to be the set of all $\mu \in M_1^+(X)$ such that $\varphi\mu \in M_{\varphi(x)}^+(\partial_e S)$. Equivalently, $M_x^+(\partial_A X)$ consists of all $\mu \in M_1^+(\partial_A X)$ such that

$$a(x) = \int a d\mu \quad \text{all } a \in A,$$

i.e. μ represents x with respect to A . Also we denote by $M_x^+(X)$ the set of probability measures on all of X which represents x in this way. Similarly we denote by $M_p^+(S)$ the set of probability measures on S with barycenter p . The annihilator of A in $M(X)$ is the set

$$A^\perp = \{ \mu \in M(X) \mid \mu(a) = 0 \quad \text{all } a \in A \}$$

Finally we shall use the symbol $\mathcal{B}(X)$ to denote the class of all complex valued bounded Borel functions on X .

2. A dominated extension theorem.

We start by proving a general dominated extension theorem, which may be of some independent interest. In this connection we give the following:

Definition 2.1. \mathcal{A} is the class of all $f \in \mathcal{B}(X)$ such that

$$(2.1) \quad \mu(f) = 0 \quad \text{all } \mu \in A^\perp$$

Clearly $A \subset \mathcal{A}$

Theorem 2.2. Let F be a closed subset of X for which $A|_F = \{a|_F \mid a \in A\}$ is closed in $\mathcal{C}_0(F)$; let $a_0 \in A|_F$ and let $\varphi: X \rightarrow \mathbb{R}^+ \cup \{\infty\}$ be a strictly positive l.s.c. function such that $|a_0(x)| < \varphi(x)$ for all $x \in F$.

Now, if there exists a function $\bar{a}_0 \in \mathcal{A}$ such that

$$(2.2) \quad \bar{a}_0|_F = a_0, \quad |\bar{a}_0(x)| < \varphi(x) \quad \text{all } x \in X$$

then there exists a function in A with the same properties.

Proof: Without lack of generality we can assume that φ is a bounded function with values in \mathbb{R}^+ , and we assume for contradiction that

$$(2.3) \quad a_0 \notin G|_F = \{a|_F \mid a \in G\},$$

where

$$(2.4) \quad G = \{a \in A \mid |a(x)| < \varphi(x)\}.$$

Since φ is l.s.c., G is an open subset of A . Since $A|_F$ is closed in $\mathcal{C}_\mathbb{C}(F)$, we may apply the Open Mapping Theorem to the restriction map $R_F: A \rightarrow A|_F$. Hence $G|_F$ is an open subset of $A|_F$. Furthermore $G|_F$ is convex and circled. By the Hahn-Banach Theorem we can find a measure $\nu \in M(X)$ with $\text{supp } \nu \subset F$ such that

$$(2.5) \quad \nu(a_0) \geq 1 \geq |\nu(b_0)| \quad \text{all } b_0 \in G|_F$$

Now we consider $\mathcal{C}_\mathbb{C}(X)$ equipped with the norm

$$(2.6) \quad \|f\|_\varphi = \sup \left\{ \frac{|f(x)|}{\varphi(x)} \mid x \in X \right\},$$

and observe that this norm is topologically equivalent with the customary, uniform norm. The dual of $(\mathcal{C}_\mathbb{C}(X), \|\cdot\|_\varphi)$ is seen to be $M(X)$ equipped with the norm $\|\mu\|_\varphi = \|\varphi\mu\|$.

It follows from (2.5) that the linear functional ξ on $(A, \|\cdot\|_\varphi)$ defined by

$$(2.7) \quad \xi(a) = \nu(R_F a) \quad \text{all } a \in A.$$

is bounded with norm $\|\xi\|_\varphi \leq 1$. Now we extend ξ with preservation of φ -norm to a bounded linear functional on $(\mathcal{C}_\mathbb{C}(X), \|\cdot\|_\varphi)$. This gives a measure $\mu \in M(X)$, such that

$$(2.8) \quad \xi(a) = \mu(a) \quad \text{all } a \in A, \quad \|\varphi\mu\| = \|\xi\|_\varphi \leq 1$$

It follows from (2.2) and (2.8) that

$$(2.9) \quad |\mu(\bar{a}_0)| = |(\varphi\mu)(\varphi^{-1}\bar{a}_0)| < 1$$

From (2.7) and (2.8) it follows that $\mu - \nu \in A^\perp$, and since $\bar{a}_0 \in \mathcal{A}$ we shall have

$$(2.10) \quad \left| \int_X \bar{a}_0 \, d\mu \right| = \left| \int_X \bar{a}_0 \, d\nu \right| = \int_F a_0 \, d\nu \geq 1$$

This contradicts (2.9) and the proof is complete.

3. Applications of the geometry of the state space.

We shall consider compact subsets F of $\partial_A X$ satisfying one or the other of the following two requirements:

$$(A.1) \quad \mu \in M(\partial_A X) \cap A^\perp \implies \mu|_F \in A^\perp$$

$$(A.2) \quad \mu \in M(\partial_A X) \cap A^\perp \implies \mu(F) = 0$$

We assume first (A.1). We also agree to write $S_F = \overline{\text{co}}(\Phi(F))$, and we observe that there is a canonical embedding ψ_F of $A|_F$ into $A_{\mathbb{C}}(S_F)$, defined by

$$(3.1) \quad \psi_F(a_0)(p) = p(a), \quad \text{all } p \in S_F$$

where $a \in A$; $a|_F = a_0$. In fact, it follows by the integral form of the Krein-Milman Theorem that p can be expressed as the barycenter of a probability measure on $\Phi(F)$, and hence that the particular choice of a is immaterial.

