ON THE STRUCTURE AND TENSOR
PRODUCTS OF JC-ALGEBRAS

Harald Hanche-Olsen
Inst. of Math., University of Oslo

ON THE STRUCTURE AND TENSOR
PRODUCTS OF JC-ALGEBRAS

Harald Hanche-Olsen

Norm closed (or weakly closed) Jordan algebras of self-adjoint
operators on a Hilbert space were initially studied by Topping,
Effros, and Størmer [15], [4], [12], [13]. These works are very
"spatial", in that the algebras are considered in one given re-
presentation. The introduction of their abstract counterparts,
the JB- and JBW-algebras, has led to an increased interest in
this subject. The author hopes this paper will support the view
that a more "space-free" approach is fruitful, even if only the
"concrete" algebras are under study. In accordance with this
view, a "JC-algebra" in this paper will mean a normed Jordan
algebra over the reals, which is isometrically isomorphic to a norm
closed Jordan algebra of self-adjoint operators.

Some of the results in this paper are closely related to, or
rewordings of, results in the above-mentioned papers. However,
I feel that the present approach is sufficiently different to be
of interest in itself. In particular, many of the technical
difficulties associated with earlier approaches are avoided.

In § 1 a spin algebra is defined as a global variant of a
spin factor, and spin algebras are shown to have no dense represen-
tation of type higher than I_2. The spin algebras complement
the "universally reversible" algebras, introduced in § 2.
(Theorem 2.5).

Lemma 2.1 has been proved in [9], but I believe my approch

is even more direct. The universally reversible JC-algebras are shown to be those not having spin factor representations of too high dimensions. The study of those is reduced to the study of antiautomorphisms of order two on C^*-algebras.

§ 3 contains an example of a simple JC-algebra that has dense representations of both real, complex, and quaternionic types. This example is closely related to the fact, proved independently by Størmer [14] and Giordano [5], that there is, up to conjugation with an automorphism, only one antiautomorphism of order two on the hyperfinite II_1-factor.

The universal C^*-algebra was first introduced as a technical tool in [2]. It is further studied in § 4. We get an exact functor C^* from the category of JB-algebras to that of C^*-algebras. It is shown how to compute it for universally reversible JC-algebras, and, as an application, we generalize Størmer's result [11] on the decomposition of a Jordan homomorphism of C^*-algebras as a sum of a homomorphism and an anti-homomorphism.

§ 5 consists of an application of the earlier material to the problem of defining a tensor product of JC-algebras. A universal candidate is studied, and it is applied to show that a more naive approach would fail. The universal tensor product is computed in a few cases.

I got the idea for the example in § 3 from a lecture in Marseille by Thierry Giordano on his uniqueness result for anti-automorphisms of the II_1 factor. Bruno Iochum should also be thanked for his invitation and generous hospitality during my visit to Marseille. Most of the work for the present paper was conducted in Toronto during the year 1980/81. My warmest thanks

Let us now turn to notation and preliminaries. Let A be a JB-algebra. A <u>dense representation</u> of A is a homomorphism of A onto a weakly dense subalgebra of a type I JBW-factor. The representation is said to be of <u>real</u>, <u>complex</u>, <u>or quaternionic type</u> if the factor is isomorphic to the algebra of self-adjoint operators on real, complex, or quaternionic Hilbert space. Up to equivalence, all dense representations arise in the following way: Consider a pure state ρ of A. Let $c(\rho)$ be its central support in A^{**}, $A_\rho = c(\rho)A^{**}$, and $\varphi_\rho : A \rightarrow A_\rho$ is the map $a \rightarrow c(\rho)a$. Then A_ρ is a type I factor, φ_ρ is a dense representation, and φ_ρ^* maps the normal state space of A_ρ isomorphically onto the minimal split face F_ρ generated by ρ. See [2; § 2]. A <u>concrete representation</u> of A is a homomorphism into the Jordan algebra $\mathcal{B}(H)_{sa}$.

If $a \in A$, we define $T_a(x) = a \circ x$, $U_a(x) = \{axa\}$. Then $U_a = 2T_a^2 - T_{a^2}$, and $U_a(x) = axa$ in $\mathcal{B}(H)_{sa}$. Two elements a and b are said to <u>operator commute</u> if T_a and T_b commute.

If B is a subset of a C^*-algebra, [B] is the C^*-algebra generated by B. If Φ is an antiautomorphism of order 2 on a C^*-algebra $\mathcal{A}$, $\mathcal{A}_{sa}^\Phi$ is the JC-algebra consisting of all elements a of $\mathcal{A}$ such that $a = a^* = \Phi(a)$.

1. Spin algebras.

A spin factor is $V_n = \mathbb{R} 1 \oplus H_n$, where H_n is a real Hilbert space of dimension $n \geq 2$. V_n is made into a JB-algebra by defining the product in such a way that 1 is the unit and $\xi \circ \eta = (\xi | \eta) 1$ if $\xi, \eta \in H_n$, and defining the norm $\| \lambda 1 \oplus \xi \| = |\lambda| + \| \xi \|$. Spin factors are simple JC-algebras, and reflexive as Banach spaces [16].

A spin algebra is a JC-algebra which has a faithful family of representations onto spin factors.

Proposition 1.1. A spin algebra has no dense representations other than onto spin factors and $\mathbb{R}$.

This is an immediate consequence of Theorem 1.2 below. The primitive ideal space $\operatorname{Prim} A$ is defined as for C^*-algebras, only with dense representations replacing the irreducible ones [7]. For a dense representation $\varphi: A \to M$, we say φ has type I_k if M is a type I_k factor, and let $\operatorname{Prim}_n(A)$ be the set of kernels of type I_k representations, where $k \leq n$.