For every $a_0 \in A|_F$ we define

$$(3.2) \quad \bar{a}_0(x) = \int_F a_0 \, d\mu_x, \quad x \in X, \quad \mu_x \in M_x^+(\partial_A X),$$

and

$$(3.3) \quad \bar{a}_0(p) = \int_{S_F} \psi_F(a_0) d\mu_p, \quad p \in S, \quad \mu_p \in M_p^+(\partial_e S)$$

and we note that these definitions are legitimate by virtue of (A.1). We also note that $\mu_p(S_F) = \mu_p(\mathfrak{F}(F))$ for all $p \in S$ and $\mu_p \in M_p^+(\partial_e S)$ [3, Lem.1].

Clearly \bar{a}_0 is an extension of a_0 to a function defined on all of X ; and if we think of \mathfrak{F} as an imbedding of X into S , then \bar{a}_0 will in turn be an extension of \bar{a}_0 to a function defined on all of S . More specifically, for every $\mu_x \in M_x^+(\partial_A X)$ the transported measure $\mathfrak{F}\mu_x$ is in $M_{\mathfrak{F}(x)}^+(\partial_e S)$ and so

$$\bar{a}_0(\mathfrak{F}(x)) = \int_{S_F} \psi_F(a_0) d(\mathfrak{F}\mu_x) = \int_F \psi_F(a_0) \circ \mathfrak{F} d\mu_x = \int_F a_0 d\mu_x,$$

which entails

$$(3.4) \quad \bar{a}_0 \circ \mathfrak{F} = \bar{a}_0$$

Lemma 3.1. If F satisfies (A.1) and $a_0 \in A|_F$, then $\bar{a}_0 \in \mathcal{A}$

Proof: Let $\lambda = \|a_0\|_F$ and define $a_1 = \operatorname{Re} \psi_F(a_0) + \lambda$, $a_2 = \operatorname{Im} \psi_F(a_0) + \lambda$. Then $a_1, a_2 \in A_{\mathbb{R}}(S_F)^+$ and for any $p \in S$ and $\mu_p \in M_p^+(\partial_e S)$

$$\bar{a}_0(p) = \int_{S_F} \psi_F(a_0) d\mu_p = \int_{S_F} a_1 d\mu_p + i \int_{S_F} a_2 d\mu_p - \lambda \mu_p(S_F) - i \lambda \mu_p(S_F)$$

At this point we shall appeal to the geometric theory of compact convex sets. It follows from the requirement (A.1) that S_F is a split face of S , and hence that

$$\bar{a}_0(p) = \widehat{a_1 \chi_{S_F}}(p) + i \widehat{a_2 \chi_{S_F}}(p) - \lambda \hat{\chi}_{S_F}(p) - i \lambda \hat{\chi}_{S_F}(p)$$

where all the functions on the right hand side are u.s.c. and affine [1, Th.II6.12], [1, Th.II.6.18] (cf. also [2, Th.3.5]). In particular \bar{a}_0 is a Borel function, and it follows from (3.4) that \bar{a}_0 is a Borel function as well. Since the barycentric calculus applies to real valued u.s.c. affine functions on S [1, Cor.II.4], we shall have:

$$(3.5) \quad \bar{a}_0(p) = \int_S \bar{a}_0 d\mu_p, \quad p \in S, \quad \mu_p \in M_p^+(S)$$

Let $\mu \in A^\perp$ be arbitrary and decompose

$$(3.6) \quad \mu = \sum_{i=1}^4 \alpha_i \mu_i,$$

where $\alpha_1 \in \mathbb{R}^+$, $\alpha_2 \in -\mathbb{R}^+$, $\alpha_3 \in i\mathbb{R}^+$, $\alpha_4 \in (-i)\mathbb{R}^+$ and $\mu_i \in M_1^+(X)$ for $i = 1, 2, 3, 4$. Let $p_i \in S$ be the barycenter of μ_i and let $\sigma_i \in M_{p_i}^+(\partial_e S)$ for $i = 1, 2, 3, 4$.

Since $\overline{\partial_e S} \subseteq \Phi(X)$ we can transport σ_i back to X by the map Φ^{-1} , and it follows that the measures $\mu_i - \Phi^{-1}\sigma_i$ are (real) orthogonal measures for $i = 1, 2, 3, 4$.

Writing

$$\tau = \sum_{i=1}^4 \alpha_i \Phi^{-1}\sigma_i$$

we obtain $\tau \in M(\partial_A X)$ and $\mu - \tau \in A^\perp$. In fact for every $a \in A$,

$$\int_X a d(\mu - \tau) = \int_S \Psi(a) d(\Phi(\mu - \tau)) = \sum_{i=1}^4 \alpha_i \int_S \Psi(a) d(\mu_i - \sigma_i) = 0$$

Since $\mu \in A^\perp$, we shall also have $\tau \in A^\perp$ and then $\tau|_F \in A^\perp$ by virtue of (A.1). Hence by (3.3), (3.4), (3.5):

$$\begin{aligned} \int_X \bar{a}_0 d\mu &= \int_X \bar{a}_0 \circ \Phi d\mu = \int_S \bar{a}_0 d(\Phi\mu) = \sum_{i=1}^4 \alpha_i \int_S \bar{a}_0 d(\mu_i) = \\ &= \sum_{i=1}^4 \alpha_i \bar{a}_0(p_i) = \sum_{i=1}^4 \alpha_i \int_{S_F} \Psi_F(a_0) d\sigma_i = \int_{S_F} \Psi_F(a_0) d(\Phi\tau) \\ &= \int_F a_0 d\tau = 0 \end{aligned}$$

Hence $\bar{a}_0 \in \mathcal{A}$, and the proof is complete.

We next turn to the less restrictive requirement (A.2). It follows by a slight modification of the proof of [1, Th.II.6.12], that the requirement (A.2) implies that S_F is a parallel face of S and hence that the function $\hat{\chi}_{S_F}$ is affine [15 Th.12].

For every $x \in X$ we define

$$(3.7) \quad \bar{\chi}_F(x) = \int_F 1 d\mu_x, \quad \mu_x \in M_x^+(\partial_A X)$$

and we note that this definition is legitimate by virtue of (A.2).

For $x \in X$ and $\mu_x \in M_x^+(\partial_A X)$ we shall have:

$$\hat{\chi}_{S_F}(\phi(x)) = \int_{S_F} 1 d(\phi\mu_x) = \int_F 1 d\mu_x = \bar{\chi}_F(x)$$

which entails

$$(3.8) \quad \hat{\chi}_{S_F} \circ \phi = \bar{\chi}_F$$

Applying (3.8) and proceeding as in the proof of Lemma 3.1, we can prove.

Lemma 3.2. If F satisfies (A.2), then $\bar{\chi}_F \in \mathcal{A}$

4. Extensions dominated by A -superharmonic functions.

We now proceed to the main theorem, but first we give some definitions.

Definition 4.1. A function $\psi: X \rightarrow \mathbb{R} \cup \{\infty\}$ is said to be A -superharmonic if it satisfies

$$(i) \quad \psi \text{ l.s.c.}$$

$$(ii) \quad \psi(x) \geq \int_X \psi d\mu_x, \quad \text{all } x \in X \text{ and } \mu_x \in M_x^+(X)$$

Definition 4.2. Let F be a compact subset of X . F has the almost norm preserving extension property, if for each $\epsilon > 0$ and $a_0 \in A|_F$ there exists a function $a \in A$ such that

$$(4.1) \quad \underline{a|_F = a_0, \quad \|a\|_X \leq \|a_0\|_F + \epsilon}$$

If ϵ can be taken to be zero in (4.1), then F has the norm preserving extension property.

We shall need a criterion for the almost norm preserving extension property, which is essentially due to Gamelin [9, p.281] (cf. also Glicksberg [10, p.420] and Curtis [8]). For the sake of completeness we present a short proof.

Lemma 4.3. A closed subset F of X has the almost norm preserving extension property if for each $\sigma \in A^\perp$:

$$(4.2) \quad \inf_{v \in (A|_F)^\perp} \|\sigma|_F + v\| \leq \|\sigma|_{X \setminus F}\|$$

Proof: The almost norm preserving extension property is tantamount to the equality of the uniform norm on $A|_F$ and the extension norm:

$$\|a_0\|_{\text{ext.}} = \inf \{ \|a\|_X \mid a \in A, \quad a|_F = a_0 \}.$$

In this norm $A|_F$ is isometrically isomorphic to the quotient space A/F^\perp where $F^\perp = \{a \in A \mid a \equiv 0 \text{ on } F\}$; and we are to prove that the canonical imbedding $\rho: A/F^\perp \rightarrow A|_F$ is an isometry from the quotient norm to the uniform norm. By duality (i.e. by Hahn-Banach) we may as well prove that the transposed map ρ^* is an isometry. Representing the occurring functionals by measures, we can translate this statement into

$$(4.3) \quad \inf_{\sigma \in A^\perp} \|\mu + \sigma\| = \inf_{\nu \in (A|_F)^\perp} \|\mu + \nu\|, \quad \text{all } \mu \in M(F)$$

To prove that (4.2) implies (4.3), we consider measures $\mu \in M(F)$, $\sigma \in A^\perp$ and an arbitrary $\epsilon > 0$. Also we can choose $\nu_0 \in (A|_F)^\perp$ such that

$$\|\sigma|_F - \nu_0\| \leq \inf_{\nu \in (A|_F)^\perp} \|\sigma|_F - \nu\| + \epsilon \leq \|\sigma|_{X \setminus F}\| + \epsilon$$

Then

$$\begin{aligned} \|\mu - \sigma\| &= \|\mu - \sigma|_F\| + \|\sigma|_{X \setminus F}\| \geq \|\mu - \nu_0\| - \|\nu_0 - \sigma|_F\| + \|\sigma|_{X \setminus F}\| \\ &\geq \|\mu - \nu_0\| - \epsilon \geq \inf_{\nu \in (A|_F)^\perp} \|\mu - \nu\| - \epsilon, \end{aligned}$$

which completes the proof.

We remark for later purposes that for $\mu \in M(F)$:

$$(4.4) \quad \sup \{ \left| \int_F a_0 d\mu \right| \mid \|a_0\|_F \leq 1, a_0 \in A|_F \} = \inf_{\nu \in (A|_F)^\perp} \|\mu - \nu\|$$

Proposition 4.4. If F is a compact subset of $\partial_A X$ satisfying (A.1), then F has the almost norm preserving extension property.