By Theorem 1.2, there is an ideal J of A such that a dense representation φ of A is onto $\mathbb{R}$ or a spin factor iff $\ker \varphi \supseteq J$. If A is a spin algebra, J must be 0, and the conclusion of Proposition 1.1 follows.

Theorem 1.2. If A is a unital JB-algebra and $n < \infty$, $\operatorname{Prim}_n A$ is closed in $\operatorname{Prim} A$.

Proof. For any pure state ρ, consider the dense representation $\varphi_\rho: A \to A_\rho$. Clearly, A_ρ is of type I_k ($k \leq n$) iff for each $b \in A_\rho$, the powers $1, b, \ldots, b^n$ are linearly dependent.

But φ_ρ^* maps the normal state space of A_ρ isomorphically onto F_ρ, and the pure normal states of A_ρ separate points. Therefore, $\ker \varphi_\rho \in \mathrm{Prim}_n A$ iff for each $a \in A$ and $\rho_0, \ldots, \rho_n \in \partial_e F_\rho$ the $(n+1)$ by $(n+1)$ matrix $(\langle a^i, \rho_j \rangle)_{i,j=0}^n$ is singular.

Choose $\rho \in \partial_e S(A)$ with $\ker \varphi_\rho \notin \mathrm{Prim}_n A$, and pick $a \in A$, $\rho_0, \ldots, \rho_n \in \partial_e F_\rho$ such that $(\langle a^i, \rho_j \rangle)$ is non-singular. Choose neighbourhoods (in the relative weak*-topology) V_j of ρ_j in $\partial_e K$ such that whenever $\sigma_j \in V_j$, $(\langle a^i, \sigma_j \rangle)$ is non-singular. Let W be the intersection of the sets $\{\ker \varphi_\sigma : \sigma \in V_j\}$ for $j = 0, \ldots, n$. Then W is a neighbourhood of $\ker \varphi_\rho$ in $\mathrm{Prim}\, A$ [7; Thm. 4.1].

Assume that $\varphi : A \to M$ is a dense representation, and $\ker \varphi \in W \cap \mathrm{Prim}_n A$. Then there is $\sigma_j \in V_j$ such that $\ker \varphi = \ker \varphi_{\sigma_j}$. Since M is a spin factor or finite dimensional, φ is onto M. Therefore, φ_{σ_j} is equivalent to φ for $j = 0, \ldots, n$. In particular, $\sigma_j \in \partial_e F_{\sigma_0}$. Since the matrix $(\langle a^i, \sigma_j \rangle)$ is non-singular, then $\ker \varphi \notin \mathrm{Prim}_n A$, a contradiction. $\qquad \square$

2. Reversibility.

Recall from [12] that a JC-algebra A contained in $\mathcal{B}(H)_{sa}$ is called _reversible_ if $a_1, \ldots, a_n \in A \Rightarrow a_1 \cdots a_n + a_n \cdots a_1 \in A$. We call A _universally reversible_ if $\pi(A)$ is reversible for each concrete representation $\pi : A \to \mathcal{B}(H)_{sa}$.

Lemma 2.1 [9] Of all spin factors, only V_2, V_3 and V_5 admit reversible representations, and only V_2 and V_3 are universally reversible.

Proof. That V_3 is universally reversible is proved in [2; Lemma 4.1]. The same proof holds for V_2 as well.

Assume that $V_n \subseteq \mathcal{B}(H)_{sa}$ is reversible, $n \geq 4$. Choose an orthonormal set $s_1, s_2, s_3, s_4 \in H_n$. Equivalently, s_j is a symmetry in V_n, and $s_i \circ s_j = 0$ if $i \neq j$. Let $s_5 = s_1 s_2 s_3 s_4 = \frac{1}{2}(s_1 s_2 s_3 s_4 + s_4 s_3 s_2 s_1)$. Then $s_5 \in V_n$ is a symmetry, anticommuting with s_1, s_2, s_3, s_4. So $s_1, \ldots, s_5$ is an orthonormal set in H_n, and this is impossible if $n = 4$. If $n \geq 6$, choose a new unit vector s_6 in H_n, orthogonal to $s_1, \ldots, s_5$. Then s_6 is a symmetry which both anticommutes and commutes with $s_5 = s_1 s_2 s_3 s_4$; again, this is a contradiction.

$V_5 \cong M_2(\mathbb{H})_{sa}$ has a reversible representation π in $M_4(\mathbb{C})_{sa}$. Then, for some orthonormal basis $s_1, \ldots, s_5$ of H_5, $\pi(s_1)\pi(s_2)\pi(s_3)\pi(s_4) = \pi(s_5)$. Define a representation π' by $\pi'(s_i) = \pi(s_i)$ if $1 \leq i \leq 4$ and $\pi'(s_5) = -\pi(s_5)$. Then $\pi \oplus \pi'(V_5)$ is not reversible. $\square$

Theorem 2.2. A JC-algebra is universally reversible iff it has no spin factor representations other than onto V_2 and V_3.

Proof. The "only if" part is immediate from Lemma 2.1. The proof of [2; Thm. 4.6] proves the "if" part, with the minor modification that in [2; Lemma 4.5] $s_0\bar{A}$ splits into two parts, one a "global V_2" and one a "global V_3". $\square$

The study of universally reversible algebras can be reduced to the study of *-antiautomorphisms of C^*-algebras:

Proposition 2.3. If A is a universally reversible JC-algebra then there is a C^*-algebra $\mathcal{A}$ and a *-antiautomorphism Φ of $\mathcal{A}$ of order two such that A is isomorphic to $\{a \in \mathcal{A} : a = a^* = \Phi(a)\}$.