Proof: By Lemma 4.3 and the above remark (4.4), it suffices to prove that for every $\sigma \in A^\perp$:

$$\sup \{ \left| \int_F a_0 d\sigma \right| \mid \|a_0\|_F \leq 1, a_0 \in A|_F \} \leq \|\sigma|_{X \setminus F}\|.$$

Let $\sigma \in A^\perp$, and $a_0 \in A|_F$ with $\|a_0\|_F \leq 1$. Applying Lemma 3.1 we obtain

$$0 = \sigma(\bar{a}_0) = \int_F a_0 d\sigma + \int_{X \setminus F} \bar{a}_0 d\sigma,$$

such that

$$\left| \int_F a_0 \, d\sigma \right| = \left| \int_{X \setminus F} \bar{a}_0 \, d\sigma \right| \leq \|\sigma\|_{X \setminus F},$$

which completes the proof.

If F is a compact subset of $\partial_A X$ satisfying (A.1), then $A|_F$ is a closed subspace of $\mathcal{C}_c(F)$. In fact, $A|_F$ is isometrically isomorphic to A/F^\perp .

We are now able to state and prove the main theorem. The proof of this theorem is essentially based upon Theorem 2.1 and the technique developed by Bai Andersen [3].

Theorem 4.5. Let F be a compact subset of $\partial_A X$ satisfying (A.1), i.e.

$$\underline{\mu \in M(\partial_A X) \cap A^\perp \implies \mu|_F \in A^\perp.}$$

Let $a_0 \in A|_F$ and let ψ be a strictly positive A -superharmonic function on X such that $|a_0(x)| \leq \psi(x)$ for all $x \in F$. Then there exists a function $a \in A$ such that

$$(i) \quad \underline{a|_F = a_0},$$

$$(ii) \quad \underline{|a(x)| \leq \psi(x) \quad \text{all } x \in X}.$$

Proof: Without loss of generality we may assume ψ to be bounded. Since F satisfies the requirement (A.1), $A|_F$ is closed and $\bar{a}_0 \in A$.

Thus by Theorem 2.1 we can extend a_0 to a function $a'_0 \in A$ such that $|a'_0(x)| < \varphi(x)$ for all $x \in X$, whenever φ is a bounded l.s.c. function on X such that

$|\bar{a}_0(x)| < \varphi(x)$ for all $x \in X$.

Applying this to the function $\varphi_1 = 2\psi$, we can extend a_0 to a function $a_1 \in A$ such that $|a_1(x)| < 2\psi(x)$ for all $x \in X$.

Now define

$$\varphi_2 = 2\psi \wedge [2^2(\psi - 2^{-1}|a_1|)] .$$

The function φ_2 is strictly positive on all of X . For $x \in F$ we have $\varphi_2(x) = 2\psi(x)$, and hence for an arbitrary $x \in X$:

$$\begin{aligned} |\bar{a}_0(x)| &= \left| \int_F a_0 d\mu_x \right| \leq \int_F |a_0| d\mu_x \leq \int_F \psi d\mu_x < \int_X 2^2(\psi - 2^{-1}|a_1|) d\mu_x \\ &= 2^2 \left(\int_X \psi d\mu_x - 2^{-1} \int_X |a_1| d\mu_x \right) \leq 2^2(\psi(x) - 2^{-1}|\int_X a_1 d\mu_x|) \\ &= 2^2(\psi(x) - 2^{-1}|a_1(x)|) . \end{aligned}$$

Hence $|\bar{a}_0(x)| < \varphi_2(x)$ all $x \in X$.

By Theorem 2.1 we can choose $a_2 \in A$ such that

$$|a_2| < \varphi_2, \quad a_2|_F = a_0$$

Assume for induction that extensions $a_1, \dots, a_n \in A$ have been constructed such that

$$|a_p| < 2\psi \wedge [2^p(\psi - \sum_{r=1}^{p-1} 2^{-r}|a_r|)] = \varphi_p, \quad p = 2, \dots, n,$$

and define

$$\varphi_{n+1} = 2\psi \wedge [2^{n+1}(\psi - \sum_{r=1}^n 2^{-r}|a_r|)] .$$

The function φ_{n+1} is strictly positive by induction hypothesis. For $x \in F$ we shall have

$$2^{n+1}(\psi(x) - \sum_{r=1}^n 2^{-r}|a_0(x)|) \geq 2^{n+1}(\psi(x) - \sum_{r=1}^n 2^{-r}\psi(x)) = 2\psi(x)$$

such that $\varphi_{n+1}(x) = 2\psi(x)$. Hence for an arbitrary $x \in X$:

$$\begin{aligned}
 |\bar{a}_0(x)| &= \left| \int_F a_0 d\mu_x \right| \leq \int_F \psi d\mu_x < \int_X 2^{n+1} \left(\psi - \sum_{r=1}^n 2^{-r} |a_r| \right) d\mu_x \\
 &\leq 2^{n+1} \left(\int_X \psi d\mu_x - \sum_{r=1}^n 2^{-r} \left| \int a_r d\mu_x \right| \right) \leq 2^{n+1} \left(\psi(x) - \sum_{r=1}^n 2^{-r} |a_r(x)| \right).
 \end{aligned}$$

Hence $|\bar{a}_0(x)| < \varphi_{n+1}(x)$ for all $x \in X$.