Proof. If $A \subseteq \mathcal{B}(H)_{sa}$, choose a transposition on $\mathcal{B}(H)$ corresponding to some orthonormal basis, and identify A with $\{a \oplus a^t : a \in A\} \subseteq \mathcal{B}(H) \oplus \mathcal{B}(H)$. Let $\mathcal{C}$ be the C^*-algebra generated by A, with the $*$-antiautomorphism Φ given by $\Phi(a \oplus b) = b^t \oplus a^t$.

If $z \in \mathcal{C}$ then z belongs to the closed real linear span of elements of the form $z' = x_1 \ldots x_n + iy_1 \ldots y_n$; where $x_i, y_i \in A$. If $z = z^* = \Phi(z)$ then $z = \frac{1}{4}(z + z^* + \Phi(z) + \Phi(z)^*)$ can be approximated by linear combinations of elements of the form $\frac{1}{4}(z' + z'^* + \Phi(z') + \Phi(z')^*) = \frac{1}{2}(x_1 \ldots x_n + x_n \ldots x_1) \in A$. Therefore $z \in A$, and the non-trivial part of the proof is finished. $\square$

Finally, the study of general JC-algebras can be reduced to the study of spin algebras, universally reversible algebras, and certain extensions.

Lemma 2.4. Let I be a closed ideal in a JB-algebra A, and M a JBW-algebra. Any homomorphism $I \to M$ extends to a homomorphism $A \to M$.

Proof. Identify I^{**} with the weak closure in A^{**} of I. Then $I^{**} = eA^{**}$, for a central projection e in A^{**} [3, Thm. 3.3]. The homomorphism $I \to M$ extends to a normal homomorphism $I^{**} \to M$ [2; p. 270]. Compose with the map $a \to ae$ of A into I^{**}. $\square$

Theorem 2.5. Any JC-algebra A has an ideal I which is universally reversible and such that A/I is a spin algebra.

Proof. Let I be the intersection of all kernels of spin factor representations. Then A/I is a spin algebra. By Lemma 2.4 and Theorem 2.2, I is universally reversible. $\square$

3. A simple JC-algebra.

One might hope that the topology of Prim A will separate, say, the real representations from the complex ones. This section contains an example to effectively crush any such hope: There exists a simple JC-algebra possessing representations of three different types.

First, however, we prove some auxiliary results. The symbol $\perp$ below signifies positive annihilator in A^*, resp. A:

Lemma 3.1. Let A be a JB-algebra and B a closed quadratic ideal in A. Then $B^+ = B^\perp {}_\perp$

Proof. The weak closure $\bar{B}$ of B in A^{**} is a quadratic ideal, and $B = \bar{B} \cap A$. By [3] $\bar{B} = \{pA^{**}p\}$ for some projection $p \in A^{**}$. If $b \in A^+$, $b \notin B$ then $\{(1-p)b(1-p)\} \neq 0$. If $\rho \in A^{*+}$ does not annihilate this element, then $\{(1-p)\rho(1-p)\} \in B^\perp$ but does not annihilate b. Hence $b \notin B^\perp {}_\perp$, and the proof is complete. □

The next two results (for Jordan ideals) are contained in [4], but the proofs below are shorter and more direct.

Proposition 3.2. If A is a JC-algebra, $A \subseteq \mathcal{B}(H)_{sa}$, and B is a closed quadratic ideal of A, then $B = A \cap [B]$.

Proof. If $a \in A^+$ but $a \notin B$ there is a positive linear functional annihilating B but not a. (Lemma 3.1). Extend to a positive linear functional on $\mathcal{B}(H)$. By the Cauchy-Schwarz inequality, the extension annihilates [B], so $a \notin [B]$. □

Proposition 3.3. Let $\mathcal{A}$ be a C*-algebra, $A \subseteq \mathcal{A}_{sa}$ a JC-algebra generating $\mathcal{A}$ and J an ideal of A. Then [J] is an ideal of $\mathcal{A}$.

Proof. We prove $AJ \subseteq [J]$. Indeed, J is generated by its squares, so it is enough to prove $ab^2 \in [J]$ when $a \in A, b \in J$. But $ab^2 = (a \circ b)b - b(a \circ b) + a \circ b^2$ proves that. Now $A[J] \subseteq [J]$ follows, and since A generates $\mathcal{A}$, $\mathcal{A}[J] \subseteq [J]$. Similarly, $[J] \mathcal{A} \subseteq [J]$. $\square$

Theorem 3.4. There exists a simple JC-algebra which admits both real, complex, and quaternionic dense representations.

Proof. Let $\mathcal{A} = \overset{\infty}{\underset{n=1}{\otimes}} M_2(C)$, the CAR algebra. Let $t_2 : M_2(C) \to M_2(C)$ be transposition and $u_2 : M_2(C) \to M_2(C)$ the "quaternionic flip":

$$u_2 \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$$

Now consider the two antiautomorphisms $t = t_2 \otimes t_2 \otimes \ldots$ and $u = u_2 \otimes t_2 \otimes t_2 \otimes \ldots$ on $\mathcal{A}$. We claim that t, u are conjugate, i.e. there is an automorphism α of $\mathcal{A}$ such that $u = \alpha^{-1} t \alpha$. Indeed, if $\sim$ denotes conjugacy, we have $u_2 \otimes u_2 \sim t_2 \otimes t_2$ on $M_4(C)$, so

$$u = u_2 \otimes (t_2 \otimes t_2) \otimes (t_2 \otimes t_2) \otimes \ldots$$
$$\sim u_2 \otimes (u_2 \otimes u_2) \otimes (u_2 \otimes u_2) \otimes \ldots$$
$$= (u_2 \otimes u_2) \otimes (u_2 \otimes u_2) \otimes \ldots$$
$$\sim (t_2 \otimes t_2) \otimes (t_2 \otimes t_2) \otimes \ldots = t.$$

It follows that there is an automorphism of $\mathcal{A}$ carrying $\mathcal{A}_{sa}^t$ onto $\mathcal{A}_{sa}^u$. Below we shall construct a real and a complex

representation of $\mathcal{O}_{sa}^{t}$, and a quaternionic representation of $\mathcal{O}_{sa}^{u}$.