Again by Theorem 2.1 we can choose $a_{n+1} \in A$ such that

$$|a_{n+1}| < \varphi_{n+1}, \quad a_{n+1}|_F = a_0.$$

Continuing in this way we obtain a sequence $\{a_n\}_{n=1}^\infty \subseteq A$ such that for $n = 1, 2, \dots$

$$(i) \quad a_n|_F = a_0,$$

$$(ii) \quad \psi(x) - \sum_{r=1}^n 2^{-r} |a_r(x)| > 0, \quad \text{all } x \in X,$$

$$(iii) \quad \|a_n\| \leq 2 \sup_{x \in X} \psi(x).$$

By (iii) the sequence $\sum_{r=1}^\infty 2^{-r} a_r$ is uniformly convergent and $a = \sum_{r=1}^\infty 2^{-r} a_r \in A$. Clearly $a|_F = a_0$ and it follows from (ii) that $|a(x)| \leq \psi(x)$ for all $x \in X$. This completes the proof.

Taking $\psi \equiv 1$ in Theorem 4.5 we obtain the following:

Corollary 4.6. Let F be a compact subset of $\partial_A X$ satisfying (A.1), i.e.

$$\underline{\mu \in M(\partial_A X) \cap A^\perp \implies \mu|_F \in A^\perp},$$

then F has the norm preserving extension property.

Remark. In the proof of Theorem 4.5 we have actually proved slightly more than was stated. The A -superharmonicity of the function ψ was used just once, namely in the verification that $|\bar{a}_0(x)| < \varphi_{n+1}(x)$ for $n = 1, 2, \dots$ and all $x \in X$. However, if x is a point of X such that

$$\mu \in M_X^+(\partial_A X) \implies \mu_X(F) = 0 ,$$

then by definition $\bar{a}_0(x) = 0$, and there is nothing to verify.

Hence, Theorem 4.5 subsists if $\psi: X \rightarrow \mathbb{R}^+ \cup \{\infty\}$ is allowed to be a l.s.c. function such that

$$\psi(x) \geq \int \psi d\mu_x ,$$

for all points $x \in X$ for which $\mu_x(F) \neq 0$ for some

$\mu_x \in M_X^+(\partial_A X)$.

5. A peak set theorem

In this section we shall deal with compact subsets F of $\partial_A X$ satisfying the requirement (A.2). For such an F we define the function $\bar{\chi}_F$ as in (3.7).

Proposition 5.1 If F is a compact subset of $\partial_A X$ satisfying (A.2), then the A -convex hull of F is equal to the set of all $x \in X$ such that $\bar{\chi}_F(x) = 1$.

Proof: By definition, the A -convex hull of F is the set

$$(5.1) \quad F^\wedge = \{x \in X \mid |a(x)| \leq \|a\|_F, \text{ all } a \in A\}$$

We first assume that $\bar{\chi}_F(x) = 1$ i.e. $\mu_x(F) = 1$ for $\mu_x \in M_x^+(\partial_A X)$. Then we obtain for every $a \in A$,

$$|a(x)| = \left| \int_X a \, d\mu_x \right| \leq \int_F |a| \, d\mu_x \leq \|a\|_F$$

such that $x \in F^\wedge$.

Next assume that $\bar{\chi}_F(x) < 1$. This implies that $\phi(x) \notin S_F$. Hence we can separate $\phi(x)$ and S_F by a w^* -continuous linear functional on A^* i.e. there exists a function $a \in A$ and an $\alpha \in \mathbb{R}$ such that

$$\operatorname{Re} \psi(a)(\phi(x)) > \alpha > \operatorname{Re} \psi(a)(S_F) \geq 0,$$

and hence again

$$\operatorname{Re} a(x) > \alpha > \operatorname{Re} a(F) \geq 0.$$

Now, for sufficiently large $\delta \in \mathbb{R}^+$, the function $a + \delta \in A$ satisfies

$$|a(x) + \delta| > \delta + \alpha > |a(y) + \delta| \quad \text{all } y \in F.$$

In fact, it suffices to take

$$\delta > \frac{\gamma^2 + \beta^2 - \alpha\beta}{\alpha - \beta},$$

where

$$\beta = \max \{ \operatorname{Re} a(y) \mid y \in F \} < \alpha, \quad \gamma = \max \{ |\operatorname{Im} a(y)| \mid y \in F \}$$

Hence

$$\|a + \delta\|_F < |a(x) + \delta|$$

i.e. $x \notin F^\wedge$, which completes the proof.

Lemma 5.2. Let F be a compact subset of $\partial_A X$ satisfying (A.2), for which $A|_F$ is closed in $\mathcal{C}_0(F)$. Let ψ be a strictly positive A -superharmonic function on X such that $1 \leq \psi(x)$ for all $x \in F$.

Then there exists a function $a \in A$ such that

$$(5.2) \quad \underline{a|_F = 1, \quad |a(x)| \leq \psi(x) \quad \text{all } x \in X}$$

Proof: Since $\bar{\chi}_F$ is an element of \mathcal{A} and $A|_F$ is assumed to be closed in $\mathcal{C}_0(F)$, we can use Theorem 2.1 with $a_0 \in A|_F$, $a_0 \equiv 1$. Now using the same technique as in the proof of Theorem 4.5 we obtain a function $a \in A$ satisfying (5.2).