We start with a real representation of $\mathcal{O}_{sa}^{t}$. Let ρ_2 be the pure state on $M_2(C)$ given by $\langle a, \rho_2 \rangle = a_{11}$. Then $\rho = \rho_2 \otimes \rho_2 \otimes \ldots$ is a pure state on $\mathcal{O}$. Let (e_1, e_2) be the natural basis of C^2 and denote by $\overset{\infty}{\underset{n=1}{\otimes}}(C^2, e_1) = H$ the Hilbert space direct limit of the sequence $H_n = \overset{n}{\underset{k=1}{\otimes}} C^2$ where H_n is identified with the subspace $H_n \otimes e_1$ of H_{n+1}. Let J_2 be coordinatewise conjugation in C^2. Then $J = J_2 \otimes J_2 \otimes \ldots$ is well defined on H, $J^2 = 1$, the GNS-representation π_ρ of $\mathcal{O}$ can be thought of as acting in H (with representing vector $e_1 \otimes e_1 \otimes \ldots$) and

$$\pi_\rho(t(a)) = J\pi_\rho(a)^* J \qquad (a \in \mathcal{O}).$$

It follows that the restriction of π_ρ to $\mathcal{O}_{sa}^{t}$ is a dense representation in the JBW-factor $\{a \in \mathcal{B}(H) : a = a^* = JaJ\}$, which can be identified with $\mathcal{B}(K)_{sa}$, where K is the real Hilbert space $\{\xi \in H : J\xi = \xi\}$.

Next, consider the mapping $j_2 : C^2 \to C^2$ defined by

$$j_2 \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} \bar{\mu} \\ -\bar{\lambda} \end{pmatrix},$$

and note that $j = j_2 \otimes J_2 \otimes J_2 \otimes \ldots$ is a conjugate linear unitary (an "antiunitary") on H, with $j^2 = -1$. We find, for $a \in \mathcal{O}$,

$$\pi_\rho(u(a)) = -j\,\pi_\rho(a)^* j.$$

Therefore, the restriction of π_ρ to $\mathcal{O}_{sa}^{u}$ is a dense representation in the JW-factor $\{a \in \mathcal{B}(H) : a = a^* \text{ and } aj = ja\}$. H can be given a structure of quaternionic Hilbert space in which j is multiplication with the unit quaternion usually called j, so the above JW-factor consists of all quaternionic linear mappings. (see also the proof of [2; Thm. 3.1]).

Finally, choose pure states $\sigma_2 \neq \tau_2$ on $M_2(C)$ such that $\sigma_2 = \tau_2 \circ t_2$. Let $\sigma = \sigma_2 \otimes \sigma_2 \otimes \ldots$, $\tau = \tau_2 \otimes \tau_2 \otimes \ldots$. Then σ, τ are inequivalent pure states on $\mathcal{O}\!\mathcal{L}$, and we get an antihomomorphism β of $\mathcal{B}(H_\sigma)$ onto $\mathcal{B}(H_\tau)$ such that

$$\pi_\tau t(a) = \beta \pi_\sigma(a) \qquad (a \in \mathcal{O}\!\mathcal{L}).$$

Here π_σ and π_τ are the GNS-representations in H_σ, H_τ corresponding to σ and τ. Let $\pi = \pi_\sigma \oplus \pi_\tau$. Since π_σ and π_τ are inequivalent and irreducible, $\pi(\mathcal{O}\!\mathcal{L})$ is dense in $\mathcal{B}(H_\sigma) \oplus \mathcal{B}(H_\tau)$. Define the antiautomorphism Φ on $\mathcal{B}(H_\sigma) \oplus \mathcal{B}(H_\tau)$ by $\Phi(a \oplus b) = \beta^{-1}(b) \oplus \beta(a)$. Then $\Phi^2 = 1$, and the restriction of π to $\mathcal{O}\!\mathcal{L}_{sa}^t$ is a dense representation in the JW-factor $\{c \in \mathcal{B}(H_\sigma) \oplus \mathcal{B}(H_\tau) : \Phi(c) = c = c^*\}$, which is isomorphic with $\mathcal{B}(H_\sigma)$.

We have proved that $\mathcal{O}\!\mathcal{L}_{sa}^t$ has representations of all three types. That it is simple is an easy consequence of the propositions 3.2 and 3.3. $\qquad \square$

4. The universal C^*-algebra.

Consider a JB-algebra A. To A we associate a C^*-algebra $C^*(A)$, which is the unique C^*-algebra with the following properties: There is a homomorphism $\psi_A : A \to C^*(A)_{sa}$ such that $\psi_A(A)$ generates $C^*(A)$ as a C^*-algebra. Whenever $\mathcal{B}$ is a C^*-algebra and $\pi : A \to \mathcal{B}_{sa}$ is a homomorphism, π lifts to a *-homomorphism $\hat{\pi} : C^*(A) \to \mathcal{B}$ such that $\pi = \hat{\pi} \circ \psi_A$.

$$
\begin{array}{ccc}
C^*(A) & \xrightarrow{\;\hat{\pi}\;} & \mathcal{B} \\[4pt]
\psi_A \big\uparrow & & \big\uparrow \\[4pt]
A & \xrightarrow[\pi]{} & \mathcal{B}_{sa}
\end{array}
$$

The existence of $C^*(A)$ is proved in [2; Thm. 5.1]. A need not have a unit (drop all references to the unit in the proof), but if it does, then $\psi_A(1)$ is a unit of $C^*(A)$. It may be that $C^*(A) = 0$, for example if $A = M_3^8$. In general, the kernel of ψ_A is the exceptional ideal of [1].

If A, B are JB-algebras and $\varphi: A \to B$ is a homomorphism, we get a $*$-homomorphism $C^*(\varphi) = (\psi_B \circ \varphi)^\wedge : C^*(A) \to C^*(B)$:

$$
\begin{array}{ccc}
C^*(A) & \xrightarrow{\;C^*(\varphi)\;} & C^*(B) \\
\psi_A \uparrow & & \uparrow \psi_B \\
A & \xrightarrow[\varphi]{} & B
\end{array}
$$

In this way, C^* becomes a functor from the category of JB-algebras to the category of C^*-algebras.

Theorem 4.1. The functor C^* is exact.

Proof. This means that if J is a closed Jordan ideal in a JB-algebra A, then the top row in the commutative diagram below is exact:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C^*(J) & \xrightarrow{\;C^*(j)\;} & C^*(A) & \xrightarrow{\;C^*(q)\;} & C^*(A/J) & \longrightarrow & 0 \\
& & \psi_J \uparrow & & \psi_A \uparrow & & \psi_{A/J} \uparrow & & \\
0 & \longrightarrow & J & \xrightarrow[j]{} & A & \xrightarrow[q]{} & A/J & \longrightarrow & 0
\end{array}
$$

Exactness at $C^*(A/J)$ is trivial, since $\psi_A(A)$ generates A and $C^*(A/J)$ is generated by $\psi_{A/J}(A/J) = \psi_{A/J}(q(A)) = C^*(q)\psi_A(A)$.

Next, $C^*(q)C^*(j) = C^*(qj) = 0$, so half the exactness at $C^*(A)$ follows. Let $\mathcal{J} = C^*(j)(C^*(J))$. Then $\mathcal{J}$ is generated $\cdot$

by $\psi_A(J)$, which is an ideal in $\psi_A(A)$. By Proposition 3.3, $\mathcal{J}$ is an ideal in $C^*(A)$. But the composition $A \to C^*(A) \to C^*(A)/\mathcal{J}$ annihilates J, so it factors through $C^*(A/J)$.

$$A \xrightarrow{\psi_A} C^*(A) \longrightarrow C^*(A)/\mathcal{J}$$

with $C^*(q)$ and $C^*(A/J)$

Then, in this diagram, the circumference commutes, and so does the triangle on the left, by definition. Since $\psi_A(A)$ generates $C^*(A)$, the triangle on the right commutes. Therefore, the kernel of $C^*(q)$ is contained in the image $\mathcal{J}$ of $C^*(j)$, and exactness at $C^*(A)$ is proved.

To prove exactness at $C^*(J)$, let $\pi: C^*(J) \to \mathfrak{B}(H)$ be a faithful representation. Then $\pi\psi_J$ is a homomorphism of J into $\mathfrak{B}(H)_{sa}$. By Lemma 2.4, this extends to a homomorphism of A into $\mathfrak{B}(H)_{sa}$. Factor the extension through $C^*(A)$:

$$C^*(J) \xrightarrow{C^*(j)} C^*(A)$$
$$\psi_J \uparrow \qquad \downarrow$$
$$J \xrightarrow{\pi\psi_J} \mathfrak{B}(H)$$

But then the composition $C^*(J) \to C^*(A) \to \mathfrak{B}(H)$ must be π, which is injective, so $C^*(j)$ is injective. $\square$

By [2; Cor. 5.2] $C^*(A)$ admits a unique antiautomorphism Φ of order 2, which is the identity on $\psi(A)$. If A is a JC-algebra then ψ_A is injective; then we will identify A with $\psi(A)$.

Lemma 4.2. A JC-algebra A is universally reversible iff it is reversible in $C^*(A)$ or, equivalently, iff $A = C^*(A)_{sa}^\Phi$.

Proof: The first two conditions are trivially equivalent. So are the last two, see the proof of Proposition 2.3. □

Lemma 4.3. Let A be a universally reversible JC-algebra. If $\mathcal{J}$ is a Φ-invariant ideal of $C^*(A)$, then $\mathcal{J}$ is generated by $\mathcal{J} \cap A$.

Proof. Let (e_μ) be a self-adjoint, bounded approximate unit for $\mathcal{J}$. Then $(\frac{1}{2}(e_\mu + \Phi(e_\mu)))$ is another such, so we may as well assume that $e_\mu = \Phi(e_\mu)$. Then $e_\mu \in A$, so $\mathcal{J} \cap A$ generates $\mathcal{J}$ (as an ideal, and hence as a C^*-algebra by Proposition 3.3). □

We now characterize the universal C^*-algebra of a universally reversible JC-algebra.

Theorem 4.4. Assume A is a universally reversible JC-algebra, that $\mathcal{B}$ is a C^*-algebra, and that $\theta : A \to \mathcal{B}_{sa}$ is an injective homomorphism such that $\theta(A)$ generates $\mathcal{B}$. If $\mathcal{B}$ admits an antiautomorphism φ such that $\varphi\theta = \theta$, then $\hat\theta$ is a *-isomorphism of $C^*(A)$ onto $\mathcal{B}$.