Lemma 5.3. Let F be a compact subset of $\partial_A X$ satisfying (A.2), and let G be a compact subset of $X \setminus F^\wedge$. Then there exists an A -superharmonic function ψ on X such that:

$$(i) \quad \underline{\psi(x) = 1 \quad \text{for all } x \in F^\wedge}$$

$$(ii) \quad \underline{|\psi(x)| < 1 \quad \text{for all } x \in G}$$

$$(iii) \quad \underline{0 < \psi(x) \leq 1 \quad \text{for all } x \in X}.$$

Proof: We write $S_G = \overline{\text{co}}(\Phi(G))$ and claim that $S_F \cap S_G = \emptyset$.

To prove this, we assume for contradiction that there exists a $p_0 \in S_F \cap S_G$, and we recall that $\hat{\chi}_{S_F}$ is u.s.c. and affine (since S_F is a parallel face) and that χ_{S_F} is related to $\bar{\chi}_F$ by formula (3.8). Now we obtain

$$1 = \hat{\chi}_{S_F}(p_0) = \max_{p \in S_G} \hat{\chi}_{S_F}(p) = \max_{p \in \Phi(G)} \hat{\chi}_{S_F}(p) = \max_{p \in G} \bar{\chi}_F(p) .$$

By Proposition 5.1, this contradicts the hypothesis $G \cap F^\wedge = \emptyset$, and the claim is proved.

Now there exists a number δ such that

$$\max_{p \in S_G} \hat{\chi}_{S_F}(p) < \delta < 1 ,$$

and hence we can define two disjoint convex subsets of $A^* \times \mathbb{R}$ by the formulas:

$$(5.3) \quad F_0 = \{(p, \alpha) \mid p \in S, \alpha \in \mathbb{R}, 0 \leq \alpha \leq \hat{\chi}_{S_F}(p)\}$$

$$(5.4) \quad F_1 = \{(p, \alpha) \mid p \in S_G, \alpha \in \mathbb{R}, \delta \leq \alpha\}$$

The set F_0 is compact and the set F_1 is closed. Hence we can use Hahn-Banach separation to obtain a function $b \in A$ such that

$$\hat{\chi}_{S_F}(p) < \text{Re } \Psi(b)(p) , \quad \text{all } p \in S ,$$

and

$$\text{Re } \Psi(b)(p) < \delta < 1 , \quad \text{all } p \in S_G .$$

The function $\psi = \text{Re } (b) \wedge 1$ is A -superharmonic and satisfies (i), (ii) and (iii).

Theorem 5.4. Let X be a metrizable compact Hausdorff space and let F be a compact subset of $\partial_A X$ which satisfies (A.2) i.e.

$$\underline{\mu \in M(\partial_A X) \cap A^\perp \implies \mu(F) = 0},$$

and for which $A|_F$ is closed. Then there exists a function $a \in A$ such that

$$(5.5) \quad \underline{a|_{F^\wedge} = 1, \quad |a(x)| < 1 \quad \text{all } x \in X \setminus F^\wedge},$$

i.e. the A -convex hull of F is a peak set.

Proof: By metrizability F^\wedge is a G_δ -set, and we can write $X \setminus F^\wedge = \bigcup_{n=1}^{\infty} K_n$, where K_n is closed.

Now we use Lemma 5.3 to obtain strictly positive A -superharmonic functions ψ_n on X such that

$$\psi_n(x) = 1 \quad \text{for all } x \in F^\wedge, \quad \psi_n(x) < 1 \quad \text{for all } x \in K_n, \\ n = 1, 2, \dots$$

and $\psi_n(x) \leq 1$ for all $x \in X$. It follows from Lemma 5.2 that there exist functions $a_n \in A$ such that $a_n|_F = 1$ and $|a_n(x)| \leq \psi_n(x)$ for all $x \in X$.

Now the function

$$a = \sum_{n=1}^{\infty} 2^{-n} a_n$$

satisfies (5.5) and the proof is complete.

Remark: Actually the conclusion of Theorem 5.4 subsists under more general assumptions. The metrizability of X was only invoked to make F^\wedge a G_δ -set. In particular we shall have the following:

Corollary 5.5. Let $x \in \partial_A X$ be a G_δ - point satisfying (A.2),
i.e.

$$\underline{\mu \in M(\partial_A X) \cap A^\perp \implies \mu(\{x\}) = 0},$$

then x is a peak point for A .

Finally we remark that if X is a metrizable compact Hausdorff space and F is a compact subset of $\partial_A X$ satisfying the stronger condition (A.1) then the A - convex hull of F is a peak set.

6. Relations to the Bishop-Rudin-Carleson Theorem.

In the present chapter we shall consider a compact subset F of X satisfying the requirement

$$(B) \quad \mu \in A^\perp \implies \mu|_F = 0.$$

Clearly (B) is more restrictive than (A.1), and a fortiori than (A.2). Note also that (B) implies $F \subset \partial_A X$ since $M_X^+(X) = \{\epsilon_x\}$ for all $x \in F$.

If $x \notin F$ and $\mu_x \in M_X^+(X)$, then $\epsilon_x - \mu_x \in A^\perp$. Now the requirement (B) implies $(\epsilon_x - \mu_x)|_F = 0$, such that $\mu_x(F) = 0$. By the definition (3.2) we shall have $\bar{a}_0(x) = 0$. Hence

$$(6.1) \quad \bar{a}_0 = a_0 \cdot \chi_F$$

Transferring to the state space and making use of (3.8), we observe that the function $\hat{\chi}_{S_F}$ takes the value zero on $\Phi(X \setminus F)$. Geometrically, this means that the canonical embedding $\Phi: X \rightarrow S$ maps F into the (compact) split face $S_F = \overline{\text{co}}(\Phi(F))$, and $X \setminus F$ into the complementary (G_δ -) face S'_F (cf. [2, Cor.1.2]).