Proof. Since $\theta(A)$ generates $\mathcal{B}$, $\hat\theta$ is onto. The composition $\varphi\hat\theta\Phi$ is a *-homomorphism of $C^*(A)$ to $\mathcal{B}$ extending θ. Therefore, by the uniqueness of such extensions, $\varphi\hat\theta\Phi = \hat\theta$, or $\varphi\hat\theta = \hat\theta\Phi$. So the kernel of $\hat\theta$ is a Φ-invariant ideal, whose intersection with A is O. By Lemma 4.3, $\hat\theta$ is injective. □

If $\mathcal{O}$ is a C^*-algebra, denote by $[\mathcal{O},\mathcal{O}]$ the commutator ideal of $\mathcal{O}$, and by $\mathcal{O}^\circ$ the opposite C^*-algebra. There is a

*-anti-isomorphism a → a° of $\mathcal{O}l$ onto $\mathcal{O}l°$.

Corollary 4.5. If $\mathcal{O}l$ is a C^*-algebra then $C^*(\mathcal{O}l_{sa})$ can be identified with

$$\mathcal{B} = \{a \oplus b° \in \mathcal{O}l \oplus \mathcal{O}l° : a-b \in [\mathcal{O}l,\mathcal{O}l]\},$$

with $\mathcal{O}l_{sa}$ identified with $\{a \oplus a° : a \in \mathcal{O}l_{sa}\}$.

Proof. Clearly $\mathcal{B}$ is a norm closed *-subspace of $\mathcal{O}l \oplus \mathcal{O}l°$. If $a \oplus b°$ and $c \oplus d°$ are elements of $\mathcal{B}$ then $(a \oplus b°)(c \oplus d°) = ac \oplus (db)°$, and $ac - db = a(c-d) + (ad-da) + d(a-b) \in [\mathcal{O}l,\mathcal{O}l]$, so $\mathcal{B}$ is a C^*-algebra.

Let $\mathcal{B}_0$ be the C^*-subalgebra of $\mathcal{B}$ generated by $\{(a \oplus a°) : a \in \mathcal{O}l_{sa}\}$. If $a,b \in \mathcal{O}l$ then $(a \oplus a°)(b \oplus b°) - (ba) \oplus (ba)° = (ab-ba) \oplus 0$, so $\{x \in \mathcal{O}l : x \oplus 0 \in \mathcal{B}_0\}$ contains all commutators. Since this set is an ideal of $\mathcal{O}l$, it contains $[\mathcal{O}l,\mathcal{O}l]$, and so $[\mathcal{O}l,\mathcal{O}l] \oplus 0 \subseteq \mathcal{B}_0$. It follows that $\mathcal{B} \subseteq \mathcal{B}_0$, so $\mathcal{B}_0 = \mathcal{B}$.

In $\mathcal{B}$ we have the antiautomorphism $a \oplus b° \to b \oplus a°$, leaving $\{a \oplus a° : a \in A\}$ pointwise fixed. Theorem 4.4 completes the proof. $\square$

Corollary 4.6. (cf. [11]) If $\mathcal{O}l, \mathcal{B}$ are C^*-algebras, and $\mathcal{O}l$ has no one-dimensional representations, then any Jordan *-homomorphism of $\mathcal{O}l$ into $\mathcal{B}$ is a sum of a *-homomorphism and a *-anti-homomorphism.

Proof. In this case $[\mathcal{O}l,\mathcal{O}l] = \mathcal{O}l$, and so $C^*(\mathcal{O}l_{sa}) = \mathcal{O}l \oplus \mathcal{O}l°$. $\square$

If $\mathcal{O}l$ has one-dimensional representations, the conclusion of Corollary 4.6 may be false. Indeed, let $\mathcal{O}l = \{f \in C([-1,1],M_2(\mathbb{C})) : f(0) \in \mathbb{C}1\}$, and define $\varphi : \mathcal{O}l \to \mathcal{O}l$ by $\varphi f(x) = f(x)$ if $x \geq 0$, $\varphi f(x) = f(x)^t$ if $x < 0$. Then φ is

not a sum of a *-homomorphism and a *-anti-homomorphism.

We also remark that the conclusion of Corollary 4.6 holds if $\mathcal{O}$ is a von Neumann algebra, which may have one-dimensional representations. For then $\mathcal{O}$ is a direct sum of an abelian part, which offers no problems, and a non-abelian part, which has no one-dimensional representations. (Not even non-normal ones!).

We can use Theorem 4.4 to compute the universal C^*-algebras of some particular JC-algebras. For example, $C^*(M_n(\mathbb{R})_s) = M_n(C)$, and $C^*(M_n(\mathbb{H})_{sa}) = M_{2n}(C)$ if $n \geq 3$. The analoguous results hold equally well in infinite dimension, whether we work with all bounded or only all compact operators.

It should also be noted here that $C^*(V_n)$ may be computed explicitly for $2 \leq n < \infty$, and we get $C^*(V_{2n}) = M_{2^n}(C)$, $C^*(V_{2n+1}) = M_{2^n}(C) \oplus M_{2^n}(C)$. See [8; p. 276]. Note that C^* preserves direct limits. It then follows that the universal C^*-algebra of the infinite dimensional separable spin factor is the infinite tensor product $\otimes M_2(C)$, a well-known fact which motivates the name "CAR-algebra".

5. The universal tensor product of JC-algebras.

What should be expected of a "good" tensor product of two JC-algebras A and B? Assuming that A and B have units (we shall keep this assumption throughout), the tensor product C should contain copies of A and of B such that any element a of A and any element b of B operator commute.

We would like to identify C with the closure of $A \otimes B$ in some norm. But we shall see below that this is rarely possible.

We define the _universal tensorproduct_ of JC-algebras A, B to be the JC-subalgebra of $C^*(A) \otimes_{max} C^*(B)$ generated by the subspace $A \otimes B$. We denote this JC-algebra $A \tilde{\otimes} B$. Before proceeding, we need a lemma on operator commutativity.