It follows from (6.1) that $\bar{\chi}_F = \chi_F$ and by Proposition 5.1 we obtain $F = \hat{F}$. Moreover, it follows from Proposition 4.4 that $A|_F$ is a closed subspace of $\mathcal{C}_0(F)$, and it follows from (B) that $(A|_F)^\perp = (0)$. Hence $A|_F = \mathcal{C}_0(F)$. Also it follows from the results of chapter 5 that if F is a G_δ , then it is a peak set.

In other words: If F satisfies (B) then it is an interpolation set; and if in addition it is a G_δ , then it is a peak-interpolation set.

Finally we note that we may apply Theorem 4.5 in the form stated in the Remark at the end of §4, to obtain:

Theorem 6.1. (Bishop-Rudin-Carleson) Let F be a compact subset of X satisfying (B), i.e.

$$\mu \in A^\perp \implies \mu|_F = 0 ;$$

let $f_0 \in \mathcal{C}_0(F)$, and let $\psi: X \rightarrow \mathbb{R}^+ \cup \{\infty\}$ be a strictly positive l.s.c. function such that $|f_0(x)| \leq \psi(x)$ for all $x \in F$. Then there exists an $a \in A$ such that $a|_F = f_0$ and $|a(x)| \leq \psi(x)$ for all $x \in X$.

Remark: Theorem 6.1 is the most general form of the Bishop-Rudin-Carleson Theorem. Originally Bishop stated and proved this theorem for a continuous function ψ and strict inequality sign [4]. Appealing to the inductive construction of Pelczynski [12], Semadeni improved it to the form stated above [16]. (Cf. also Michael - Pelczynski [11, p. 569]).

7. The sup-norm algebra case.

In this section we shall assume that A is a sup-norm algebra, and we shall consider two new requirements on a compact subset F of $\partial_A X$:

$$(G.1) \quad \mu \in A^\perp \implies \mu|_F \in A^\perp$$

$$(G.2) \quad \mu \in A^\perp \implies \mu|_{F^\wedge} \in A^\perp$$

Clearly (B) implies (G.1) and (G.2) , and each one of these implies (A.1). In fact, (G.2) implies (A.1) since $\mu|_{F^\wedge} = \mu|_F$ for every $\mu \in M(\partial_A X)$ [3, Lem.1.].

In [9] and [10] Gamelin and Glicksberg have dealt with the requirement (G.1), and from their works we shall adopt the following:

Definition 7.1. Let F be a compact subset of X and let $t > 0$. $A|_F$ is said to have the property E_t if the following conditions holds:

Given $f \in A|_F$ with $\|f\|_F < 1$ and a compact subset G of $X \setminus F$, there exists an extension $g \in A$ of f such that

$$\|g\|_X < \max\{1, t\} , \quad |g(x)| < t \quad \text{all } x \in G .$$

The extension constant $e(A, F)$ of F associated with $A|_F$ is defined by the formula:

$$(7.1) \quad e(A, F) = \inf\{t \mid A|_F \text{ has property } E_t\}$$

If $A|_F$ has property E_t for no t , then we define $e(A, F) = \infty$.

The connection between the extension constant and the requirement (G.1) is expressed in the following:

Theorem 7.2. (Gamelin-Glicksberg). Let F be a compact subset of X . Then the following conditions are equivalent:

- (i) $\mu \in A^\perp \implies \mu|_F \in A^\perp$
- (ii) $e(A, F) = 0$
- (iii) F is an intersection of peak sets for A .

Proof: See [9] and [10].

Proposition 7.3. Let F be a sup-norm algebra over X and let F be a compact subset of $\partial_A X$ satisfying the requirement (A.1). Also let G be a compact subset of $X \setminus F^\wedge$ and let $\epsilon > 0$. Then there exists a function $a \in A$ such that

- (i) $a(x) = 1$ for all $x \in F^\wedge$
- (ii) $|a(x)| < \epsilon$ for all $x \in G$
- (iii) $\|a\|_X = 1$

Proof: Choose ψ as in Lemma 5.3 and let $a_0 \in A|_F$, $a_0 \equiv 1$. Using Theorem 4.5 we obtain a function $b \in A$ such that

$$b|_F = 1, \quad |b(x)| \leq \psi(x) \quad \text{for all } x \in X.$$

Clearly $b(x) = 1$ for all $x \in F^\wedge$ and $|b(x)| < 1$ for all $x \in G$. Now choose a natural number n such that $\|b\|_G^n < \epsilon$ and define $a = b^n$. The proof is complete.

We are now able to clarify the connection between (A.1) and the extension constant of F^\wedge .

Theorem 7.4. Let A be a sup-norm algebra over X and let F be a compact subset of $\partial_A X$. Then $e(A, F^\wedge) = 0$ if and only if F satisfies (A.1) i.e.

$$\underline{\mu \in M(\partial_A X) \cap A^\perp \implies \mu|_F \in A^\perp}$$

Proof: By virtue of Theorem 7.2 and the fact that $\mu|_{F^\wedge} = \mu|_F$ for every $\mu \in M(\partial_A X)$, it follows that $e(A, F^\wedge) = 0$ implies (A.1).