Lemma 5.1. _Let A be a JC-subalgebra of a C^*-algebra $\mathcal{O}l$. Two elements in A operator communte in A iff they commute in $\mathcal{O}l$._

Proof. The "if" part is evident. To prove the converse, assume a and b operator commute in A. We may assume A generates $\mathcal{O}l$ in what follows. For any $c \in A$ we have $4[T_a, T_b](c) = [[a,b],c]$. Hence $[a,b]$ commutes with every $c \in A$, and therefore $[a,b]$ belongs to the center of $\mathcal{O}l$. For any irreducible representation π of $\mathcal{O}l$, $[\pi(a), \pi(b)]$ must be a scalar multiple of the identity. It is well known that then $[\pi(a), \pi(b)] = 0$, so $[a,b] = 0$. $\square$

$A \tilde{\otimes} B$ is characterized by a universal property, and its universal C^*-algebra is identified in the next result.

Proposition 5.2. _If A,B, and C are unital JC-algebras, and $\varphi: A \to C$ and $\psi: B \to C$ are unital homomorphisms with $\varphi(a)$ operator commuting with $\psi(b)$ for all $a \in A, b \in B$, there is a unique homomorphism χ of $A \tilde{\otimes} B$ into C such that $\chi(a \otimes b) = \varphi(a) \cdot \psi(b)$. Moreover, the universal C^*-algebra of $A \tilde{\otimes} B$ is $C^*(A) \otimes_{max} C^*(B)$._

Proof. We may assume $C \subseteq \mathcal{E}_{sa}$, for a C^*-algebra $\mathcal{E}$.

Consider $\hat{\varphi}$: $C^*(A) \to \mathcal{E}$ and $\hat{\psi}$: $C^*(A) \to \mathcal{E}$. By Lemma 5.1, $\hat{\varphi}$ and $\hat{\psi}$ have commuting ranges, so there exists a *-homomorphism $\hat{\chi}$ of $C^*(A) \underset{max}{\otimes} C^*(B)$ into $\mathcal{E}$ such that $\hat{\chi}(a \otimes b) = \hat{\varphi}(a) \hat{\psi}(b)$. Let χ be the restriction of $\hat{\chi}$ to $A \widetilde{\otimes} B$. Clearly, $\chi(A \widetilde{\otimes} B) \subseteq C$, and $\chi(a \otimes b) = \varphi(a) \circ \psi(b)$.

To prove the final statement, let χ: $A \widetilde{\otimes} B \to \mathcal{E}_a$ be a homomorphism, for any C^*-algebra $\mathcal{E}$. Let $\varphi(a) = \chi(a \otimes 1)$, $\psi(b) = \chi(1 \otimes b)$. Repeat the above discussions to get an extension of χ to a *-homomorphism $\hat{\chi}$ on $C^*(A) \underset{max}{\otimes} C^*(B)$. Clearly, $C^*(A) \underset{max}{\otimes} C^*(B)$ is generated by $A \otimes B$, and therefore by $A \widetilde{\otimes} B$. Then it must equal $C^*(A \widetilde{\otimes} B)$. $\square$

The following result, when combined with Theorem 4.4, enables one to compute $A \widetilde{\otimes} B$ in many cases. Indeed, if Φ_A is the anti-automorphism of $C^*(A)$ leaving A pointwise fixed, $\Phi_A \otimes \Phi_B$ is an antiautomorphism of $C^*(A) \underset{max}{\otimes} C^*(B)$ leaving $A \widetilde{\otimes} B$ pointwise fixed. Therefore, if $A \underset{max}{\otimes} B$ is universally reversible, it is exactly the self-adjoint fixed points of $\Phi_A \otimes \Phi_B$, the tensor product of the canonical antiautomorphisms associated with A and B.

Proposition 5.3. $A \widetilde{\otimes} B$ is universally reversible unless one of A, B has a scalar representation and the other has a representation onto a spin factor V_n, where $n \geq 4$.

Proof. Assume χ: $A \widetilde{\otimes} B \to V_n$ is a representation onto V_n. Then $\chi(A \otimes 1)$ and $\chi(1 \otimes B)$ are operator commuting subalgebras of V_n, whose union generates V_n. Then one subalgebra must consist of the scalars only, and the other equals V_n. Indeed, if $s \in V_n$ is a non-trivial symmetry (i.e. a unit vector of H_n) then

the only elements of V_n operator commuting with s are the linear combinations of s and 1. Theorem 2.3 completes the proof. ☐

As an application of the above Proposition and the remarks preceding it, we here show how to compute a few universal tensor products. For instance, if V_∞ is the infinite dimensional separable spin factor, $V_\infty \tilde\otimes V_\infty$ is isomorphic to the simple JC-algebra considered in § 3. For $C^*(V_\infty)$ is just the CAR algebra, and Φ_{V_∞} is the transposition on the CAR algebra.

An antiautomorphism α of $\mathcal{B}(H)$ of order 2 is induced by an antiunitary j with $j^2 = \pm 1$. If $j^2 = +1$, α is called real, while if $j^2 = -1$, α is called quaternionic. Now the behaviour of tensor product of antiautomorphisms on $\mathcal{B}(H)$ can be summarized in: real ⊗ real = real, real ⊗ quaternionic = quaternionic, quaternionic ⊗ quaternionic = real. As a result, we get the following table for $\tilde\otimes$:

$\tilde\otimes$	$M_n(\mathbb{R})_s$	$M_n(\mathbb{H})_{sa}, n \geq 3$
$M_m(\mathbb{R})_s$	$M_{mn}(\mathbb{R})_s$	$M_{mn}(\mathbb{H})_{sa}$
$M_m(\mathbb{H})_{sa}, n \geq 3$		$M_{4mn}(\mathbb{R})_{sa}$.

The computation of universal tensor products involving non-reversible spin factors requires more care: The canonical anti-automorphisms of their universal C^*-algebras must be analyzed. Their behavior turns out to be cyclic, depending on n mod 8. We just state here, without proof, that the universal tensor product of $V_5 = M_2(\mathbb{H})_{sa}$ with itself is a direct sum of four copies of $M_{16}(\mathbb{R})_s$.

<u>Lemma 5.4.</u> If $\mathcal{O}$ is a C^*-algebra with no scalar represen-tations and B is a JC-algebra then $\mathcal{O}_{sa} \tilde{\otimes} B \cong (\mathcal{O} \underset{max}{\otimes} C^*(B))_{sa}$.

<u>Proof.</u> We have $C^*(\mathcal{O}_{sa}) = \mathcal{O} \oplus \mathcal{O}^O$, with the antiautomor-phism $\Phi_{\mathcal{O}_{sa}}(a \oplus b^O) = (b \oplus a^O)$. Hence $C^*(\mathcal{O}_{sa} \tilde{\otimes} B) = \mathcal{O} \otimes C^*(B) \oplus \mathcal{O}^O \otimes C^*(B)$, where the canonical antiautomorphism interchanges the two summands. Hence the self-adjoint fixed point algebra is isomorphic to $(\mathcal{O} \underset{max}{\otimes} C^*(B))_{sa}$. $\qquad \square$

We conclude with a result showing that a more naive approach to defining tensor products of JC-algebras is bound to fail.

<u>Theorem 5.5.</u> Assume A is a unital JC-algebra and that $A \otimes M_2(C)_{sa}$ is a JC-algebra with some product satisfying

(i) $(a \otimes 1)^2 = a^2 \otimes 1$

(ii) $(1 \otimes a)^2 = 1 \otimes a^2$

(iii) $(a \otimes 1) \circ (1 \otimes a) = a \otimes a$

(iv) $[T_{1 \otimes a}, T_{a \otimes 1}] = 0$.

Then A is isomorphic to the self-adjoint part of a C^*-algebra.

<u>Proof.</u> By Lemma 5.4, $A \tilde{\otimes} M_2(C)_{sa}$ is the self-adjoint part of a C^*-algebra. By Proposition 5.2, $A \otimes M_2(C)_{sa}$ is a quotient of $A \tilde{\otimes} M_2(C)_{sa}$ and therefore also the self-adjoint part of a C^*-algebra. (Note that by Propositions 3.2 and 3.3, a Jordan ideal in the self-adjoint part of a C^*-algebra is the self-adjoint part of a two-sided ideal).

By (ii), (iii) and (iv) we get $U_{1 \otimes \beta}(a \otimes a) = U_{1 \otimes \beta} T_{a \otimes 1}(1 \otimes a) = T_{a \otimes 1} U_{1 \otimes \beta}(1 \otimes a) = T_{a \otimes 1}(1 \otimes \{\beta a \beta\}) = a \otimes \{\beta a \beta\}$.

If β is a one-dimensional projection in $M_2(C)$, then $\beta a \beta$ is a scalar, so $U_{1 \otimes \beta}(A \otimes M_2(C)_{sa}) = A \otimes \beta$. By (i) and (ii) $a \rightarrow a \otimes \beta$ is a Jordan homomorphism, so $A \cong U_{1 \otimes \beta}(A \otimes M_2(C)_{sa})$. But cutting down the self-adjoint part of a C^*-algebra with a projection $(1 \otimes \beta)$ we get the self-adjoint part of a C^*-algebra, so the proof is complete. $\square$

References

1. E.M. Alfsen, F.W. Shultz, E. Størmer, A Gelfand-Neumark theorem for Jordan algebras, Adv. Math. 28 (1978), 11-56.

2. E.M. Alfsen, H. Hanche-Olsen, F.W. Shultz, State spaces of C^*-algebras, Acta Math. 144 (1980), 267-305.

3. C.M. Edwards, Ideal theory in JB-algebras, J. London Math. Soc. (2) 16 (1977), 507-513.

4. E. Effros, E. Størmer, Jordan algebras of self-adjoint operators, Trans. Amer. Math. Soc. 127 (1967), 313-316.

5. T. Giordano, Antiautomorphismes involutifs des facteurs de von Neumann injectifs, thesis (1981).

6. H. Hanche-Olsen, A note on the bidual of a JB-algebra, Math. Zeitschr. 175 (1980), 29-31.

7. H. Hanche-Olsen, Split faces and ideal structure of operator algebras, Math. Scand. 48 (1981), 137-144.

8. A.A. Kirillov, "Elements of the theory of representations", Springer-Verlag Berlin Heidelberg New York (1976).

9. A.G. Robertson, Automorphisms of spin factors and decomposition of positive maps, preprint (1981).

10. F.W. Shultz, <u>On normed Jordan algebras which are Banach</u> dual spaces, J. Funct. Anal. <u>31</u> (1979), 360-376.

11. E. Størmer, On the Jordan structure of C^*-algebras, Trans. Amer. Math. Soc. <u>120</u> (1965), 438-447.

12. E. Størmer, <u>Jordan algebras of type I</u>, Acta Math. <u>115</u> (1966), 165-184.

13. E. Størmer, <u>Irreducible Jordan algebras of self-adjoint operators</u>, Trans. Amer. Math. Soc. <u>130</u> (1968), 153-166.

14. E. Størmer, <u>Real structure in the hyperfinite factor</u>, Duke Math. J. <u>47</u> (1980), 145-154.

15. D. Topping, <u>Jordan algebras of self-adjoint operators</u>, Mem. Amer. Math. Soc. <u>53</u> (1965).

16. D. Topping, <u>An isomorphism invariant for spin factors</u>, J. Math. Mech. <u>15</u> (1966), 1055-1064.