Now assume (A.1) and let $a_0 \in A|_{F^\wedge}$ with $\|a_0\|_{F^\wedge} = \|a_0\|_F < 1$. Let G be a compact subset of $X \setminus F^\wedge$ and let $\epsilon > 0$. We choose $b \in A$ such that $\|b\|_X = \|a_0\|_F$ and $b|_F = a_0|_F$ according to Corollary (4.6), and we choose $h \in A$ according to Proposition (7.3) i.e.

$$h|_{F^\wedge} = 1, \quad |h(x)| < \epsilon \quad \text{for all } x \in G$$

and $\|h\|_X = 1$. Then we define $a = h \cdot b \in A$. Now, a is a norm preserving extension of a_0 and $|a(x)| < \epsilon$ for all $x \in G$. Hence $A|_{F^\wedge}$ has property E_ϵ for all $\epsilon > 0$, and so we have proved that $e(A, F^\wedge) = 0$.

Thus we see how the requirements (A.1), (G.1) and (G.2) are related for sup norm algebras. (A.1) and (G.2) are always equivalent for every compact subset F of $\partial_A X$, and if in addition F is A -convex, then they are equivalent to (G.1). This is not always the case even if A is an algebra and F satisfies (A.1), as can be seen from the following example

Example 7.5. (The "Tomato Can Algebra").

Let $X \subset \mathbb{R} \times \mathbb{C}$ be defined as $\{(t, z) \mid t \in [0, 1], |z| \leq 1\}$;
let A be the sup-norm algebra consisting all functions $f \in \mathcal{C}_0(X)$
such that $f(0, z)$ is analytic for $|z| < 1$; and let
 $F = \{(0, z) \mid |z| = 1\}$. Then F satisfies (A.1) and
 $F^\wedge = \{(0, z) \mid |z| < 1\}$.

Proof: We first note that:

$$\partial_A X = \{(t, z) \mid t \in]0, 1], |z| \leq 1 \text{ or } t = 0, |z| = 1\}$$

Hence the Shilov boundary $\partial_S A = \overline{\partial_A X}$ is all of X , and it also follows that X is the maximal ideal space M_A of A .

If G is a compact subset of $X \setminus \{(0, z) \mid |z| \leq 1\}$, then G is a peak interpolation set for A and $A|_G = \mathcal{C}_0(G)$. Hence if $\mu \in A^\perp$ then $\mu|_G = 0$. In other words $\text{supp}(\mu) \subset \{(0, z) \mid |z| \leq 1\}$ for all $\mu \in A^\perp$.

Now assume $\mu \in M(\partial_A X) \cap A^\perp$. Then $\mu|_F = \mu \in A^\perp$. Hence F satisfies (A.1) but trivially $F^\wedge = \{(0, z) \mid |z| < 1\}$; and the proof is complete.

This example shows also that (A.1) and (G.1) need not be equivalent even if we consider A as a sup-norm algebra over the maximal ideal space or the Shilov boundary.

Finally we remark that if X is a compact subset of \mathbb{C} and $A = R(X)|_{\partial X}$ then the two conditions (A.1) and (G.1) are equivalent since $F = F^\wedge$ for every compact subset F of $\partial_A X$.

References.

- [1] E.M. Alfsen, Compact convex sets and boundary integrals, Ergebnisse der Mathematik, Springer Verlag, Germany, (to appear).
- [2] E.M. Alfsen and T.B. Andersen, Split faces of compact convex sets, to appear in Proc. London Math. Soc.
- [3] T.B. Andersen, On dominated extensions of continuous affine functions on split faces, (to appear).
- [4] E. Bishop, A general Rudin-Carleson theorem, Proc. Amer. Math. Soc. 13 (1962) p. 140-143.
- [5] J.E. Bjørk, Private communication.
- [6] J.E. Bjørk, Interpolation on closed linear subspaces of $C(X)$, (to appear).
- [7] E. Briem, Restrictions of subspaces of $C(X)$, Århus Univ. Math. Inst. Preprint Ser. 1969/70 No. 26.
- [8] P.C. Curtis, Topics in function spaces, Lecture notes, Univ. of Århus 1969/70.
- [9] T.W. Gamelin, Restrictions of subspaces of $C(X)$, Trans. Amer. Math. Soc. 112 (1964) p. 278-286.
- [10] I. Glicksberg, Measures orthogonal to algebras and sets of antisymmetri, Trans. Amer. Math. Soc. 105 (1962) p. 415-435.
- [11] E. Michael and A. Pelczynski, A linear extension theorem, Illinois Journal of Mathematics 11 (1967) p. 563-579.
- [12] A. Pelczynski, Supplement to my paper "On simultaneous extensions of continuous functions", Studia. Math. 25 (1964) p. 157-161.
- [13] R.R. Phelps, Lectures on Choquet's theorem, Van Nostrand, New York, 1966.
- [14] M. Rao, Measurable selections of representing measures, Århus Univ. Math. Inst. Preprint Ser. 1969/70 No. 24.

- [15] M. Rogalski, Topologies faciales dans les convexes compacts; calcul fonctionnel et décomposition spectrale dans le centre d'un espace $A(X)$, Seminaire Choquet 1969-70 No. 4.
- [16] Z. Semadini, Simultaneous extensions and projections in spaces of continuous functions, Lecture Notes, Univ. of Århus 1965.